On Spector's Bar Recursion

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We show that Spector's "restricted" form of bar recursion is sufficient (over system T) to define Spector's search functional. This new result is then used to show that Spector's restricted form of bar recursion is in fact as general as the supposedly more general form of bar recursion. Given that these two forms of bar recursion correspond to the (explicitly controlled) iterated products of selection function and quantifiers, it follows that this iterated product of selection functions is T-equivalent to the corresponding iterated product of quantifiers.

1 Introduction

Let $\mathbb{N}$ denote the ground type of system T, and X, R range over arbitrary finite types. In his seminal paper [15], when producing a realiser for the dialectica interpretation [1, 7] of the classical countable choice, Spector singled out two forms of bar recursion. The first, which he clearly considered to be the *general bar recursion* of type (X, R) is defined as

$$\mathsf{gBR}^{X,R}_s(\omega)(\phi)(q) \stackrel{R}{=} \begin{cases} q_s & \text{if } \omega(\hat{s}) < |s| \\ \phi_s(\lambda x^X.\mathsf{gBR}_{s*x}(\omega)(\phi)(q)) & \text{if } \omega(\hat{s}) \geqslant |s|, \end{cases} \quad (1)$$

where $s * x$ denotes concatenation of strings, and $\hat{s} \colon X^{\mathbb{N}}$ is some fixed extension of the finite sequence $s \colon X^*$ into an infinite sequence, say $\hat{s} = s * \mathbf{0}^{X^{\mathbb{N}}}$. The types of ω and q_s and ϕ_s are:

$$\begin{aligned} \omega &: X^{\mathbb{N}} \to \mathbb{N} \\ q_s &: R \\ \phi_s &: (X \to R) \to R. \end{aligned}$$

We think of functionals of type $X^* \to Y$ as a family of objects of type Y indexed by the finite sequences $s \colon X^*$, and hence write $f \colon X^* \to Y$ as $f_s \colon Y$. In Equation (1) both gBR and q and ϕ are viewed in this way.

When actually constructing a *dialectica* realizer for the double negation shift, Spector shows that only the following *restricted bar recursion* is needed

$$\mathsf{rBR}^{X}_s(\omega)(\varepsilon) \stackrel{X^{\mathbb{N}}}{=} s \mathbin{@} \begin{cases} \mathbf{0}^{X^{\mathbb{N}}} & \text{if } \omega(\hat{s}) < |s| \\ \mathsf{rBR}_{s*a_s}(\omega)(\varepsilon) & \text{if } \omega(\hat{s}) \geqslant |s|, \end{cases} \quad (2)$$

where $a_s = \varepsilon_s(\lambda x^X.\mathsf{rBR}_{s*x}(\omega)(\varepsilon))$, and $s \mathbin{@} \alpha$ denotes the infinite sequence consisting of s following by $\lambda n.\alpha(n+|s|)$, the types of s and ω are as above, and ε_s is of type $(X \to X^{\mathbb{N}}) \to X$.

The simplest way to understand bar recursion is as a recursive definition of a function Φ_s whose domain is the set of nodes of a well-founded tree $s \in T$. First define the value of the function on the leaves of the tree, and then

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show how the values of immediate children Φ_{s*x} can be used to calculate the value on a particular node Φ_s. It is then clear that bar recursion is the computational analogue of the logical principle of bar induction.

Spector observed that for the computational (dialectica) interpretation of countable choice the appropriate well-founded tree to consider is

$$T = \{s : \forall t \preceq s(\omega(\hat{t}) \geqslant |t|)\},$$

where $t \preceq s$ stands for: t is a prefix of s. T is a well-founded tree whenever for any sequence α there exists a point n such that $\omega(\overline{\alpha, n}) < n$, where $\overline{\alpha, n}$ is some canonical extension of the finite initial segment of α of length n. This happens to hold whenever ω is either a continuous functional [13] or a majorizable functional [2, 16].

Spector defines rBR in section 10 (page 15) of [15], simply saying that this is a "restricted form" of gBR, and that rBR is all one needs to interpret the double negation shift and hence (classical) countable choice. Subsequent uses of bar recursion, however, have mainly made use of the more general form gBR (cf. [8]). This raises the question of whether rBR, which is exactly what one needs to dialectica-interpret countable choice, is indeed strictly more restrictive than gBR, which, on the other hand, has been used for other purposes. This paper shows that this "restricted" form of bar recursion is in fact as general as "general" bar recursion. We do this by constructing a term in T + rBR which satisfies (provably in T + rBR) the equation for gBR (Theorem 5.2).

Remark 1.1 Following the notation of [4], we call $\varepsilon_s : (X \to R) \to X$ *selection functions*, whereas $\phi_s : (X \to R) \to R$ are called *quantifiers*. In the following, the type of selection functions $(X \to R) \to X$ will be abbreviated $J_R X$, and the type of quantifiers $(X \to R) \to R$ will be abbreviated as $K_R X$. It has been shown in [3, 5] (cf. also Section 6) that gBR is equivalent to an iterated product of quantifiers, whereas rBR is equivalent to a corresponding product of selection functions.

