Amalgamation, Groupoids, Symmetries in Finite Structures

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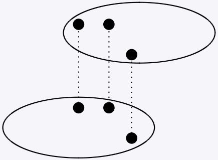
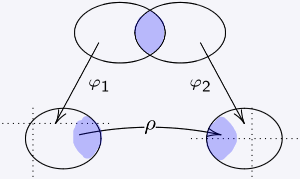
Global Finite Realisations of Local Specifications

examples of local views & specifications

- **exploded view of a hypergraph**



- **coordinate charts for manifolds**



- **decomposition and synthesis** of graphs, hypergraphs, . . .
- **implicit specifications** of (macro-)bisimulation types $p \rightarrow \Diamond q$
- **i.e., guarded extension properties** $\forall \mathbf{x} \big(\theta(\mathbf{x}) \rightarrow \exists \mathbf{y} \theta'(\mathbf{xy}) \big)$

local ——————— global

atlas of local maps — **manifolds**
with changes of coordinates

distinguished substructures — **relational structures**
with overlap specifications

hyperedges — **hypergraphs**
with overlap specifications

partial isomorphisms — **automorphism groups**
with composition in overlaps

local specifications? — **global realisations?**

the role of groupoids/inverse semigroups

two 'equivalent' algebraic formats for
composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

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groups capture global symmetries

groupoids capture local/partial symmetries

the role of hypergraphs

hypergraph: $\mathcal{A} = (A, S)$ with sets $\begin{cases} A \text{ of vertices} \\ S \subseteq \mathcal{P}(A) \text{ of hyperedges} \end{cases}$

examples:

- **combinatorial patterns of structural decompositions**
- **hypergraphs of guarded subsets of relational structures**

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intersection graph of $\mathcal{A}$:

$\mathrm{I}(\mathcal{A}) := (S, E)$ where $E = \{(s, s') \colon s \neq s', s \cap s' \neq \emptyset\}$

records pairwise overlaps between hyperedges s $\in$ S

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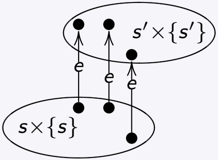
records pairwise overlaps between hyperedges $\mathbf{s} \in \mathbf{S}$

exploded view of $\mathcal{A}$ based on $\mathrm{I}(\mathcal{A})$

the disjoint union of the hyperedges $s \in S$
with partial bijections ρ_e for $e = (s, s') \in E$



$\rightsquigarrow$ **format of local overlap specifications**

overview

(I) specification & realisation of overlap patterns

(II) reduced products with groupoids (core results)

(III) from local to global symmetries

(IV) further applications

(I) abstract specification & realisation

incidence pattern $\mathrm{I} = (S, (E[s,s'])_{s,s' \in S})$

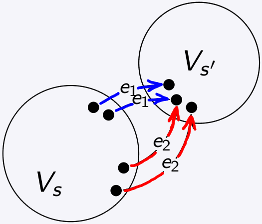
multi-graph with vertices $s \in S$ (sorts)
directed edges $e \in E[s,s']$ from s to s' with $e^{-1} \in E[s',s]$

- **fixed bisimulation type for pairwise overlaps**

I-**graph** $\mathrm{H} = \big(V, (V_s)_{s \in S}, (\rho_e)_{e \in E}\big)$

vertex set V partitioned into sorts V_s for $s \in S$
ρ_e a partial* bijection between V_s and $V_{s'}$ for $e \in E[s,s']$

- **an exploded view of the desired pairwise overlaps**



realisation:

a realisation of $\mathrm{H} = (V, (V_s), (\rho_e))$ is a
hypergraph $\mathcal{A} = (A, \tilde{S})$ with projection $\pi \colon \tilde{S} \longrightarrow S$
and an atlas of bijections $\pi_{\tilde{s}} \colon \tilde{s} \to V_{\pi(\tilde{s})}$ for $\tilde{s} \in \tilde{S}$ s.t.

