Some remarks on the Chambery topos and its effective variants

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The Chambery topos $\mathcal{K}$ is boolean, i.e. $\Omega \cong 2 = 1 + 1$. Therefore, we may define a canonical map $i_2 : \Omega \rightarrowtail \Delta(2)$ as $[0, 1]$, i.e. the source tupling of the global points $0, 1 : 1 \to \Delta(2)$, and one easily sees that i_2 is monic with characteristic predicate $(i = 0) \vee (i = 1)$. But i_2 is not an iso since $\mathcal{K}$ validates the proposition that $\Delta(2)$ is infinite. But we will show that

Theorem 1 *The map $\mathcal{K}(1, i_2) : \mathcal{K}(1, \Omega) \to \mathcal{K}(1, \Delta(2))$ is a bijection.*

telling us that the existence property heavily fails for $\mathcal{K}$ because the valid proposition that $\Delta(2)$ contains elements different from 0 and 1 is not witnessed by a global element.

Since $\mathcal{K}$ is a subtopos of $\mathcal{E} = \mathbf{RT}(D, \mathsf{PL})$ and in $\mathcal{E}$ the nno $N_\mathcal{E}$ is the countable coproduct of 1 the nno $N_\mathcal{K}$ in $\mathcal{K}$ is also the countable coproduct of 1 since the sheafification functor $\mathcal{E} \to \mathcal{K}$ preserves colimits and finite limits. We write $n_\mathcal{E}$ and $n_\mathcal{K}$ for the n-fold copower of 1 in $\mathcal{E}$ and $\mathcal{K}$, respectively.

From these considerations it follows that

Lemma 1
If f is a morphism from $N_\mathcal{K}$ or $n_\mathcal{K}$ to $\Delta(2)$ then f factors through i_2.

Proof: Let $i_k : 1 \to N_\mathcal{K}$ be the inclusion of the k-th summand. Thus f factors through i_2 iff all $f \circ i_k$ factor through i_2. But by Th.1 all morphisms $1 \to \Delta(2)$ factor through i_2. □

Theorem 2 *For $n > 2$ there is no monomorphism $n_\mathcal{K} \rightarrowtail \Delta(2)$ and thus no monomorphism $N_\mathcal{K} \rightarrowtail \Delta(2)$.*

Proof: If $m : n_\mathcal{K} \rightarrowtail \Delta(2)$ then $m = i_2 \circ f$ for some $f : n_\mathcal{K} \to \Omega$ which is also monic. Thus $\mathcal{K}$ validates the proposition that f is 1-1. But for $n > 2$ it holds in the internal logic of $\mathcal{K}$ that f is not 1-1. Contradiction!

If there were a mono $N_{\mathcal{K}} \rightarrowtail \Delta(2)$ then there also were a mono $3_{\mathcal{K}} \rightarrowtail \Delta(2)$ because $3_{\mathcal{K}} \rightarrowtail N_{\mathcal{K}}$ contradicting the observation above. □

It has been shown that $\mathcal{K}$ validates the propositions that $\Delta(2)$ has infinitely many elements and that there even exists a 1-1-map from $N_{\mathcal{K}}$ to $\Delta(2)$. But these existential statements hold only in the sense of the internal logic of $\mathcal{K}$ and are not witnessed by global elements which forecefully illustrates Miquel's warning that one should never trust a realizer of an existential statement!

Notice that these results are independent from the answer to the open question whether $\Omega_{\mathcal{K}}$ is 2-valued. But Theorem 1 says that Ω has as many global elements as $\Delta(2)$. Thus, if Ω and thus $\Delta(2)$ has n different global elements $a_1, \ldots, a_n$ then by source tupling we obtain a morphism $n_{\mathcal{K}} \to \Delta(2)$ which, however, cannot be a monomophism if $n > 2$. Similarly, if $\Omega_{\mathcal{K}}$ and thus $\Delta(2)$ has $\aleph_0$ many different global elements then by source tupling we obtain a map $f : N_{\mathcal{K}} \to \Delta(2)$ which, however, cannot be a monomorphism. Notice that this hypothetical map $f : N_{\mathcal{K}} \to \Delta(2)$ will not be monic although $\Gamma(f) = \mathcal{K}(1, f) : \mathcal{K}(1, N_{\mathcal{K}}) \to \mathcal{K}(1, \Delta(2))$ is one-to-one.

