Decision Problems for Two-Variable Logics

Martin Otto
RWTH Aachen
otto@informatik.rwth-aachen.de

Bruxelles, 6-99

Outline

- **Two-Variable Logics**
  – motivation, examples, tentative classification

- **Satisfiability in the Vicinity of FO^2**
  – the classical decision problem
  – the robustness of modal logic

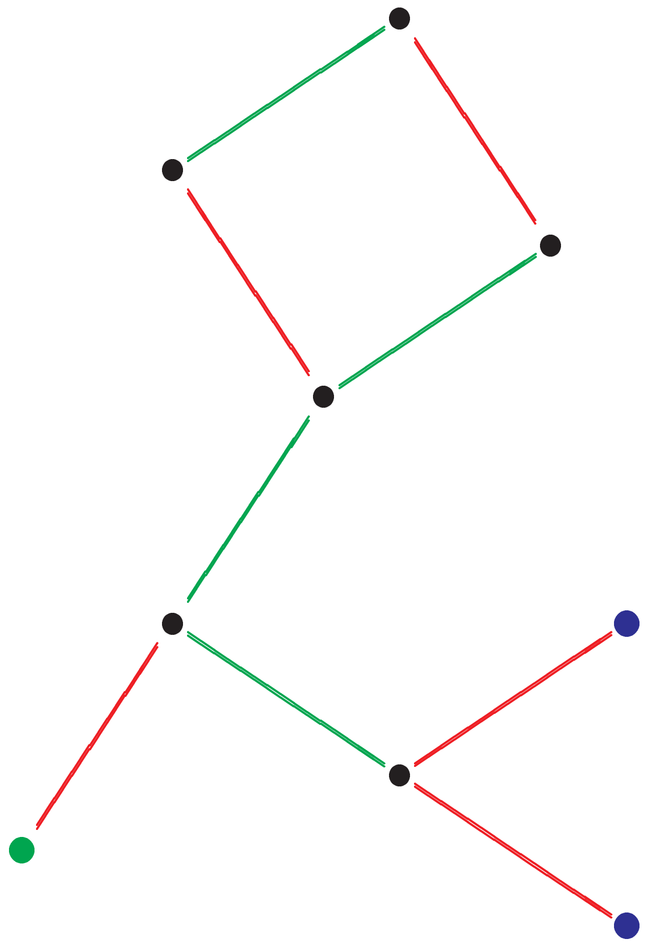
  Undecidability Results

  Decidability Results

- **Other Issues in Algorithmic Model Theory**
  – semantic completeness
  – recursion and the Boundedness Problem

2 Variables — a kind of locality

looking only at
two elements at a time:
what can you discern?



Two-variable phenomena

$$\Diamond_{\text{red}} \Diamond_{\text{green}} \cdots \Diamond_{\text{red}} \Diamond_{\text{green}} \text{Blue}$$

"$\exists$ path ______"

$$\Box_{\text{red}} \Diamond_{\text{green}} \cdots \Box_{\text{red}} \Diamond_{\text{green}} \text{Blue}$$

"$\exists$ strategy ______"

purely modal

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but also

$\Diamond_{\text{red} \cap \text{green} \cap \neq} \cdots$	modified basic relations equality
$\forall x \forall y \ldots$	global quantification
$\Diamond_{\leqslant 3} \ldots$	counting graded modalities

examples of two-variable logics

modal logic, ML

$\Diamond$, $\Box$, $\wedge$, $\vee$, $\neg$

infinitary variant ML_∞ with infinite $\bigvee$, $\bigwedge$

two-variable first-order logic, FO^2

with variables x, y relational, with $=$

infinitary variant FO^2_∞ with infinite $\bigvee$, $\bigwedge$

$$\mathsf{ML}_{(\infty)} \subsetneq \mathsf{FO}^2_{(\infty)}$$

$$\Diamond\varphi(x) \equiv \exists y(Exy \wedge \varphi(y))$$

$$\Box\varphi(x) \equiv \forall y(Exy \to \varphi(y))$$

computation tree logic, CTL

$$\mathsf{ML} + \begin{array}{l}\text{quantification over paths} \\ \text{linear temporal operators}\end{array}$$

