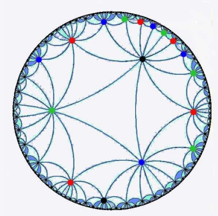
# Bisimulation and Coverings for Graphs and Hypergraphs

Martin Otto
Dpt of Mathematics
TU Darmstadt



**ICLA 13, Chennai**



## questions

- **what is modal/graph bisimulation good for?**
- **how does it generalise from graphs to hypergraphs?**
- **what is guarded/hypergraph bisimulation good for?**
- **which features and applications generalise?**

$\longrightarrow$ **logic vs combinatorial challenges**

## bisimulation: the quintessential back & forth

### state-transition systems

transition systems: coloured directed graphs

Kripke structures: possible worlds, accessibility relations

temporal structures: states, flow of time

epistemic structures: knowledge states, uncertainty equivalences

**game graphs: positions and possible moves**

### notions of behaviour

sequences of transitions (between observable states)

interactive behaviour: challenge/response instead of traces

embeddable trees of action sequences (up to multiplicities)

**bisimulation classes**



## the bisimulation game

**back & forth in graph-like structures**

with binary (transition) relations $\mathbf{R} = (R_1, \dots)$
and unary (state) predicates $\mathbf{P} = (P_1, \dots)$

**two players on two structures:** $\begin{cases} \mathcal{A} = (A, (\mathbf{R}^{\mathcal{A}}), (\mathbf{P}^{\mathcal{A}})) \\ \mathcal{A}' = (A', (\mathbf{R}^{\mathcal{A}'}), (\mathbf{P}^{\mathcal{A}'})) \end{cases}$

**game positions:** $(a, a') \in A \times A'$

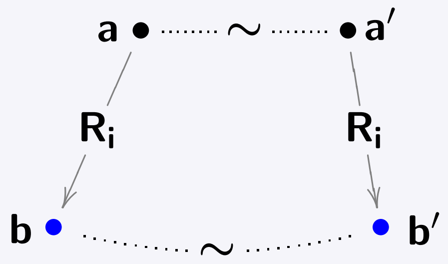
pebbles on $a$ in $\mathcal{A}$ and $a'$ in $\mathcal{A}'$

**single round, challenge/response:**

player **I** makes a transition from $a$ or from $a'$
player **II** needs to match this transition on opposite side

## the bisimulation game

**single round (challenge/response):**




**winning/losing:**

- player **II** needs to maintain local equivalence between states
- player **I** or **II** lose when stuck

**winning strategies for player II:**

| | |
|---|---|
| $\mathcal{A}, a \sim^{\ell} \mathcal{A}', a'$ | player **II** has winning strategy in $\ell$-round game from position $(a, a')$ |
| $\mathcal{A}, a \sim \mathcal{A}', a'$ | player **II** has winning strategy in unbounded game from position $(a, a')$ |



## model-theoretic applications to modal logics

**on graph-like structures**

with binary (transition) relations $\mathbf{R} = (R_1, \ldots)$ ⇝ modalities $\Diamond_i / \Box_i$
and unary (state) predicates $\mathbf{P} = (P_1, \ldots)$ ⇝ basic propositions $p_i$

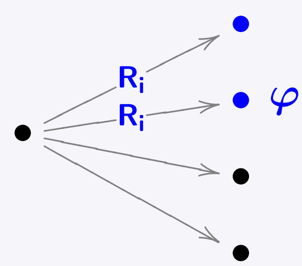
**basic modal logic ML**

with atomic $\bot, \top, p_i$, closed under booleans, and
**modal quantification:**

$$\Diamond_i \varphi \equiv \exists y \big( R_i xy \wedge \varphi(y) \big)$$

$$\Box_i \varphi \equiv \forall y \big( R_i xy \rightarrow \varphi(y) \big)$$

relativised FO quantification




**modal logics ML ··· $\mathrm{ML}_\infty$ preserved under bisimulation**

## bisimulation equivalence: modal Ehrenfeucht–Fraïssé

### the modal Ehrenfeucht–Fraïssé thm

$\mathcal{A}, a \sim^{\ell} \mathcal{A}', a' \quad \Leftrightarrow \quad \mathcal{A}, a \equiv^{\ell}_{\mathrm{ML}} \mathcal{A}', a'$ (ML$^{\ell}$ equiv./depth $\ell$)

