

The World's Simplest Proof of Moens' Lemma

Let \mathbb{B} be a category with finite limits and $P : \mathbb{X} \rightarrow \mathbb{B}$ be a fibration of finite limit categories with internal sums. For $I \in \mathbb{B}$ let $\varphi_I : 1_I \rightarrow \Delta(I)$ be a cocartesian arrow from 1_I (terminal object in the fibre \mathbb{X}_I) over the terminal projection $I \rightarrow 1$ in \mathbb{B} . For $u : J \rightarrow I$ in \mathbb{B} let $\Delta(u) : \Delta(J) \rightarrow \Delta(I)$ be the unique vertical arrow making the diagram

$$\begin{array}{ccc} 1_J & \xrightarrow{\varphi_J} & \Delta(J) \\ \downarrow 1_u & (1) & \downarrow \Delta(u) \\ 1_I & \xrightarrow{\varphi_I} & \Delta(I) \end{array}$$

commute. This gives rise to a functor $\Delta : \mathbb{B} \rightarrow \mathbb{X}_1$.

A fibration P as above is called *internally extensive* iff every commuting diagram with α and β vertical

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & U \\ \downarrow \alpha & & \downarrow \beta \\ 1_I & \xrightarrow{\varphi_I} & \Delta(I) \end{array}$$

is a pullback iff φ is cocartesian. Obviously, if P is internally extensive then the functor $\varphi_I^* : \mathbb{X}_1/\Delta(I) \rightarrow \mathbb{X}_I/1_I \cong \mathbb{X}_I$ is an equivalence and due to (1) the diagram

$$\begin{array}{ccc} \mathbb{X}_J & \xleftarrow[\cong]{\varphi_J^*} & \mathbb{X}_1/\Delta(J) \\ \uparrow u^* & & \uparrow \Delta(u)^* \\ \mathbb{X}_I & \xleftarrow[\varphi_I^*]{\cong} & \mathbb{X}_1/\Delta(I) \end{array}$$

commutes up to isomorphism for all $u : J \rightarrow I$ in \mathbb{B} and thus $P \simeq P_\Delta = \Delta^* P_{\mathbb{X}_1}$ (where $P_{\mathbb{X}_1} = \partial_1 : \mathbb{X}_1^2 \rightarrow \mathbb{X}_1$, the fundamental fibration of \mathbb{X}_1).

Finally we show that $\Delta : \mathbb{B} \rightarrow \mathbb{X}_1$ preserves finite limits. Obviously Δ preserves 1. Recall that a functor $F : \mathbb{B} \rightarrow \mathbb{C}$ between categories with pullbacks preserves pullbacks iff $F^* P_{\mathbb{C}}$ has internal sums.¹ Thus, since $P \simeq \Delta^* P_{\mathbb{X}_1}$ has internal sums it follows that Δ preserves pullbacks. Since Δ preserves 1 and pullbacks it follows that Δ preserves finite limits.

¹see Lemma 13.2 on pp.45-46 of my notes on *Fibred Categories*