1.1 System T

Let T denote Gödel's calculus of primitive recursive functionals with weak extensionality (cf. [16], section 1.6.12). For convenience we shall assume that T contains not only function types $X \to Y$ but also product $X \times Y$, finite product X^*, and co-product types $X + Y$. These are easily definable with the help of $X \to Y$ and the basic type $\mathbb{N}$ of natural numbers. Given $x : X$ and $s, t : X^*$ and $\alpha : X^{\mathbb{N}}$ and $q : X^* \to R$, we shall use the following notation:

- $\mathbf{n}^X$ stands for the constant n functional of type X, e.g. $\mathbf{0}^X$ or $\mathbf{1}^X$.
- $s * x$ and $s * t$ and $s * \alpha$ denotes the concatenation of words, finite or infinite.
- $s \,@\, \alpha$ denotes the infinite sequence starting with s followed by the tail of α from point $|s|$.
- $s \,@\, t$ denotes the finite sequence starting with s followed by, if $|t| > |s|$, the tail of t from point $|s|$.
- $\hat{s} : X^{\mathbb{N}}$ denotes the extension of s with $\mathbf{0}$'s into an infinite sequence.
- $\overline{\alpha}(n) : X^*$ denotes the finite word consisting of the first n values of α.
- $\overline{\alpha, n} : X^{\mathbb{N}}$ denotes the infinite sequence consisting of the first n values of α followed by $\mathbf{0}$'s.
- Given $a : X$ and $b : Y$ we denote by $[a]_X$ and $[b]_Y$ the injections of a and b into the type $X + Y$.

1.2 Spector's search functional

Let us call *Spector's search functional* the functional

$$\mu_{\mathsf{Sp}}(\omega)(\alpha) = \text{least } n \text{ such that } \omega(\overline{\alpha, n}) < n. \tag{3}$$

Or, more formally, the functional μ_{Sp} defined by

$$\underbrace{\omega(\overline{\alpha, \mu_{\mathsf{Sp}}(\omega)(\alpha)}) < \mu_{\mathsf{Sp}}(\omega)(\alpha) \ \wedge \ (\omega(\overline{\alpha, n}) < n \to n \geqslant \mu_{\mathsf{Sp}}(\omega)(\alpha))}_{\mathsf{SP}(\omega, \alpha, \mu_{\mathsf{Sp}})}. \tag{4}$$

Howard[1] showed (cf. [8], Lemma 3C) how μ_{Sp} is definable over system T from the general form of bar recursion gBR. Clearly μ_{Sp} does not exist[2] in all models of system T (e.g. the full set-theoretic model) and hence it is not T-definable. The next lemma, however, shows that, quite surprisingly, the "totalisation" of Spector's search functional is T-definable:

Lemma 1.2 *There exists a term $t \in T$ such that, for all n, the following is provable in T*

$$\omega(\overline{\alpha, n}) < n \to \mathsf{SP}(\omega, \alpha, t).$$

P r o o f. We show how the unbounded search μ_{Sp} can be turned into a bounded search, assuming an n satisfying $\omega(\overline{\alpha, n}) < n$ exists. Consider the following construction, given $\alpha \colon X^{\mathbb{N}}$ and $\omega \colon X^{\mathbb{N}} \to \mathbb{N}$ define α^ω as

$$\alpha^\omega(i) = \begin{cases} \mathbf{0}^X & \text{if } \exists k \leqslant i+1\, (\omega(\overline{\alpha, k}) < k) \\ \alpha(i) & \text{otherwise.} \end{cases}$$

Under the assumption $\exists n(\omega(\overline{\alpha, n}) < n)$, let n be the least number such that $\omega(\overline{\alpha, n}) < n$ holds. Clearly $n > 0$. By the minimality of n, $\alpha^\omega(i) = \alpha(i)$, for $i < n-1$, and $\alpha(i) = \mathbf{0}$, for $i \geqslant n-1$. Hence, $\alpha^\omega = \overline{\alpha, n-1}$. Because n is the least, we must have that $\omega(\alpha^\omega) = \omega(\overline{\alpha, n-1}) \geqslant n-1$, and hence $n \leqslant \omega(\alpha^\omega) + 1$. Therefore, $\omega(\alpha^\omega) + 1$ serves as an upper bound on the search μ_{Sp}. □

Remark 1.3 The search for the least $n > N$ satisfying $\omega(\overline{\alpha, n}) < n$ can be defined from μ_{Sp} as

$$\mu_{\mathsf{Sp}}(\underbrace{\lambda\beta.\omega(\overline{\alpha}(N) * \beta) \dot{-} N}_{\hat{\omega}})(\underbrace{\lambda i.\alpha(i+N)}_{\hat{\alpha}}),$$

where $n \dot{-} m$ denotes cut-off subtraction. That follows since the least n such that $\hat{\omega}(\overline{\hat{\alpha}, n}) < n$ is also the least $n > 0$ satisfying $\omega(\overline{\alpha, n+N}) < n + N$, as

$$n > \hat{\omega}(\overline{\hat{\alpha}, n}) = \omega(\overline{\alpha}(N) * \overline{\hat{\alpha}, n}) \dot{-} N = \omega(\overline{\alpha, n+N}) \dot{-} N$$

implies $\omega(\overline{\alpha, n+N}) < n + N$.