- "$\forall$paths (eventually Blue)"

modal μ-calculus, L_μ

ML + monadic least fixed points

- $\mu_X \big(\mathsf{Blue} \vee \Box X\big)$

- $\mu_X \big(\mathsf{Blue} \vee \Box_{\mathsf{red}} \Diamond_{\mathsf{green}} X\big)$

$$\begin{array}{ccccccc}
\mathsf{FO}^2 & & & & & & \mathsf{FO}^2_\infty \\
\not\cup & & & & & & \not\cup \\
\mathsf{ML} & \subsetneq & \mathsf{CTL} & \subsetneq & \mathsf{L}_\mu & \subsetneq & \mathsf{ML}_\infty \\
 & & & & & \uparrow &
\end{array}$$

over finite structures

graded modal logic, ML_c

with graded modalities $\Diamond^{\leqslant m}$, $\Diamond^{\geqslant m}$, $\Diamond^{=m}$

two-variable counting logic, C^2

with counting quantifiers $\exists^{\leqslant m}$, $\exists^{\geqslant m}$, $\exists^{=m}$

$$\boxed{\mathrm{ML} \subsetneqq \mathrm{ML}_c \subsetneqq \mathrm{C}^2 \subsetneqq \mathrm{FO}^2}$$

$\longrightarrow$ other process logics
temporal logics
description logics
etc

Characterization Theorems

ML_∞-eq. —— bisimulation game
van Benthem, Milner, Park, ...

FO^2_∞-eq. —— 2-pebble game
Barwise, Poizat, Immerman

C^2_∞-eq. —— 2-pebble game with counting
Immerman, Lander

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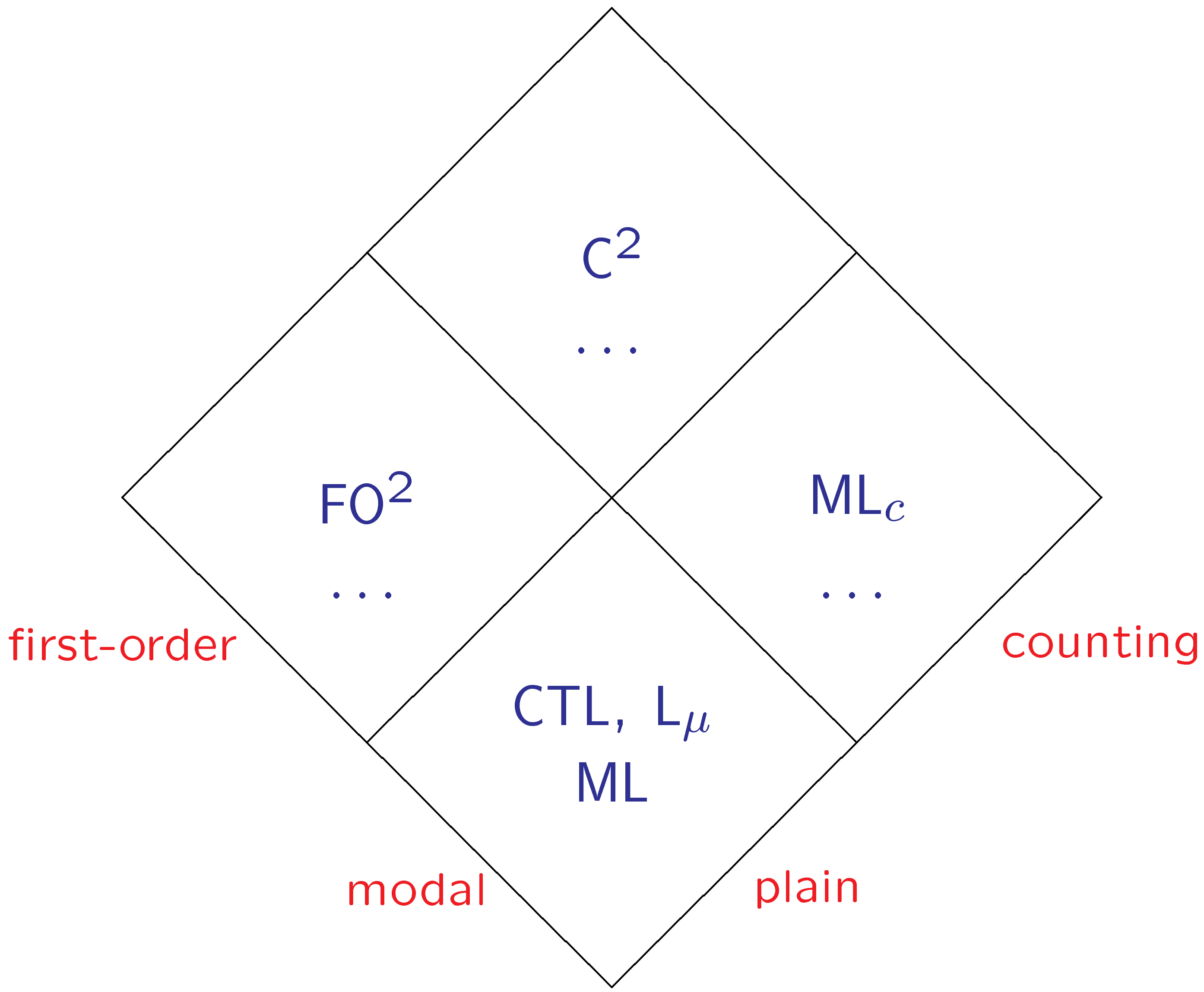
e.g., L is a $\left\{ \begin{array}{l} \text{modal} \\ \text{first-order} \end{array} \right\}$ two-variable logic

