Tractable Finite Models

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Tractability can mean many things

computational/algorithmic tractability:

- *tractable* instances of (hard) problems
  e.g. satisfiability/formula evaluation/theories

  $\rightsquigarrow$ tractable (fragments of) logics,
  tractable classes of structures

here is one more:

model-theoretic tractability (of structures):

- low-level logic determines high-level behaviour

  e.g. determination of FO_∞-theory by FO-theory

  . . . and in finite model theory ?

A: Tractability – model-theoretic

classical example: ω-saturated models

FO-types determine FO_∞-types

shorthand

FO	▶	FO_∞
$\equiv$	▶	$\simeq_{\mathrm{part}}$

elementary equivalence class determines partial isomorphy class

tractable representatives of $\equiv$-classes

available by compactness

sample application: **expressive completeness proofs**
like Lyndon–Tarski, Łos–Tarski, ...

expressive completeness proofs, example

Lyndon–Tarski Theorem $\quad \mathrm{FO}/_{\mathrm{hom}} \equiv \mathrm{pos}\,\exists\,\mathrm{FO}$

crux (modulo compactness):

φ preserved under homomorphisms $\mathfrak{A} \xrightarrow{\mathrm{hom}} \mathfrak{B}$	$\Longrightarrow$	φ preserved under pos $\exists$ FO transfer $\mathfrak{A} \Rightarrow_{\mathrm{pos}\,\exists} \mathfrak{B}$

to upgrade $\Rightarrow_{\mathrm{pos}\,\exists}$ to $\xrightarrow{\mathrm{hom}}$ can use

tractability: pos $\exists$ FO ▶ pos $\exists$ FO_∞ in ω-saturated models

$$\mathfrak{A} \overset{\mathrm{pos}\,\exists}{\Longrightarrow} \mathfrak{B} \qquad \mathfrak{A}, \mathfrak{B} \text{ ctbl}$$

$$\mathfrak{A} \xrightarrow{\mathrm{hom}} \mathfrak{B}^* \quad (\mathfrak{B} \preceq \mathfrak{B}^*) \qquad \mathfrak{B}^* \ \omega\text{-saturated}$$

aside: related expressive completeness results in fmt

Lyndon–Tarski: Rossman (2005)

$\mathrm{FO}/_{\mathrm{hom}} \equiv \mathrm{pos}\,\exists\,\mathrm{FO}$ in the sense of finite model theory

$\mathrm{FO}^m/_{\mathrm{hom}} \equiv \mathrm{pos}\,\exists\,\mathrm{FO}^m$ classical + parameter-awareness (!)

Lyndon–Tarski: Atserias–Dawar–Kolaitis (2004)

$\mathrm{FO}/_{\mathrm{hom}} \equiv \mathrm{pos}\,\exists\,\mathrm{FO}$ in restricted classes of finite models

Łos–Tarski: Atserias–Dawar–Grohe (2005)

$\mathrm{FO}/_{\mathrm{ext}} \equiv \exists\,\mathrm{FO}$ in restricted classes of finite models

fails over the class of all finite structures: Gurevich–Tait

plan for parts B & C

B: Tree-like models & tractability

(1) graphs & trees

(2) hypergraphs & tree decompositions

C: Approximations in finite covers

(1) finite graph covers

(2) finite hypergraph covers

$\rightarrow$ tractability w.r.t. modal logics (1) and guarded logics (2)

B: Tree-like models and tractability

(1) transition systems, game graphs & trees

computational behaviour of transition systems strategy analysis in game graphs	$\rightarrow$	bisimilar unfolding into *infinite* trees

bisimulation equivalence $\sim$ (= modal back&forth equivalence) has

tree models as **tractable representatives**

- well understood algorithmic model theory
- automata and MSO over trees
- tractability of modal logics

expressive completeness proof, example

Janin–Walukiewicz (1996)

$\mathrm{MSO}/\!\sim \;\equiv\; \mathrm{L}_\mu$

crux: preservation under $\sim$ $\Rightarrow$ preservation under $\equiv_\mu^m$

without compactness (!)
need finite index approx.

tractability

L_μ ▶ MSO
$\equiv_\mu$ ▶ $\equiv_{\mathrm{MSO}}$
$\equiv_\mu^{f(k)}$ ▶ $\equiv_{\mathrm{MSO}}^k$

