The computational content of Zorn's lemma

Thomas Powell

University of Innsbruck

Proof, Computation, Complexity (PCC 2016)
special session dedicated to Ulrich Berger

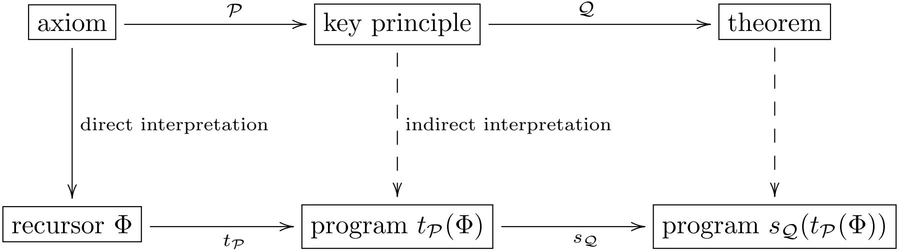
LMU Munich

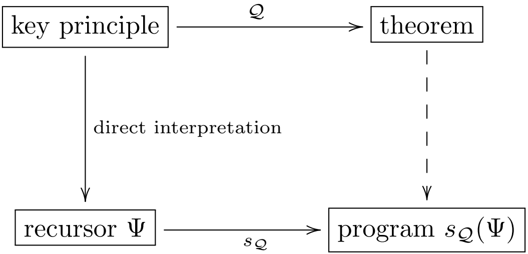
Germany

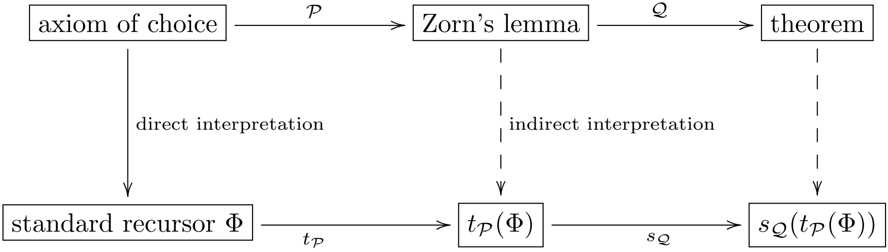
5 May 2016

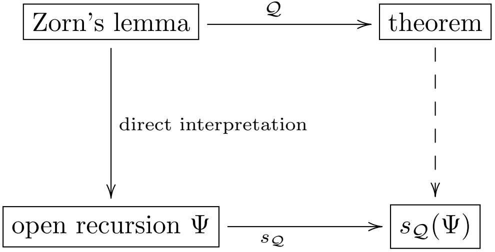
Outline

1. Past: The realizability interpretation of open induction
2. Present: A functional interpretation of open induction
3. Future: Open questions on open induction


axiom
$\mathcal{P}$
key principle
$\mathcal{Q}$
theorem
direct interpretation
indirect interpretation
recursor Φ
$t_{\mathcal{P}}$
program $t_{\mathcal{P}}(\Phi)$
$s_{\mathcal{Q}}$
program $s_{\mathcal{Q}}(t_{\mathcal{P}}(\Phi))$


key principle
$\mathcal{Q}$
theorem
direct interpretation
recursor Ψ
$s_{\mathcal{Q}}$
program $s_{\mathcal{Q}}(\Psi)$


axiom of choice
$\mathcal{P}$
Zorn's lemma
$\mathcal{Q}$
theorem
direct interpretation
indirect interpretation
standard recursor Φ
$t_{\mathcal{P}}$
$t_{\mathcal{P}}(\Phi)$
$s_{\mathcal{Q}}$
$s_{\mathcal{Q}}(t_{\mathcal{P}}(\Phi))$


Zorn's lemma
$\mathcal{Q}$
theorem
direct interpretation
open recursion Ψ
$s_{\mathcal{Q}}$
$s_{\mathcal{Q}}(\Psi)$

A partial order $(X, \prec)$ is *chain complete* if every non-empty chain γ has a greatest lower bound $\bigwedge \gamma$.

