Modal Characterisation Theorems over Special Classes of Frames

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FO and bisimulation invariance

- semantic characterisations of modal logics
- effective syntax for undecidable invariances
- expressive completeness for semantic tasks

van Benthem's Theorem

$\mathbf{ML} \equiv \mathbf{FO} \,/\, \sim$

for $\varphi \in \mathrm{FO}$:

φ **bisimulation-invariant**
$\Leftrightarrow$ φ **expressible in ML**

bisimulation equivalence

between (pointed) transition systems / Kripke structures / coloured graphs

$$\mathfrak{A} = (A, (R_i^{\mathfrak{A}}), (P_j^{\mathfrak{A}}))$$

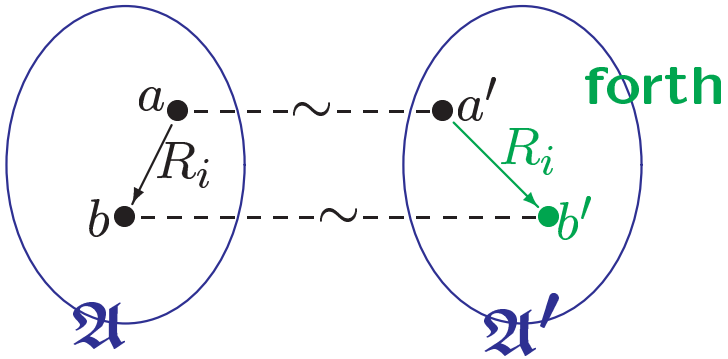
behavioural equivalence

back & forth along edges

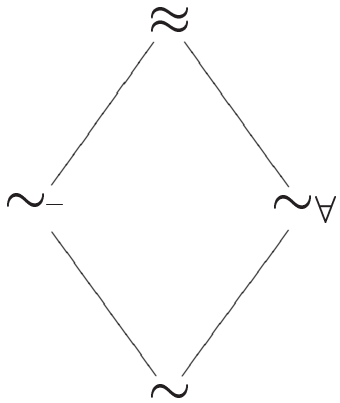
modal Ehrenfeucht-Fraïssé

$Z : \mathfrak{A} \sim \mathfrak{A}'$

- $\emptyset \neq Z \subseteq A \times A'$
- respects unary predicates
- back & forth along edges



variations: two-way and global bisimulations



modal logics and bisimulation preservation

relativised quantification along edges/transitions

$$\Box_i\varphi(x) \equiv \forall y\big(R_i xy \to \varphi(y)\big)$$

basic modal logic ML atomic props, $\wedge$, $\vee$, $\neg$, $\Box_i$, $\Diamond_i$

extensions $\mathrm{ML}^{\forall}$, ML^{-}, $\mathrm{ML}^{-\forall}$ with universal/inverse modalities

bisimulation invariance (preservation):

$\varphi \in$ **ML**: $\quad \mathfrak{A}, a \sim \mathfrak{A}', a' \;\Rightarrow\; \big(\mathfrak{A}, a \models \varphi \;\Leftrightarrow\; \mathfrak{A}', a' \models \varphi\big)$

conversely (expressive completeness):

$\varphi \in$ **FO**: $\quad \varphi$ **$\sim$-invariant** $\Rightarrow$ φ **expressible in ML**

main point of the characterisation theorem

similarly for $\mathrm{ML}^{\forall}$ / $\sim_{\forall}$, ML^{-} / $\sim_{-}$, $\mathrm{ML}^{-\forall}$ / $\approx$

variation of the underlying class of frames

e.g. finite vs. classical model theory

from van Benthem to Rosen (Rosen 97)

$\mathbf{FO}/{\sim} \equiv \mathbf{ML}$ (all frames) **$\mathbf{FO}/{\sim} \equiv \mathbf{ML}$ (all finite frames)**

compactness and
saturated models

model constructions that
work for finite frames, too

good for any elementary class

new proof (O_ LICS'02):

uniformly based on locality and games

various ramifications, in particular:

$$\begin{aligned} \mathrm{FO}/{\sim_\forall} &\equiv \mathrm{ML}^\forall \\ \mathrm{FO}/{\sim_-} &\equiv \mathrm{ML}^- \\ \mathrm{FO}/{\approx} &\equiv \mathrm{ML}^{-\forall} \end{aligned}$$

good for any $\approx$-closed class of (finite) frames

crux of new game based proofs

compactness property for
finite approximations $\leftrightharpoons^\ell$ of full back-and-forth equivalence $\leftrightharpoons$