} in ω-tree unfoldings

expressive completeness proof, example

van Benthem's Theorem $\mathsf{FO}/\!\sim\ \equiv\ \mathsf{ML}$

crux (modulo compactness): preservation under $\sim$ $\Rightarrow$ preservation under $\equiv_{\mathsf{ML}}$

tractability (classically) modal saturation/ω-saturation give

ML ▶ ML_∞

$\equiv_{\mathsf{ML}}$ ▶ $\sim$

alternatively ML ▶ FO in tree-like models

$\equiv_{\mathsf{ML}}^{f(k)}$ ▶ $\equiv_{\mathsf{FO}}^{k}$

yields fmt analogue (Rosen 1997) $\rightarrow$ part C

B: Tree-like models and tractability

(2) hypergraphs & tree decompositions

hypergraph: $\mathbf{H} = (A, S)$, $S \subseteq \mathcal{P}(A)$: set of hyperedges

with induced graph $\mathbf{G}(\mathbf{H}) = \big(A, \{e \in \mathcal{P}_2(A) \colon e \subseteq s \in S\}\big)$

example: **hypergraph of guarded sets**

of a relational structure $\mathfrak{A} = (A, R^{\mathfrak{A}})$

$$\mathbf{H}(\mathfrak{A}) = \big(A, \{\ [\mathbf{a}]\ :\ \mathbf{a} \in R^{\mathfrak{A}}\}\big)$$

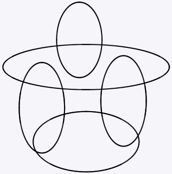
$[\mathbf{a}] := \{a_i \colon 1 \leqslant i \leqslant k\}$ if $\mathbf{a} = (a_1, \ldots, a_k)$

whose induced graph is the

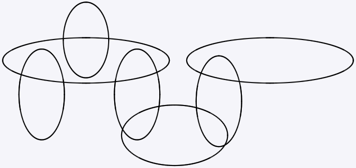
Gaifman graph $\mathbf{G}(\mathfrak{A})$ of $\mathfrak{A}$

hypergraph acyclicity

compare



versus



hypergraph acyclicity = tree decomposability

Graham's decomposition:
- delete simply covered vertices
- delete $\subseteq$-covered hyperedges

hypergraph acyclicity = conformality + chordality

- conformal: every Gaifman clique covered by some hyperedge
- chordal: no chordless cycles in Gaifman graph

hypergraph acyclicity

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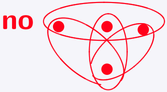
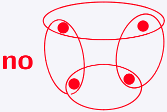




acyclicity achievable in *infinite* guarded unfoldings
(tree unfoldings w.r.t. intersection graph of $\mathrm{H}(\mathfrak{A})$)

tractable representatives up to **guarded bisimulation** $\approx$

guardedness

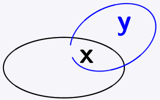
the guarded fragment (Andreka, van Benthem, Nemeti, 1998)

restriction/relativisation of FO-quantification
to guarded tuples induces the *guarded fragment*

$$\mathbf{GF} \subseteq \mathbf{FO}$$

with quantification patterns

$$\varphi(\mathbf{x}) = \forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \psi(\mathbf{xy})\big)$$
$$\varphi(\mathbf{x}) = \exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \psi(\mathbf{xy})\big)$$



a generalisation of modal logic

with further variations like **CGF**: clique guarded fragment

guardedness and GF

properties of guarded logics

- finite model property for GF (and CGF)
  using Herwig's EPPA construction
  $\rightarrow$ Hrushovski–Herwig–Lascar
- decidability of SAT (= FINSAT)
- expressive completeness

$$\mathrm{FO}/\!\approx \;\equiv\; \mathrm{GF}$$

classical: via saturation (Andreka, van Benthem, Nemeti)

open in fmt until recently $\rightarrow$ part C

C: Approximations to tree-likeness in finite covers

(1) graph structures

- tree unfoldings of cyclic graph structures are infinite
- **finite locally acyclic bisimilar covers**
  obtainable as products with Cayley groups of large girth

every finite graph $\mathfrak{A} = (A, E^{\mathfrak{A}})$ admits a finite bisimilar cover

$$\pi : \mathfrak{A}^* \xrightarrow{\sim} \mathfrak{A}$$

by a finite graph $\mathfrak{A}^*$ that is N-locally acyclic.

- suitable Cayley groups: combinatorial group action on regularly coloured acyclic graphs (Biggs, Alon)

example: tractability in locally acyclic covers

van Benthem–Rosen: $\mathsf{FO}/{\sim} \equiv \mathsf{ML}$

uniform proof based on locally acyclic covers (of high branching)

crux: $\varphi \in \mathrm{FO}^k$ preserved under $\sim$ $\Rightarrow$ φ preserved under $\equiv_{\mathrm{ML}}^{f(k)}$

tractability $\mathrm{ML}^{f(k)}$ ▶ FO^k

$\equiv_{\mathrm{ML}}^{f(k)}$ ▶ $\equiv^k$ in covers $\pi\colon \mathfrak{A}^* \xrightarrow{\sim} \mathfrak{A}$

tractability argument provides this upgrading, **classically and fmt parameter-aware (!)**

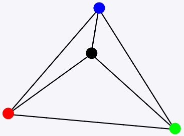
$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\;\equiv_{\mathrm{ML}}^{f(k)}\;} & \mathfrak{B} \\ \big| & & \big| \\ \sim & & \sim \\ \big| & & \big| \\ \mathfrak{A}^* & \text{———}\ \equiv^{\mathbf{k}}\ \text{———} & \mathfrak{B}^* \end{array}$$

