

Model theoretic tools for deciding boundedness of fixed points

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partly based on joint work with Stephan Kreutzer
and Nicole Schweikardt (ICALP 07)

boundedness: expressiveness, decidability

static versus dynamic properties:

- FO expresses only ‘static’ and ‘local’ properties
 - extension by fixedpoint recursion/monotone inductive dfns
- cf. ML \longrightarrow L_μ in the modal world

fixedpoints of monotone operators capture
transitive closures, wellfoundedness, ...

boundedness of fixedpoint process:

spurious recursion, static (and FO) after all

detection of (un)boundedness:

- generalisation of SAT, for fragments of FO

explore – borderline ‘static’ vs. ‘dynamic’
– the possible dynamics of fixedpoints
– FO-definability within richer formalisms

relational fixed point recursion and boundedness

monadic least fixed point induction on $\varphi(X, x)$, positive in X

least fixpoint $(\mu_X \varphi)[\mathfrak{A}] = \bigcap \{ P \subseteq A : \varphi[\mathfrak{A}, P] = P \}$

inductive stages: $(\mu_X \varphi)[\mathfrak{A}] = \bigcup_{\alpha} X^{\alpha}[\mathfrak{A}]$ where

$$\begin{aligned} X^0[\mathfrak{A}] &= \emptyset \\ X^{\alpha+1}[\mathfrak{A}] &= \varphi[\mathfrak{A}, X^{\alpha}[\mathfrak{A}]] \\ X^{\lambda}[\mathfrak{A}] &= \bigcup_{\alpha < \lambda} X^{\alpha}[\mathfrak{A}] \end{aligned}$$

closure ordinal $\gamma[\varphi, \mathfrak{A}] = \min_{\alpha} (X^{\alpha+1}[\mathfrak{A}] = X^{\alpha}[\mathfrak{A}])$

always $\gamma[\varphi, \mathfrak{A}] < |\mathfrak{A}|^+$

$\gamma[\varphi, \mathfrak{A}] \leq \omega$ for $\varphi \in \exists^*$ -FO [esp. Datalog]

$\gamma[\varphi, \mathfrak{A}] \leq \omega$ for ω -saturated \mathfrak{A}

no bound across all structures in general: look at $\mu_X \square X$

relational fixedpoint recursion and boundedness

$\varphi(X, x)$ bounded: $\exists n \in \mathbb{N}$ s.t. $\gamma[\varphi, \mathfrak{A}] < n$ for all \mathfrak{A}

for $\varphi(X, x) \in \mathbf{FO}$:

φ bounded \Rightarrow recursion spurious
 $\mu_X \varphi$ uniformly FO

φ unbounded \Rightarrow all finite stage increments can be non-trivial
by compactness there are \mathfrak{A} s.t. $\gamma[\varphi, \mathfrak{A}] \geq \omega$
and even $\gamma[\varphi, \mathfrak{A}] = \omega$

boundedness and definability

Barwise–Moschovakis theorem

for any X -positive FO formula $\varphi(X, x)$

the following are equivalent:

- (i) $\mu_X \varphi$ **bounded**
- (ii) $\mu_X \varphi$ **uniformly FO definable**
- (iii) $\mu_X \varphi[\mathfrak{A}]$ **FO definable in each \mathfrak{A}**

relativises to natural fragments: \forall^* , \exists^* , FO^k , ML, GF, ...

relativises to elementary/projective classes:

acyclic, treewidth k , ...

boundedness as a decision problem

for a class \mathcal{F} of FO formulae:

BDD(\mathcal{F})

given $\varphi(X, x) \in \mathcal{F}$

decide if $\mu_X \varphi$ is bounded

- **SAT reducible to BDD** for natural fragments \mathcal{F}
- **BDD a generalised SAT problem:** $(\varphi^{n+1} \wedge \neg \varphi^n)$ for all $n \in \mathbb{N}$
- **few decidable cases**, even for monadic recursion

decidability vs. undecidability for monadic BDD

undecidable	decidable
\exists^* and even $\exists_+^* (\neq)$ existential, positive with inequality Gaifman, Mairson, Sagiv, Vardi 87	\exists_+^* [Datalog] pure existential positive Cosmadakis, Gaifman, Kanellakis, Vardi 95
FO^2 two variables Kolaitis, O_ 98	ML modal O_ 98, improved 06
\forall^* and even $\forall_-^* (=)$ • universal, mixed polarities or with equality O_ 06	\forall_-^* universal, single polarities without equality O_ 06