1 Proof of Theorem 1

Both Ω and $\Delta(2)$ have the same underlying set $2 = \{0, 1\}$. For distinct $i, j \in 2$ we have $eq_\Omega(i, j) = \{\top\} = eq_{\Delta(2)}(i, j)$ but for $i \in 2$ we have $eq_\Omega(i, i) = \uparrow\{\top, \underline{i}\}$ whereas $eq_{\Delta(2)}(i, i) = \uparrow\{\top, \underline{0}\}$.

A partial element of Ω is given by a predicate $P : 2 \to \Omega$ such that $\bigcap\limits_{i,j \in 2} P(i) \to P(j) \to eq_\Omega(i, j)$ contains a proof-like element whereas a partial element of $\Delta(2)$ is given by a predicate $P : 2 \to \Omega$ such that $\bigcap\limits_{i,j \in 2} P(i) \to P(j) \to eq_{\Delta(2)}(i, j)$ contains a proof-like element.

We first will show that partial elements of $\Delta(2)$ are in 1-1-correspondence with partial elements of Ω via pullback along $i_2 : \Omega \rightarrowtail \Delta(2)$. Clearly, pullback along i_2 exhibits $\mathcal{K}(1, \widetilde{\Omega})$ as a retract of $\mathcal{K}(1, \widetilde{\Delta(2)})$ since every partial element of Ω may be considered as a partial element of $\Delta(2)$ and $i_2^* \circ \Sigma_{i_2}$ is isomorphic to identity.[1] Thus, it suffices to show that i_2^* reflects equality of partial elements.

Lemma 2 *If P and Q are subsingleton predicates on $\Delta(2)$ then P and Q are equal whenever $P \circ i_2$ and $Q \circ i_2$ are equal.*

Proof: A partial element of $\Delta(2)$ is given by a map $P : 2 \to \Omega$ such that

$$P(i) \to P(j) \to eq_{\Delta(2)}(i, j)$$

is realized uniformly in $i, j \in 2$ by some proof-like $t_P \in D$, i.e. for $t \in P(i)$ and $t' \in P(j)$ it holds that $t_P t t' \in eq_{\Delta(2)}(i, j) = \uparrow(\{\top\} \cup \{\underline{0} \mid i = j\})$.

[1] As usual we write $\widetilde{A}$ for the partial map classifier for A.

Obviously, w.l.o.g. it suffices to consider realizers t_P with $t_P tt' \in \{\top, \underline{0}, \bot\}$ for all $t, t' \in D$ and we may assume t_P to be commutative[2]. Notice, moreover, that $\delta_P = \lambda x. t_P xx$ is proof-like, too.

Let A be the set of minimal elements x with $\delta_P(x) \neq \bot$. Notice that $\{[x, t_P(x)] \mid x \in A\}$ is the trace of δ_P and, thus, by stability of δ_P the set A is an antichain, i.e. coherent elements of A are equal. If $e \in A$ and $x, y \sqsupseteq e$ with $x \in P(0)$ and $y \in P(1)$ then $\delta_P(e) = \top$ since $\delta_P(e) = t_P(e, e) \sqsubseteq t_P(x, y) = \top$ and thus $e \notin \mathsf{PL}$ since δ_P is proof-like. Thus, by contraposition for $e \in A \cap \mathsf{PL}$ for all $x, y \in P(0) \cup P(1)$ with $x, y \sqsupseteq e$ we have $x \in P(0)$ iff $y \in P(0)$. Let $A \cap {\downarrow}(P(0) \cup P(1)) = A_0 \dot\cup A_1 \dot\cup A_2$ such that ${\uparrow}A_0 \cap P(1) = \emptyset = {\uparrow}A_1 \cap P(0)$ and $A_2 \subseteq {\downarrow}P(0) \cap {\downarrow}P(1)$. Then there exists a $d_P \in \mathsf{PL}$ sending all elements of A_0 to $\underline{0}$, all elements of A_1 to $\underline{1}$ and all elements of A_2 to $\top$.