2 Restricted Bar Recursion from General Bar Recursion

When drawing an anology between the well-known principle of bar induction, and the novel form of recursion (to be called bar recursion) Spector [15] first arrived at the general form of bar recursion gBR. Then, when constructing a dialectica realizer for the double negation shift, and a posteriori for the classical axiom of countable choice, Spector observed that only the "restricted" form rBR of gBR was necessary. We presume that Spector believed rBR was strictly weaker than gBR, as otherwise he would not have bothered to give rBR a label "BR".

He then states the restricted form of bar recursion rBR, without feeling the need to give an indication as to how rBR can be obtained from gBR. Although this is rather simple indeed, let us in this section briefly show how rBR follows from gBR over T, as a warm-up for the results that are to follow (this is also shown in [11, 12]).

Theorem 2.1 *There exists a term Φ of $T + \mathsf{gBR}^{X,X^{\mathbb{N}}}$, using a single instance of $\mathsf{gBR}^{X,X^{\mathbb{N}}}$, such that $\Phi_s(\omega)(\phi)(q)$ satisfies the equation for rBR^X, provably in $T + \mathsf{gBR}^{X,X^{\mathbb{N}}}$.*

P r o o f. For each selection function $\varepsilon_s \colon J_{X^{\mathbb{N}}} X$, define a quantifier $\phi_s \colon K_{X^{\mathbb{N}}} X$ as

$$\phi_s(p^{X \to X^{\mathbb{N}}}) \stackrel{X^{\mathbb{N}}}{=} p(\varepsilon_s p). \tag{5}$$

Given the parameters of rBR, i.e. $s \colon X^*$ and $\varepsilon_s \colon J_{X^{\mathbb{N}}} X$ and $\omega \colon X^{\mathbb{N}} \to \mathbb{N}$, we claim that

$$\Phi_s(\omega)(\varepsilon) \stackrel{X^{\mathbb{N}}}{=} \mathsf{gBR}_s(\omega)(\phi)(\lambda t.\hat{t}) \tag{6}$$

does the job, where ϕ is defined from ε as in (5). We argue by cases:

If $\omega(\hat{s}) < |s|$ then

[1] Howard in fact attributes the construction in the proof of Lemma 3C to Kreisel

[2] i.e. is not uniquely determined by equation (4).

$$\begin{aligned}\Phi_s(\omega)(\varepsilon) &\overset{(6)}{=} \mathsf{gBR}_s(\omega)(\phi)(\lambda t.\hat{t}) \\ &\overset{(1)}{=} \hat{s}.\end{aligned}$$

If $\omega(\hat{s}) \geqslant |s|$ then

$$\begin{aligned}\Phi_s(\omega)(\varepsilon) &\overset{(6)}{=} \mathsf{gBR}_s(\omega)(\phi)(\lambda t.\hat{t}) \\ &\overset{(1)}{=} \phi_s(\lambda x^X.\mathsf{gBR}_{s*x}(\omega)(\phi)(\lambda t.\hat{t})) \\ &\overset{(5)}{=} \mathsf{gBR}_{s*a_s}(\omega)(\phi)(\lambda t.\hat{t}) \\ &\overset{(6)}{=} \Phi_{s*a_s}(\omega)(\varepsilon)\end{aligned}$$

where $a_s \overset{(5)}{=} \varepsilon_s(\lambda x.\mathsf{gBR}_{s*x}(\omega)(\phi)(\lambda t.\hat{t})) \overset{(6)}{=} \varepsilon_s(\lambda x.\Phi_{s*x}(\omega)(\varepsilon))$. □

3 Kreisel's Trick via the Restricted Bar Recursion

Our main result in this paper is to show the converse of Theorem 2.1. The main step in our proof will be to show that Kreisel/Howard construction of [8] (Lemma 3C) can already be done in $T + \mathsf{rBR}$, i.e. μ_{Sp} is already definable in $T + \mathsf{rBR}$.

Theorem 3.1 *There exists a term t of $T + \mathsf{rBR}^X$ such that $T + \mathsf{rBR}^X$ proves for all $\omega \colon X^{\mathbb{N}} \to \mathbb{N}$ and $\alpha \colon X^{\mathbb{N}}$*

$$\exists n \leqslant t(\omega)(\alpha) \; \big(\omega(\overline{\alpha, n}) < n\big).$$

P r o o f. Assume $\omega \colon X^{\mathbb{N}} \to \mathbb{N}$ and $\alpha \colon X^{\mathbb{N}}$ are given. We use rBR for the type X. Define a family of selection functions $\varepsilon_s \colon (X \to X^{\mathbb{N}}) \to X$ as

$$\varepsilon_s p \overset{X}{=} \begin{cases} \mathbf{0}^X & \text{if } \exists j \leqslant |s| \, (\omega(\overline{\alpha, j}) < j) \\ \mathbf{1}^X + p(\alpha(|s|))(|s|+1) & \text{otherwise.} \end{cases} \tag{7}$$