A partial order $(X, \prec)$ is *chain complete* if every non-empty chain γ has a greatest lower bound $\bigwedge \gamma$.

A predicate O is *open* if $O(\bigwedge \gamma) \to \exists x \in \gamma O(x)$.

Equivalently, $\forall x \in \gamma \neg O(x) \to \neg O(\bigwedge \gamma)$ i.e. the set $S := \{x \in X \,:\, \neg O(x)\}$ is chain complete.

A partial order $(X, \prec)$ is *chain complete* if every non-empty chain γ has a greatest lower bound $\bigwedge \gamma$.

A predicate O is *open* if $O(\bigwedge \gamma) \to \exists x \in \gamma O(x)$.

Equivalently, $\forall x \in \gamma \neg O(x) \to \neg O(\bigwedge \gamma)$ i.e. the set $S := \{x \in X : \neg O(x)\}$ is chain complete.

By Zorn's lemma, if $S \neq \emptyset$ then S has a minimal element:

$$\exists x \neg O(x) \to \exists y (\neg O(y) \wedge \forall z \prec y O(z)).$$

A partial order $(X, \prec)$ is *chain complete* if every non-empty chain γ has a greatest lower bound $\bigwedge \gamma$.

A predicate O is *open* if $O(\bigwedge \gamma) \to \exists x \in \gamma O(x)$.

Equivalently, $\forall x \in \gamma \neg O(x) \to \neg O(\bigwedge \gamma)$ i.e. the set $S := \{x \in X : \neg O(x)\}$ is chain complete.

By Zorn's lemma, if $S \neq \emptyset$ then S has a minimal element:

$$\exists x \neg O(x) \to \exists y(\neg O(y) \wedge \forall z \prec y O(z)).$$

The contrapositive of this is the principle of *open induction* is given by

$$\forall y(\forall z \prec y O(z) \to O(y)) \to \forall x O(x).$$

Open induction is a generalisation of induction to non well-founded relations $\prec$, but it now only works for open formulas.

Consider $(X^{\mathbb{N}}, <_{\text{lex}})$ where

$$\alpha <_{\text{lex}} \beta :\equiv \exists n(\bar{\alpha}n = \bar{\beta}n \wedge \alpha(n) < \beta(n)).$$

The relation $<_{\text{lex}}$ is not well-founded, but it is chain-complete.

Call a predicate O on $X^{\mathbb{N}}$ piecewise if $O(\alpha) :\equiv \exists k \varphi(\bar{\alpha}k)$ for φ quantifier-free. Then O is open w.r.t. $<_{\text{lex}}$.

Consider $(X^{\mathbb{N}}, <_{\text{lex}})$ where

$$\alpha <_{\text{lex}} \beta :\equiv \exists n(\bar{\alpha}n = \bar{\beta}n \wedge \alpha(n) < \beta(n)).$$

The relation $<_{\text{lex}}$ is not well-founded, but it is chain-complete.

Call a predicate O on $X^{\mathbb{N}}$ piecewise if $O(\alpha) :\equiv \exists k \varphi(\bar{\alpha}k)$ for φ quantifier-free. Then O is open w.r.t. $<_{\text{lex}}$.

Lexicographic open induction OI_{lex} is defined to be open induction over $<_{\text{lex}}$ where O ranges over piecewise formulas i.e.

$$\mathsf{OI}_{\text{lex}} \; : \; \forall \alpha(\forall n, x < \alpha(n), \beta \, O(\bar{\alpha}n * x \,@\, \beta) \rightarrow O(\alpha)) \rightarrow \forall \alpha O(\alpha).$$

where

$$\bar{\alpha}n * x \,@\, \beta := \alpha(0), \ldots, \alpha(n-1), x, \beta(n+1), \beta(n+2), \ldots$$

Consider $(X^{\mathbb{N}}, <_{\text{lex}})$ where

$$\alpha <_{\text{lex}} \beta :\equiv \exists n(\bar{\alpha}n = \bar{\beta}n \wedge \alpha(n) < \beta(n)).$$

The relation $<_{\text{lex}}$ is not well-founded, but it is chain-complete.

