A constructive interpretation of open induction

Thomas Powell

University of Innsbruck

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A map of the talk:

1. A *very* quick introduction to (a) open induction (b) Gödel's functional (Dialectica) intepretation;

2. A short discussion on the functional interpretation of induction principles;

3. A list of questions and directions for future research!

A partial order $(X, \prec)$ is *chain complete* if every non-empty chain γ has a greatest lower bound $\bigwedge \gamma$.

A predicate O is *open* if $O(\bigwedge \gamma) \rightarrow \exists x \in \gamma O(x)$.

The principle of *open induction* is given by

$$\forall y(\forall z \prec y O(z) \rightarrow O(y)) \rightarrow \forall x O(x),$$

which is classically equivalent to the minimum principle

$$\exists x \neg O(x) \rightarrow \exists y(\neg O(y) \wedge \forall z \prec y O(z))$$

Open induction is just Zorn's lemma in disguise

Let $S := \{x \in X \ : \ \neg O(x)\}$. Openness of O means that every chain in S has a lower bound **in S**. Then $\exists x \neg O(x)$ implies S is non-empty, therefore has a minimal element y.

Chain complete partial orders include:

- $(X, <)$ for is well-founded $<$ (then any formula on X is open).
- $(\mathbb{N} \rightharpoonup X, \sqsupset)$ for partial functions with usual domain-theoretic ordering.
- $(\mathbb{N} \to X, <_{\text{lex}})$ for sequences with lexicographic ordering.

Nash-Williams' proof is just open induction (over $<_{\text{lex}}$) in disguise

$$u \in X^{\mathbb{N}} \text{ is good} \equiv \exists i < j(u_i \leq u_j)$$

depends only on a finite initial segment of u, so an open property.

$$\underbrace{\exists u(u \text{ is bad}) \to \exists v(v \text{ is bad} \wedge \forall w {<_{\text{lex}}} v(w \text{ is good}))}_{\text{minimal bad sequence argument}}$$

Let $v_i = v_i' * x_i$. By Ramsey's theorem $x_{i_0} < x_{i_1} < \ldots$ for some subsequence (x_{i_j}). Then

$$w := v_0, \ldots, v_{i_0 - 1}, v'_{i_0}, v'_{i_1}, \ldots$$

satisfies $w {<_{\text{lex}}} v$ and thus w is good. But this implies v cannot be bad.

What is the computational content of Higman's lemma?

↑

What is the computational content of open induction?

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What is the computational content of Zorn's lemma?

What is the computational meaning of a Π_3-theorem?

$$A :\equiv \forall x^X \exists y^Y \forall z^Z P(x, y, z)$$

In general we cannot hope to produce a direct computable witness for $\exists y$. But suppose we double negate and Skolemize:

$$\begin{aligned} \neg A &\leftrightarrow \exists x \forall y \exists z \neg P(x, y, z) \\ &\leftrightarrow \exists x, \phi^{Y \to Z} \forall y \neg P(x, y, \phi(y)) \\ \neg\neg A &\leftrightarrow \forall x, \phi \exists y \neg\neg P(x, y, \phi(y)) \\ &\leftrightarrow \forall x, \phi \exists y \; P(x, y, \phi(y)) \end{aligned}$$

We can typically extract some *indirect* computable witness

$$F \colon X \to (Y \to Z) \to Y$$

for $\exists y$ in $\neg\neg A$, i.e.

$$\forall x, \phi \; P(x, F_{x,\phi}, \phi(F_{x,\phi})).$$

In the statement

$$A \equiv \forall x \exists y \forall z P(x, y, z),$$

y is an ideal object which works for *all* z. On the other hand, in the statement

$$A' \equiv \forall x, \phi \exists y P(x, y, \phi(y))$$

y is a finitary approximation to ideal object, which works for just $\phi(y)$. The function ϕ can be seen as determining the size, or 'quality', of this approximation.

Over classical logic $A \leftrightarrow A'$, but A' can be directly realized, unlike A.