Let

$$\beta(i) \overset{\mathbb{N}}{=} \mathsf{rBR}_{\overline{\alpha}(i)}(\omega)(\varepsilon)(i)(\mathbf{0}^Y), \tag{8}$$

where $X = Y \to \mathbb{N}$. We claim that

$$\exists i \leqslant \beta(0) \, (\omega(\overline{\alpha, i}) < i). \tag{9}$$

First, let us show the following:

$$\beta(i) \overset{X}{=} \begin{cases} 0 & \text{if } \exists j \leqslant i(\omega(\overline{\alpha, j}) < j) \\ 1 + \beta(i+1) & \text{otherwise.} \end{cases} \tag{10}$$

We prove (10) by cases: Fix i:

a) If $\omega(\overline{\alpha, i}) < i$ then

$$\beta(i) \overset{(8)}{=} \mathsf{rBR}_{\overline{\alpha}(i)}(\omega)(\varepsilon)(i)(\mathbf{0}^Y) \overset{(2)}{=} (\overline{\alpha}(i) \;@\; \mathbf{0}^{X^{\mathbb{N}}})(i)(\mathbf{0}^Y) = 0.$$

b) If $\omega(\overline{\alpha, i}) \geqslant i$ and $\exists j \leqslant i \, (\omega(\overline{\alpha, j}) < j)$ then

$$\beta(i) \overset{(8)}{=} \mathsf{rBR}_{\overline{\alpha}(i)}(\omega)(\varepsilon)(i)(\mathbf{0}^Y) \overset{(2)}{=} \varepsilon_{\overline{\alpha}(i)}(\lambda x.\mathsf{rBR}_{\overline{\alpha}(i)*x}(\omega)(\varepsilon))(\mathbf{0}^Y) \overset{(7)}{=} 0.$$

c) If $\omega(\overline{\alpha, i}) \geqslant i$ and $\forall j \leqslant i \, (\omega(\overline{\alpha, j}) \geqslant j)$

$$\begin{aligned} \beta(i) &\overset{(8)}{=} \mathsf{rBR}_{\overline{\alpha(i)}}(\omega)(\varepsilon)(i)(\mathbf{0}^Y) \\ &\overset{(2)}{=} \varepsilon_{\overline{\alpha(i)}}(\lambda x.\mathsf{rBR}_{\overline{\alpha(i)}*x}(\omega)(\varepsilon))(\mathbf{0}^Y) \\ &\overset{(7)}{=} (\mathbf{1} + \mathsf{rBR}_{\overline{\alpha(i+1)}}(\omega)(\varepsilon)(i+1))(\mathbf{0}^Y) \\ &\overset{(8)}{=} 1 + \beta(i+1). \end{aligned}$$

Therefore, by a), b) and c) we can conclude (10). Hence,

$$\beta(i) \neq 0 \wedge j \leqslant i \rightarrow \beta(j) = 1 + \beta(j+1).$$

By induction, taking $j = 0$ to $j = i - 1$, it follows that

$$\beta(i) \neq 0 \rightarrow \beta(0) = i + \beta(i).$$

Taking $i = \beta(0)$ we have that $\beta(\beta 0) = 0$. Hence, by (10), we obtain (9), and we can take $t = \beta(0)$. □

Combining Lemma 1.2 and Theorem 3.1 we have that there exists a term already in Gödel system T such that, provably in $T + \mathsf{rBR}$, this term computes Spector's search functional.

Corollary 3.2 *There exists a term $t \in T$ such that $T + \mathsf{rBR}^X$ proves $\omega(\overline{\alpha, n}) < n$ where $n = t(\omega)(\alpha)$.*

4 Restricted Bar Recursion for Finite Sequences

Consider the following useful variant of rBR where the infinite sequence type $X^{\mathbb{N}}$ is replaced by the type of finite sequences X^*. Let us call this *restricted bar recursion for finite sequences*

$$\mathsf{fBR}^X_s(\omega)(\varepsilon) \overset{X^*}{=} s \,@ \begin{cases} s & \text{if } \omega(\hat{s}) < |s| \\ \mathsf{fBR}_{s*a_s}(\omega)(\varepsilon) & \text{if } \omega(\hat{s}) \geqslant |s|, \end{cases} \tag{11}$$

where $a_s = \varepsilon_s(\lambda x^X.\mathsf{fBR}_{s*x}(\omega)(\varepsilon))$ and ε_s is of type $(X \to X^*) \to X$. Let us show, using Theorem 3.1 that this is T-definable from rBR.

Lemma 4.1 *There exists a term Φ of $T + \mathsf{rBR}^X$, using a single instance of rBR^X, such that $\Phi_s(\omega)(\varepsilon)$ satisfies the equation for fBR^X, provably in $T + \mathsf{rBR}^X$.*

Proof. For any α and k let us denote by $n^{\alpha,k}$ the least $i \geqslant k$ such that $\omega(\overline{\alpha, i}) < i$, where ω will be fixed throughout the proof. By Corollary 3.2, $n^{\alpha,k}$ is T-definable provably in $T + \mathsf{rBR}^X$. Note also that, if $\omega(\hat{s}) \geqslant |s|$ then

$$n^{s*\alpha,|s|} = n^{s*\alpha,|s|+1}. \tag{12}$$

Now, for each selection function $\varepsilon_s \colon J_{X*}X$, define the following selection function $\tilde{\varepsilon}_s \colon J_{X^{\mathbb{N}}}X$

$$\tilde{\varepsilon}_s(p^{X \to X^{\mathbb{N}}}) \overset{X}{=} \varepsilon_s(\lambda x.\overline{p(x)}(n^{px,|s|})). \tag{13}$$