Call a predicate O on $X^{\mathbb{N}}$ piecewise if $O(\alpha) :\equiv \exists k \varphi(\bar{\alpha}k)$ for φ quantifier-free. Then O is open w.r.t. $<_{\text{lex}}$.

Lexicographic open induction OI_{lex} is defined to be open induction over $<_{\text{lex}}$ where O ranges over piecewise formulas i.e.

$$\mathsf{OI}_{\text{lex}} \ : \ \forall \alpha(\forall n, x < \alpha(n), \beta \, O(\bar{\alpha}n * x \,@\, \beta) \to O(\alpha)) \to \forall \alpha O(\alpha).$$

where

$$\bar{\alpha}n * x \,@\, \beta := \alpha(0), \ldots, \alpha(n-1), x, \beta(n+1), \beta(n+2), \ldots$$

OI_{lex} is classically equivalent to countable dependent choice, but can be used to give very short and elegant proofs to many theorems in mathematics e.g. Kruskal's theorem.

Berger (2004):

Define open recursion to be recursion over $<_{\text{lex}}$, where the output type is restricted to being $\mathbb{N}$:

$$\mathsf{OR}^F_{\text{lex}}(\alpha) =_{\mathbb{N}} F_\alpha(\lambda n, x, \beta \,.\, \mathsf{OR}^F_{\text{lex}}(\bar{\alpha}n * x \mathbin{@} \beta) \text{ if } x < \alpha(n)).$$

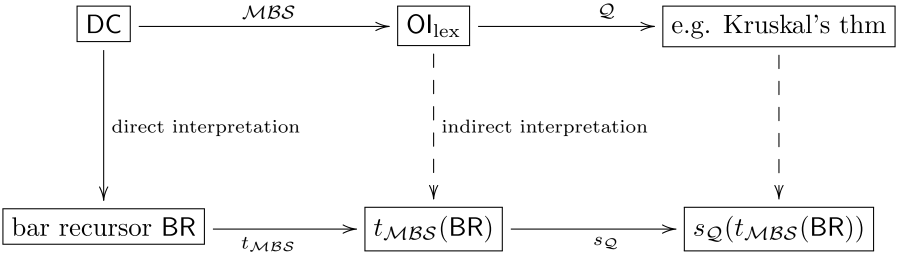
Then there is a term Φ definable in $T + \mathsf{OR}_{\text{lex}}$ such that $\Phi \; mr \; \mathsf{OI}_{\text{lex}}$, provably in the model of continuous functionals $\mathcal{C}^\omega$.

Berger (2004):

Define open recursion to be recursion over $<_{\text{lex}}$, where the output type is restricted to being $\mathbb{N}$:

$$\mathsf{OR}^F_{\text{lex}}(\alpha) =_{\mathbb{N}} F_\alpha(\lambda n, x, \beta \,.\, \mathsf{OR}^F_{\text{lex}}(\bar{\alpha}n * x \mathbin{@} \beta) \text{ if } x < \alpha(n)).$$

Then there is a term Φ definable in $T + \mathsf{OR}_{\text{lex}}$ such that $\Phi\ mr\ \mathsf{OI}_{\text{lex}}$, provably in the model of continuous functionals $\mathcal{C}^\omega$.

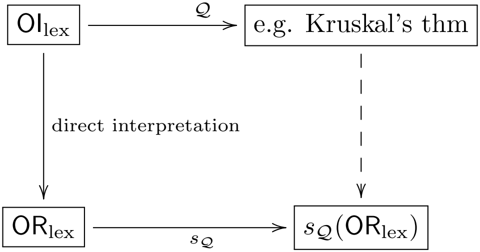


Berger (2004):

Define open recursion to be recursion over $<_{\text{lex}}$, where the output type is restricted to being $\mathbb{N}$:

$$\mathsf{OR}^F_{\text{lex}}(\alpha) =_{\mathbb{N}} F_\alpha(\lambda n, x, \beta \,.\, \mathsf{OR}^F_{\text{lex}}(\bar{\alpha}n * x \,@\, \beta) \text{ if } x < \alpha(n)).$$

Then there is a term Φ definable in $T + \mathsf{OR}_{\text{lex}}$ such that $\Phi \; mr \; \mathsf{OI}_{\text{lex}}$, provably in the model of continuous functionals $\mathcal{C}^\omega$.



