Various Ways of Splitting Fibrations and Equality of Objects

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Fibrations and Semantics of Type Theory

Contexts and substitutions are modelled by a category $\mathbb{C}$ which typically has finite limits.

Types and maps between them are modelled by a *Grothendieck fibration* $P : \mathbb{X} \to \mathbb{C}$.

For modelling **judgemental equality of types** one assumes P to be a **split** fibration.

NB Only superficially this contradicts the principles of HoTT which identifies **propositional** equality of types with **isomorphism / weak equivalence** because only **judgemental equality of types** will be interpreted as good old **equality of objects**.

Left and Right Adjoint Splitting of Fibrations

Let $\mathbf{Fib}(\mathbb{C})$ be the 2-category of fibrations over $\mathbb{C}$ and cartesian functors and $\mathbf{Sp}(\mathbb{C})$ the 2-category of split fibrations and split cartesian functors.
We write $U : \mathbf{Sp}(\mathbb{C}) \to \mathbf{Fib}(\mathbb{C})$ for the obvious 2-functor forgetting the splittings.

Already in J. Giraud's 1971 book *Cohomologie non-abelienne* (reporting on work from the 1960s) one finds left and right 2-adjoints of U which we denote by L and R, respectively.

Unfortunately, it is not so easy to decode Giraud's writings and thus it may be useful to present his constructions in a more modern and unbureaucratic manner.

The right adjoint splitting 1

was popularized by J. Bénabou (1980) and used by M. Hofmann (1994) for splitting models of type theory.

It is based on the following observation: for a fibration $P : \mathbb{X} \to \mathbb{C}$ the fiber $P(I)$ is equivalent to $\mathbf{Fib}(\mathbb{C})(\underline{I}, P)$ where $\underline{I} = \partial_0 : \mathbb{C}/I \to \mathbb{C}$ is the discrete fibration corresponding to $y(I)$.

The right adjoint R of $U : \mathbf{Sp}(\mathbb{C}) \to \mathbf{Fib}(\mathbb{C})$ is given by

$$R(P)(I) = \mathbf{Fib}(\mathbb{C})(\underline{I}, P) \qquad R(P)(u) = \mathbf{Fib}(\mathbb{C})(\underline{u}, P)$$

where for $u : J \to I$ in $\mathbb{C}$ the functor $\underline{u} : \underline{J} \to \underline{I}$ is given by postcomposition with u usually denoted as $\Sigma_u : \mathbb{C}/J \to \mathbb{C}/I$.

Notice that $R(P) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a presheaf of categories, i.e. a category internal to $\mathbf{SET}^{\mathbb{C}^{\mathrm{op}}}$, which is identified with the associated split fibration.

The right adjoint splitting 2

The counit $E_P : U(R(P)) \to P$ in $\mathbf{Fib}(\mathbb{C})$ maps $F \in \mathbf{Fib}(\mathbb{C})(\underline{I}, P)$ to $F_I(\mathrm{id}_I) \in P(I)$.

Obviously, the cartesian functor E_P is full and faithful. Thus E_p is a weak equivalence iff it is essentially surjective.
Elements $F \in \mathbf{Fib}(\mathbb{C})(\underline{I}, P)$ correspond to choices $C(u, X) : u^*X \to X$ of cartesian morphisms over u with $C(\mathrm{id}_I, X) = \mathrm{id}_X$ where $X = F_I(\mathrm{id}_I)$.

Using **global choice**, i.e. AC for classes, one can show that E_P is essentially surjective: for every $X \in P(I)$ and $u : J \to I$ choose a cartesian morphism $C(u, X) : u^*X \to X$ over u. Using LEM one can arrange the choice in such a way that $C(\mathrm{id}_I, X) = \mathrm{id}_X$.

The right adjoint splitting 3

As shown by Bénabou and described in my notes on Fibred Categories the 2-category **Fib**($\mathbb{C}$) can be obtained from **Sp**($\mathbb{C}$) by **freely inverting** all split cartesian functors F which are **weak equivalences** in the sense that all fibres F_I are (weak) equivalences in the 1-categorical sense that they are full and faithful and essentially surjective.

Notice that weak equivalences in **Sp**($\mathbb{C}$) do in general not have split quasi-inverses (since in $\mathbf{Set}^{\mathbb{C}^{op}}$ not all epis are split!).

Thus, the step from **Sp**($\mathbb{C}$) to **Fib**($\mathbb{C}$) may be understood as identifying *weak* equivalences with *strong* equivalences which is reminiscent of Voevodsky's Univalence Axiom.

The left adjoint splitting 1

For $F : \mathbb{B} \to \mathbb{C}$ the change of base functor $F^* : \mathbf{Fib}(\mathbb{C}) \to \mathbf{Fib}(\mathbb{B})$ has left and right adjoints $\coprod_F$ and $\prod_F$, respectively. As described in my notes on fibrations canonical choices of $\coprod_F$ and $\prod_F$ actually produce split fibrations. Moreover, the right adjoint splitting is nothing but $\prod_{\mathrm{Id}_{\mathbb{C}}}$.

