Bisimulation Invariance over Transitive Kripke Structures

Dagstuhl 08: beyond the finite

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based on recent joint work with Anuj Dawar
expressive completeness results for modal logics

van Benthem–Rosen

\[ \text{FO}/\sim \equiv \text{ML} \quad \text{over the class of} \quad \begin{cases} 
\text{all Kripke structures} \\
\text{all finite Kripke structures}
\end{cases} \]

Hafer–Thomas, Moller–Rabinovich

\[ \text{MSO}^\varphi/\sim \equiv \text{CTL}^* \quad \text{over the class of all (unranked) trees} \]

Janin–Walukiewicz

\[ \text{MSO}/\sim \equiv \text{L}_\mu \quad \text{over the class of all Kripke structures} \]

classical model theory (compactness)/decomposition techniques/
automata theory/explicit model transformations

commom thread: upgradings between game-based equivalences
example: van Benthem–Rosen

classical upgrading versus upgrading through explicit
compactness & saturation model transformations

crux (in either setting)

\[
\text{FO/}\sim \equiv \bigcup_{\ell} \text{FO/}\sim^{\ell} (\equiv \text{ML})
\]

\[
\sim^{\ell} \begin{cases} 
\ell\text{-bisimulation equivalence} \\
\ell\text{-round bisimulation game} \\
equivalence in ML_{\ell}
\end{cases}
\]
\[
\text{classical argument: compactness}
\]

upgrading \( \sim^\omega := \bigcap_{\ell} \sim^\ell \) to \( \sim \)

\[
\text{as } \sim^\omega = \sim \text{ on saturated elementary extensions:}
\]

\[
\begin{array}{ccc}
\overline{A} & \overset{\sim^\omega}{\sim} & \overline{B} \\
\equiv & & \equiv \\
\overline{A^*} & \overset{\sim}{\sim} & \overline{B^*}
\end{array}
\]

arbitrarily long finite games

infinite game
**upgrading through explicit model transformation**

from 
concrete level $\sim^\ell$ of approximate bisimulation equivalence 
to 
concrete level $\equiv$ of approximate target equivalence

\[
\begin{array}{c}
A \quad \sim^\ell \quad B \\
\downarrow \quad \sim \quad \downarrow \\
A^* \quad \equiv \quad B^*
\end{array}
\]

→ van Benthem–Rosen with $\ell = 2^{\text{FO} - \text{qr}}$ via locality argument
classical argument vs. explicit model transformation
orthogonal approaches in expressive completeness proofs
for generic (game-)equivalences \( \equiv / \equiv^\omega / \equiv^\ell \) & approximate \( \equiv \)

via full \( \equiv \) to full \( \equiv \)

\[
\begin{array}{c}
A \equiv^\omega B \\
\equiv \\
A^* \equiv B^*
\end{array}
\]

upgrading via \( \omega \)-saturation
relativises to
elementary classes

via full \( \equiv \) to approximate \( \equiv \)

\[
\begin{array}{c}
A \equiv^\ell B \\
\equiv \\
A^* \equiv B^*
\end{array}
\]

direct upgrading
relativises to classes with
suitable closure properties
generalises to MSO
examples from D/O LICS 05: **locality based**

upgrading $\sim^\ell$ to some level of Gaifman equivalence

- $\text{FO}/\sim \equiv \text{ML}$ on (finite) multi-modal equivalence frames
- $\text{FO}/\sim \equiv \text{ML}[\forall]$ on (finite) rooted frames

examples from D/O LICS 05: **decomposition based**

upgrading $\sim^\ell$ to $\equiv_q$ through path decomposition & pumping arguments

- $\text{FO}/\sim \equiv \text{ML}$ on (finite) transitive $\prec$-trees (irreflexive)
- $\text{FO}/\sim \equiv \text{ML}$ on (finite) transitive $\preceq$-trees (reflexive)
the rest of this talk

• **FO path decomposition** & pumping argument on $\prec$-trees

• **extension via interpretation** & upgrading to $\preceq$-trees and other transitive frames finite and infinite