Let's take another look at the open recursor:

$$\mathsf{OR}^F_{\mathrm{lex}}(\alpha) =_{\mathbb{N}} F_\alpha(\lambda n, x, \beta\ .\ \mathsf{OR}^F_{\mathrm{lex}}(\bar{\alpha}n * x \mathbin{@} \beta) \text{ if } x < \alpha(n)).$$

Why does this make sense, even though $<_{\mathrm{lex}}$ is not well-founded?

Let's take another look at the open recursor:

$$\mathsf{OR}^F_{\rm lex}(\alpha) =_{\mathbb{N}} F_\alpha(\lambda n, x, \beta \, . \, \mathsf{OR}^F_{\rm lex}(\bar{\alpha}n * x \,@\, \beta) \text{ if } x < \alpha(n)).$$

Why does this make sense, even though $<_{\rm lex}$ is not well-founded?

Because in continuous models, totality of $\mathsf{OR}^F(\alpha)$ only depends on a finite initial segment of $\bar{\alpha}n$ of α, i.e. totality is a piecewise and therefore an *open* formula.

So just as well-founded recursion can be justified via well-founded induction, open recursion $\mathsf{OR}_{\rm lex}$ can be justified by open induction $\mathsf{OI}_{\rm lex}$ *plus* continuity.

IMPORTANT REMARK. If the outcome type of $\mathsf{OR}_{\rm lex}$ is not discrete e.g. $\mathsf{OR}_{\rm lex}(\alpha) \, : \, \mathbb{N} \to \mathbb{N}$, then $\mathsf{OR}_{\rm lex}$ is not well-defined in general.

Outline

1. Past: The realizability interpretation of open induction
2. Present: A functional interpretation of open induction
3. Future: Open questions on open induction

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i|P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y\exists k|P|^k_z \to \exists j|P|^j_y) \to \forall x\exists i|P|^i_x$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i|P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y\exists k|P|^k_z \to \exists j|P|^j_y) \to \forall x\exists i|P|^i_x$$
$$\rightsquigarrow\forall y(\exists\phi\forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x\exists i|P|^i_x$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i|P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y(\exists \phi \forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i |P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y(\exists \phi \forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y, \phi \exists j(\forall z < y |P|^{\phi z}_z \to |P|^j_y) \to \forall x \exists i |P|^i_x$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i |P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y(\exists \phi \forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi \exists j(\forall z < y |P|^{\phi z}_z \to |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \exists F \forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{F y \phi}_y) \to \forall x \exists i |P|^i_x$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i|P|^i_x$. Realizability takes the following approach:

$$\begin{aligned}
&\forall y(\forall z < y \exists k |P|^k_z \to \exists j|P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall y(\exists \phi \forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall y, \phi(\forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall y, \phi \exists j(\forall z < y|P|^{\phi z}_z \to |P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\exists F \forall y, \phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall F[\forall y, \phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i |P|^i_x]
\end{aligned}$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i|P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y\exists k|P|^k_z \to \exists j|P|^j_y) \to \forall x\exists i|P|^i_x$$
$$\rightsquigarrow\forall y(\exists\phi\forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x\exists i|P|^i_x$$
$$\rightsquigarrow\forall y,\phi(\forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x\exists i|P|^i_x$$
$$\rightsquigarrow\forall y,\phi\exists j(\forall z < y|P|^{\phi z}_z \to |P|^j_y) \to \forall x\exists i|P|^i_x$$
$$\rightsquigarrow\exists F\forall y,\phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x\exists i|P|^i_x$$
$$\rightsquigarrow\forall F[\forall y,\phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x\exists i|P|^i_x]$$
$$\rightsquigarrow\forall F[\forall y,\phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \exists f\forall x|P|^{fx}_x]$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i |P|^i_x$. Realizability takes the following approach:

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y(\exists \phi \forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi \exists j(\forall z < y |P|^{\phi z}_z \to |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \exists F \forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall F[\forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i |P|^i_x]$$

$$\rightsquigarrow \forall F[\forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \exists f \forall x |P|^{fx}_x]$$

$$\rightsquigarrow \exists \Phi \big(\forall F[\forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x |P|^{\Phi F x}_x]\big)$$

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i|P|^i_x$. Realizability takes the following approach:

$$
\begin{aligned}
&\forall y(\forall z < y \exists k|P|^k_z \to \exists j|P|^j_y) \to \forall x \exists i|P|^i_x \\
\rightsquigarrow &\forall y(\exists \phi \forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x \exists i|P|^i_x \\
\rightsquigarrow &\forall y, \phi(\forall z < y|P|^{\phi z}_z \to \exists j|P|^j_y) \to \forall x \exists i|P|^i_x \\
\rightsquigarrow &\forall y, \phi \exists j(\forall z < y|P|^{\phi z}_z \to |P|^j_y) \to \forall x \exists i|P|^i_x \\
\rightsquigarrow &\exists F \forall y, \phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i|P|^i_x \\
\rightsquigarrow &\forall F[\forall y, \phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i|P|^i_x] \\
\rightsquigarrow &\forall F[\forall y, \phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \exists f \forall x|P|^{fx}_x] \\
\rightsquigarrow &\exists \Phi\big(\forall F[\forall y, \phi(\forall z < y|P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x|P|^{\Phi Fx}_x]\big)
\end{aligned}
$$

IDEA: Recursively define $\Phi F x =_{\mathbb{N}} Fx(\lambda z < x \,.\, \Phi F z)$.

Suppose we want to give a computational interpretation to induction over some well-founded relation $<$, for a Σ^0_1-formula $\exists i |P|^i_x$. Realizability takes the following approach:

$$
\begin{aligned}
&\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall y(\exists \phi \forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall y, \phi \exists j(\forall z < y |P|^{\phi z}_z \to |P|^j_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\exists F \forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i |P|^i_x \\
\rightsquigarrow &\forall F[\forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x \exists i |P|^i_x] \\
\rightsquigarrow &\forall F[\forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \exists f \forall x |P|^{fx}_x] \\
\rightsquigarrow &\exists \Phi \big(\forall F[\forall y, \phi(\forall z < y |P|^{\phi z}_z \to |P|^{Fy\phi}_y) \to \forall x |P|^{\Phi F x}_x]\big)
\end{aligned}
$$

Idea: Recursively define $\Phi F x =_{\mathbb{N}} Fx(\lambda z < x \,.\, \Phi F z)$.

The output type of Φ is the same as the type of i, so if $i\colon \mathbb{N}$ (as for a piecewise open formula) then $\Phi F x \ : \ \mathbb{N}$. Roughly speaking, this is how we extend well-founded induction to open induction (although there plenty of details)!

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\forall y(\forall z < y \exists k |P|_z^k \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\begin{aligned} &\forall y(\forall z < y \exists k |P|_z^k \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i \\ \rightsquigarrow &\forall y, \phi(\forall z < y |P|_z^{\phi z} \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i \end{aligned}$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y, \phi \exists j, z([z < y \to |P|^{\phi z}_z] \to |P|^j_y) \to \forall x \exists i |P|^i_x$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi \exists j, z([z < y \to |P|^{\phi z}_z] \to |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \exists F \forall y, \phi([F_1 y \phi < y \to |P|^{\phi(F_1 y \phi)}_{F_1 y \phi}] \to |P|^{F_0 y \phi}_y) \to \forall x \exists i |P|^i_x$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall y, \phi \exists j, z([z < y \to |P|^{\phi z}_z] \to |P|^j_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \exists F \forall y, \phi([F_1 y\phi < y \to |P|^{\phi(F_1 y\phi)}_{F_1 y\phi}] \to |P|^{F_0 y\phi}_y) \to \forall x \exists i |P|^i_x$$