In words, for each x we only look at the initial segment of the infinite sequence $p(x)$ up to the point $i \geqslant |s|$ where Spector's condition first holds. Given the parameters of fBR, i.e. $s \colon X^*$ and $\varepsilon_s \colon J_{X*}X$ and $\omega \colon X^{\mathbb{N}} \to \mathbb{N}$, we claim that

$$\Phi_s(\omega)(\varepsilon) \overset{X^*}{=} \overline{\mathsf{rBR}_s(\omega)(\tilde{\varepsilon})}(n^{\mathsf{rBR}_s(\omega)(\tilde{\varepsilon}),|s|}) \tag{14}$$

does the job. We argue by cases:

If $\omega(\hat{s}) < |s|$ then

$$\begin{aligned}\Phi_s(\omega)(\varepsilon) &\stackrel{(14)}{=} \overline{\mathsf{rBR}_s(\omega)(\tilde{\varepsilon})}(n^{\mathsf{rBR}_s(\omega)(\tilde{\varepsilon}),|s|}) \\ &\stackrel{(2)}{=} \overline{\hat{s}}(n^{\hat{s},|s|}) \\ &= s,\end{aligned}$$

since $n^{\hat{s},|s|} = |s|$.
If $\omega(\hat{s}) \geqslant |s|$ then ($\tilde{a}_s = a_s$ is shown below)

$$\begin{aligned}\Phi_s(\omega)(\varepsilon) &\stackrel{(14)}{=} \overline{\mathsf{rBR}_s(\omega)(\tilde{\varepsilon})}(n^{\mathsf{rBR}_s(\omega)(\tilde{\varepsilon}),|s|}) \\ &\stackrel{(2)}{=} \overline{(s \ @ \ \mathsf{rBR}_{s*\tilde{a}_s}(\omega)(\tilde{\varepsilon}))}(n^{\mathsf{rBR}_{s*\tilde{a}_s}(\omega)(\tilde{\varepsilon}),|s|}) \\ &= \overline{(s \ @ \ \mathsf{rBR}_{s*a_s}(\omega)(\tilde{\varepsilon}))}(n^{\mathsf{rBR}_{s*a_s}(\omega)(\tilde{\varepsilon}),|s|}) \\ &= s \ @ \ \overline{\mathsf{rBR}_{s*a_s}(\omega)(\tilde{\varepsilon})}(n^{\mathsf{rBR}_{s*a_s}(\omega)(\tilde{\varepsilon}),|s*a_s|}) \\ &\stackrel{(14)}{=} s \ @ \ \Phi_{s*a_s}(\omega)(\varepsilon),\end{aligned}$$

where

$$\begin{aligned}\tilde{a}_s &\stackrel{(2)}{=} \tilde{\varepsilon}_s(\lambda x.\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon})) \\ &\stackrel{(13)}{=} \varepsilon_s(\lambda x.\overline{\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon})}(n^{\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon}),|s|})) \\ &\stackrel{(12)}{=} \varepsilon_s(\lambda x.\overline{\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon})}(n^{\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon}),|s*x|})) \\ &\stackrel{(14)}{=} \varepsilon_s(\lambda x.\Phi_{s*x}(\omega)(\varepsilon)) \\ &\stackrel{(11)}{=} a_s.\end{aligned}$$

□

Remark 4.2 It will follow from the main result of the next section that fBR is in fact equivalent to rBR. Nevertheless, we believe fBR to be much more powerful in practice, since it already embodies the Spector search μ_{Sp}, as

$$\mu_{\mathsf{Sp}}(\omega)(\alpha) = |\mathsf{fBR}_{\langle\rangle}(\omega)(\lambda s \lambda p.p(\alpha(|s|)))|.$$

Moreover, rather than computing a solution to Spector's equations

$$\begin{aligned}\alpha(n) &= \varepsilon_{\overline{\alpha}(n)} p_{\overline{\alpha}(n)} \\ q\alpha &= p_{\overline{\alpha}(n)}(\varepsilon_{\overline{\alpha}(n)} p_{\overline{\alpha}(n)})\end{aligned}$$

for all $n \leqslant \omega\alpha$ as one normally does with rBR, with fBR we can in fact compute a *finite sequence* s such that

$$\begin{aligned}s_n &= \varepsilon_{\overline{s}(n)} p_{\overline{s}(n)} \\ q\hat{s} &= p_{\overline{s}(n)}(\varepsilon_{\overline{s}(n)} p_{\overline{s}(n)})\end{aligned}$$

for all $n < |s|$, with the extra guarantee that $\omega\hat{s} < |s|$ (so in particular we have the equations for $n \leqslant \omega\hat{s}$).