### the modal Karp thm

$\mathcal{A}, a \sim \mathcal{A}', a' \quad \Leftrightarrow \quad \mathcal{A}, a \equiv^{\infty}_{\mathrm{ML}} \mathcal{A}', a'$ (inf. equiv. in ML$^{\infty}$)

**and further classical model-theoretic consequences:**

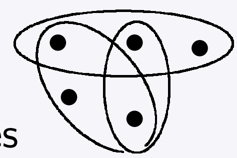
- bisimulation invariance/preservation (!)
- tree model property of modal logics (!!)
- Hennessy–Milner thms for saturated structures
- classical expressive completeness proofs (van Benthem)



## guardedness

### observable configurations in relational structures

**examples:**



- tuples in relational database
- clusters of variables in CSP and conjunctive queries
- higher-arity roles (as in description logics)

### essence of the generalisation: from graphs to hypergraphs

transition systems/graphs $\longrightarrow$ relational structures/hypergraphs

modal logic $\longrightarrow$ guarded logic

**modal bisimulation for graph structures** $\longrightarrow$ **guarded bisimulation for hypergraph structures**

## hypergraph of guarded subsets

of a relational structure $\mathcal{A} = (A, (R^{\mathcal{A}})_{R \in \tau})$:

$$\mathbf{H(\mathcal{A}) = (A, S[\mathcal{A}])}$$

with hyperedges generated by subsets $[\mathbf{a}] \subseteq A$ for $\mathbf{a} \in R^{\mathcal{A}}$, $R \in \tau$
closed under subsets & singleton sets

**hypergraph terminology:**

- $\mathbf{H = (A, S)}$, $S \subseteq \mathcal{P}(A)$ the set of hyperedges
- $\mathbf{G(H) = (A, E)}$, associated graph: hyperedges $\rightsquigarrow$ cliques
- $\mathbf{G(\mathcal{A}) = G(H(\mathcal{A}))}$, the Gaifman graph of $\mathcal{A}$

| | | |
|---|---|---|
| **relational structure** | = | **hypergraph link structure (topology)**<br>**+ local relational content** |



## the hypergraph bisimulation game

**hypergraph bisimulation** $\mathbf{H, s \sim H', s'}$ **and** $\mathbf{H, s \sim^{\ell} H', s'}$

**idea: moves between hyperedges respecting the overlap**

**positions in game on $\mathbf{H = (A, S)}$ vs. $\mathbf{H' = (A', S')}$:**

bijections $s \leftrightarrow s'$, $s \in S, s' \in S'$

**single round, challenge/response:**

player **I** selects $t \in S$ or $t' \in S'$
player **II** needs to complete to new bijection $t \leftrightarrow t'$
compatible with $s \leftrightarrow s'$ on $s \cap t$ (on $s' \cap t'$)

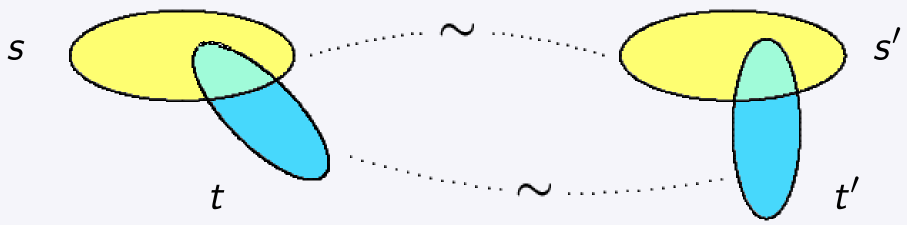
**II loses when stuck**

## the hypergraph bisimulation game

**hypergraph bisimulation** $\mathbf{H, s \sim H', s'}$ **and** $\mathbf{H, s \sim^{\ell} H', s'}$

**single round, challenge/response:**

player **I** selects $t \in S$ or $t' \in S'$
player **II** needs to complete to new bijection $t \leftrightarrow t'$
compatible with $s \leftrightarrow s'$ on $s \cap t$ (on $s' \cap t'$)




challenge/response
move from • to •



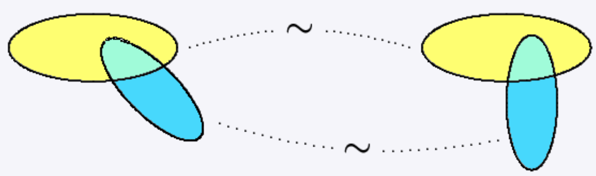
## the hypergraph bisimulation game

**guarded bisimulation** $\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}} \mathcal{A}', \mathbf{a}'$ **and** $\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}}^{\ell} \mathcal{A}', \mathbf{a}'$