$$\rightsquigarrow \forall F, x\Big(\forall y, \phi([F_1 y\phi < y \to |P|^{\phi(F_1 y\phi)}_{F_1 y\phi}] \to |P|^{F_0 y\phi}_y) \to \exists i |P|^i_x\Big)$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\forall y(\forall z < y \exists k |P|^k_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y, \phi(\forall z < y |P|^{\phi z}_z \to \exists j |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall y, \phi \exists j, z([z < y \to |P|^{\phi z}_z] \to |P|^j_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \exists F \forall y, \phi([F_1 y \phi < y \to |P|^{\phi(F_1 y \phi)}_{F_1 y \phi}] \to |P|^{F_0 y \phi}_y) \to \forall x \exists i |P|^i_x$$
$$\rightsquigarrow \forall F, x \big(\forall y, \phi([F_1 y \phi < y \to |P|^{\phi(F_1 y \phi)}_{F_1 y \phi}] \to |P|^{F_0 y \phi}_y) \to \exists i |P|^i_x\big)$$
$$\rightsquigarrow \forall F, x \exists i, y, \phi\big(([F_1 y \phi < y \to |P|^{\phi(F_1 y \phi)}_{F_1 y \phi}] \to |P|^{F_0 y \phi}_y) \to |P|^i_x\big)$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$
\begin{aligned}
&\forall y(\forall z < y \exists k |P|_z^k \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i \\
\rightsquigarrow &\forall y, \phi(\forall z < y |P|_z^{\phi z} \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i \\
\rightsquigarrow &\forall y, \phi \exists j, z([z < y \to |P|_z^{\phi z}] \to |P|_y^j) \to \forall x \exists i |P|_x^i \\
\rightsquigarrow &\exists F \forall y, \phi([F_1 y\phi < y \to |P|_{F_1 y\phi}^{\phi(F_1 y\phi)}] \to |P|_y^{F_0 y\phi}) \to \forall x \exists i |P|_x^i \\
\rightsquigarrow &\forall F, x\Big(\forall y, \phi([F_1 y\phi < y \to |P|_{F_1 y\phi}^{\phi(F_1 y\phi)}] \to |P|_y^{F_0 y\phi}) \to \exists i |P|_x^i\Big) \\
\rightsquigarrow &\forall F, x \exists i, y, \phi\Big(([F_1 y\phi < y \to |P|_{F_1 y\phi}^{\phi(F_1 y\phi)}] \to |P|_y^{F_0 y\phi}) \to |P|_x^i\Big) \\
\rightsquigarrow &\exists \Phi \forall F, x \begin{cases} ([F_1(\Phi_{1,2}Fx)\phi < \Phi_1 Fx \to |P|_{F_1(\Phi_{1,2}Fx)}^{(\Phi_2 Fx)(F_1(\Phi_{1,2}Fx))}] \to |P|_{\Phi_1 Fx}^{F_0(\Phi_{1,2}Fx)}) \\ \to |P|_x^{\Phi_0 Fx} \end{cases}
\end{aligned}
$$

Now consider the functional interpretation of well-founded induction (it's not necessary to follow all of the details!)...

$$\forall y(\forall z < y \exists k |P|_z^k \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i$$
$$\rightsquigarrow \forall y, \phi(\forall z < y |P|_z^{\phi z} \to \exists j |P|_y^j) \to \forall x \exists i |P|_x^i$$
$$\rightsquigarrow \forall y, \phi \exists j, z([z < y \to |P|_z^{\phi z}] \to |P|_y^j) \to \forall x \exists i |P|_x^i$$
$$\rightsquigarrow \exists F \forall y, \phi([F_1 y\phi < y \to |P|_{F_1 y\phi}^{\phi(F_1 y\phi)}] \to |P|_y^{F_0 y\phi}) \to \forall x \exists i |P|_x^i$$
$$\rightsquigarrow \forall F, x\big(\forall y, \phi([F_1 y\phi < y \to |P|_{F_1 y\phi}^{\phi(F_1 y\phi)}] \to |P|_y^{F_0 y\phi}) \to \exists i |P|_x^i\big)$$
$$\rightsquigarrow \forall F, x \exists i, y, \phi\big(([F_1 y\phi < y \to |P|_{F_1 y\phi}^{\phi(F_1 y\phi)}] \to |P|_y^{F_0 y\phi}) \to |P|_x^i\big)$$
$$\rightsquigarrow \exists \Phi \forall F, x \begin{cases} ([F_1(\Phi_{1,2}Fx)\phi < \Phi_1 Fx \to |P|_{F_1(\Phi_{1,2}Fx)}^{(\Phi_2 Fx)(F_1(\Phi_{1,2}Fx))}] \to |P|_{\Phi_1 Fx}^{F_0(\Phi_{1,2}Fx)}) \\ \to |P|_x^{\Phi_0 Fx} \end{cases}$$