Gödel's Dialectica interpretation is a systematic way of doing this for arbitrary A: A map $A \mapsto \exists x \forall y A^*(x, y)$ for quantifier-free A^* such that

$$\mathcal{P} \vdash A \Rightarrow \mathcal{P}' \vdash \forall y A^*(t, y)$$

where t is a term extracted from the proof $\mathcal{P}$.

Take a very simple case of 'open' induction: $\prec$ well-founded, P decidable.

$$\exists x \neg P(x) \to \exists y(\neg P(y) \wedge \forall z \prec y P(z)).$$

This has a functional interpretation given by

$$\begin{aligned}
&\exists x \neg P(x) \to \neg\neg\exists y \forall z(\neg P(y) \wedge (z \prec y \to P(z))) \\
\rightsquigarrow &\exists x \neg P(x) \to \forall \phi \exists y(\neg P(y) \wedge (\phi y \prec y \to P(\phi y))) \\
\rightsquigarrow &\boxed{\forall x, \phi \exists y(\neg P(x) \to \neg P(y) \wedge (\phi y \prec y \to P(\phi y)))}
\end{aligned}$$

"For all x, ϕ there exists some y such that whenever $\neg P(x)$ then $\neg P(y)$ and y is approximately minimal with respect to ϕy"

We can witness y in x and ϕ using well-founded recursion over $\prec$:

$$y := \begin{cases} x & \text{if } \phi x \prec x \to P(\phi x) \\ \phi x & \text{if } \phi^{(2)} x \prec \phi x \to P(\phi^{(2)} x) \\ \phi^{(2)} x & \text{if } \phi^{(3)} x \prec \phi^{(2)} x \to P(\phi^{(3)} x) \\ \vdots & \vdots \end{cases}$$

Remark: For Σ_1-formulas this becomes a lot more complex!

$$\exists x \neg \exists i P(x,i) \to \exists y (\neg \exists j P(y,j) \wedge \forall z \prec y \exists k P(z,k)))$$

has a functional interpretation

$$\forall x, \phi \exists i, y, g [\neg P(x,i) \to \neg P(y, \phi_0 y g) \wedge (\phi_1 y g \prec y \to P(\phi_1 y g, g(\phi y g)))]$$

Solution is given by (cf. Schwichtenberg 2008)

$$\begin{aligned} i &:= \phi_0 x g_x^{\phi_0} \\ y &:= \begin{cases} x, g_x^{\phi_0} & \text{if } \tilde{\phi}_1 x \prec x \to P(\tilde{\phi}_1 x, g_x(\tilde{\phi} x)) \\ \tilde{\phi} x, g_{\tilde{\phi} x}^{\phi_0} & \text{if } \tilde{\phi}_1^{(2)} x \prec \tilde{\phi} x \to P(\tilde{\phi}_1^{(2)} x, g_x(\tilde{\phi}^{(2)} x)) \\ \tilde{\phi}^{(2)} x, g_{\tilde{\phi}^{(2)} x}^{\phi_0} & \text{if } \tilde{\phi}_1^{(3)} x \prec \tilde{\phi}^{(2)} x \to P(\tilde{\phi}_1^{(3)} x, g_x(\tilde{\phi}^{(3)} x)) \\ \vdots & \vdots \end{cases} \\ \tilde{\phi} x &:= \phi x g_x^{\phi_0} \\ g_x^{\phi_0} &:= \lambda y \prec x \,.\, \phi_0 y g_y^{\phi_0} \end{aligned}$$

Let's consider a simple case of open induction for non well-founded orders. Let $(\mathbb{N} \rightharpoonup X, \sqsupset)$ be the chain complete set of partial functions $\mathbb{N} \to X$ with the usual domain theoretic ordering, and $P(n, x)$ a predicate on $\mathbb{N} \times X$.

