

# Symmetry-Preserving Finite Synthesis & Amalgamation

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Berkeley, Nov. 2016

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- amalgamation: overlap specifications
- local vs. global consistency
- local vs. global symmetry

finite model constructions with a touch of  
combinatorial geometry and group theory

## amalgamation?

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- **synthesis & amalgamation:**

hypergraphs & relational structures: (de)composition patterns  
model theory & algebra: homogeneity, Fraïssé limits  
geometry: atlases of charts & changes of co-ordinates

- **local/partial symmetries:**

logic & model theory: partial isomorphisms  
geometry & algebra: groupoids, inverse semigroups

- **local consistency:**

logic & combinatorics: constraint satisfaction  
logic & databases: constraints, dependencies  
logic & computation: e.g. quantum information

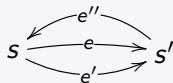
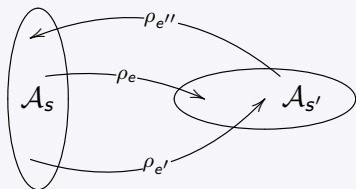
... calling for global solutions, closures, completions

## a generic free amalgamation construction

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from finite families

$$\left. \begin{array}{l} (\mathcal{A}_s)_{s \in S} \text{ of relational structures} \\ (\rho_e : \mathcal{A}_s \xrightarrow{\text{part}} \mathcal{A}_{s'})_{e \in E[s, s']} \text{ of partial isomorphisms} \end{array} \right\} (*)$$



$$I = (S, E)$$

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find natural free amalgam of disjoint copies

$(\mathcal{A}_s, w) \simeq \mathcal{A}_s$  tagged by walks  $w$  in  $I = (S, (E[s, s']))$ ,  
with  $(\mathcal{A}_s, w)$  and  $(\mathcal{A}_{s'}, w \cdot e)$  overlapping according to  $\rho_e$

$$(((\mathcal{A}_s), (\rho_e)) \otimes I^*) / \approx \quad \text{based on multi-sorted monoid } I^* \text{ of walks in } I$$

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- realises the amalgamation pattern  $(*)$
- is free in a universal algebraic sense
- is generic but infinite (unless  $I$  is acyclic)

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- ... finite analoga?
- ... based on what instead of  $I^*$  ?
- ... equally homogeneous?
- ... of controlled acyclicity?

## realisation of $H = ((\mathcal{A}_s), (\rho_e))$ in general:

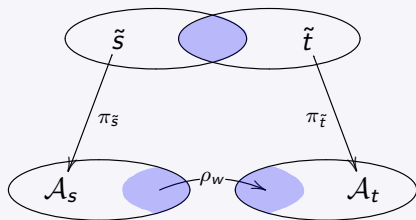
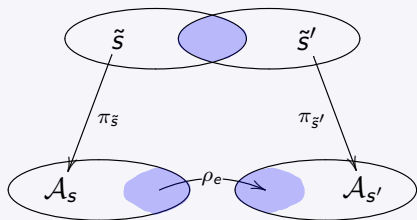
a relational structure  $\mathcal{A}$  with an atlas given by superimposed hypergraph structure  $(A, \tilde{S})$  of charts  $\pi_{\tilde{s}}: \mathcal{A} \upharpoonright \tilde{s} \simeq \mathcal{A}_{\pi(\tilde{s})}$  s.t.

- **locally, all  $\rho_e$ -overlaps are realised:**

each  $\pi_{\tilde{s}}^{-1}(\mathcal{A}_s)$  overlaps with some  $\pi_{\tilde{s}'}^{-1}(\mathcal{A}_{s'})$  according to  $\rho_e$

- **globally, no incidental overlaps occur:**

if  $\tilde{s} \cap \tilde{t} \neq \emptyset$ , then this is due to a composition of  $\rho_{e_i}$  for some *single* walk  $w = e_1 \cdots e_m$  from  $\pi(\tilde{s})$  to  $\pi(\tilde{t})$





## realisation of $H = ((\mathcal{A}_s), (\rho_e))$ in general:

a relational structure  $\mathcal{A}$  with an atlas given by superimposed hypergraph structure  $(A, \tilde{S})$  of charts  $\pi_{\tilde{s}}: \mathcal{A} \upharpoonright \tilde{s} \simeq \mathcal{A}_{\pi(\tilde{s})}$  s.t.