The output type of Φ must now encode i, y, ϕ i.e. is at least a function type.

How do we extend the realizability interpretation of OI_{lex} via OR_{lex} to the setting of the functional interpretation?

First challenge. The recursive equations that must be solved are completely different:

$$\exists\Phi\big(\forall F[\forall y,\phi(\forall z<y|P|_z^{\phi z}\to|P|_y^{Fy\phi})\to\forall x|P|_x^{\Phi Fx}]\big)$$

vs

$$\exists\Phi\forall F,x\begin{cases}([F_1(\Phi_{1,2}Fx)\phi<\Phi_1Fx\to|P|_{F_1(\Phi_{1,2}Fx)}^{(\Phi_2Fx)(F_1(\Phi_{1,2}Fx))}]\to|P|_{\Phi_1Fx}^{F_0(\Phi_{1,2}Fx)})\\ \to|P|_x^{\Phi_0Fx}\end{cases}$$

Second challenge. The output type of the open recursor will now be >0, so open recursion OR_{lex} will no longer be valid in $\mathcal{C}^\omega$.

Suppose that $G : Y^{\mathbb{N}} \to \mathbb{N}$ is continuous. Then for any $f : Y^{\mathbb{N}}$ there is some point n such that

$$G(\overline{f, n}) < n$$

where $\overline{f, n} = f(0), \ldots, f(n-1), 0, 0, 0, \ldots$.

Suppose that $G : Y^{\mathbb{N}} \to \mathbb{N}$ is continuous. Then for any $f : Y^{\mathbb{N}}$ there is some point n such that

$$G(\overline{f,n}) < n$$

where $\overline{f,n} = f(0), \ldots, f(n-1), 0, 0, 0, \ldots$.

Let $N_{G,f}$ be the least such n. Then $\overline{f, N_{G,f}}$ is primitive recursively definable as

$$\overline{f, N_{G,f}} = \lambda m. \begin{cases} f(m) & \text{if } \forall k \leq m (G(\overline{f,k}) \geq k) \\ 0_Y & \text{if } \exists k \leq m (G(\overline{f,k}) < k) \end{cases}$$

Suppose that $G : Y^{\mathbb{N}} \to \mathbb{N}$ is continuous. Then for any $f : Y^{\mathbb{N}}$ there is some point n such that

$$G(\overline{f,n}) < n$$

where $\overline{f,n} = f(0), \ldots, f(n-1), 0, 0, 0, \ldots$.

Let $N_{G,f}$ be the least such n. Then $\overline{f, N_{G,f}}$ is primitive recursively definable as

$$\overline{f, N_{G,f}} = \lambda m. \begin{cases} f(m) & \text{if } \forall k \leq m (G(\overline{f,k}) \geq k) \\ 0_Y & \text{if } \exists k \leq m (G(\overline{f,k}) < k) \end{cases}$$

Define the explicitly controlled open recursor by

$$\mathsf{OR}^{F,G}_{\exp}(\alpha) =_Z F(\overline{\underbrace{[\alpha \mid \lambda n, x < \alpha(n), \beta \,.\, \mathsf{OR}^{F,G}_{\exp}(\bar{\alpha}n * x \mathbin{@} \beta)]}_{f}, N_{G,f}}).$$

This is similar to Berger's OR_{lex}, but now totality is controlled explicitly using an additional functional G. As a result, the output type Z can be arbitrary.