The following is essentially Σ_1-comprehension over numbers:

Theorem. $\exists v \forall n([n \in \mathrm{dom}(v) \to P(n, vn)] \wedge [n \notin \mathrm{dom}(v) \to \forall x \neg P(n, x)])$.

Proof.

Define the open predicate on $\mathbb{N} \rightharpoonup X$ by

$$O(u) :\equiv \exists n \in \mathrm{dom}(u) \neg P(n, un)$$

Then $\neg O(\emptyset)$ trivially holds, therefore by open induction there exists some $\sqsupset$-minimal ($\sqsubset$-maximal!) v satisfying $\neg O(v) \wedge \forall w \sqsupset v O(w)$, which implies

$$\forall n \in \mathrm{dom}(v) P(n, vn) \wedge \forall m(m \notin \mathrm{dom}(v) \to \forall x \neg P(m, x))$$

i.e.

$$\forall n([n \in \mathrm{dom}(v) \to P(n, vn)] \wedge [n \notin \mathrm{dom}(v) \to \forall x \neg P(n, x)])$$

□

The functional interpretation of the theorem is given by

$$\neg\neg\exists v \forall n, x([n \in \mathrm{dom}(v) \to P(n, vn)] \wedge [n \notin \mathrm{dom}(v) \to \neg P(n, x)])$$
$$\rightsquigarrow \forall \varphi, \phi \exists v([\varphi v \in \mathrm{dom}(v) \to P(\varphi v, v(\varphi v))] \wedge [\varphi v \notin \mathrm{dom}(v) \to \neg P(\varphi v, \phi v)])$$

This is interpreted by recursion over the (non-well-founded) order $\sqsupset$ as

$$v := \begin{cases} \emptyset =: v_0 & \text{if } \varphi v_0 \notin \mathrm{dom}(v_0) \to \neg P(\varphi v_0, \phi v_0) \\ [\varphi v_0 \mapsto \phi v_0] =: v_1 & \text{if } \varphi v_1 \notin \mathrm{dom}(v_1) \to \neg P(\varphi v_1, \phi v_1) \\ [\varphi v_0 \mapsto \phi v_0, \varphi v_1 \mapsto \phi v_1] =: v_2 & \text{if } \varphi v_2 \notin \mathrm{dom}(v_2) \to \neg P(\varphi v_2, \phi v_2) \\ \vdots & \vdots \end{cases}.$$

If we assume continuity of φ (it only looks at a finite part of its input), then at some point we have $\varphi v_i \in \mathrm{dom}(v_i)$, so the recursion is well-founded.

Remark: Continuity ensures that the predicate *"recursion is well-founded"* is also an open property.

What is the functional interpretation of open induction over $<_{\text{lex}}$ for predicates of the form $\exists i P([u](i))$ where $[u](i)$ the initial segment of $u \in X^{\mathbb{N}}$ of length i? Naturally we have $\exists n \exists i < j \leq n(u_i \leq u_j)$ in mind!

$$\exists u \forall i \neg P([u](i)) \to \exists v (\forall j \neg P([v](j)) \wedge \forall w (\underbrace{w <_{\text{lex}} v}_{\exists n (w <_{\text{lex},n} v)} \to \exists k P([w](k)))).$$

This is given by

$$\forall u, \varphi, \phi \exists i, v, g \begin{cases} \neg P([u](i)) \to \\ \neg P([v](\varphi v g)) \wedge (\phi_1 v g <_{\text{lex}, \phi_0 v g} v \to P([\phi_1 v g](g(\phi v g)))) \end{cases}$$

"For all u, φ and ϕ there exists some i, v and g such that: if u is i-bad then v is $\varphi v g$-bad and whenever $\phi_1 v g$ is lexicographically less than v at point $\phi_0 v g$ then $\phi_1 v g$ is good at point $g(\phi v g)$"

Remark: A realizing term must have type

$$X^{\mathbb{N}} \times (X^{\mathbb{N}} \times (\mathbb{N} \times X^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N} \times \mathbb{N} \times X^{\mathbb{N}}) \to \mathbb{N} \times X^{\mathbb{N}} \times (\mathbb{N} \times X^{\mathbb{N}} \to \mathbb{N})$$

Questions

Q1 (Proof theory/WQO theory)

How should we realize the functional interpretation of open induction over $<_{\text{lex}}$ for Σ_1-formulas (and more generally), and thus give a computational interpretation to the classical proofs of Higman's lemma/Kruskal's theorem?