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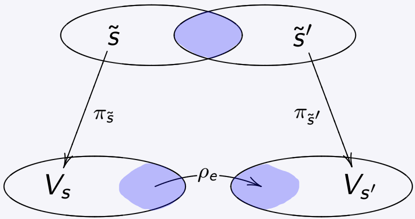
- **all specified overlaps are realised:**
  for $e \in E[s, s']$, ρ_e is realised at every $\tilde{s} \in \pi^{-1}(s)$
  by an actual overlap with some $\tilde{s}' \in \pi^{-1}(s')$



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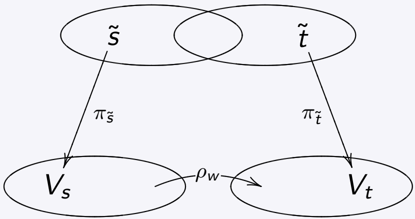
- **no further, incidental overlaps occur:**
  all actual overlaps of $\mathrm{I}(\mathcal{A})$ are induced by
  compositions ρ_w of partial bijections ρ_e in H



realisations vs. exploded views

the exploded view of hypergraph $\mathcal{A} = (A, S)$
is an I-graph $\mathrm{H}(\mathcal{A})$ w.r.t. $\mathrm{I}(\mathcal{A}) = (S, E)$

$\mathcal{A}$ is a realisation of $\mathrm{H}(\mathcal{A})$ obtained as a quotient $\mathrm{H}(\mathcal{A})/{\approx}$
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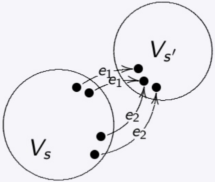
$\approx$ may even collapse individual $\mathbf{V}_s$



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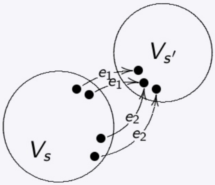

idea: try local unfolding in products of H with . . . ?

(II) reduced products with groupoids

I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

- **I-groupoids come with Cayley graphs that are I-graphs**

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reduced products as candidate realisations:

$\rightsquigarrow$ $\mathrm{H} \times \mathbb{G}$ natural direct product (of I-graphs)

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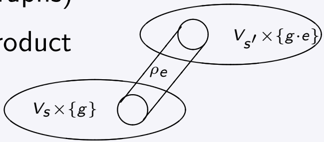

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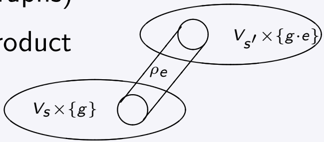
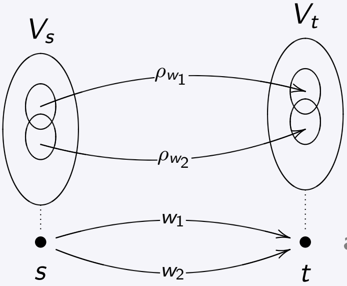


when is this a realisation of H?

obstructions to simple realisations

- $\mathbb{H}$ may fail to be *coherent* (as seen before):
  a lack of path-independence in $\mathbb{H}$, with conflicting identifications collapsing individual V_s

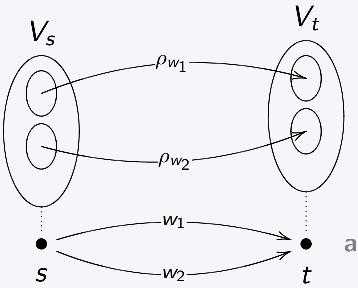


a bad cycle in $\mathbb{G}$?

can be overcome by relatively simple pre-processing

obstructions to simple realisations

- $\mathbb{H}$ and $\mathbb{G}$ may fail to be *confluent* in the product: causing incidental overlaps (with potential conflicts at the relational level)

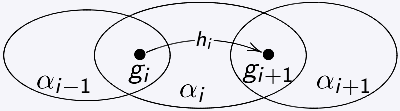


a bad cycle in $\mathbb{G}$?