P. (Still in a drawer)

There is a term Φ definable in $T + \mathsf{OR}_{\mathrm{exp}}$ which solves the functional interpretation of $\mathsf{OI}_{\mathrm{lex}}$.

P. (Still in a drawer)

There is a term Φ definable in $T + \mathsf{OR}_{\text{exp}}$ which solves the functional interpretation of OI_{lex}.

The proof uses open induction, but does not explicitly require continuity, just the condition that

$$G(\overline{f, n}) < n$$

for some n. As a result, the interpretation is valid in both $\mathcal{C}^\omega$ and the model of majorizable functionals $\mathcal{M}^\omega$.

P. (Still in a drawer)

There is a term Φ definable in $T + \mathsf{OR}_{\text{exp}}$ which solves the functional interpretation of OI_{lex}.

The proof uses open induction, but does not explicitly require continuity, just the condition that

$$G(\overline{f,n}) < n$$

for some n. As a result, the interpretation is valid in both $\mathcal{C}^\omega$ and the model of majorizable functionals $\mathcal{M}^\omega$.

The algorithmic behaviour of the term Φ

The open recursive realizer Φ can be viewed as a learning procedure which computes a 'minimal element' in the chain complete partial order $\{\alpha \in X^{\mathbb{N}} \, : \, \neg O(\alpha)\}$.

Learning and the functional interpretation will form the theme of my talk in M4C next week...

Outline

Q1 (Applied proof theory)

Can we utilise either the realizability or functional interpretations of $\mathsf{OI}_{\mathrm{lex}}$ to extract short, intuitive open recursive programs from proofs in mathematics that use Zorn's lemma in this form?

Potential candidates:

WQO theory: Higman's lemma and more generally Kruskal's theorem.

Abstract algebra: Krull's theorem, Wiener's theorem.

Q2 (Higher order computability)

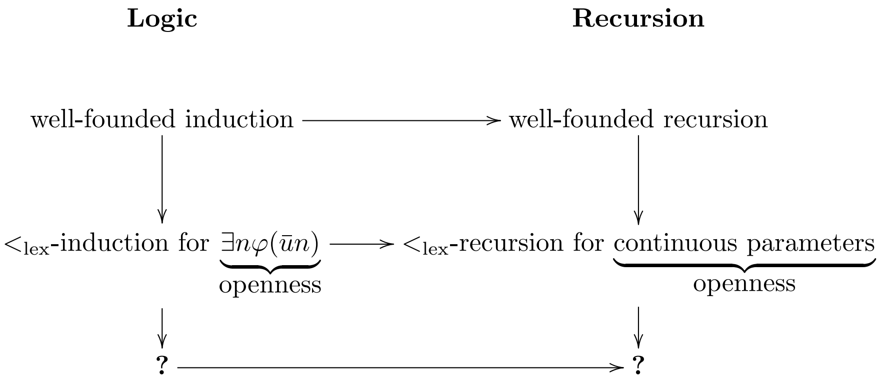
What is the strength of explicitly controlled open recursion OR_{exp}?

Open recursion in the sense of Berger is not S1-S9 computable in the Kleene-Kreisel continuous functionals, as it defines the Gandy-Hyland Γ-functional (P. 2014), and is thus stronger than bar recursion in the sense of Spector.

Claim. OR_{exp} is S1-S9 computable, and primitive recursively equivalent to Spector's bar recursion. In particular, the type 1 lexicographically recursive functions are precisely the provably recursive functions of $\mathsf{PA} + \mathsf{DC}$.

Q3 (Proof theory/Topology)

Can we give a functional interpretation to instances of open induction/Zorn's lemma over chain-complete partial orders more complex than $<_{\text{lex}}$?



- Berger's computational interpretation of open induction contains a number highly novel ideas, in particular a new form of higher-type recursion and a new approach to extracting programs from proofs which use Zorn's lemma.

- These ideas have since been lifted to several different settings, and the related forms of recursion studied.

- More importantly, they have prompted many interesting open questions in mathematics, computer science, proof theory and computability theory.