Open induction of this form is provable from dependent choice, which in turn has a functional interpretation in the *bar recursive* functionals (Spector 1962). Can extract a term using a higher-type form of bar recursion in the standard way (P. 2013), but this is hugely complex and the realizer does not behave like recursion over $<_{\text{lex}}$!

U. Berger 2004 has shown that the modified realizability (essentially BHK) interpretation of open induction can be witnessed by a form of recursion over $<_{\text{lex}}$ called *open recursion*. Can we adapt this idea for the (very different) functional interpretation?

Can we use this to extract **natural** programs from the Nash-Williams proof of Kruskal's theorem?

Q2 (Higher order computability)

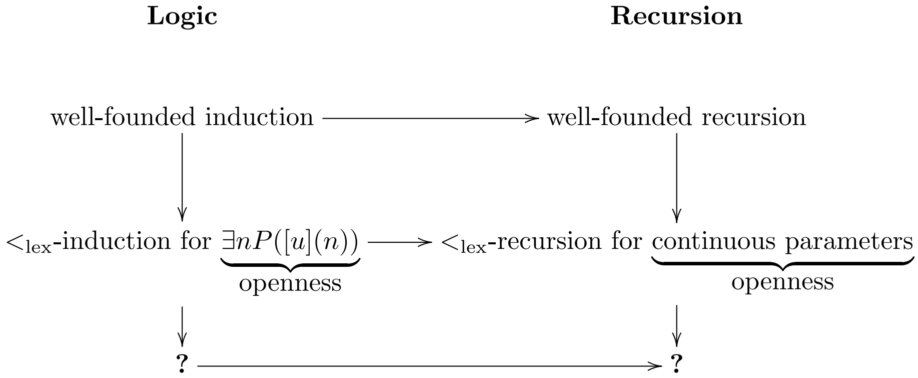
Can we give an alternative characterisation of Spector's class of bar recursive functionals in terms of lexicographic recursion? Then in particular the type 1 lexicographically recursive functions will be exactly the provably recursive functions of $\mathsf{PA} + \mathsf{DC}$.

Open recursion in the sense of Berger is not S1-S9 computable in the Kleene-Kreisel continuous functionals, as it defines the Gandy-Hyland Γ-functional (P. 2014), and is thus stronger than bar recursion (in the sense of Spector).

Can we define an S1-S9 computable form of induction over $<_{\text{lex}}$ which gives a natural computational interpretation to open induction but is still primitive recursively equivalent to Spector's bar recursion?

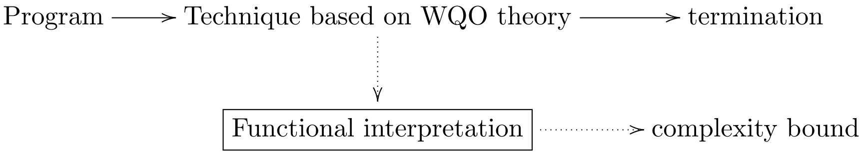
Q3 (Proof theory/Topology)

Can we give a functional interpretation to instances of open induction/Zorn's lemma over chain-complete partial orders more complex than $<_{\mathrm{lex}}$?



Q4 (Applications in computer science)

By studying weak forms of open induction used to prove termination of programs, can we use the functional interpretation to extract useful bounds on the complexity of these programs?



Proof theoretic analyses not only yield a complexity bound, but reveal on a deep, structural level *why* the termination technique produces this bound. This in turn encourages a general way of thinking about these techniques and how they can be modified and extended. Can the functional interpretation add something new?

Thank you!