iff L does not distinguish between

$\left\{ \begin{array}{l} \text{bisimulation eq.} \\ \text{two-pebble eq.} \end{array} \right\}$ structures



most classical process logics: plain modal

description logics / terminological logics:

often first-order, with counting

Satisfiability SAT(L)

a common concern in many applications

satisfiability of specifications / concepts
test for implication / subsumption
test for equivalence, ...

The Classical Decision Problem

classify fragments $L \subseteq FO$ w.r.t.

decidability / complexity of

SAT(L)

given: $\varphi \in L$,
decide: does φ have a model ?

FINSAT(L)

given: $\varphi \in L$,
decide: does φ have a finite model ?

classification complete w.r.t. **prefix classes**

→ Börger, Grädel, Gurevich

max. decidable		**versus**	**min. undecidable**	
$\exists^*\forall\exists^*$	Bernays-Schönfinkel		$\forall^2\exists$	Gödel$_=$
$\exists^*\forall^*$	Ackermann		$\forall\exists\forall$	Kahr

prefix classes unsatisfactory for applications

other criteria

- **number of variables**
  also a computational resource
  excellent closure properties
- restricted forms of quantification
  → guarded fragments

two variables: threshold for SAT in FO

FO^3 undecidable: e.g. $\forall\exists\forall$ (Kahr)

Theorem Mortimer'75
Grädel,Kolaitis,Vardi'97

- FO^2 has **finite model property**
  $\Rightarrow$ $SAT(FO^2) = FINSAT(FO^2)$ decidable
- a **small model property** gives
  $SAT(FO^2)$ in Nexptime (complete)

the robustness of modal logic (Vardi)

modal extensions like CTL, L_μ
go beyond FO but retain

- **tree model property** $(*)$
  by bisimulation invariance

- **finite model property (f.m.p.)**
  CTL: Fischer,Ladner L_μ: Kozen

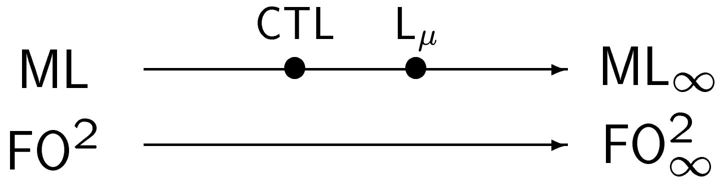
- **decidability**
  CTL: Fischer,Ladner L_μ: Emerson,Streett