$\rightsquigarrow$ **need substantial acyclicity conditions on $\mathbb{G}$**

an appropriate notion of acyclicity

- not just short cycles in the Cayley graph of $\mathbb{G}$, but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$



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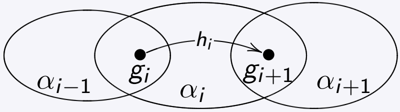
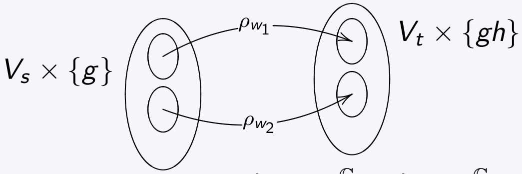
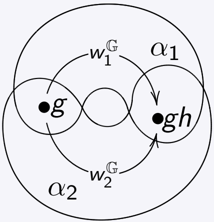


- in particular, need to avoid certain coset cycles of length 2



$$h_1 = w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}} = h_2^{-1}$$



any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

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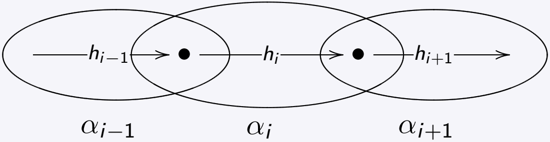
construction by inductive interleaving:

- groupoidal action on I-graphs
- use of amalgamation chains of $\mathrm{I}[\alpha]$-graphs (local unfoldings) to eliminate short cycles



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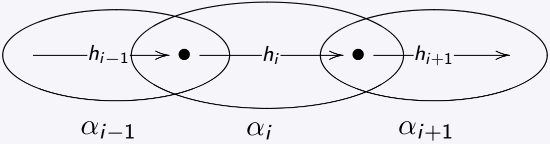

cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification H (an I-graph), obtain realisations $\mathrm{H} \otimes \mathbb{G}$ (as reduced products with finite I-groupoids $\mathbb{G}$) that

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- admit transitive automorphisms in the second factor
- respect all symmetries of the specification H

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symmetric realisations

corollary

every finite hypergraph admits, for N $\in \mathbb{N}$, finite coverings that

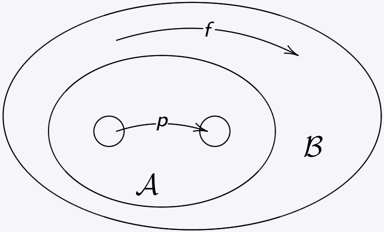
- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

(III) from local to global symmetries

extension property for partial automorphisms (EPPA):
how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$ s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$



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theorem (Herwig–Lascar 00)

same, as a *finite model property* over any class $\mathcal{C}$ defined by finitely many forbidden homomorphisms

if $\mathcal{A} \in \mathcal{C}_{\mathrm{fin}}$ has any EPPA extension in $\mathcal{C}$ then it also has a finite one in $\mathcal{C}_{\mathrm{fin}}$

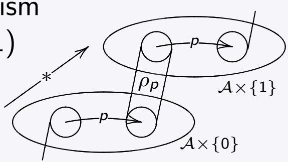
a “naive” idea towards EPPA for single p

for a single partial isomorphism p of $\mathcal{A} = (A, R)$ find

a free infinite EPPA extension as a reduced product $(\mathcal{A} \times \mathbb{Z})/\approx$ where $\approx$ is induced by the partial bijections

$$\begin{aligned} \rho_p^{i,i-1} \colon \mathcal{A} \times \{i\} &\longrightarrow \mathcal{A} \times \{i-1\} \\ (a, i) &\longmapsto (p(a), i-1) \end{aligned}$$

p in $\mathcal{A} \simeq \mathcal{A} \times \{0\}$ extends to the automorphism induced by the shift $* \colon (a, i) \longmapsto (a, i+1)$



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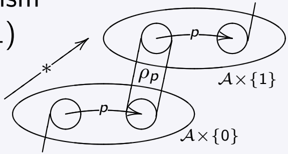

