A Constructive Proof of Higman’s Lemma

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CL&C’12
University of Warwick, 8 July 2012
Overview of paper

‘Applying Gödel’s Dialectica interpretation to obtain a constructive proof of Higman’s lemma.’
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2. Extract a program for finding embedded pairs in sequences of words, and attempt to understand how it works.
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3. Present a case study in which formal program extraction is carried out intuitively - output presented as a mathematical proof.
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4. Provide some insight into constructive aspects of WQO theory.
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- Statement of the extracted program.
- A comparison with programs extracted using other methods.
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Well-Quasi-Orders

A preorder $\leq_X$ on $X$ is a reflexive, transitive binary relation. Define a sequence $(x_i)_{i \in \mathbb{N}}$ in $X$ to be bad if we have $x_i \not\leq_X x_j$ for all $i < j$. It is good otherwise.
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**WQO (Definition 1).** A preorder $(X, \leq_X)$ is a well-quasi-order (WQO) if all sequences in $X$ are good i.e. for all sequences $(x_i)_{i \in \mathbb{N}}$ we have $x_i \leq_X x_j$ for some $i < j$. 
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- A is a WQO whenever $A$ is finite: in any infinite sequence at least one element of $A$ appears twice.
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- $A$ is a WQO whenever $A$ is finite: in any infinite sequence at least one element of $A$ appears twice.
- $(\mathbb{N}, \leq)$ is a WQO: by well foundedness of $\mathbb{N}$ there can be no infinite decreasing chains $x_0 > x_1 > \ldots$. 
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- \( A \) is a WQO whenever \( A \) is finite: in any infinite sequence at least one element of \( A \) appears twice.
- \((\mathbb{N}, \leq)\) is a WQO: by well foundedness of \( \mathbb{N} \) there can be no infinite decreasing chains \( x_0 > x_1 > \ldots \).
- \((\mathbb{N}, |)\) is not a WQO: The prime numbers 2, 3, 5, \ldots form an infinite bad sequence.
There are many alternative ways to characterise WQOs:

**WQO (Definition 2).** $(X, \leq_X)$ is a WQO iff all sequences $(x_i)_{i \in \mathbb{N}}$ in $X$ have an infinite increasing subsequence $x_{g_0} \leq x_{g_1} \leq x_{g_2} \leq \ldots$. 
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\[ x_{g0} \leq x_{g1} \leq x_{g2} \leq \ldots. \]

- For \( A \) finite, by the infinite pigeonhole principle for any infinite sequence in \( A \) at least one element appears infinitely often.

- Given \( (x_i)_{i \in \mathbb{N}} \) in \( \mathbb{N} \), define \( g0 \) such that
  \[ x_{g0} := \min\{x_k : k \in \mathbb{N}\} \]

  Define \( g(i + 1) > gi \) such that \( x_{g(i+1)} := \min\{x_k : k > gi\} \).

  Then we must have \( x_{g0} \leq x_{g1} \leq \ldots \).
Building new WQOs

**Theorem.** If \((X, \leq_X), (Y, \leq_Y)\) are WQOs, then so is \((X \times Y, \leq_{X \times Y})\).
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**Proof.** Take \((x_i, y_i)_{i \in \mathbb{N}}\) in \(X \times Y\).
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\(X\) a WQO \(\Rightarrow\) \((x_i)_{i \in \mathbb{N}}\) has an infinite increasing subsequence 
\(x_{g_0} \leq_X x_{g_1} \leq_X x_{g_2} \leq \ldots\).

\(Y\) a WQO \(\Rightarrow\) the sequence \((y_{g_i})_{i \in \mathbb{N}}\) has \(y_{g_i} \leq_Y y_{g_j}\) for some \(i < j\).