$(*)$ $\Rightarrow$ applications of **automata theory**

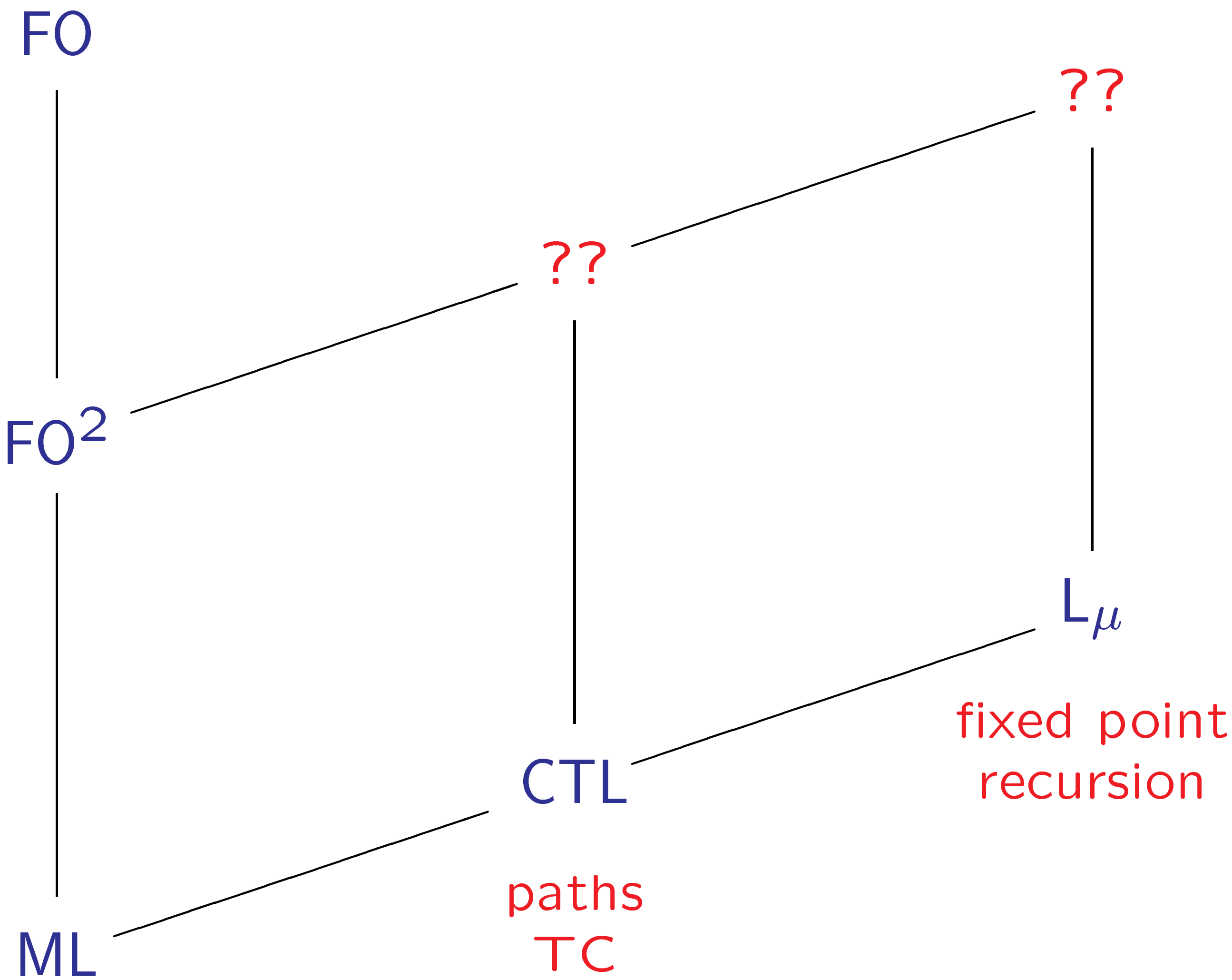
- CTL, L_μ **Exptime complete**
  CTL: Emerson,Jutla L_μ: Emerson,Halpern

programme to determine the borderline

- combine modal extns with FO closure props
- compare e.g.