• **extension to cover MSO:**
  collapse results for MSO/$\sim$ over classes of transitive frames de Jongh–Sambin–Smorynski & Janin–Walukiewicz

special interest: “**path-finite transitive frames**”
transitive frames without infinite strict (or irreversible) paths

no infinite nested chain of generated subframes
a quasi-wellordering property
<-trees: FO path decomposition & pumping argument

in wide <-tree companions \( s_q(\mathcal{A}, \alpha) := TC((\mathcal{A} \otimes q)^*_\alpha) \)

\[
\begin{array}{ccc}
\text{from} & \cdot \alpha & \text{to} \\
\downarrow & & \downarrow \\
\cdot a & & \cdot a \\
\end{array}
\quad
\begin{array}{ccc}
\text{to} & \cdot \alpha & \text{to} \\
\uparrow & & \uparrow \\
\cdot a & & \cdot a \\
\end{array}
\]

colours: \( \equiv_{q-1} \)-classes of subtrees

**pumping lemma/Ehrenfeucht–Fraïssé**; **finiteness property**

bound on length of relevant words realised and sub-word closure property in \( s_q(\mathcal{A}, \alpha) \) (!)

\[ \rightarrow \text{ (non-elementary) bound on } \ell \text{ for } \sim^\ell \text{ that governs } \equiv_q \]
the (harmless) extension to $\preceq$-trees

the natural quantifier-free interpretation: $\mathbf{A}_\preceq, \alpha \mapsto \mathbf{A}_\preceq, \alpha$

upgrading:

$\mathbf{A}_\preceq, \alpha \sim 2\ell + 1 \sim 2\ell + 1 \mathbf{B}_\preceq, \beta$

$\mathbf{A}_\preceq, \alpha \sim 2\ell + 1 \sim \mathbf{B}_\preceq, \beta$

$\mathbf{A}_\preceq, \alpha \sim \ell \mathbf{B}_\preceq, \beta$ $\Rightarrow \mathbf{A}_\preceq, \alpha \equiv_q \mathbf{B}_\preceq, \beta$

insertion of copies of reflexive nodes
if reflexivity is not prescribed: transitive tree-like frames

\[ \varphi(x) = \exists y (E_{xy} \land E_{yy}) \]

- \(\sim\) invariant over finite/path-finite transitive frames
- not \(\sim\) invariant over transitive frames with infinite paths
- not \(\sim^\ell\) invariant for any \(\ell\) over all finite transitive frames

examples:

\[ \Rightarrow \text{FO/}\sim \not\equiv \text{ML over the class of finite transitive frames} \]
a new modality: $\Diamond^* \varphi \equiv \exists y (E_{xy} \land E_{yy} \land \varphi(y))$

with associated $\sim_* / \sim^*_{\ell}$

$\mathcal{A}, a \sim \mathcal{B}, b \Rightarrow \mathcal{A}, a \sim_* \mathcal{B}, b$ on path-finite (!) transitive frames

upgrading in interpretation: $\mathcal{A}, \alpha \mapsto \mathcal{A}^\bullet, \alpha$

\[
\begin{array}{ccc}
\mathcal{A}, \alpha & \sim^L & \mathcal{B}, \beta \\
\sim_* & \sim & \\
\sim & \sim & L = \ell^2 + \ell + 1 \\
\hline
\tilde{\mathcal{A}}, \alpha & \sim^L & \tilde{\mathcal{B}}, \beta \\
\sim_* & \sim & \\
\tilde{\mathcal{A}}^\bullet, \alpha & \equiv_q & \tilde{\mathcal{B}}^\bullet, \beta \\
\end{array}
\]

non-trivial game analysis

bullet-expansions with marker predicate for reflexive nodes
results for \(\text{FO/}\sim\) over transitive frames

- \(\text{FO/}\sim \equiv \text{ML}\) on \((\text{finite})\) transitive \(<\)-trees/\(\preccurlyeq\)-trees

- \(\text{FO/}\sim \equiv \text{ML}[\Diamond^*]\) on
  \[
  \begin{cases}
  \text{finite transitive tree-like frames} \\
  \text{finite transitive frames} \\
  \text{path-finite transitive tree-like frames} \\
  \text{path-finite transitive frames}
  \end{cases}
  \]

translation transitive \(\longrightarrow\) transitive tree-like via FO-interpretations