can encode tilings

decidable via tree codings

example: undecidability via tiling for \forall^* -FO

reduction of the tiling problem for tiling systems \mathcal{D} to (un)boundedness of $\varphi^{\mathcal{D}}(X, x)$ in \forall^* -FO with equality

$$\varphi^{\mathcal{D}}(X, x) = \varphi_0^{\mathcal{D}} \wedge \varphi_1(X, x)$$

$\varphi_0^{\mathcal{D}}$ (static): H and V are the graphs of partial functions, commuting where defined;
 $(P_d)_{d \in \mathcal{D}}$ compatible with tiling constraints

$$\varphi_1(X, x): \quad a \notin X^n[\mathfrak{A}] \quad \Rightarrow \quad h: (n \times n)\text{-grid} \xrightarrow{\text{hom}} (A, H, V)$$
$$(0, 0) \longmapsto a$$

then \mathcal{D} tiles $\mathbb{N} \times \mathbb{N}$ -grid \Leftrightarrow \mathcal{D} tiles arbitrarily large $(n \times n)$ -grids
 $\Leftrightarrow \varphi^{\mathcal{D}}$ unbounded

example: undecidability via tiling for \forall^* -FO

ctd.

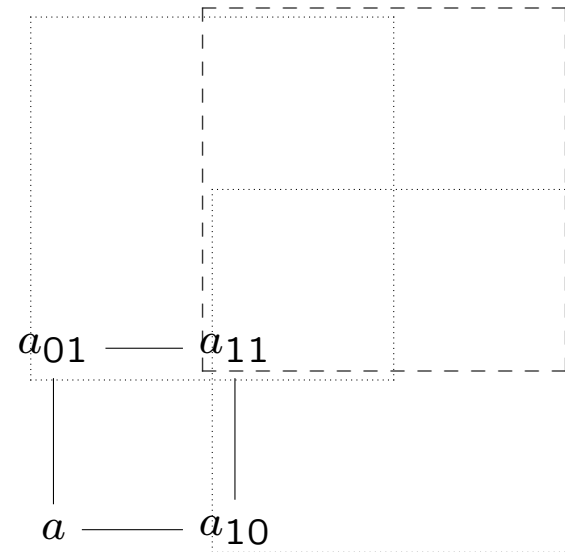
φ_1 can be chosen in universal ML:

$$\varphi_1(X) := \Box_H X \vee \Box_V X \vee \Box_H \Box_V X \vee \Box_V \Box_H X$$

$$\begin{array}{l}
 a \notin \varphi_1^{n+1}[\mathcal{A}] \\
 \mathcal{A} \models \varphi_0
 \end{array}
 \Rightarrow
 \begin{array}{ccc}
 a_{01} & \xrightarrow{H} & a_{11} \\
 \uparrow V & & \uparrow V \\
 a & \xrightarrow{H} & a_{10}
 \end{array}
 \quad \text{for some } a_{10}, a_{01}, a_{11} \notin \varphi_1^n[\mathcal{A}]$$

glue homomorphisms
of three $(n \times n)$ -grids at a_{ij}
to homomorphism of
 $((n + 1) \times (n + 1))$ -grid at a

compatibility guaranteed by φ_0



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can encode tilings

decidable via tree codings

example: decidability via the MSO-theory of trees for ML

Barwise–Moschovakis for modal fixedpoints:

$$\varphi(X) \in \text{ML bounded} \Leftrightarrow \mu_X \varphi \in \text{ML}$$

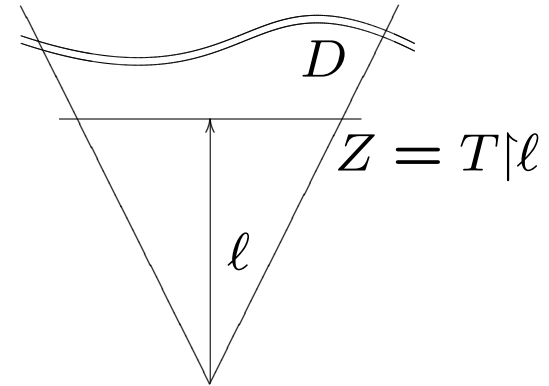
as φ and $\psi := \mu_X \varphi$ are invariant under bisimulation:

- consider (countable) tree-models
- over trees, capture definability in ML by a locality criterion
- crux: locality criterion in MSO

tree-locality of $\psi \in \text{MSO}$

$\exists \ell \in \mathbb{N}$ such that for all trees T ,
and all initial $D \subseteq T$ with $D \supseteq T \upharpoonright \ell$:

$$T \models \psi \text{ iff } T \upharpoonright D \models \psi$$



towards a reduction to the MSO-theory of T_ω

Z initial and for all I and all initial D : $\left. \begin{array}{l} Z \subseteq D \longrightarrow (\psi[I] \leftrightarrow \psi[I \upharpoonright D]) \end{array} \right\} \eta(Z) \in \text{MSO}$

ψ tree-local iff $T_\omega \models \exists Z ($ Z bounded $\wedge \eta(Z))$

not MSO

König's lemma for regular expansions of T_ω

for regular (T_ω, Z) (finite number of subtrees up to \simeq)
with initial $Z \subseteq T_\omega$ t.f.a.e.:

- (i) **Z path-finite** (no infinite path within Z)
- (ii) **Z bounded** ($Z \subseteq T \upharpoonright \ell$ for some $\ell \in \mathbb{N}$)

tree-locality criterion in $\text{MSO-Th}(T_\omega)$:

$$\begin{aligned} T_\omega &\models \exists Z (\varphi_{\text{path-fin}}(Z) \wedge \eta(Z)) \\ \Leftrightarrow (T_\omega, Z) &\models \varphi_{\text{path-fin}}(Z) \wedge \eta(Z) \quad \text{for some } Z \subseteq T_\omega \\ \Leftrightarrow (T_\omega, Z) &\models \varphi_{\text{path-fin}}(Z) \wedge \eta(Z) \quad \text{for some regular } (T_\omega, Z) \\ \Leftrightarrow T_\omega &\models \exists Z (Z \text{ bounded} \wedge \eta(Z)) \end{aligned}$$

corollary

tree-locality decidable for $\psi(x) \in \text{MSO}$

back to example: decidability of BDD(ML)

for any $\psi(x) \in \text{MSO}$ that is preserved under bisimulation, t.f.a.e.:

- (i) $\psi(x)$ tree-local (at radius ℓ)
- (ii) $\psi(x)$ expressible in ML (at nesting depth ℓ)

\Rightarrow locality-testing for $\psi = \mu_X \varphi$ decides boundedness of $\varphi(X, x)$.

Remark: modulo some pre-processing,
and using ideas from guarded logic,
this idea essentially lifts to a procedure for
deciding boundedness for $\varphi \in \forall_-^*$ -FO (LICS 06)

undecidable	decidable
\exists^* and even $\exists_+^*(\neq)$ existential, positive with inequality	\exists_+^* pure existential positive
FO^2 two variables	ML modal
\forall^* and even $\forall_-^*(=)$ universal, mixed polarities or with equality	\forall_-^* universal, single polarities without equality
can encode tilings	decidable via tree codings

change of perspective:

rather than **fragments of FO** over all structures,
consider all of **FO** over **restricted classes** of structures

decidability for all of FO over acyclic structures

theorem

Kreutzer, O., Schweikardt (ICALP 07)

let \mathcal{C} be (any FO-definable subclass of) the class of all acyclic structures; then the following is decidable:

given an X -positive $\varphi(X, x) \in \text{FO}$,

decide whether $\begin{cases} \varphi(X, x) & \text{is bounded over } \mathcal{C} \\ \mu_X \varphi(X, x) & \text{is FO over } \mathcal{C} \end{cases}$

NB: these are equivalent by Barwise–Moschovakis

technical ingredients: Gaifman locality and phase analysis

locality-testing for *phases* of the fixedpoint process on φ ,
rather than on the resulting $\mu_X \varphi$

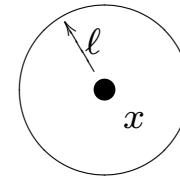
first: Gaifman preprocessing of X -positive $\varphi(X, x) \in \text{FO}$

Gaifman's locality theorem

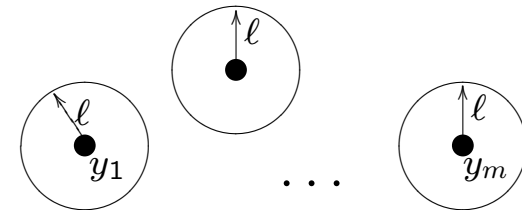
here: in graphs

$\varphi(x) \in \text{FO}$ equivalent to boolean combination of FO-formulae of two types

(L) $\chi^{(\ell)}(x)$ asserting properties of $N^\ell(x)$



(S) assertions about existence of ℓ -scattered tuples y_1, \dots, y_m with $\bigwedge_i \chi^{(\ell)}(y_i)$



how about respecting positivity in X ?