- bisimulation of hypergraphs of guarded subsets that locally respects relations
- pebble game with guarded pebble configurations

the two are equivalent

**both captured by a bisimulation game on associated transition system of guarded subsets/tuples (Grädel–Hirsch–O_)**




challenge/response

## model-theoretic applications to guarded logics

on relational structures $\mathcal{A} = (A, \mathbf{R})$

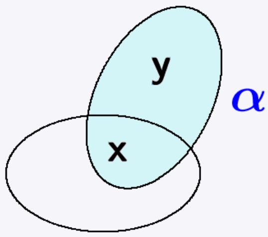
**the guarded fragment GF:**

atomic formulae of FO, closed under booleans, and
**guarded quantification**

$\exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \varphi(\mathbf{xy})\big)$

$\forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})\big)$

guard atom $\alpha$: $\mathrm{free}(\varphi) \subseteq \mathrm{var}(\alpha)$




quantification relativised
to guarded tuples

**ML $\subsetneq$ GF $\subsetneq$ FO** the natural extension of ML
to relations of arbitrary arity



## guarded bisimulation and GF

**the guarded Ehrenfeucht–Fraïssé thm**

$\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}}^{\ell} \mathcal{A}', \mathbf{a}' \quad \Leftrightarrow \quad \mathcal{A}, \mathbf{a} \equiv_{\mathsf{GF}}^{\ell} \mathcal{A}', \mathbf{a}'$ (GF$^{\ell}$-equiv./depth $\ell$)

**the guarded Karp thm**

$\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}} \mathcal{A}', \mathbf{a}' \quad \Leftrightarrow \quad \mathcal{A}, \mathbf{a} \equiv_{\mathsf{GF}}^{\infty} \mathcal{A}', \mathbf{a}'$ (inf. equiv. in GF$^{\infty}$)

**in striking analogy with ML, find:**

- finite model property and decidability
- generalised tree model property (Grädel)
- invariance/preservation under guarded bisimulation
- classical expressive completeness
  (Andreka–van Benthem–Nemeti)

## aside/example: expressive completeness results

**theorem** (van Benthem, Rosen)

$\mathrm{FO}/{\sim} \equiv \mathrm{ML}$ (classically and fmt)

i.e., equivalent for $\varphi(x) \in \mathrm{FO}$:

– $\varphi$ is $\sim$-invariant: $\mathcal{A}, a \sim \mathcal{B}, b \;\Rightarrow\; (\mathcal{A}, a \models \varphi \;\Leftrightarrow\; \mathcal{B}, b \models \varphi)$

– $\varphi \equiv \varphi' \in \mathrm{ML}$

ML is expressively complete for $\sim$-invariant FO-properties,
— in two readings (a priori independent!)

**theorem** (Andreka–van Benthem–Nemeti, O_)

$\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv \mathrm{GF}$ (classically and fmt)



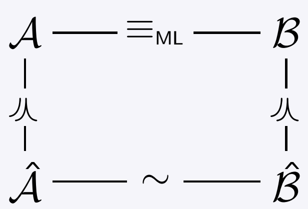
## aside/example: expressive completeness results

**crux: a compactness property**

for $\varphi \in \mathrm{FO}$ (over all or just finite structures):

invariance under $\sim/\sim_{\mathrm{g}}$ $\Rightarrow$
invariance under $\sim^{\ell}/\sim^{\ell}_{\mathrm{g}}$ for some $\ell$

**classical model theory:** use saturated elementary extensions,
e.g., to boost $\mathcal{A} \equiv_{\mathrm{ML}} \mathcal{B}$ to $\hat{\mathcal{A}} \sim \hat{\mathcal{B}}$:



## aside/example: expressive completeness results

for constructive fmt approach:

**model transformations $\mathcal{A} \mapsto \hat{\mathcal{A}}$, $\mathcal{B} \mapsto \hat{\mathcal{B}}$**

$$\begin{array}{ccc} \mathcal{A} & \text{---}\ \sim^{\ell}\ \text{---} & \mathcal{B} \\ | & & | \\ \sim & & \sim \\ | & & | \\ \hat{\mathcal{A}} & \text{---}\ \equiv^{q}\ \text{---} & \hat{\mathcal{B}} \end{array}$$

- need to respect $\sim / \sim_{\mathrm{g}}$
- need to determine $\mathrm{FO}^q$ by $\mathrm{ML}^\ell/\mathrm{GF}^\ell$: such that $\sim^\ell/\sim^\ell_{\mathrm{g}}$ implies $\equiv^q$
- need to avoid all obstacles to $\equiv^q$ that are *not* controlled by $\sim^\ell$ / $\sim^\ell_{\mathrm{g}}$
- need to avoid small cyclic configurations, small multiplicities

**want to use bisimilar coverings:**

homomorphisms $\pi\colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ that induce a bisimulation



## the rest of this talk

**attention to link structure: graphs/hypergraphs**

**attention to control of cycles in coverings**

**focus on passage from graphs to hypergraphs**

- **N-acyclic graph coverings**
  $\to$ Cayley groups of large girth
- **N-acyclic hypergraph coverings**
  $\to$ hypergraph acyclicity
  $\to$ stronger Cayley groups and groupoids

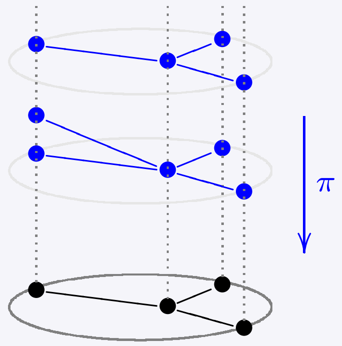
## bisimilar graph coverings

**graph covering:**

$\pi\colon (\hat{V},\hat{E}) \xrightarrow{\sim} (V,E)$

homomorphism (*forth*) with *back*-property:

for $e$ incident with $\pi(\hat{v})$ there is $\hat{e} \in \hat{E}$ incident with $\hat{v}$ s.t. $\pi(\hat{e}) = e$




**N-acyclic:** no cycles of length up to $N$,
$\ell$-locally acyclic for $\ell = \lfloor (N-1)/2 \rfloor$



## N-acyclic graph coverings

**Cayley groups and graphs:**

group $\mathrm{G}$ with involutive generators $e \in E$ induces homogeneous regular graph on vertex set $\mathrm{G}$ with edges $(g, g \cdot e)$

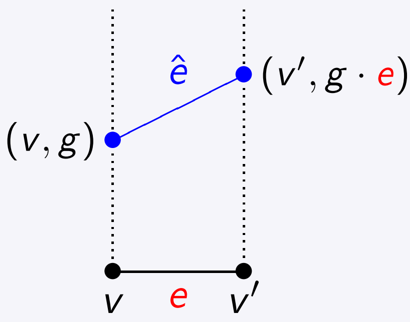
**of girth $\geqslant$ N:** no short generator sequence representing $1 \in \mathrm{G}$

**available from group action on suitable E-coloured graphs**

**N-acyclic covers by products:**

natural products $(V,E) \otimes \mathrm{G}$
with Cayley graphs of girth $\geqslant N$
and involutive generators $e \in E$

provide $N$-acyclic coverings
$\pi\colon (V,E) \otimes \mathrm{G} \xrightarrow{\sim} (V,E)$



## bisimilar hypergraph coverings

**hypergraph covering:** $\pi \colon (\hat{A}, \hat{S}) \xrightarrow{\sim} (A, S)$

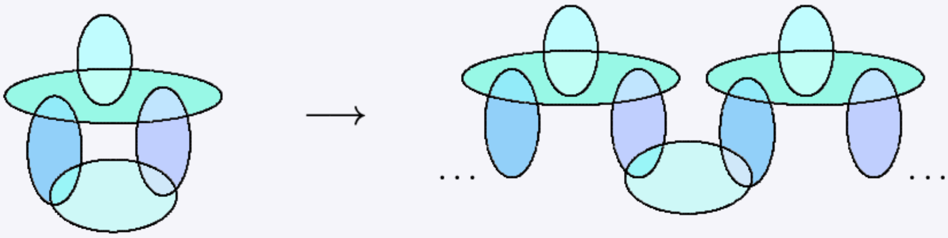
hypergraph homomorphism (locally bijective on $\hat{s} \in \hat{S}$, *forth*)
with *back*-property:

for $s = \pi(\hat{s})$ and $t \in S$ there is $\hat{t} \in \hat{S}$ such that $\pi(\hat{s} \cap \hat{t}) = s \cap t$