5 Restricted Bar Recursion is not Restricted

In this section we show the converse of Theorem 2.1, i.e. that "restricted" bar recursion is as general as "general" bar recursion. Because of Lemma 4.1, it is enough to show that gBR is T-definable from fBR. The intuition behind our construction is as follows. General bar recursion gBR can be thought of as assigning an R-value V_s to each node s of Spector's tree. At the leaves of the tree the value is simply $V_s = q_s$, whereas at intermediate nodes s the value is computed by the quantifier as $V_s = \phi_s(\lambda x.V_{s*x})$ querying the R-value of potentially several children nodes V_{s*x}. The restricted bar recursion (for finite sequences) performs a similar exhaustive search, but instead of computing an R-value for each node s, it instead computes a path extending s. Therefore, in order to simulate $\mathsf{gBR}^{X,R}$ we will use fBR^{X+R} so that we can encode the R-value of node s as the first element in the path extending s. We can easily recover this R-value because the path extension computed by fBR is a finite sequence which we can search for the first R-value.

Let us first define a few (T-definable) constructions:

1. Let $\check{s}$ be the standard embedding of $s \colon X^*$ into $(X + R)^*$.
2. Given an $\alpha \colon (X + R)^{\mathbb{N}}$ define $\tilde{\alpha} \colon X^{\mathbb{N}}$ as
$$\tilde{\alpha}(k) \stackrel{X}{=} \begin{cases} \mathbf{0}^X & \text{if } \alpha(i) \in R \\ \alpha(i) & \text{if } \alpha(i) \in X. \end{cases}$$
3. Given a quantifier $\phi_s \colon K_R X$ and $q \colon (X + R)^* \to R$ define the selection function $\varepsilon_s \colon J_{(X+R)^*}(X + R)$ as
$$\varepsilon_s(p^{X+R\to(X+R)^*}) \stackrel{X+R}{=} [\phi_{\check{s}}(\lambda x.q(p([x]_X)))]_R. \tag{15}$$
4. Given $q \colon X^* \to R$ define $\tilde{q} \colon (X + R)^* \to R$
$$\tilde{q}s \stackrel{R}{=} \begin{cases} q(\tilde{s}) & \text{if } \forall i < |s|(s_i \in X) \\ s_i & \text{for least } i < |s| \text{ such that } s_i \in R. \end{cases}$$
5. Given $\omega \colon X^{\mathbb{N}} \to R$ define $\tilde{\omega} \colon (X + R)^{\mathbb{N}} \to \mathbb{N}$
$$\tilde{\omega}\alpha \stackrel{\mathbb{N}}{=} \omega(\tilde{\alpha})$$

Lemma 5.1 *The following is easily derivable from the definitions above:*

- $\omega(\hat{s}) < |s|$ *if and only if* $\tilde{\omega}(\check{s} * \mathbf{0}) < |\check{s}|$.
- $\tilde{q}(\check{s} * [a]_R * \beta) = a$.

Theorem 5.2 *There exists a term Φ of $T + \mathsf{fBR}^{X+R}$, using a single instance of fBR^{X+R}, such that $\Phi_s(\omega)(\phi)(q)$ satisfies the equation for $\mathsf{gBR}^{X,R}$, provably in $T + \mathsf{fBR}^{X+R}$.*

Proof. Let $s \colon X^*$ and $\phi_s \colon K_R X$ and $\omega \colon X^{\mathbb{N}} \to \mathbb{N}$ and $q \colon X^* \to R$ be given. We claim that Φ_s defined as
$$\Phi_s(\omega)(\phi)(q) \stackrel{R}{=} \tilde{q}(\mathsf{fBR}_{\check{s}}(\tilde{\omega})(\varepsilon^{\phi,q})) \tag{16}$$
satisfies the equation for gBR, where $\tilde{q}$ and $\tilde{\omega}$ are defined from s as above. Let us argue by cases.

If $\omega(\hat{s}) < |s|$, by Lemma 5.1, we also have $\tilde{\omega}(\check{s} * \mathbf{0}) < |\check{s}|$. Therefore,
$$\Phi_s(\omega)(\phi)(q) \stackrel{(16)}{=} \tilde{q}(\mathsf{fBR}_{\check{s}}(\tilde{\omega})(\varepsilon^{\phi,q})) \stackrel{(2)}{=} \tilde{q}(\check{s}) = q_s.$$

If $\omega(\hat{s}) \geqslant |s|$, again by Lemma 5.1, $\tilde{\omega}(\check{s} * \mathbf{0}) \geqslant |\check{s}|$. Hence, we have
$$\begin{aligned} \Phi_s(\omega)(\phi)(q) &\stackrel{(16)}{=} \tilde{q}(\mathsf{fBR}_{\check{s}}(\tilde{\omega})(\varepsilon^{\phi,q})) \\ &\stackrel{(11)}{=} \tilde{q}(\check{s} * [a_{\check{s}}]_R \mathbin{@} \mathsf{fBR}_{\check{s}*a_{\check{s}}}(\tilde{\omega})(\varepsilon^{\phi,q})) \\ &\stackrel{\text{L5.1}}{=} a_{\check{s}} \\ &\stackrel{(*)}{=} \phi_s(\lambda x^X.\tilde{q}(\mathsf{fBR}_{(\check{s*x})}(\tilde{\omega})(\varepsilon^{\phi,q}))) \\ &\stackrel{(16)}{=} \phi_s(\lambda x.\Phi_{s*x}(\omega)(\phi)(q)) \end{aligned}$$

where $(*)$ $[a_{\check{s}}]_R = \varepsilon_s^{\phi,q}(\lambda x^{X+R}.\mathsf{fBR}_{\check{s}*x}(\tilde{\omega})(\varepsilon^{\phi,q}))$ so that

$$\begin{aligned} a_{\check{s}} &\overset{(15)}{=} \phi_s(\lambda x^X.\tilde{q}(\mathsf{fBR}_{\check{s}*[x]_X}(\tilde{\omega})(\varepsilon^{\phi,q}))) \\ &= \phi_s(\lambda x^X.\tilde{q}(\mathsf{fBR}_{(\check{s*x})}(\tilde{\omega})(\varepsilon^{\phi,q}))) \\ &\overset{(16)}{=} \phi_s(\lambda x^X.\Phi_{s*x}(\omega)(\phi)(q)). \end{aligned}$$