- strengthen decidabilty results, and/or
- understand modal robustness by contrast



two-variable fixed-point logic FP^2

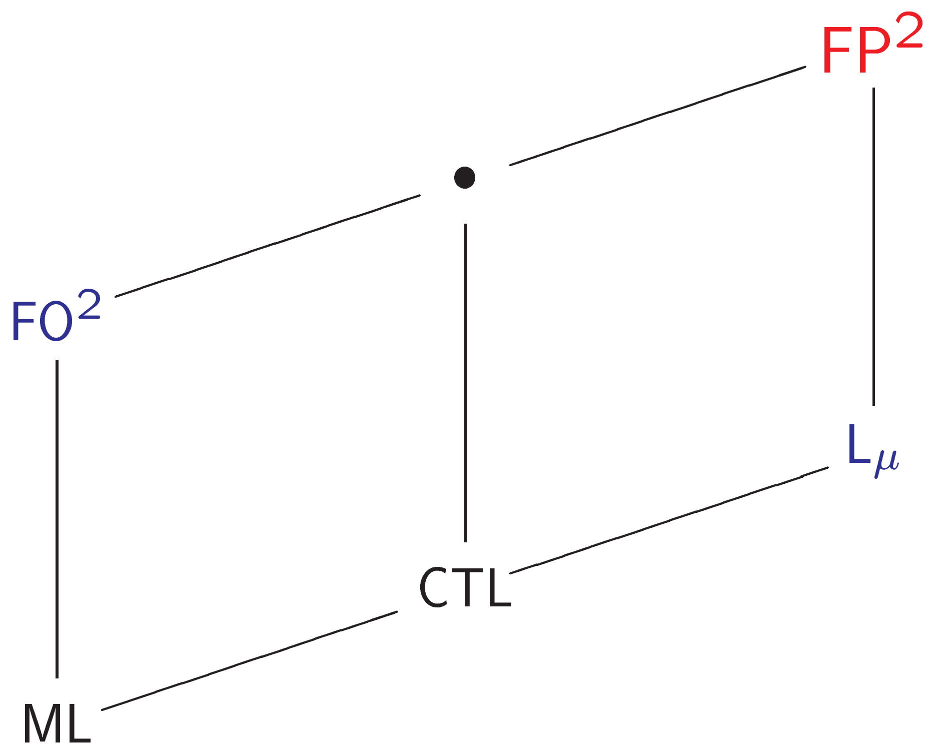
FP^2 : FO^2 + monadic least fixed points

$\mu_{X,x}\varphi(X,x)$, φ positive in X

defining the $\subseteq$-least P s.t. $P = \{a \colon \varphi(P,a)\}$

– a first-order two-variable logic

– the least common extn of FO^2 and L_μ



expressive power of $\mathbf{FP^2}$

- forbidding infinite (back-)paths, well-foundedness

  $\psi_{\mathsf{WF}}(R) = \forall x \big[\mu_{X,x} \forall y \big(Ryx \to Xy\big)\big](x)$

- not f.m.p.: $\psi_{\mathsf{WF}}(R) \wedge \forall x \exists y Rxy$

- axiomatize well-orderings (!)
  $\psi_{\mathsf{WF}}(<) \wedge \forall x \forall y \big(x = y \vee x < y \vee y < x\big)$

Theorem (Grädel,O_,Rosen, STACS'97)
SAT(FP^2), FINSAT(FP^2) undecidable
SAT even Σ^1_1-hard

Proof: reduction of **domino tiling problems**

two-variable transitive closure logic TC^2
two-variable path logic CL^2

TC^2 : FO^2 + transitive closures $\mathrm{TC}\varphi(x, y)$

$\mathrm{TC}\varphi(x, y)$: "there is a φ-path from x to y"

CL^2 : FO^2 + reachability $\langle\varphi(x, y)\rangle\psi(y)$

$\langle\varphi(x, y)\rangle\psi(y)$: "there is a φ-path from x to a y s.t. $\psi(y)$"

