Computational complexity theory for spaces of integrable functions

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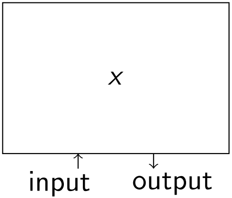
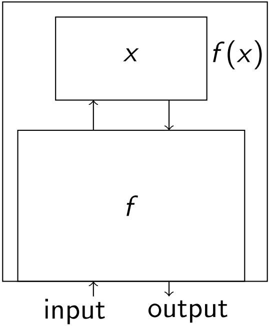
The model of computation

Representation: $\xi : {\Sigma^*}^{\Sigma^*} \to X$.
$\varphi \in \xi^{-1}(x)$: name of x.

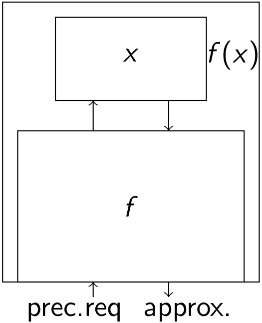




How to implement a continuous function

Consider $f \in C([0,1])$.

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$



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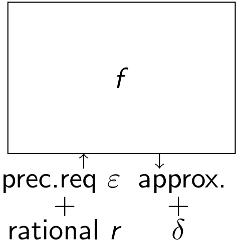


Computing on continuous functions

Theorem (Kawamura and Cook (2013))

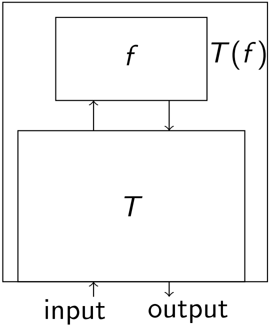
This is the least set about information about a function such that evaluation is polytime.

Theorem (folklore (2003); formally: Ziegler et al (2014))

The norm on $C([0,1])$ is exponential- but not polynomial-time computable.

Theorem ("")

Integration is exponential- but not polynomial-time computable.

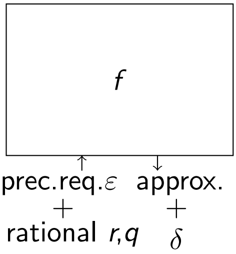


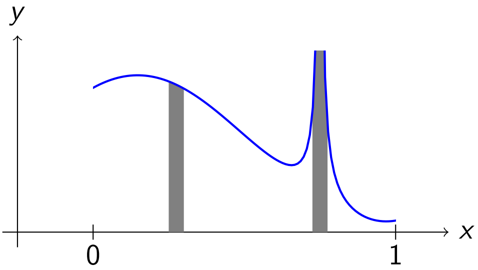
One dimension

Integrable functions:

- Not everywhere defined, no meaningful point evaluation.
- No modulus of continuity.

$$|h| \leq \delta \Rightarrow \left| \int_x^{x+h} f d\lambda \right| < \varepsilon$$





approx. to $\int_r^q f(t)dt$.

Also: higher dimensions and more general Ω.

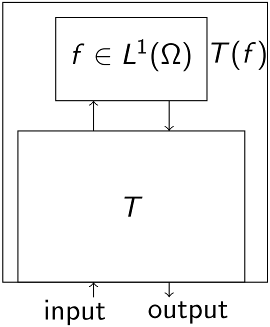
Results

Theorem

This is the least set about information about a function such that integration is polytime.

Theorem

Neither the metric nor the norm is computable. Not even continuous.



For one dimension there is a simpler description.

$L^p([0,1])$:

Functions $f : [0,1] \to \mathbb{R}$ such that

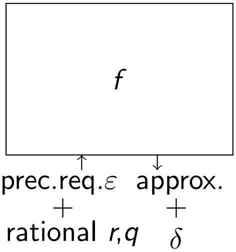
$$\|f\|_p := \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}} < \infty$$

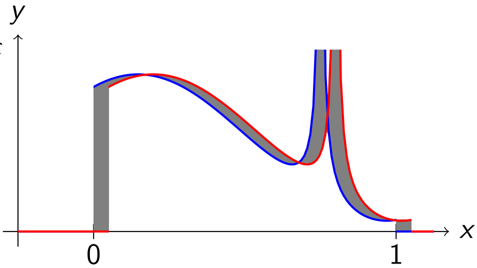
$\|f\|_p$ is a norm and $L^p([0,1])$ a Banach space.
(if we identify functions that coincide almost everywhere).

$L^\infty([0,1])$: bounded functions with supremum norm.

$$C([0,1]) \subsetneq L^\infty([0,1]) \subsetneq L^p([0,1]) \subsetneq L^1([0,1])$$

$$|h| \leq \delta \Rightarrow \|\tilde{f} - \tau_h \tilde{f}\|_{1p} < \varepsilon$$





L^p-modulus of f $\rightsquigarrow$ Frechet Kolmogorov for bounded sets.

approx. to $\int_r^q f(t)dt$.
Also: higher dimensions

Consider $f \in L^1([0,1])$
Consider $f \in L^p([0,1])$

Theorem

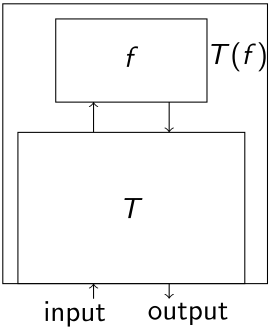
Integration is possible in polynomial time.

Theorem

The norm on L^p is exponential- but not polynomial-time computable.

Theorem

Not the smallest set of information to allow fast integration.

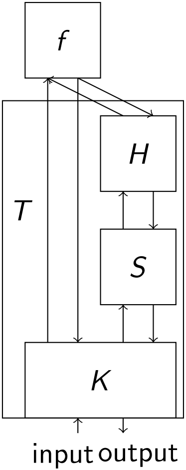


Weihrauch reductions

$$T \leq_W S. \quad T \leq_P S.$$

Weihrauch reductions:

- Compare operators.
- Only one application of S to compute T.
- H: Pre-processor.
- K: Post-processor.



Examples

Computing the norm on L^p is reducible to integration on the continuous functions.

Multiplication operators

Theorem

The mapping $L^1 \to L^1, f \mapsto x \cdot f$ is polytime Weihrauch equivalent to integration of a continuous function.

partial integration:

$$\int_a^b t \cdot f(t)dt = b\int_0^b f(t)dt - a\int_0^a f(t)dt - \int_a^b \int_0^t f(s)dsdt.$$

Remains true on L^p.

Theorem

The integration operator is polynomial-time Weihrauch reducible to the pairing of L^p with L^q.

Laplaces equation

Theorem (partially previously proven by Kawamura)

The following are polynomial-time Weihrauch equivalent:

- *Integration.*
- *Integrating a bivariate function in the second variable.*
- $\#\mathrm{EXIST}$.
- *Applying any of the above a countable number of times in parallel.*

Theorem

Integration is polynomial-time Weihrauch equivalent to the solution operator for the Dirichlet problem for Poisson's equation on the unit ball.

Thanks

Thanks!

References

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