The fact that Φ only makes use of a single instance of fBR^{X+R} follows since (provably in $T + \mathsf{fBR}^X$) Spector's search function μ_{Sp} is T-definable (Corollary 3.2). □

Combining Lemma 4.1 and Theorems 2.1 and 5.2 we can conclude the following:

Corollary 5.3 $\mathsf{gBR}, \mathsf{fBR}$ *and* rBR *are inter-definable over system* T*. Moreover,*

- *a single instance of* $\mathsf{gBR}^{X,X^{\mathbb{N}}}$ *defines* rBR^X *provably in* $T + \mathsf{gBR}^{X,X^{\mathbb{N}}}$*, and*
- *a single instance of* rBR^{X+R} *defines* $\mathsf{gBR}^{X,R}$ *provably in* $T + \mathsf{rBR}^{X+R}$*.*

Remark 5.4 Howard [9] shows that semi-terms (terms with at most function parameters) of type 0 in $T_1 + \mathsf{gBR}^{\mathbb{N},\mathbb{N}^{\mathbb{N}}}$ have the associated ordinal ε_0. Schwichtenberg [14] shows that when X has type level 0 or 1 (for R arbitrary) and ϕ, ω and q are T-definable then the function $\lambda s.\mathsf{gBR}_s^{X,R}(\omega)(\phi)(q)$ is also T-definable. Let T_i be the fragment of system T with primitive recursion of type level $\leqslant i$ only. Kohlenbach [10] uses this result combined with a normalization argument, showing, in particular, that all type 2 definable functionals of $T_1 + \mathsf{gBR}^{\mathbb{N},\mathbb{N}^{\mathbb{N}}}$ are T-definable, and that this is no longer the case for either $T_2 + \mathsf{gBR}^{\mathbb{N},\mathbb{N}^{\mathbb{N}}}$ or $T_0 + \mathsf{gBR}^{\mathbb{N}^{\mathbb{N}},\mathbb{N}^{\mathbb{N}}}$ or $T_0 + \mathsf{gBR}^{\mathbb{N},\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}}$. From the negative results of Kohlenbach one immediately concludes that there are functionals of type 2 in $T_2 + \mathsf{rBR}^{\mathbb{N}}$ or $T_0 + \mathsf{rBR}^{\mathbb{N}^{\mathbb{N}}}$ which are not T-definable. Note that, by Corollary 5.3, in order to define gBR of lowest type (i.e. $\mathsf{gBR}^{\mathbb{N},\mathbb{N}^{\mathbb{N}}}$) we have used $\mathsf{fBR}^{\mathbb{N}^{\mathbb{N}}}$. But, $\mathsf{fBR}^{\mathbb{N}^{\mathbb{N}}}$ already suffices to define $\mathsf{gBR}^{\mathbb{N}^{\mathbb{N}},\mathbb{N}^{\mathbb{N}}}$, which is much stronger than $\mathsf{gBR}^{\mathbb{N},\mathbb{N}^{\mathbb{N}}}$. So, although rBR and gBR are T-equivalent as a schema over all types, Spector was right in making it explicit that his solution only uses rBR since his proof shows that $\mathsf{rBR}^{\mathbb{N}}$ is enough to interpret arithmetical comprehension. This is not implied by the fact that $\mathsf{gBR}^{\mathbb{N},\mathbb{N}^{\mathbb{N}}}$ also suffices for this. We would like to thank the anonymous referee for observing this and communicating it to us.

6 Product of Selections Functions and Quantifiers

Given a family of selection functions $\varepsilon_s \colon (X \to R) \to X$, consider the following "product" of these selection functions

$$\widetilde{\mathsf{EPS}}_s^{X,R}(\omega)(\varepsilon)(q) \overset{X^{\mathbb{N}}}{=} s \ @ \begin{cases} \mathbf{0} & \text{if } \omega(\hat{s}) < |s| \\ \widetilde{\mathsf{EPS}}_{s*a_s}(\omega)(\varepsilon)(q) & \text{if } \omega(\hat{s}) \geqslant |s|, \end{cases} \tag{17}$$

where $a_s \overset{X}{=} \varepsilon_s(\lambda x^X.q(\widetilde{\mathsf{EPS}}_{s*x}(\omega)(\varepsilon)(q)))$. This is a slight modification of the explicitly controlled product of selection functions EPS as it has actually been defined in [4, 6], since there the functional q also changes during the recursive calls so that EPS only needs to compute the "continuation" of s (cf. Remark 6.3 below).