φ $\leftrightharpoons$-**invariant** $\Rightarrow$ φ $\leftrightharpoons^\ell$-**invariant for some** ℓ

this suffices: $\leftrightharpoons^\ell$ has finite index,
$\leftrightharpoons^\ell$ classes definable (Ehrenfeucht-Fraïssé)

technique:

upgrading $\leftrightharpoons^\ell$ to approximations $\doteq$ of $\equiv$
in $\leftrightharpoons$-preserving model construction

$$\begin{array}{ccc} \mathfrak{A}, a & \overset{\leftrightharpoons^\ell}{\text{———}} & \mathfrak{B}, b \\ \leftrightharpoons \Big| & & \Big| \leftrightharpoons \\ \widehat{\mathfrak{A}}, \widehat{a} & \overset{\doteq}{\text{———}} & \widehat{\mathfrak{B}}, \widehat{b} \end{array}$$

φ preserved under $\leftrightharpoons$
φ preserved under $\doteq$

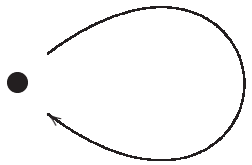
$\Rightarrow$ φ preserved under $\leftrightharpoons^\ell$

all finite frames still too many?

e.g., counter-intuitive reliance on invariance under disjoint unions (inaccessible parts)

observation:

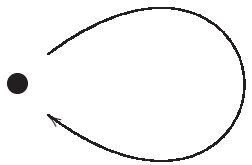
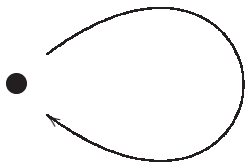
$\sim$-classes of finite frames not ML definable



not characterised by $\begin{cases} \Diamond\top & \text{too weak} \\ \forall\Diamond\top & \text{too strong} \end{cases}$

but by $\nu_X(\Diamond\top \wedge \Box X)$

intended semantics often based on "connected" frames, but

 $\approx$  $+$ 'anything' elsewhere

some natural classes of (finite) frames

not closed under $\approx$

rooted frames: all states reachable from root
not elementary

symmetric frames: all edges bi-directional
elementary / canonical reductions

transitive frames: one transitive transition relation
elementary / non-trivial in fmt: locality fails

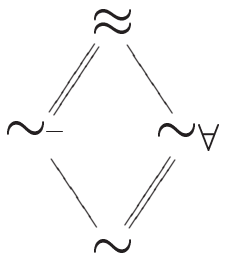
equivalence frames: several equivalence relations (S5)
elementary / non-trivial locality

transitive trees: partial order on trees
not elementary / locality fails

main results — and two main techniques

note that $\sim \;=\; \sim_\forall$

$\sim_- \;=\; \approx$ over rooted frames



locality based:

$\mathrm{FO}/{\sim} \equiv \mathrm{ML}^\forall$ (finite) rooted frames

$\mathrm{FO}/{\sim_\forall} \equiv \mathrm{ML}^\forall$ (finite) equivalence frames

decomposition based:

$\mathrm{FO}/{\sim} \equiv \mathrm{ML} \equiv \mathrm{ML}^\forall$ (finite) trees (= p.o. trees)

$\mathrm{FO}/{\sim} \equiv \mathrm{ML} \equiv \mathrm{ML}^\forall$ (finite) rooted transitive frames

locality based: adaptations of covers (from O_02)

cover of $\mathfrak{A}$: homomorphism $\pi\colon \widehat{\mathfrak{A}} \xrightarrow{\approx} \mathfrak{A}$
and two-way global bisimulation

every finite $\mathfrak{A}$ admits a cover $\pi\colon \hat{\mathfrak{A}} \to \mathfrak{A}$ by a finite ℓ-acyclic $\hat{\mathfrak{A}}$

upgrading finite levels of bisimilarity
to local approximations of $\equiv$ in covers:

$\equiv^{(\ell)}_{q,n}$ graded by locality parameters

$\bigcap_{\ell,q,n} \equiv^{(\ell)}_{q,n}$ is $\equiv$ by Gaifman's Theorem

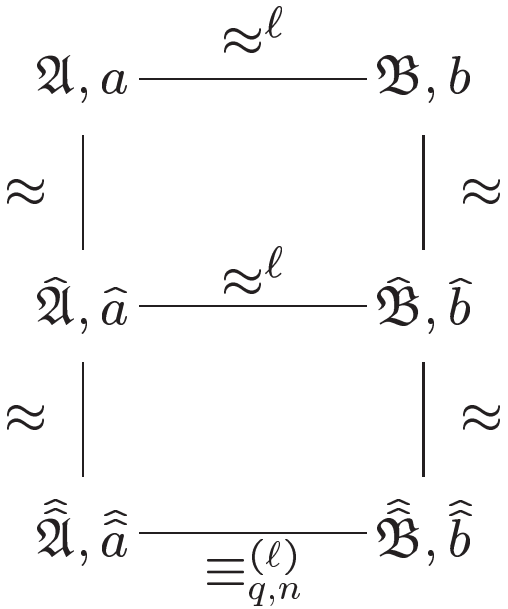
works for
- (finite) rooted frames
- (finite) equivalence frames

e.g., for equivalence frames (S5):

simple S5 frame: any two equivalence classes intersect in at most one element (no multiple edges)

ℓ-acyclicity: no non-trivial cycles in ℓ-neighbourhoods

every finite S5 frame admits a cover by a finite simple ℓ-acyclic S5 frame



$\leftarrow$ simple and ℓ-acyclic

$\leftarrow$ simple, ℓ-acyclic, (q, n)-saturated

decomposition based: upgrading in saturated p.o. trees

rooted transitive frame $\mathfrak{A} \overset{\sim_\forall}{\longmapsto} s(\mathfrak{A})$ p.o. tree structure

bisimilar unfolding with boosted multiplicities

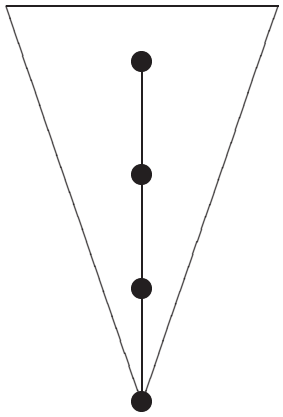
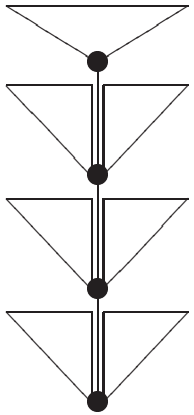
upgrading $\sim^\ell$ to $\equiv_r$ for suitable $\ell = f(r)$

$$\begin{array}{ccc} \mathfrak{A}, a & \overset{\sim^\ell}{\text{———}} & \mathfrak{B}, b \\ \sim_\forall \Big| & & \Big| \sim_\forall \\ s(\mathfrak{A}) & \overset{\equiv_r}{\text{———}} & s(\mathfrak{B}) \end{array} \quad \leftarrow \text{ saturated p.o. trees}$$

in suitably saturated finite p.o. trees:

establish $\equiv_r$ via games and path decompositions

decomposition & game argument — from trees to words

from  to  to

pumping lemma (Ehrenfeucht-Fraïssé; Feferman-Vaught):

bound on length of relevant words realised in $s(\mathfrak{A})$
finiteness property

$\longrightarrow$ inductive bound on ℓ
for which $\sim^\ell$ governs $\equiv_r$

summary

natural modal characterisations of bisimulation invariance over:

(finite) rooted frames

(finite) equivalence frames

(finite) transitve frames

(finite) rooted partial orderings (trees)

in contrast:

finite linear orderings (words) : $\mathrm{FO}/{\sim} \equiv \mathrm{FO} \not\equiv \mathrm{FO}^2 \supseteq \mathrm{ML}^{\neg\forall}$