examples

- $\Big(\exists\text{path}\,(\,\text{eventually}\,P)\Big)(x)$

  $\equiv\ \exists y\Big(Py \wedge [\mathrm{TC}Exy](x, y)\Big)$

  $\equiv\ \langle Exy\rangle Py$

- not f.m.p.: cycle-freeness / well-foundedness

expressive power of $\mathbf{TC}^2$ and $\mathbf{CL}^2$

- $\mathrm{CTL} \subseteq \mathrm{TC}^2$ over finite structures

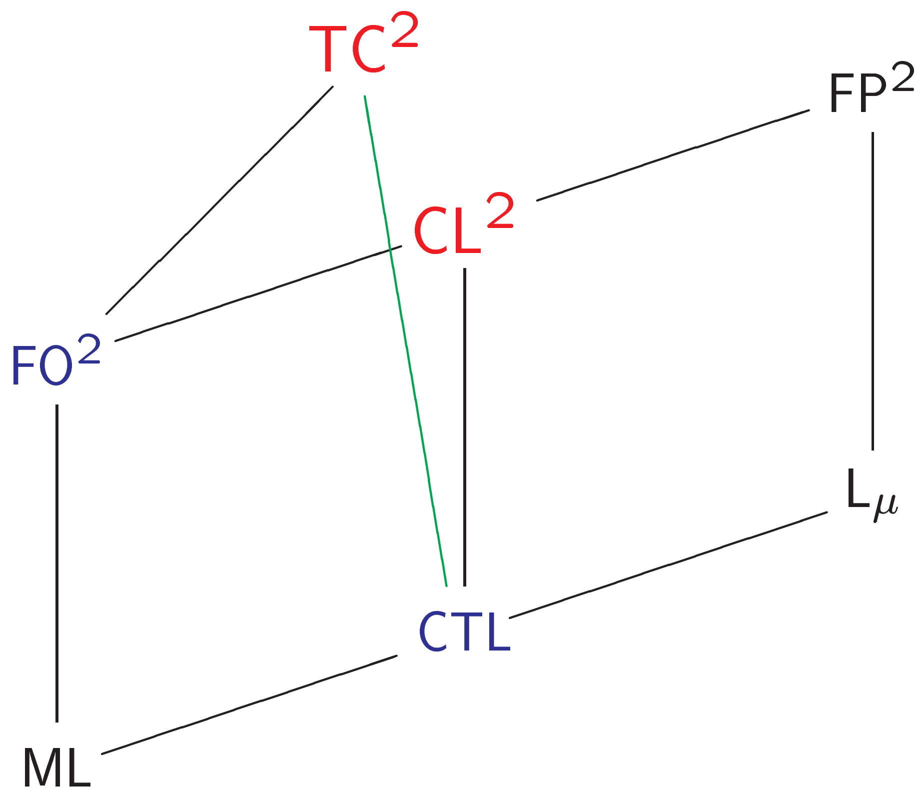
(Immerman,Vardi)

advocated for model checking
not a two-variable logic

- $\mathrm{CTL} \subsetneq \mathrm{CL}^2 \subsetneq \mathrm{FP}^2$

$\mathrm{CL}^2 \subsetneq \mathrm{TC}^2$ over finite structures

CL^2 the least common extn of FO^2 and CTL
and a first-order two-variable logic



more undecidability results

despite strong conjectures to the contrary:

Theorem (Grädel,O_,Rosen, STACS'97)
SAT(TC^2), FINSAT(TC^2) undecidable

Theorem (O_'98)
SAT(CL^2), FINSAT(CL^2) undecidable

again: SAT even Σ^1_1-hard

Proof: reduction of **domino tiling problems**

domino tiling problems

$\mathcal{D} = (D, R_h, R_v)$ $\quad$ D finite, $R_{h/v} \subseteq D \times D$

hor./vert. adjacency

domino system

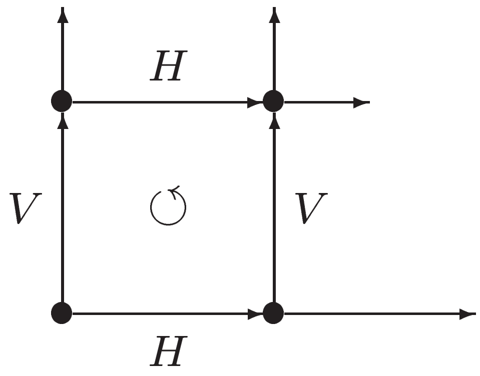
$t : \mathbb{N} \times \mathbb{N} \longrightarrow D$ $\quad$ $(t(n,m), t(n+1,m)) \in R_h$

$(t(n,m), t(n,m+1)) \in R_v$

$\mathcal{D}$-tiling

is described by an expansion of

$$\mathfrak{G}_{\mathbb{N}} = \big(\mathbb{N} \times \mathbb{N}, H, V\big)$$



with colours P_d, $d \in D$ satisfying

$$\begin{aligned} \psi_{\mathcal{D}} = \ & \forall x \dot{\bigvee}_{d \in D} P_d x \\ & \wedge \forall x \forall y \big(Hxy \rightarrow \bigvee_{(d,d') \in R_h} (P_d x \wedge P_{d'} y)\big) \\ & \wedge \forall x \forall y \big(Vxy \rightarrow \ldots\big) \end{aligned}$$