Obviously, this d_P realizes $P(i) \to eq_\Omega(i, i)$ uniformly in i. Thus, we can find proof-like realizers for the equivalence of the predicates P and $\lambda i{:}2.\, P(i) \wedge eq_\Omega(i, i)$. The latter, obviously, represents the restriction of P along i_2.

Thus, if P and Q represent partial elements of $\Delta(2)$ whose restriction along i_2 is equal, i.e. $\lambda i{:}2.\, P(i) \wedge eq_\Omega(i, i)$ and $\lambda i{:}2.\, Q(i) \wedge eq_\Omega(i, i)$ are equivalent in $\mathcal{K}$, then also P and Q are equivalent in $\mathcal{K}$ as desired. □

Now we are ready to give the

Proof (of Theorem 1) :
In every boolean topos the partial map classifier $\widetilde{A}$ is given by $A + 1$. The subobject $\iota_1 : A \rightarrowtail A{+}1$ is classified by the map $c_A = [\mathit{true}_A, \mathit{false}] : A{+}1 \to \Omega$. Thus $a : 1 \to A + 1$ is a global element of A iff $c_A \circ a = \mathit{true}$.

Notice, moreover, that for $f : B \to A$ we have $c_A \circ (f + 1) = c_B$.

Lemma 2 says that every $a : 1 \to \Delta(2){+}1$ factors through the mono $i_2{+}1$ via a unique $a' : 1 \to \Omega + 1$. We have $c_\Omega \circ a' = c_{\Delta(2)} \circ (i_2 + 1) \circ a' = c_{\Delta(2)} \circ a = \mathit{true}$ for which reason a is a global element of $\Delta(2)$ iff a' is a global element of Ω. □

2 Some General Remarks

Boolean valued models can be characterized as those Grothendieck toposes where all epis split. One says that an elementary topos $\mathcal{E}$ validates the (internal) axiom of choice iff e^A is an epi whenever e is an epi. Already in Johnstone's 1977 book on *Topos Theory* one finds the theorem that in an elementary topos $\mathcal{E}$ every epi splits iff it validates the (internal) axiom of choice and *supports split*, i.e. every epi to a subterminal object has a section. This latter property is a natural generalization of the *existence property* claiming that every epi to a terminal object has a section.

Boolean Grothendieck toposes can be characterized as localic boolean toposes over the *Schanuel topos* $\mathcal{S}$ of all continuous actions of the topological group G_0

[2] since there is a bistrict stable binary map $\wedge$ on $\{\top, \underline{0}, \bot\}$ with $\underline{0} \wedge \underline{0} = \underline{0}$ and $\top = x \wedge y$ whenever $\{\top\} \subseteq \{x, y\} \subseteq \{\top, \underline{0}\}$

of permutations of $\mathbb{N}$ on sets.[3] Thus, boolean Grothendieck toposes are those equivalent to $\mathcal{S}(B)$ for some complete boolean algebra B in $\mathcal{S}$. In set theory these models are known as *symmetric boolean valued models* devised for the purpose of refuting the axiom of choice.[4] Thus, boolean Grothendieck toposes may be understood as the synthesis of the ideas of boolean valued models and permutation models (where $\mathcal{S}$ is the prototypical permutation model).

It is well known (see e.g. Jech's 1973 book on *Axiom of Choice*) that symmetric boolean valued models may refute countable choice since there may exist infinite sets S which are not Dedekind infinite, i.e. not admitting an injection of $\mathbb{N}$ into S.

It is unlikely that classical realizability models include symmetric boolean valued models in which case it were immediate that classical realizability does not in general validate countable choice.

Using bar recursion one can show that the Chambery topos validates dependent and thus countable choice. I hope that $\mathcal{K}_{\mathsf{eff}}$, the effective version of the Chambery topos, does not because it might validate the proposition that

$$\forall f : N_{\mathcal{K}} \to \Delta(2). \neg \forall n, m : N_{\mathcal{K}}.\ f(n) = f(m) \to n = m$$

i.e. that no $f : N_{\mathcal{K}} \to \Delta(2)$ is one-to-one.