**hypergraph acyclicity:** several equivalent characterisations

- tree-decomposable with hyperedges as bags
- decomposable through elementary deletion steps (Graham)
- **conformality and chordality** (of associated Gaifman graph)

**example:** infinite tree-like unfoldings are acyclic coverings





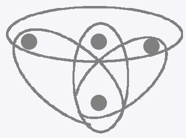
## hypergraph terminology

for hypergraph $\mathrm{H} = (A, S)$ and associated Gaifman graph

$\mathrm{G(H)} = (A, E) = \bigcup_{s \in S} K[s]$ **a clique for each s ∈ S**

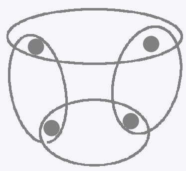
- **conformality:** every clique in G(H) is contained in some $s \in S$

no



- **chordality:** every cycle of length $> 3$ in G(H) has a chord

no



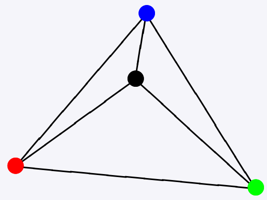
**N-acyclicity = N-conformality + N-chordality:**

acyclicity of induced sub-configurations of size up to $N$

## examples & limitations

### the facets of the 3-simplex/tetrahedron

uniform width 3 hypergraph on 4 vertices

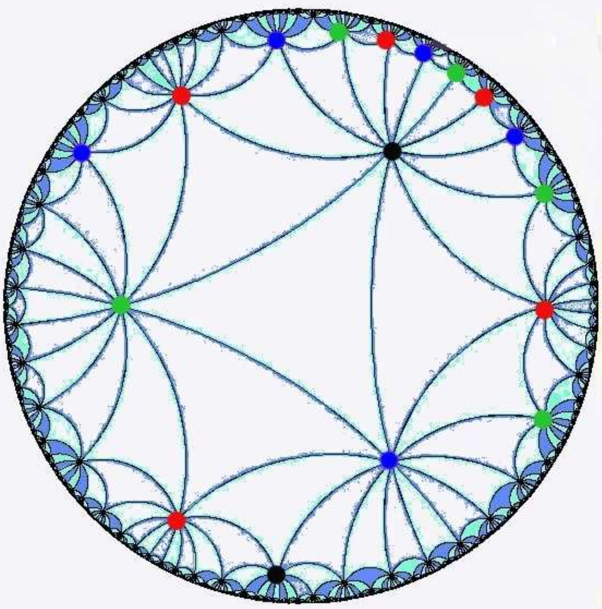


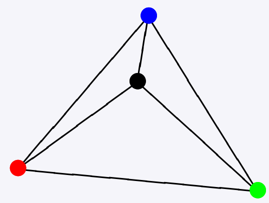
- chordal but not conformal
- finite coverings cannot be 1-locally acyclic
- admits locally finite coverings without short chordless cycles
- also admits simple finite 5-acyclic covering in which every induced sub-configuration on up to 5 vertices is acyclic



## example ctd

### a locally finite covering of the tetrahedron





conformal; shortest chordless cycles have length 12
here by isometric tesselation in hyperbolic geometry

## finite N-acyclic hypergraph coverings

**thm** (O_ 10, improved 12)

every finite hypergraph $\mathrm{H} = (V, S)$ admits, for every $N \in \mathbb{N}$,

finite $N$-acyclic coverings $\pi \colon \hat{\mathrm{H}} \xrightarrow{\sim} \mathrm{H}$,

**method:**

- highly acyclic Cayley groups, local–global constructions, ...
- (new approach) reduced products with highly acyclic Cayley groupoids

