The World’s Simplest Proof of Moens’ Lemma

Let $\mathcal{B}$ be a category with finite limits and $P : \mathcal{X} \to \mathcal{B}$ be a fibration of finite limit categories with internal sums. For $I \in \mathcal{B}$ let $\varphi_I : 1_I \to \Delta(I)$ be a cocartesian arrow from $1_I$ (terminal object in the fibre $\mathcal{X}_I$) over the terminal projection $I \to 1$ in $\mathcal{B}$. For $u : J \to I$ in $\mathcal{B}$ let $\Delta(u) : \Delta(J) \to \Delta(I)$ be the unique vertical arrow making the diagram

\[
\begin{array}{ccc}
1_J & \xrightarrow{\varphi_J} & \Delta(J) \\
\downarrow 1_u & (1) & \downarrow \Delta(u) \\
1_I & \xrightarrow{\varphi_I} & \Delta(I)
\end{array}
\]

commute. This gives rise to a functor $\Delta : \mathcal{B} \to \mathcal{X}_1$.

A fibration $P$ as above is called extensive iff every commuting diagram with $\alpha$ and $\beta$ vertical

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\varphi} & U \\
\downarrow \alpha & & \downarrow \beta \\
1_I & \xrightarrow{\varphi_I} & \Delta(I)
\end{array}
\]

is a pullback iff $\varphi$ is cocartesian. Obviously, if $P$ is extensive then the functor $\varphi_I^* : \mathcal{X}_1/\Delta(I) \to \mathcal{X}_I/1_I \cong \mathcal{X}_I$ is an equivalence and due to (1) the diagram

\[
\begin{array}{ccc}
\mathcal{X}_J & \xleftarrow{\varphi_J^*} & \mathcal{X}_1/\Delta(J) \\
\downarrow u^* & & \downarrow (\Delta(u))^* \\
\mathcal{X}_I & \xleftarrow{\varphi_I^*} & \mathcal{X}_1/\Delta(I)
\end{array}
\]

commutes up to isomorphism for all $u : J \to I$ in $\mathcal{B}$ and thus $P \cong \Delta^* P_{\mathcal{X}_1}$ (where $P_{\mathcal{X}_1} = \partial_1 : \mathcal{X}_2^1 \to \mathcal{X}_1$, the fundamental fibration of $\mathcal{X}_1$).

Finally we show that $\Delta : \mathcal{B} \to \mathcal{X}_1$ preserves finite limits. Obviously $\Delta$ preserves 1. Recall that a functor $F : \mathcal{B} \to \mathcal{C}$ between categories with pullbacks preserves pullbacks iff $F^* P_{\mathcal{C}}$ has internal sums. Thus, since $P \cong \Delta^* P_{\mathcal{X}_1}$ has internal sums it follows that $\Delta$ preserves pullbacks. Since $\Delta$ preserves 1 and pullbacks it follows that $\Delta$ preserves finite limits.

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1 see Lemma 13.2 on pp.45-46 of my notes on Fibred Categories