3 A variant of $\mathcal{K}$ which is not 2-valued

The variant of $\mathcal{K}$ we have in mind differs only w.r.t. its notion of proof-like object, namely the restriction $\mathsf{PL}_{\mathsf{eff}}$ of PL to its effective elements. Obviously, the set $\mathsf{PL}_{\mathsf{eff}}$ is again closed under application and contains all constants. It also contains the least fixpoint operator and thus also the bar recursor.

For a partial function f from $\mathbb{N}$ to 2 let k_f be the element of D^ω whose n-th component is $\underline{f(n)}$ if $f(n)$ is defined and $\bot$ otherwise.

Let f now be a non-effective function from $\mathbb{N}$ to 2 and A_f be the proposition $\{k_f\}^{\perp\!\!\!\perp}$ which is disjoint from PL since $k_f \in \mathsf{PL}^\omega$. Thus, in particular, we have $A_f \cap \mathsf{PL}_{\mathsf{eff}} = \emptyset$. We will show that $\neg A_f \cap \mathsf{PL}_{\mathsf{eff}}$ is also empty establishing that the classical realizability model arising from $\mathsf{PL}_{\mathsf{eff}}$ is not 2-valued.

Notice that A_f contains a least element d_f whose trace is the set $T_f = \{[k_e, \top] \mid e \subseteq_{\mathsf{fin}} f\}$. Suppose there exists an effective proof-like $p : D \to \Sigma$ mapping all elements of T_f to $\top$, i.e. $\mathsf{tr}(p) \supseteq T_f \times \{\top\}$. Let $q \subseteq p$ with $\mathsf{tr}(q) = \mathsf{tr}(p) \cap \{[[k_e, \top], \top] \mid e$ is a finite partial function from $\mathbb{N}$ to $2\}$. This q is an effective, linear and proof-like element of $\neg A_f$, i.e. $\mathsf{tr}(q) \supseteq T_f \times \{\top\}$. If $[[e_1, \top], \top]$ and $[[e_2, \top], \top]$ are in $\mathsf{tr}(q)$ then $e_1 \supset\!\!\!\subset e_2$. Thus, for $[[e', \top], \top] \in \mathsf{tr}(q)$ we have

[3] The topology on G_0 is the subspace topology induced by $G_0 \subseteq \mathbb{N}^\mathbb{N}$ where $\mathbb{N}^\mathbb{N}$ carries the Baire topology.

[4] Symmetric boolean valued models may be understood from a categorical point of view as toposes of the form $\mathcal{C}(G)(B)$ where G is a topological group, $\mathcal{C}(G)$ is the topos of continuous actions of G on **Set** and B is a complete boolean algebra in $\mathcal{C}(G)$. For further discussion see Blass and Scedrov's book *Freyd's Models for the Independence of the Axiom of Choice* from 1989.

$e' \supset e$ for all $e \subseteq_{\mathsf{fin}} f$ from which it follows that $e' \subseteq f$. Thus, we have shown that $\mathsf{tr}(q) = T_f \times \{\top\}$ which is r.e. since q is effective. But then $\{e \mid e \subseteq_{\mathsf{fin}} f\}$ is r.e. and thus f recursive contradicting the assumption on f. Thus, we have shown that there is no effective element in $\neg A_f \cap \mathsf{PL}$, i.e. $\neg A_f \cap \mathsf{PL}_{\mathsf{eff}} \cap \emptyset$.

Let $f : \mathbb{N} \to 2$ and $\Delta(f) : \Delta(\mathbb{N}) \to \Delta(2)$. Let $i_{\mathbb{N}} : N \rightarrowtail \Delta(\mathbb{N})$ be the subobject whose characteristic predicate is the usual inductive definition of N. The map $\Delta(f) \circ i_N : N \to \Delta(2)$ factors through i_2, i.e. $\forall n^N.\,(\Delta(f)(n) = 0) \vee (\Delta(f)(n) = 1)$, iff f is computable. For example for $f : N \to 2$ with $f(n) = 1$ iff $\exists k.T(n,n,k)$ the predicates $eq_{\Delta(2)}(f(n),0)$ and $\forall k^N.\neg T(n,n,k)$ are non-equivalent predicates on N in $\mathbf{RT}(D,\mathsf{PL}_{\mathsf{eff}})$.