consider, e.g., $\varphi(X, x) := \exists y (x \neq y \wedge Xy)$

respecting positivity?

example

$$\varphi(X, x) = \exists y (x \neq y \wedge Xy) \equiv \begin{cases} (\neg Xx \wedge \exists y Xy) \\ \vee \\ (Xx \wedge \exists^{\geq 2} y Xy) \end{cases}$$

$$\not\equiv \bigvee_i (\varphi_i^{(\ell)}(X, x) \wedge \psi_i(X)) \quad \text{with } X\text{-positive } \ell\text{-local } \varphi_i^{(\ell)}$$

look at	⊙	•	⊨	φ	
say	⊙	•	⊨	$\varphi_1^{(\ell)}(X, x) \wedge \psi_1(X)$	
then		•	○	⊨	$\psi_1(X)$ (by \simeq)
		⊙	○	⊨	$\varphi_1(X, x)$ (by positivity/locality)
implies	⊙	•	○	⊨	$\varphi_1^{(\ell)}(X, x) \wedge \psi_1(X) \quad \#$

respecting positivity?

- X -positive $\varphi(X, x) \equiv X$ -positive b.c. of (L)/(S) ? **no**
 X -positive type (L) may not suffice
- $\varphi(X)$ X -positive $\equiv X$ -positive b.c. of (S) ? **yes**
Dawar/Grohe/Kreutzer/Schweikardt LICS 06

in between:

for X -positive $\varphi(X, x)$:

unrestricted (L)-parts + only X -pos. (S)-parts

$$\varphi(\mathbf{X}, \mathbf{x}) = \bigvee_i (\psi_i(\mathbf{X}) \wedge \varphi_i^{(\ell)}(\mathbf{X}, \mathbf{x}))$$

$\varphi_i^{(\ell)}(X, x)$: local about x , but not necessarily X -positive

$\psi_i(X)$: X -positive, monotone guards for local components

analysis of phases: how can unboundedness occur?

(generic) example:

$$\varphi(X, x) = (\psi_1(X) \wedge \varphi_1^{(\ell)}(X, x)) \vee (\psi_2(X) \wedge \varphi_2^{(\ell)}(X, x))$$

phase patterns & their triggers	iteration on	
$\psi_1(\emptyset) \wedge \psi_2(\emptyset)$	$\varphi_1 \vee \varphi_2$	(L)
$\psi_1(\emptyset) \wedge \neg\psi_2(\emptyset) \wedge \neg\psi_2(\mu_X\varphi)$	φ_1 only	(L)
$\psi_1(\emptyset) \wedge \neg\psi_2(\emptyset) \wedge \psi_2(\mu_X\varphi)$	two phases: first on φ_1 , then $\varphi_1 \vee \varphi_2$	(?)

- two issues:
- $\neg\psi_2(\mu_X\varphi)$ not FO
 - separation in mixed phase patterns

analysis of phases: how can unboundedness occur? ctd.

phase patterns & their triggers iteration on

$$\psi_1(\emptyset) \wedge \psi_2(\emptyset) \qquad \varphi_1 \vee \varphi_2 \qquad (1)$$

$$\psi_1(\emptyset) \wedge \neg\psi_2(\emptyset) \wedge \neg\psi_2(\mu_X\varphi) \qquad \varphi_1 \text{ only} \qquad (2)$$

$$\psi_1(\emptyset) \wedge \neg\psi_2(\emptyset) \wedge \psi_2(\mu_X\varphi) \qquad \text{two phases:} \\ \text{first on } \varphi_1, \text{ then } \varphi_1 \vee \varphi_2 \qquad (3)$$

in (2): replace $\neg\psi_2(\mu_X\varphi)$ by condition

$\neg\psi_2(P)$ for any fixedpoint P of φ (projective FO)

in (3): resolution in further case distinction:

(3a) φ_1 -phase unbounded on its own: subsumed in (2)

(3b) φ_1 -phase bounded: replace by static initialisation

limitations/(counter-)examples

sensitivity to underlying classes
due to interplay

MSO (locality testing)

vs.

classical FO arguments (B–M)

for instance:

- bdd over trees $\not\equiv$ bdd over acyclic structures;
bdd over finite acyclic $\not\equiv$ bdd over all acyclic.
- (B–M) fails on (finite) trees (reachability)
on finite acyclic structures (WF)
- correspondence between FO-definability and locality
breaks down in treewidth 2

open: finite acyclic; trees; *treewidth* k

treewidth k case could explain ‘all known’ positive + more