Therefore \((x_{g_i}, y_{g_i}) \leq_{X \times Y} (x_{g_j}, y_{g_j})\). \qed
Higman’s lemma

Given a preorder \((X, \leq_X)\) we can define a preorder \((X^*, \leq_{X^*})\) on \(words\) over \(X\) via the embeddability relation:

\[
\langle x_0, \ldots, x_{m-1} \rangle \leq_{X^*} \langle x'_0, \ldots, x'_{n-1} \rangle
\]

if there is a strictly increasing map \(f : [m] \rightarrow [n]\) with \(x_i \leq_X x'_{f(i)}\) for all \(i < m\).
Higman’s Lemma  (Higman, 1952). If \((X, \leq_X)\) is a WQO then so is \((X^*, \leq_{X^*})\).
**Proof (Nash-Williams, 1963).** Suppose that \((u_i)_{i \in \mathbb{N}}\) is a bad sequence in \(X^*\). Using dependent choice, construct a minimal bad sequence \((v_i)_{i \in \mathbb{N}}\) as follows:
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2. given \(v_0, \ldots, v_{n-1}\), choose \(v_n\) such that \(v_0, \ldots, v_{n-1}, v_n\) extends to a bad sequence, but \(v_0, \ldots, v_{n-1}, y\) does not for any prefix \(y \triangleleft v_n\).
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\((v_i)_{i \in \mathbb{N}}\) bad sequence, minimal under the lexicographic ordering on \((X^*)^\omega\).
Each $v_i$ must be non-empty, so we can write $v_i = \tilde{v}_i \ast x_i$. 
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$X$ a WQO $\Rightarrow (x_i)_{i \in \mathbb{N}}$ has an infinite increasing subsequence $x_{g_0} \leq x x_{g_1} \leq x x_{g_2} \leq \ldots$.
Higman’s lemma

Classical proof of Higman’s lemma

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\( X \) a WQO \( \Rightarrow (x_i)_{i \in \mathbb{N}} \) has an infinite increasing subsequence
\( x_{g0} \leq x \ x_{g1} \leq x \ x_{g2} \leq \cdots \).

But then the sequence
\[ v_0, \ldots, v_{g0-1}, \tilde{v}_{g0}, \tilde{v}_{g1}, \tilde{v}_{g2}, \ldots \]

is bad, contradicting minimality of \((v_i)_{i \in \mathbb{N}}\). \( \square \)
Bounds for the length bad sequences

Given a WQO \((X, \leq_X)\) can we produce an explicit functional \(\Gamma\) satisfying

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\forall x \in X^\omega \exists i < j \leq \Gamma(x)(x_i \leq_X x_j)\?
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- For any \((x_i)_{i \in \mathbb{N}}\) in \(\mathbb{N}\) we have \(x_i \leq x_j\) for some \(i < j \leq x_0 + 2\) (the maximum length of a bad sequence \(x_0 > x_1 > \ldots\) is \(x_0 + 1\)).
Given a WQO $(X, \leq_X)$ can we produce an explicit functional $\Gamma$ satisfying
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**Challenge:** Analyse the classical proof of Higman’s lemma to extract a program $\Gamma_{X^\omega}$ bounding bad sequences in $(X^\omega, \leq^\omega)$, for arbitrary WQOs $(X, \leq_X)$?
Why Higman’s Lemma?

- It has a short, elegant classical proof based on a non-trivial combinatorial idea.
- Minimal bad sequence argument important building block in theory of WQOs, lies behind Kruskal's theorem.
- Higman's lemma has practical implications - termination proofs in rewriting systems.
- Extensively studied in logic and proof theory.
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1 Higman's lemma

2 The computational content of Nash-William's proof
Methods of program extraction

Negative translation and Dialectica interpretation

Maps formulas $A$ to (classically equivalent) formulas $\exists x \forall y A_D(x, y)$. If $\text{PA}_\omega \vdash A$ then there exists closed term $t \in T$ s.t. $T \vdash A_D(t, y)$. 
Methods of program extraction

**Inductive definitions** Reformulation of Nash-Williams’ proof using inductive definition of WQO by Coquand and Fridlender (1993), extended to Kruskal’s theorem by Seisenberger (2001).
Methods of program extraction


A-translation and realizability Formal program extraction carried out by Murthy (1990), improved and implemented in Minlog by Seisenberger (2003).

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Dialectica interpretation

\[ \forall x \exists y (A(x, y) \iff \text{ND} \mapsto f X \rightarrow Y) \]  
\[ \forall x A(x, fx) \]  

Can directly extract programs from classical proofs of \( \Pi_2 \) theorems.