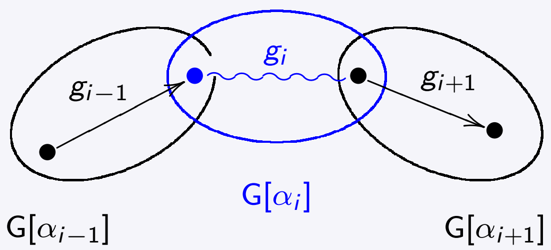


## acyclicity in Cayley group(oid)s, strengthened

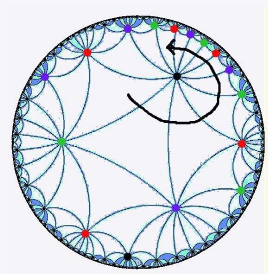
e.g., for $\mathbb{G}$ with generators $e \in E$:

- large girth (no short generator cycles):
  $e_1 \circ e_2 \circ \cdots \circ e_k \neq 1$ for small $k$
- stronger notion (no short coset cycles):
  $g_1 \circ g_2 \circ \cdots \circ g_k \neq 1$ for small $k$

  where $g_k \in \mathrm{G}[\alpha_k]$ for colour classes $\alpha_k \subseteq E$ such that corresponding cosets *locally* overlap *without shortcuts*




**motivation:**



**N-acyclic Cayley group(oid)s:** no such cycles for $k \leqslant N$

## N-acyclic Cayley group(oid)s

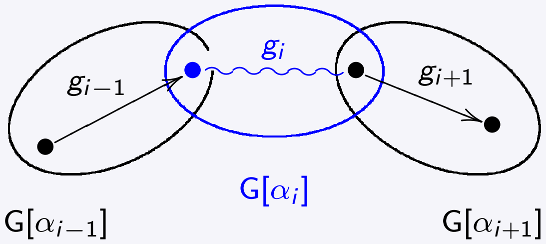
**thm** (O_ 10/12)

obtain $N$-acyclic groups and groupoids,
for every finite set $E$ of generators and $N \in \mathbb{N}$

**inductive construction:**

combinatorial group action & amalgamation of Cayley graphs
on $E$-coloured graphs | $\mathrm{G}[\alpha]$ for smaller $\alpha \subsetneq E$

to avoid short coset cycles in $\mathrm{G}[\alpha]$ for increasingly large $\alpha \subseteq E$






## hypergraph coverings by reduced products

**towards hypergraph coverings:**

reduced products of $\mathrm{H} = (A, S)$ with Cayley groupoids G
with generators $e = (s, s')$ for $s \cap s \neq \emptyset$

$$\mathrm{H} \otimes \mathrm{G} : \begin{cases} \text{quotient } (\mathrm{H} \times \mathrm{G})/\approx \\ (v, g) \approx (v, g') \quad \text{if} \quad g \circ (g')^{-1} \in \mathrm{G}[\alpha] \\ \qquad\qquad\qquad\qquad \text{for } \alpha = \{(s, s') \colon v \in s \cap s'\} \end{cases}$$

**thm** (O_ 12)

reduced products with $N$-acyclic groupoids G are $N$-acyclic

## further (new) results

similar reduced product constructions
with $N$-acyclic groupoids yield

**generic solutions for finite closures/realisations of**

- abstract specifications of local overlap patterns
- abstract specifications of complete GF-types
- extension properties for partial isomorphisms
  (in the sense of Hrushovski/Herwig/Herwig–Lascar)

these highly regular & symmetric constructions
are compatible with automorphisms of the given data
(preserve symmetries of the sepecification)

. . . **and why groupoids?**



## summary: attention to link structure (topology)

**analogies and generlisations**

**discrete mathematics:** graphs $\rightsquigarrow$ hypergraphs

**databases:** transition systems $\rightsquigarrow$ relational databases

**logic/model theory:** modal $\rightsquigarrow$ guarded logics

**e.g., tree-decompositions and tree unfoldings**
**& finite coverings with control over cycles**

**how far do these analogies carry?**

## summary: how far do the analogies carry?

- infinite tree unfoldings as fully acyclic coverings:
  a complete analogy, good for most classical purposes
- finite coverings meet different combinatorial challenges
  w.r.t. control of cycles and local-global-distinctions
- gain considerable extensions of the analogies between
  graphs/hypergraphs & modal/guarded logics
- especially through new hypergraph constructions
  via reduced products with suitable groupoids

**the end**
**questions?**