Theorem 6.1 *The restricted form of bar recursion* rBR *is* T*-equivalent to* $\widetilde{\mathsf{EPS}}$*. More precisely,*

- *a single instance of* $\widetilde{\mathsf{EPS}}^{X,X^{\mathbb{N}}}$ *defines* rBR^X *provably in* $T + \widetilde{\mathsf{EPS}}^{X,X^{\mathbb{N}}}$*, and*
- *a single instance of* rBR^X *defines* $\widetilde{\mathsf{EPS}}^{X,R}$ *provably in* $T + \mathsf{rBR}^X$*.*

This in particular implies that the type R *in* $\widetilde{\mathsf{EPS}}^{X,R}$ *is not too important, as* $\widetilde{\mathsf{EPS}}^{X,R}$ *for arbitrary types* R *is* T*-equivalent to* $\widetilde{\mathsf{EPS}}^{X,X^{\mathbb{N}}}$*.*

Proof. Clearly $\widetilde{\mathsf{EPS}}^{X,X^{\mathbb{N}}}$ defines rBR^X, taking q as the identify functional. For the other direction, we claim that $\widetilde{\mathsf{EPS}}^{X,R}$ is definable in $T + \mathsf{rBR}^X$. Given ω, ε and q, define

$$\tilde{\varepsilon}_s(p^{X\to X^{\mathbb{N}}}) \stackrel{X}{=} \varepsilon_s(\lambda x^X.q(px)). \tag{18}$$

We claim that

$$\Phi_s(\omega)(\varepsilon)(q) \stackrel{X^{\mathbb{N}}}{=} \mathsf{rBR}_s(\omega)(\tilde{\varepsilon}) \tag{19}$$

satisfies the equation for $\widetilde{\mathsf{EPS}}$.
If $\omega(\hat{s}) < |s|$ then

$$\Phi_s(\omega)(\varepsilon)(q) \stackrel{(19)}{=} \mathsf{rBR}_s(\omega)(\tilde{\varepsilon}) \stackrel{(2)}{=} \hat{s}.$$

If $\omega(\hat{s}) \geqslant |s|$ then

$$\begin{aligned}
\Phi_s(\omega)(\varepsilon)(q) &\stackrel{(19)}{=} \mathsf{rBR}_s(\omega)(\tilde{\varepsilon}) \\
&\stackrel{(2)}{=} (s * \tilde{a}_s) \ @ \ \mathsf{rBR}_{s*\tilde{a}_s}(\omega)(\tilde{\varepsilon}) \\
&= (s * a_s) \ @ \ \mathsf{rBR}_{s*a_s}(\omega)(\tilde{\varepsilon}) \\
&\stackrel{(19)}{=} (s * a_s) \ @ \ \Phi_{s*a_s}(\omega)(\varepsilon)(q)
\end{aligned}$$

where $\tilde{a}_s = \tilde{\varepsilon}_s(\lambda x.\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon})) = \varepsilon_s(\lambda x.q(\mathsf{rBR}_{s*x}(\omega)(\tilde{\varepsilon}))) = \varepsilon_s(\lambda x.q(\Phi_{s*x}(\omega)(\varepsilon)(q))) = a_s$. □

Similarly, given a family of quantifiers $\phi_s : (X \to R) \to R$, consider the following "product" of these quantifiers (where again we slightly modify the construction EPQ of [4, 6] so that q is fixed throughout the recursion)

$$\widetilde{\mathsf{EPQ}}_s^{X,R}(\omega)(\phi)(q) \stackrel{R}{=} \begin{cases} q(s) & \text{if } \omega(\hat{s}) < |s| \\ \phi_s(\lambda x.\widetilde{\mathsf{EPQ}}_{s*x}(\omega)(\varepsilon)(q)) & \text{if } \omega(\hat{s}) \geqslant |s|. \end{cases} \tag{20}$$

It is obvious that $\widetilde{\mathsf{EPQ}}$ and gBR are exactly the same recursion schema. It the follows from Corollary 5.3 that the iterated product of quantifiers is T-equivalent to the iterated product of selection functions:

Corollary 6.2 *$\widetilde{\mathsf{EPQ}}$ is T-equivalent to $\widetilde{\mathsf{EPS}}$.*

Remark 6.3 Given a functional $q : X^{\mathbb{N}} \to R$ and an element $x \in X$, define $q_x : X^{\mathbb{N}} \to R$ as $q_x(\alpha) = q(x * \alpha)$. The iterated products EPS and EPQ as defined in [4, 6] are actually slightly different than $\widetilde{\mathsf{EPS}}$ and $\widetilde{\mathsf{EPQ}}$, since the functional q is not fixed throughout the recursion, but changes to a q_x at each recursive call. It is easy to show within system T that e.g. EPS defines $\widetilde{\mathsf{EPS}}$ as

$$\widetilde{\mathsf{EPS}}_s(\omega)(\varepsilon)(q) = s * \mathsf{EPS}_s(\omega)(\varepsilon)(q_s).$$

On the other hand, although it is clear that EPS is definable in $\widetilde{\mathsf{EPS}}$, we have only been able to prove this assuming continuity and bar induction. Hence, we leave it as an open question whether it can be shown that EPS is T-definable in $\widetilde{\mathsf{EPS}}$ provably in T.

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