How do we interpret ineffective lemmas used in the proof?

\[ \exists x \forall y (B(x, y) \iff \text{N} \mapsto \neg \neg \exists x \forall y B(x, y)) \]  
\[ \leftrightarrow \forall \phi (X \rightarrow Y \exists x B(x, \phi x)) \]  
\[ \text{D} \mapsto \phi \rightarrow \text{F}(X \rightarrow Y) \rightarrow X \]  
\[ \forall \phi A(F \phi, \phi (F \phi)) \]  

\( \phi \) specifies how \( x \) is going to be used in a computation and \( \text{F} \) constructs an 'approximation' to \( x \) based on \( \phi \).
Dialectica interpretation

$\Pi_2$-formulas

$$\forall x^X \exists y^Y A(x, y) \overset{ND}{\Rightarrow} f^X \rightarrow^Y \forall x A(x, fx).$$

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Dialectica interpretation

\( \Pi_2 \)-formulas

\[ \forall x^X \exists y^Y A(x, y) \quad \overset{ND}{\iff} \quad f^{X \to Y}. \quad \forall x \ A(x, fx). \]

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\( \Sigma_2 \)-formulas

\[ \exists x^X \forall y^Y B(x, y) \quad \overset{N}{\iff} \quad \neg \exists x^X \forall y^Y B(x, y) \]

\[ \iff \quad \forall \varphi^{X \to Y} \exists x \ B(x, \varphi x) \]

\[ \overset{D}{\Rightarrow} \quad F(X \to Y) \to X. \quad \forall \varphi \ A(F \varphi, \varphi(F \varphi)). \]

\( \varphi \) specifies how \( x \) is going to be used in a computation and \( F \) constructs an ‘approximation’ to \( x \) based on \( \varphi \).
In the proof of Higman’s lemma, the assumption $X$ is a WQO is used in the sense of Definition 2 i.e. the following ineffective form:

$$MS[X] : \forall x^X \omega \exists g : \mathbb{N} \to \mathbb{N} \forall k \forall i < j \leq k (g_i < g_j \land x_{g_i} \leq x_{g_j})$$

$(X, \leq_X)$ is a WQO iff there exists $G$ realizing $MS[X]$ i.e. for all sequences $(x_i)_{i \in \mathbb{N}}$ in $X$ have arbitrary high quality approximations to infinite increasing sequences.
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\[\downarrow\]

$$\forall x, \varphi : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N} \exists g \forall i < j \leq \varphi (g_i < g_j \land x_{g_i} \leq x_{g_j})$$
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$$\text{MS}[X] : \forall x^X \exists g^{\mathbb{N} \rightarrow \mathbb{N}} \forall k \forall i < j \leq k (g_i < g_j \land x_{g_i} \leq x_{g_j})$$

$$\downarrow$$

$$\forall x, \varphi^{\mathbb{N} \rightarrow \mathbb{N}} \exists g \forall i < j \leq \varphi g (g_i < g_j \land x_{g_i} \leq x_{g_j})$$

$$\downarrow$$

$$\text{MS}[X]' : \exists G \forall x, \varphi \forall i < j \leq \varphi (G^x) (G_i < G_j \land x_{G_i} \leq x_{G_j})$$
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$$

$$
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$$

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\forall x, \varphi^{\mathbb{N}^n \to \mathbb{N}} \exists g \forall i < j \leq \varphi g (g_i < g_j \land x_{g_i} \leq x x_{g_j})
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\text{MS}[X]' : \exists G \forall x, \varphi \forall i < j \leq \varphi (G^x_{\varphi}) (G_i < G_j \land x_{G_i} \leq x x_{G_j})
$$

**WQO (definition 3).** $(X, \leq_X)$ is a WQO iff there exists $G$ realizing $\text{MS}[X]'$ i.e. for all sequences $(x_i)_{i \in \mathbb{N}}$ in $X$ have arbitrary high quality approximations to infinite increasing sequences.
**Theorem.** If \((X, \leq_X), (Y, \leq_Y)\) are WQOs, then so is 
\((X \times Y, \leq_{X \times Y})\).
Theorem. If \((X, \leq_X), (Y, \leq_Y)\) are WQOs, then so is \((X \times Y, \leq_{X \times Y})\).

Proof. Take \((x_i, y_i)_{i \in \mathbb{N}}\) in \(X \times Y\).
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**Proof.** Take \((x_i, y_i)_{i \in \mathbb{N}}\) in \(X \times Y\).

1. \(X\) a WQO \(\Rightarrow \exists g \forall k \forall i < j \leq k (g_i < g_j \land x_{g_i} \leq x_{g_j})\).