C: Approximations to tree-likeness in finite covers

(2) hypergraph covers w.r.t. $\approx$

recall: **acyclic = conformal & chordal**

- acyclic bisimilar covers of hypergraphs may necessarily be 1-locally infinite



but one still does have

- finite *conformal* bisimilar covers (Hodkinson, O_'03)

  tractability: GF ▶ CGF

  model theoretic consequences:

  $\rightarrow$ reductions from CGF to GF (fmp and small finite models)
  $\rightarrow$ generalisations of Herwig–Lascar EPPA Theorem

C: Approximations to tree-likeness in finite covers

(2) hypergraph covers w.r.t. ≈

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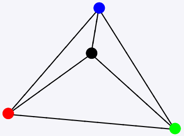
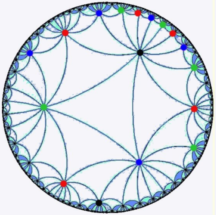


need to relax **chordality** requirement:

- no bad cycles locally (impossible)
  $\longrightarrow$ no *short* bad cycles (?)



acyclicity in finite hypergraph covers

N-acyclicity: chordality requirements for cycles up to length N

two kinds:

- **weakly N-acyclic bisimilar covers** (Barany, Gottlob, O_'10)
  guarantee 'chordality in projection' for all cycles up to length N

  tractability: GF ▶ pos ∃ FO

  model theoretic consequences:

  → fmp for GF w.r.t. classes with forbidden homomorphisms

  → optimal size bounds on small models for GF and CGF

  → polynomial canonisation & capturing $\mathrm{Ptime}/\approx$

acyclicity in finite hypergraph covers

N-acyclicity: chordality requirements for cycles up to length N

two kinds:

- **fully N-acyclic bisimilar covers** (O_'10)
  guarantee chordality in cover for all cycles up to length N

  tractability: GF ▶ FO

  model theoretic consequences:

  $\rightarrow$ fmp w.r.t. classes with forbidden cyclic substructures

  $\rightarrow$ expressive completeness proof for GF in fmt

acyclicity in finite hypergraph covers

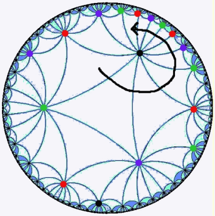
summary of tractabilities in finite hypergraph covers

kind of (finite) cover	tractability	
conformal Hodkinson, O_	GF ▶ CGF	nat & can
weakly N-acyclic Barany, Gottlob, O_	GF ▶ pos ∃ FO	nat & can polynomial
fully N-acyclic	GF ▶ FO	$*$

$*$ combinatorial challenge:
current construction does not
support EPPA generalisations

N-acyclicity in finite hypergraph covers

ingredients



finite Cayley groups of large girth
w.r.t. *discounted distance measure*

idea: count no. of non-trivial transitions
between cosets w.r.t. subgroups

used in local-to-global glueing of layers
to mend defects in partial locally finite covers

- new & universal construction of these Cayley groups:
iteration of combinatorial group action & amalgamation

aside: acyclicity in Cayley groups/graphs

G a Cayley group with involutive generators $e \in E$

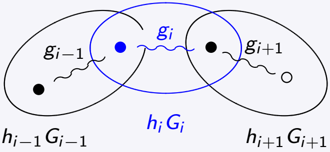
N-acyclic (= large girth)**:**

no generator cycles of length up to N;

$e_1 \circ \cdots \circ e_k \neq 1$ for short non-trivial generator sequences (e_i), $k \leqslant N$

strongly N-acyclic (new)**:**

$g_1 \circ \cdots \circ g_k \neq 1$ for short non-trivial coset sequences $(h_i G_i)$, $k \leqslant N$ where $g_i = h_{i+1} \circ h_i^{-1}$, $G_i = G(E_i)$



N-acyclicity in finite hypergraph covers

tractability of N-acyclic models

in finite, sufficiently acyclic covers (of high branching):

- convex hulls w.r.t. short chordless paths
  are of bounded size,
  – hence acyclic
  – hence controlled by GF-types

- **tractability**: GF ▶ FO
  $\equiv_{\mathrm{GF}}^{f(k)}$ ▶ $\equiv^{k}$

  yields expressive completeness of GF for FO/$\approx$ in fmt

  $$\mathrm{FO}/\!\approx \;\equiv\; \mathrm{GF}\ (\mathrm{fmt})$$

summary

- there is a purely *model-theoretic flavour of tractability,* often subsumed in notions of "well-behaved classes"
- basic idea is common in model-theoretic constructions, notably in *expressive completeness* arguments, and
- lends itself to concrete and *parameter-aware* constructions also in non-classical settings, esp. in *finite* and *algorithmic model theory.*