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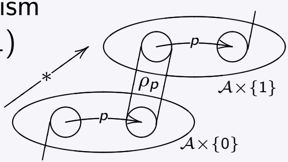

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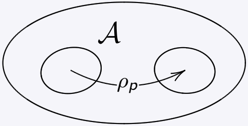
groupoidal realisations can do the trick for several p (!)

new proof of full Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I(\mathcal{A}, P)}$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\mathbf{H(\mathcal{A}, P)}$:**
$\mathrm{I}(\mathcal{A}, P)$-graph $\mathrm{H}(\mathcal{A}, P) = (A, (\rho_p)_{p \in P})$
needs to be made coherent!



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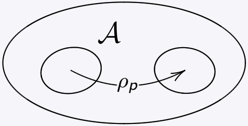

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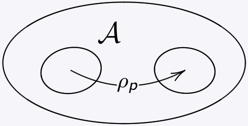

(iii) **symmetric realisations of $\mathbf{H(\mathcal{A}, P)}$ are EPPA extensions !**

(iv) **N-acyclic EPPA extensions are N-free:**
admit N-local homomorphisms into every (finite or infinite)
EPPA extension due to their N-local tree-decomposability

(IV) applications in algorithmic model theory

- **characterisation theorems** (fmt)
  for guarded logics and relatives

  using finite coverings of controlled acyclicity

- **finite model properties & finite controllability**
  for guarded logics and constraints

  using finite coverings of controlled acyclicity
  and/or Herwig–Lascar extension properties

⟶ **O_**(LICS10&JACM13)
O_(APAL13)
Bárány–Gottlob–O_(LICS10&LMCS14)
Bárány–ten Cate–O_(VLDB12)
O_(LICS13&arXiv14/15)

characterisation theorems (fmt)

theorem (O_10)

$\mathrm{GF} \equiv \mathrm{FO}/{\sim_{\mathrm{g}}}$ and $\mathrm{GF} \equiv_{\mathrm{fin}} \mathrm{FO}/{\sim_{\mathrm{g}}}$

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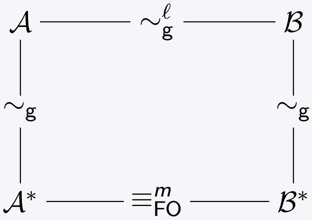
idea: show that $\sim_{\mathrm{g}}$-invariance of $\varphi \in \mathrm{FO}^m$ implies $\sim_{\mathrm{g}}^{\ell}$-invariance for some $\ell = \ell(m)$ such that over suitable locally sufficiently acyclic (finite) structures, $\sim_{\mathrm{g}}^{\ell}$ refines $\equiv_{\mathrm{FO}}^{m}$



finite model properties & finite controllability

theorem (Bárány–Gottlob–O_10/Rosati06)

finite controllability for union of conjunctive queries Q
w.r.t. constraint $\alpha \in \mathsf{GF}$:

$$\alpha \models Q \quad \Leftrightarrow \quad \alpha \models_{\mathsf{fin}} Q \qquad (\dagger)$$

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idea 1: detour via "treeification" $Q^* \in \mathsf{GF}$ of Q for which

$$Q^* \models Q \text{ and } Q \models_{\mathsf{acyc}} Q^*;$$

use locally sufficiently acyclic (finite) unfoldings to show that also

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idea 2: view ($\dagger$) as a fmp for $\alpha \in \mathsf{GF}$ within $\mathcal{C} = \mathrm{Mod}(\neg Q)$,
which follows from Herwig–Lascar EPPA (!)

summary

- **a generic construction of**
  highly acyclic finite groupoids

- **a universal & generic route to the synthesis of**
  finite realisations (and coverings) in reduced products

- **symmetry and acyclicity of realisations supports**
  extensions of local to global symmetry

- **further applications in finite model theory**

$\longrightarrow$ **Finite Groupoids, Finite Coverings**
& Symmetries in Finite Structures (arXiv 2015 (v4))