2. \(Y\) a WQO \(\Rightarrow \exists i < j \leq \Gamma_Y(y_g) (y_{g_i} \leq_Y y_{g_j})\).
Theorem. If \((X, \leq_X), (Y, \leq_Y)\) are WQOs, then so is \((X \times Y, \leq_{X \times Y})\).

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2. \(Y\) a WQO \(\Rightarrow \exists i < j \leq \Gamma_Y (y_g)(y_{g_i} \leq_Y y_{g_j})\).

Therefore \(\exists i < j \leq \Gamma_Y (y_g)((x_{g_i}, y_{g_i}) \leq_{X \times Y} (x_{g_j}, y_{g_j}))\).
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Therefore \(\exists i < j \leq \Gamma_Y (y_g) (\langle x_{g_i}, y_{g_i} \rangle \leq_{X \times Y} \langle x_{g_j}, y_{g_j} \rangle)\).

\(g\) ineffectively constructed, but only really need an approximation of \(g\) up to \(\Gamma_Y (y_g)\).
Constructive version. Given $G$ satisfying $\text{MS}[X]'$ and $\Gamma_Y$ realizing well-quasi-orderedness of $Y$ we have

$$\exists i < j \leq G^\varphi_{\Gamma_Y}(\Gamma_Y(y_{G^\varphi_{\Gamma_Y}})) \langle x_i, y_i \rangle \leq_X Y \langle x_j, y_j \rangle$$

where $\varphi := \lambda g . \Gamma_Y(y_g)$. 
Constructive version. Given $G$ satisfying $\text{MS}[X]'$ and $\Gamma_Y$ realizing well-quasi-orderedness of $Y$ we have

$$\exists i < j \leq G^X_\varphi((\Gamma_Y(y_{G^X_\varphi}))((\langle x_i, y_j \rangle \leq X \times Y \langle x_j, y_j \rangle))$$

where $\varphi := \lambda g \cdot \Gamma_Y(y_g)$.

Proof. We have

1. $\forall i < j \leq \varphi G^X_\varphi(G_i < G_j \land x_{G_i} \leq x_{G_j})$ i.e. $\forall i < j \leq \Gamma_Y(y_{G^X_\varphi})(G_i < G_j \land x_{G_i} \leq x_{G_j})$. 

Constructive version. Given $G$ satisfying MS[$X']$ and $\Gamma_Y$ realizing well-quasi-orderedness of $Y$ we have

$$\exists i < j \leq G^x_\varphi(\Gamma_Y(y_{G^x_\varphi}))((x_i, y_j) \leq_{X \times Y} (x_j, y_j))$$

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2. $\exists i < j \leq \Gamma_Y(y_{G^x_\varphi})(y_{G_i} \leq y_{G_j})$.
**Constructive version.** Given $G$ satisfying $\text{MS}[X]'$ and $\Gamma_Y$ realizing well-quasi-orderedness of $Y$ we have

$$
\exists i < j \leq G^\times_{\varphi}(\Gamma_Y(y_{G^\times_{\varphi}}))\langle x_i, y_j \rangle \leq x \times Y \langle x_j, y_j \rangle
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where $\varphi := \lambda g . \Gamma_Y(y_g)$.

**Proof.** We have

1. $$\forall i < j \leq \varphi G^\times_{\varphi}(G_i < G_j \land x_{G_i} \leq x x_{G_j})$$ i.e. $$\forall i < j \leq \Gamma_Y(y_{G^\times_{\varphi}})(G_i < G_j \land x_{G_i} \leq x x_{G_j}).$$

2. $$\exists i < j \leq \Gamma_Y(y_{G^\times_{\varphi}})(y_{G_i} \leq y_{G_j}).$$

Therefore $\langle x_{G_i}, y_{G_i} \rangle \leq x \times Y \langle x_{G_j}, y_{G_j} \rangle$ for $G_i < G_j \leq G(\Gamma_Y(y_{G^\times_{\varphi}}))$. 
Higman’s Lemma. If \((X, \leq_X)\) is a WQO then so is \((X^*, \leq_{X^*})\).
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Proof. Suppose that \(u\) is a bad sequence in \(X^*\). Using dependent choice, construct \(v^i : (X^*)^\omega\) and \(f^i : (X^*)^\omega \rightarrow \mathbb{N}\) as follows:
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Proof. Suppose that \(u\) is a bad sequence in \(X^*\). Using dependent choice, construct \(v^i: (X^*)^\omega\) and \(f^i: (X^*)^\omega \rightarrow \mathbb{N}\) as follows:

1. \(v^0\) is bad but for any \(y \prec_0 v^0\) we have 
   \[\exists i < j \leq f^0(y)(y_i \leq_{X^*} y_j)\]
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2. \([v^{n-1}](n) = [v^n](n)\) and \(v^n\) is bad, but for any \(y \prec_n v^n\) we have 
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\((v^i)\) is a bad sequence, minimal under the lexicographic ordering on \((X^*)^\omega\).
Each $v^j_i$ must be non-empty, so we can write $v^j_i = \tilde{v}^j_i \ast \bar{v}^j_i$. 
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$X$ a WQO $\Rightarrow \exists g \forall k \forall i < j \leq k (g_i < g_j \land \bar{v}_g^i \leq_X \bar{v}_g^j)$.
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Therefore the sequence $\psi_{g,v} := [v^{g_0-1}](g_0) \ast (\tilde{v}_{g_i}^{g_0})$ must be bad.
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Therefore the sequence $\psi_{g,v} := [v^{g0-1}](g0) \ast (\tilde{v}^g_i)$ must be bad.

But by minimality at $g0$, $\psi$ must have one element contained in a later one before $f^{g0}(\psi)$.
Each $v^i_j$ must be non-empty, so we can write $v^i_j = \tilde{v}^i_j \ast \overline{v}^i_j$.

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But by minimality at $g_0$, $\psi$ must have one element contained in a later one before $f[g^0](\psi)$.

This implies that $v^{f[g^0](\psi)}_g$ must have one element contained in a later one before $g(f[g^0](\psi)) \rightarrow$ contradiction.
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- $g$ up to $fg^0(\psi)$,

where $\psi_{g,\nu} := [\nu g^{0-1}](g0) \ast (\tilde{\nu}^g_{gi})$. 
Monotone sequence $g$ and minimal bad sequence $\nu, f$ ineffectively constructed, but to obtain contradiction only need

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- $\nu$ up to $g(fg^0(\psi))$ and of length $g(fg^0(\psi))$,

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- $g$ up to $fg^0(\psi)$,
- $v$ up to $g(fg^0(\psi))$ and of length $g(fg^0(\psi))$,
- $fg^0$ applied to $\psi$.

where $\psi_{g,v} := [v^{g^0-1}](g0) \ast (\tilde{v}^g_{gi})$. 
Higman’s lemma (constructive version): Given any $G$ satisfying $\text{MS}[X]'$ there exists $\Gamma_{X^*} : (X^*)^\omega \rightarrow \mathbb{N}$ satisfying

$$\forall u^{(X^*)^\omega} \exists i < j \leq \Gamma_{X^*}(u) (u_i \leq_{X^*} u_j).$$

Proof.
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Proof.

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1. Interpret minimal bad sequence argument: find procedure for producing ‘approximations’ to minimal bad sequences.

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**Proof.**

1. Interpret minimal bad sequence argument: find procedure for producing ‘approximations’ to minimal bad sequences.

2. Calibrate approximations of $g$ and minimal bad sequence required to obtain contradiction.

3. Work backwards from contradiction to obtain bound for $u$. 
Interpreting minimal bad sequence construction

Central part of program extraction! Details in paper...

System T no longer sufficient to interpret dependent choice...

dependent choice $\mapsto$ bar recursion
Higman’s lemma

The computational content of Nash-William’s proof

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dependent choice $\mapsto$ bar recursion (product of selection functions)

**Novelty:** Use recently discovered *product of selection functions*,
form of bar recursion with natural game theoretic semantics.
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**Novelty:** Use recently discovered *product of selection functions*, form of bar recursion with natural game theoretic semantics.

**Question.** Can we construct direct realizer for minimal bad sequence argument, and does it lead to a more intuitive/efficient program?
Further comments

Can we understand algorithm in qualitative terms—unwrap the syntax and appreciate its operational behaviour? How does it compare, both qualitatively and in terms of efficiency to programs extracted using other methods? It would be instructive to formalise this work in a theorem prover, and test the extracted program on some explicit examples. Does our program yield any new quantitative information i.e. new bounds for length of bad sequences? Can we interpret general minimal bad sequence argument and extract programs from more complex proofs like Kruskal’s theorem?
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Constructive proofs of Higman’s lemma

Dialectica interpretation and product of selection functions