NB: \(\Diamond^*\) is \(\sim\)-safe \emph{only} over path-finite transitive frames;
\(\Diamond^*\varphi \equiv \nu_X \Diamond (\varphi \land X)\) over path-finite transitive frames
and \(\nu_X \Diamond (\varphi \land X)\) not FO in all transitive frames
the extension to MSO/∼
subtree decomposition rather than path decomposition
upgrading: \( A, \alpha \sim^L B, \beta \rightarrow s_Q(A), \alpha \equiv^{MSO}_q s_Q(B), \beta \)
for any path-finite transitive \( \prec \)-trees \( A, \alpha, B, \beta \)

proof idea, in \( A^* := s_Q(A) = TC(A \otimes Q)^*_\alpha \):

- \( tp^{MSO}_q(A_a) \) determined by \( atp(a) \) and multiplicities of \( tp^{MSO}_q(A_b) \) at direct \( \prec \)-successors \( b \) of \( a \)

- \( tp^{MSO}_q(A^*_a) \) determined by \( atp(a) \) and the set \( \{ tp^{MSO}_q(A^*_b) : a \prec b \} \)

monotonicity \( \rightarrow \) finiteness
\[ \text{in } \mathfrak{A}^* := s_Q(\mathfrak{A}) = TC(\mathfrak{A} \otimes Q)^*_\alpha \]

\[ \xi_s, \xi_{\theta/s} \in ML_{|s|+1}: \quad \text{by induction on } |s| \]

\[
\begin{align*}
& s_Q(\mathfrak{A}^*), a \models \xi_s \iff \{ tp^\text{MSO}_q(\mathfrak{A}^*_b): a \prec b \} = s \\
& s_Q(\mathfrak{A}^*), a \models \xi_{\theta/s} \iff \{ tp^\text{MSO}_q(\mathfrak{A}^*_b): a \prec b \} = s \\
& \text{and } tp^\text{MSO}_q(\mathfrak{A}^*_a) = \theta
\end{align*}
\]

\[ s \subseteq \{ \theta: \theta \text{ an MSO}_q\text{-type} \} = \Theta(q) \]

\[
\begin{array}{ccc}
\mathfrak{A}, \alpha & \sim^L(q) & \mathfrak{B}, \beta \\
\nearrow & & \searrow \\
\sim & & \sim \\
\end{array}
\]

\[ s_Q(q)(\mathfrak{A}) \equiv^\text{MSO}_q s_Q(q)(\mathfrak{B}) \]

for \( L(q) = |\Theta(q)| + 1 \) and all sufficiently large \( Q(q) \)
results for $\text{MSO}/\sim$ over transitive frames

- $\text{MSO}/\sim \equiv \text{FO}/\sim \equiv \text{ML}$ on
  
  \[
  \begin{cases}
  \text{finite } \prec \text{-trees} \\
  \text{path-finite } \prec \text{-trees} \\
  \text{(Löb frames)}
  \end{cases}
  \]

- $\text{MSO}/\sim \equiv \text{FO}/\sim \equiv \text{ML}[\diamondsuit^*]$ on
  
  \[
  \begin{cases}
  \text{finite transitive tree-like frames} \\
  \text{finite transitive frames} \\
  \text{path-finite transitive (tree-like) frames}
  \end{cases}
  \]

collapse and de Jongh–Sambin

- $L_\mu \equiv L_\mu \cap \text{FO} \equiv \text{ML}[\diamondsuit^*]$ over the class of all path-finite transitive frames

while $L_\mu \supset L_\mu \cap \text{FO} \equiv \text{ML}$ over the class of all transitive frames
explicit model transformations and upgrading for

- characterisation of FO/$\sim$
- collapse results for MSO/$\sim$ to FO/$\sim$

over various classes of (path-finite) transitive frames

and new variants/generalisations/proofs of

- Janin-Walukiewicz: $\text{MSO}/\sim \equiv \text{ML}[\diamond^*] \subseteq L_{\mu}^1 \subseteq L_{\mu}$ on (path-)finite transitive frames
- de Jongh–Sambin: $L_{\mu} \equiv \text{ML}[\diamond^*]$ on (path-)finite transitive frames

→ ten Cate–Fontaine–Litak

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