Accordingly, left adjoint splitting is given by $\coprod_{\mathrm{Id}_{\mathbb{C}}}$. For a fibration $P : \mathbb{X} \to \mathbb{C}$ the split fibration $\coprod_{\mathrm{Id}_{\mathbb{C}}} P : \widetilde{\mathbb{X}} \to \mathbb{C}$ is constructed as follows:

The left adjoint splitting 2

objects of $\widetilde{\mathbb{X}}$ over $I \in \mathbb{C}$ are pairs (u, X) where $X \in \mathbb{X}$ and $u : I \to P(X)$

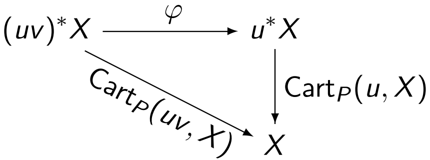
morphisms from (u, X) to (v, Y) over $w : I \to J$ in $\mathbb{C}$ are equivalence classes of spans (φ, f) in $\mathbb{X}$ where $\varphi : Z \to X$ is cartesian over u and $f : Z \to Y$ is over vw w.r.t. P and we identify (φ, f) with (φ', f') iff $f' \circ \iota = f$ for the unique vertical isomorphism ι with $\varphi' \circ \iota = \varphi$

definition of composition left as exercise

split cartesian arrows are equivalence classes of spans of the form (φ, φ)

The left adjoint splitting 3

If we invest into the (global) choice of a normalized cleavage Cart_P for P we can avoid defining morphisms of $\widetilde{\mathbb{X}}$ as equivalence classes. Instead we define $\widetilde{\mathbb{X}}\big((u,X),(v,Y)\big) = \mathbb{X}(u^*X, v^*Y)$ where $\mathrm{Cart}_P(u,X) : u^*X \to X$. Composition in $\widetilde{\mathbb{X}}$ is inherited from $\mathbb{X}$.
The fibration $L(P) : \widetilde{\mathbb{X}} \to \mathbb{C}$ sends $f : (u,X) \to (v,Y)$ to $P(f)$.
A split cleavage $\mathrm{Cart}_{L(P)}$ for $L(P)$ is given by
$\mathrm{Cart}_{L(P)}\big(v,(u,P)\big) = \varphi : (uv,X) \to (u,X)$ where
$\varphi : (uv)^*X \to u^*X$ is the unique cartesian morphism over v making



commute.

The left adjoint splitting 4

The unit of $L \dashv U$ at P is the cartesian functor $H_P : P \to U(L(P))$ sending X over I to (id_I, X) and $f : X \to Y$ to $f : (\mathrm{id}_{P(X)}, X) \to (\mathrm{id}_{P(Y)}, Y)$ which, obviously is a weak equivalence.

The left adjoint splitting of fundamental fibrations

For the particular case of (a small subfibration of) the fundamental fibration $P_{\mathbb{C}} = \mathrm{cod} : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ where $\mathbb{C}$ has pullbacks this has been described by Voevodsky in his *Notes on Type Systems*.

But in this case we can avoid choosing a cleavage for $P_{\mathbb{C}}$ by constructing a presheaf of categories $L(P_{\mathbb{C}}) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ as follows.

The objects of $L(P_{\mathbb{C}})(I)$ are cospans (u, a) where $u : I \to I_0$ and $a : A \to I_0$. With every such (u, a) we associate $E(u, a) : (\mathbb{C}/I)^{\mathrm{op}} \to \mathbf{Set}$ where $E(u, a)(v) = \{f \mid af = uv\}$ and $E(u, a)(w : vw \to v)(f) = fw$.
Morphism from (u, a) to (u', a') are just natural transformations from $E(u, a)$ to $E(u', a')$.
For $v : J \to I$ the functor $L(P_{\mathbb{C}})(v)$ sends (u, a) to (uv, a) and $\tau : E(u, a) \to E(u', a')$ to $\tau_{(\Sigma_v)^{\mathrm{op}}}$.

Relation to Bénabou's $\mathbf{set}(\mathbb{C})$

Let $\mathbf{set}(\mathbb{C})$ be the presheaf of categories over $\mathbb{C}$ where $\mathbf{set}(\mathbb{C})(I)$ is the full subcategory of $\mathbf{Set}^{(\mathbb{C}/I)^{\mathrm{op}}}$ on representable fibrations. For $u : J \to I$ the functor $\mathbf{set}(\mathbb{C})(u)$ is given by change of base along $\Sigma_u : \mathbb{C}/J \to \mathbb{C}/I$. This preserves representability since $\mathbb{C}$ is assumed to have pullbacks.

The functor $E : L(P_{\mathbb{C}}) \to \mathbf{set}(\mathbb{C})$ is a weak split equivalence but hasn't got a split quasi-inverse.