Theorem (Berger'66)

$$\left.\begin{array}{l}\textbf{D1: } \text{Ex. of } \mathcal{D}\text{-tilings} \\ \textbf{D2: } \text{Ex. of finite periodic } \mathcal{D}\text{-tilings}\end{array}\right\} \text{undecidable}$$

reduction idea

IF $\psi^*_{\mathrm{grid}} \in \mathrm{L}$ forces local grid structure

THEN $\mathcal{D} \longmapsto \psi^*_{\mathrm{grid}} \wedge \psi_{\mathcal{D}}$ reduces

D1/2 $\longrightarrow$ SAT/FINSAT(L)

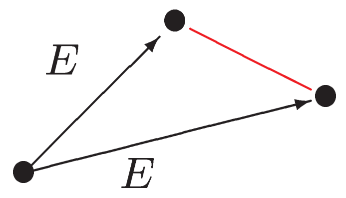
crux of the reduction

$$\left.\begin{array}{l}\text{functionality} \\ \text{commutativity}\end{array}\right\} \text{of } H \text{ and } V \textbf{ not 2-variable}$$

a simple example

transitivity — functionality for **matchings**

$$\begin{array}{l}\mathrm{dom}(E) \cap \mathrm{im}(E) = \emptyset \\ E \cup E^{-1} \cup R_{=} \text{ transitive}\end{array} \quad \Rightarrow \quad \begin{array}{l}E \text{ the graph of a} \\ \text{1-1 partial function}\end{array}$$

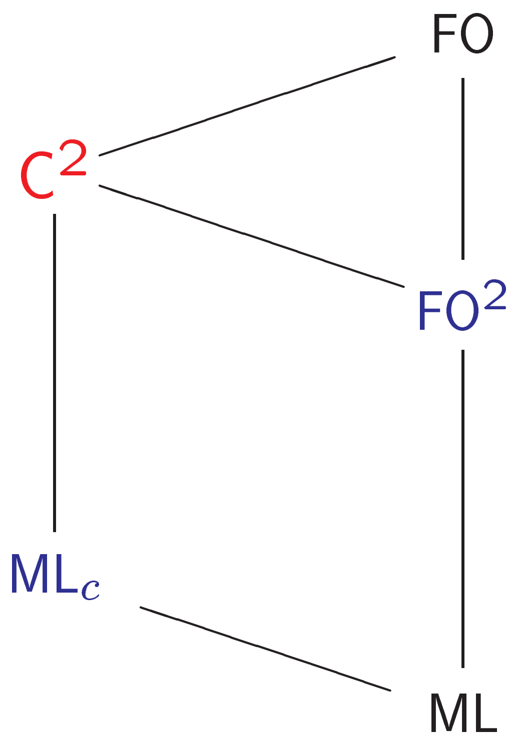


two-variable counting logic $\mathbf{C}^2$

C^2 : FO^2 with counting quantifiers $\exists^{\leqslant m}$

examples

- functionality: $\psi_{\mathsf{fct}} = \forall x \exists^{\leqslant 1} y Exy$
- $\mathsf{C}^2 \subsetneqq \mathsf{FO}$ does not have f.m.p.
- almost all finite graphs charcterized up to $\simeq$ by their C^2-descriptions
- subsumes a relevant family of description logics with "number constraints"



a decidability result

Theorem (Grädel,O_,Rosen, LICS'97)
SAT(C^2), FINSAT(C^2) decidable

natural **decidable fragment of FO without f.m.p.**

in 2-Nexptime (and Nexptime-hard)
Pacholski,Szwast,Tendera'97

Proof:

- Skolemization
- countig reduces to functionality
- combinatorial analysis
- effective descriptions of regular infinite models
- Ramsey theory

undecidability: the basic reduction idea revisited

IF $\psi^*_{\mathrm{grid}} \in \mathrm{L}$ axiomatizes local grids

get a reduction

$$\mathcal{D} \longmapsto \psi^*_{\mathrm{grid}} \wedge \psi_{\mathcal{D}}$$

$$\mathrm{D1/2} \longrightarrow \mathrm{SAT/FINSAT(L)}$$

for suitable classes $\mathcal{C}$

IF $\psi^*_{\mathcal{C}} \in \mathrm{L}$ axiomatizes $\mathcal{C}$
$\psi_{\mathrm{grid}} \in \mathrm{FO}^2$ interprets local grids over $\mathcal{C}$

get two reductions:

$$\mathcal{D} \longmapsto \psi^*_{\mathcal{C}} \wedge \psi_{\mathrm{grid}} \wedge \psi_{\mathcal{D}}$$

$$\mathrm{D1/2} \longrightarrow \mathrm{SAT/FINSAT(L)}$$

$$\mathcal{D} \longmapsto \psi_{\mathrm{grid}} \wedge \psi_{\mathcal{D}} \qquad \in \mathrm{FO}^2$$

$$\mathrm{D1/2} \longrightarrow \mathrm{SAT/FINSAT(FO^2)} \text{ over } \mathcal{C}$$

a new perspective – cf. frame conditions

$\mathcal{C}$ the class of $\left(A, E_1, \ldots, E_k, \bar{E}, \bar{P}\right)$ where the

E_i are $\begin{cases} \text{equivalence relations} & \text{FP}^2, \text{TC}^2 \\ \text{well-founded} & \text{FP}^2, \text{CL}^2 \\ \text{well-orderings} & \text{FP}^2, \text{CL}^2 \\ \text{transitive} & \text{TC}^2 \end{cases}$

Theorem (O_'97; Grädel,O_,Rosen'97; O_'98)
SAT/FINSAT(FO^2) undecidable over these $\mathcal{C}$

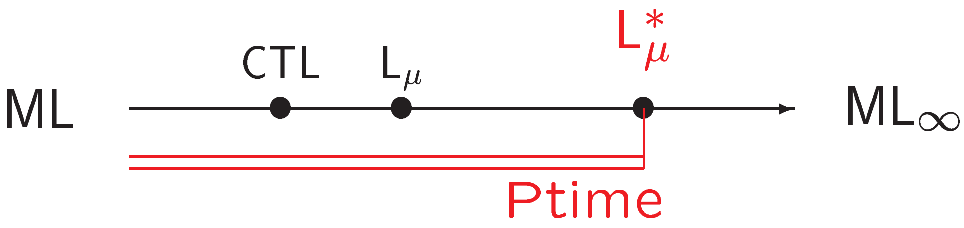
in contrast FO^2 decidable over

- functional E_i: decidability of C^2 (!)
- finite linearly ordered structures:
  FO^2 decidable as a linear temporal logic over finite structures (Nexptime)
  O_'98; cf. Etessami,Vardi,Wilke'97

other issues in the
algorithmic model theory
of the two-variable domains

Capturing Results – semantic completeness

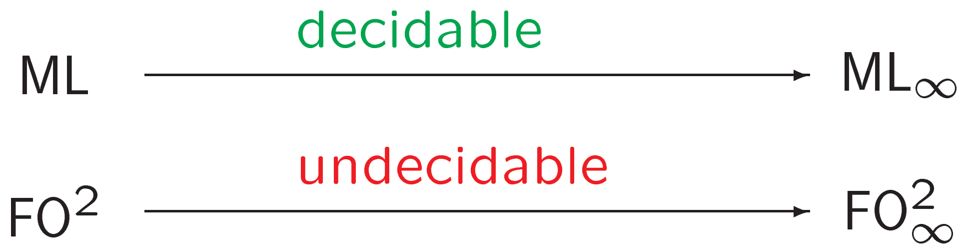
the **Ptime fragments** of all 4 two-variable domains admit **semantically complete logics**



O_ LICS'95, LFCS'97

Boundedness Problem

elimination of fixed-point recursion



O_ STACS'99; Kolaitis,O_ LICS'98

another direction: the guarded fragments

Andreka, van Benthem, Nemeti

instead of number of variables

focus on quantification pattern

generalize relativized nature of

modal quantification $\Diamond\varphi(x) \equiv \exists y\big(Exy \wedge \varphi(y)\big)$ to

$$\exists \bar{y}\big(\alpha(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{y})\big)$$

for arbitrary atoms α

that mention all free variables of φ