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**NB: the second, “no-nonsense” condition avoids potential relational inconsistencies for amalgams**

## aside: the rôle of groupoids/inverse semigroups

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### composition structure of partial bijections:

- with partial composition (as a total operation)  
     $\rightsquigarrow$  inverse semigroups
- with exact composition (as a partial operation)  
     $\rightsquigarrow$  groupoids

**groupoids capture local/partial symmetries**

... just as groups capture global symmetries

## (II) reduced products with groupoids

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**I-groupoid:**  $\mathbb{G} = (G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S})$  with

- associative compositions  $G_{st} \times G_{tu} \rightarrow G_{su}$ ,
- neutral elements  $1_s \in G_{ss}$ , inverses, ...
- designated generators  $(g_e)_{e \in E}$

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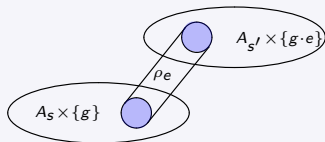
the potentially finite analogue of the infinite path monoid  $I^*$

## (II) reduced products with groupoids

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$\rightsquigarrow$  **reduced products**  $\mathbf{H} \otimes \mathbb{G}$   
as candidate realisations

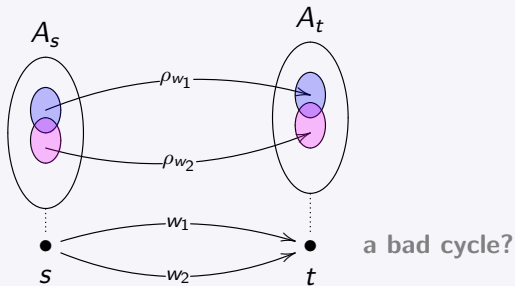


**when is this a realisation of  $\mathbf{H}$ ?**

## obstructions: wrong kind of cycles (1)

### violation of path independence for elements

- conflicting identifications may collapse individual  $A_s$

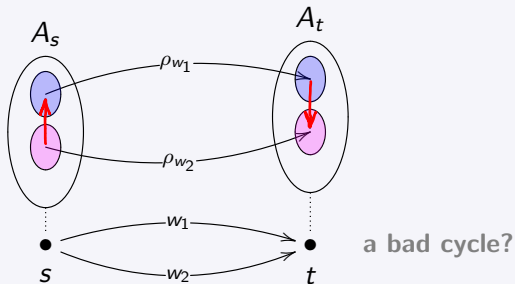


can be overcome by relatively simple pre-processing:  
replace  $I$  by finite covering  $\hat{I}$  that unfolds bad cycles

## obstructions: wrong kind of cycles (2)

### violations of path independence for tuples

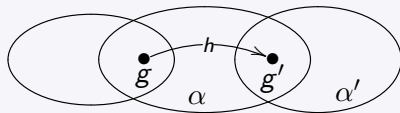
- incidental parallel identifications may cause conflicts at the relational level



⇒ substantial acyclicity conditions on suitable groupoids

## an appropriate notion of acyclicity: control coset cycles

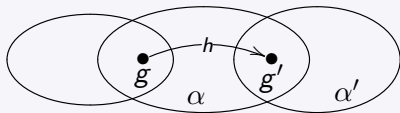
- not just short generator cycles in the Cayley graph of  $\mathbb{G}$ ,  
but short cycles of cosets  $g\mathbb{G}[\alpha]$  generated by subsets  $\alpha \subseteq E$



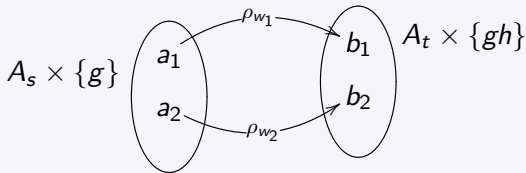


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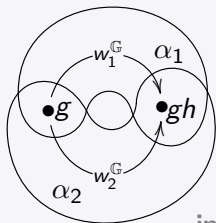
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- in particular, need to avoid certain coset cycles of length 2



$$w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}}$$

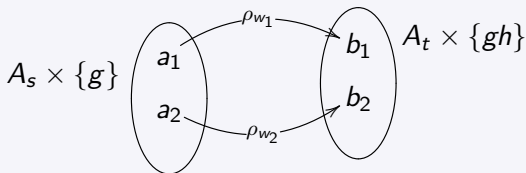


in reduced product

in  $\mathbb{G}$

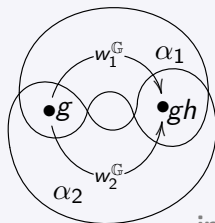
## bad coset cycles & obstruction no. (2)

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$$w_1^G = h = w_2^G$$

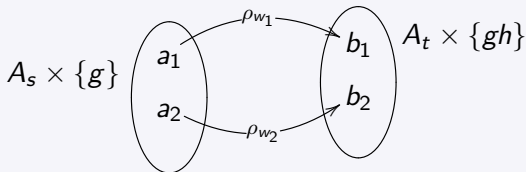
in reduced product



in  $G$

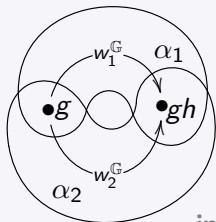
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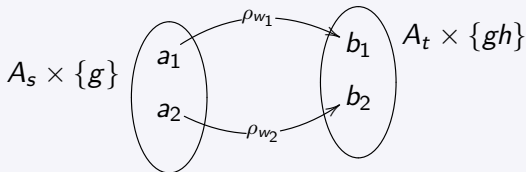


in  $G$

$\left. \begin{array}{l} \alpha_1: \text{generators carrying } a_1 \\ \alpha_2: \text{generators carrying } a_2 \end{array} \right\} \text{ want } h = w^G \text{ for } w \in \langle \alpha_1 \cap \alpha_2 \rangle$   
so that  $w$  carries both  $a_1$  and  $a_2$