Notice that $\mathbf{RT}(D,\mathsf{PL}_{\mathsf{eff}})$ validates countable and dependent choice since D contains non-computable elements. It is not clear at all whether $\mathbf{RT}(D_{\mathsf{eff}},\mathsf{PL}_{\mathsf{eff}})$ validates countable or dependent choice since it may lack the non-computable “anti-proofs” needed for proving that bar recursion validates double negation shift.

4 In the Effective Chambery Topos it holds that there is an injective function from N to $\Delta(2)$

(Krivine August 2016)

First we show that

$$\exists \nu^{\Delta(\mathbb{N})}.\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$$

i.e.

$$\neg\forall \nu^{\Delta(\mathbb{N})}.\neg\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$$

holds in the effective Chambery topos which can be seen as follows.

For $\nu \in \mathbb{N}$ every term realizing $\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$ is an extension of the minimal term s_ν with $s_\nu\underline{\nu}\underline{0} = \top$ and $s_\nu\underline{n}\top = \top$ for all $n \in \mathbb{N}$. Suppose t realizes $\forall \nu^{\Delta(\mathbb{N})}.\neg\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$. Then t sends s_0 and s_1 to $\top$. Thus, since $s_0 \supset s_1$ and t is stable we have $\top = ts_0 \sqcap ts_1 = t(s_0 \sqcap s_1)$. Notice that $s_0 \sqcap s_1$ is the minimal term f with $f\underline{n}\top = \top$ for all $n \in \mathbb{N}$. Since f is prooflike the term $\lambda x.xf$ is prooflike, too, and it sends all realizers of $\forall \nu^{\Delta(\mathbb{N})}.\neg\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$ to $\top$, i.e. $\lambda x.xf$ is a prooflike term realizing $\neg\forall \nu^{\Delta(\mathbb{N})}.\neg\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$ as desired.

Now we reason internally to the effective Chambery topos. Let $\nu \in \Delta(\mathbb{N})$ such that $\forall n^N.\,\neg\Delta(eq)(\nu,n) = 0$. Then the map $f : N \to \Delta(2)$ with $f(n) = \Delta(eq)(\nu,n)$ is injective since all $f(n) \neq 0$ and $f(n) \wedge f(m) \leq_{\Delta(2)} 0$ for all $n \neq m$ in N.

But this does not mean that the effective Chambery topos validates countable or dependent choice. So there is still some hope but it might be difficult...

5 $\Delta(2)$ can be arbitrarily big

Let κ be some arbitrary cardinal. Then we may consider the recursive type $D_\kappa \cong \Sigma^{D_\kappa^\kappa}$ in **Coh**. This again gives rise to a model of untyped λ-calculus with control and by Pitts's theorem there exists a unique $P_\kappa \subseteq D_\kappa$ consisting of all $t \in D_\kappa$ sending all elements of P_κ^κ to $\top$.

For $\alpha < \kappa$ let $\underline{\alpha}$ be the unique element t of D_κ such that $t(\vec{t}) = \top$ iff $t_\alpha = \top$. Let K be the object of $\mathbf{RT}(D_\kappa, P_\kappa)$ with underlying set κ and $eq_K(\alpha, \beta) = \uparrow\big(\{\underline{\alpha} \mid \alpha = \beta\} \cup \{\top\}\big)$.

Just like in section 4 for $\kappa = \omega$ one can show that the classical realizability model induced by D_κ, P_κ and the pole $\{t * \vec{t} \mid t(\vec{t}) = \top\}$ validates the proposition that there is an injective map from K to $\Delta(2)$.

Thus, we cannot expect that $\Delta(2)$ can be embedded into a fixed set in the cumulative hierarchy as e.g. $\mathcal{P}(\mathbb{N})$. Generally, there arises the question whether classical realizability toposes are exhausted by their cumulative hierarchy, i.e. whether every $\Delta(I)$ embeds into a set in the cumulative herarchy.