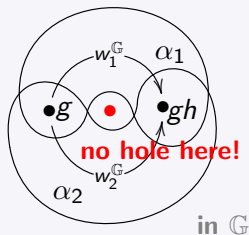
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$$\langle \alpha_1 \rangle \cap \langle \alpha_2 \rangle = \langle \alpha_1 \cap \alpha_2 \rangle$$

**avoiding a coset 2-cycle**

## any degree of acyclicity in finite groupoids

---

### **theorem** (O\_13)

---

for every  $N \in \mathbb{N}$  and incidence pattern  $I = (S, E)$  there are finite  $I$ -groupoids  $\mathbb{G}$  without coset cycles of length up to  $N$

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**idea: in an inductive construction generate  $\mathbb{G}$  from (semi)group action on amalgamation chains that unfold short cosets cycles**

**cf. constructions of acyclic Cayley graphs (Alon, Biggs)  
here lifted to more intricate adaptation for coset cycles**

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### corollary

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for any  $H = ((\mathcal{A}_s), (\rho_e))$  obtain realisations  $H \otimes \mathbb{G}$  that

- respect all symmetries of  $H$  (genericity)
- have any desired degree of (local) acyclicity

with an atlas of charts onto the  $(\mathcal{A}_s)$   
forming an  $N$ -acyclic hypergraph  
with changes of co-ordinates generated by the  $(\rho_e)$

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**generic & locally free finite realisations  
of any finite amalgamation pattern**



## applications (1): hypergraph coverings

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### corollary

---

every finite hypergraph admits, for  $N \in \mathbb{N}$ , finite coverings that

- are  $N$ -acyclic in the sense that every induced sub-hypergraph on up to  $N$  vertices is acyclic (tree decomposable,  $\alpha$ -acyclic)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

idea: generic  $N$ -acyclic finite realisations of the overlap pattern corresponding to the “exploded view” of the given hypergraph

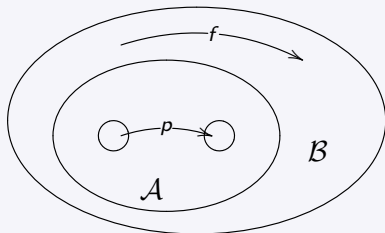
## applications (2): lifting local to global symmetries

### EPPA: extension properties for partial automorphisms

Hrushovski 92 (graphs)

Herwig 98 (relational structures)

Herwig–Lascar 00 (fmp w.r.t. forbidden homomorphisms)



## applications (2): lifting local to global symmetries

### **EPPA: extension properties for partial automorphisms**

Hrushovski 92 (graphs)

Herwig 98 (relational structures)

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### **here, as a corollary**

---

for finite  $\mathcal{A}$  and collection of partial automorphisms  $(\rho_p)$  find finite EPPA extension  $\mathcal{B} \supseteq \mathcal{A}$  as generic realisation of pattern of self-overlaps of  $\mathcal{A}$  induced by the  $\rho_p$

- with atlas of  $\mathcal{A}$ -charts forming  $N$ -acyclic hypergraph on  $B$
- hence, locally free & universal w.r.t. homomorphisms

**which implies full Herwig–Lascar result**

## applications (3): various uses in modal/guarded logics

- **finite model properties & finite controllability for guarded logics and constraints**

using finite coverings of controlled acyclicity and/or Herwig–Lascar extension properties

- **characterisation theorems (fmt & classical) for the guarded fragment GF and relatives**

using finite coverings of controlled acyclicity

- **characterisation theorems (fmt & classical) for (modal) common knowledge logic**

new, with Felix Canavoi, using (finite)  $S5$ -frames over Cayley groups with controlled coset-acyclicity

## some related references

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**Bárány–Gottlob–O** (LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

**Bárány–ten Cate–O** (VLDB 2012, arXiv:1203.0077): Queries with guarded negation

**Grädel–O** (2014): The freedoms of (guarded) bisimulation

**Hodkinson–O** (BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

**Herwig–Lascar** (Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

**O** (Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

**O** (arXiv:1404.4599): Finite groupoids, finite coverings and symmetries in finite structures

→ <http://www.mathematik.tu-darmstadt.de/~otto/>