Guarded Logics

BLC 2000 UEA

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FO / ...

modal

guarded

preservation / characterisation

model theoretic & algorithmic properties

all vocabularies relational

modal logic

over Kripke structures / transition systems

$$\mathfrak{A} = \big(A, \underbrace{R, \ldots}_{\text{binary}}, \underbrace{P, \ldots}_{\text{unary}}\big)$$

ML **basic propositional modal logic**

atoms P
Booleans $\neg$, $\vee$, $\wedge$
modal quant. $\Box_R$, $\Diamond_R$

$$\mathfrak{A}, a \models \Box_R \varphi \quad \Longleftrightarrow$$

for all b s.t. $(a, b) \in R$: $\mathfrak{A}, b \models \varphi$

- finite model property: SAT = FINSAT
- SAT(ML) decidable (Pspace)
- interpolation
- ...

ML as a fragment of FO

identify:

$$\big(\Box_R\varphi\big)(x) \quad\equiv\quad \forall y\big(Rxy \to \varphi(y)\big)$$

$$\big(\Diamond_R\varphi\big)(x) \quad\equiv\quad \exists y\big(Rxy \wedge \varphi(y)\big)$$

ML $\subsetneq$ FO

re-using x, y (inductively alternating):

ML $\subsetneq$ FO^{2} (Gabbay'81)

two-variable fragment

FO^2 has finite model property

hence decidable for SAT (Mortimer'75)

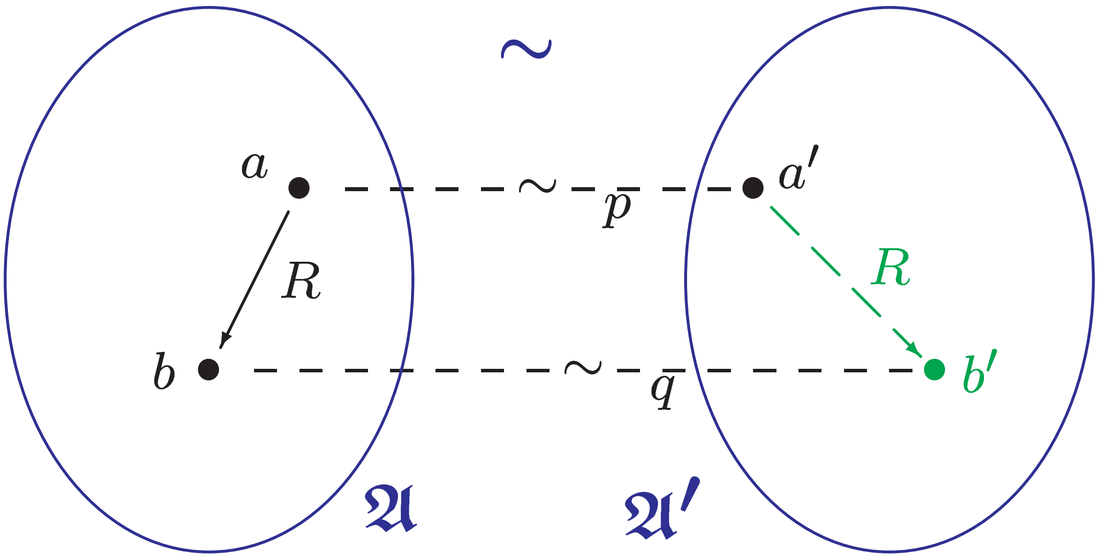
SAT(FO^2) Nexptime-complete

(Grädel/Kolaitis/Vardi'97)

bisimulation $\sim$

Ehrenfeucht-Fraïssé equivalence
associated to modal quant. $\Box$, $\Diamond$

back & forth:

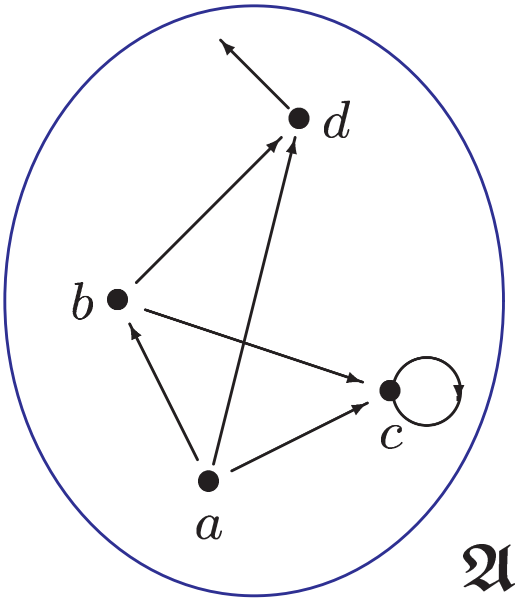


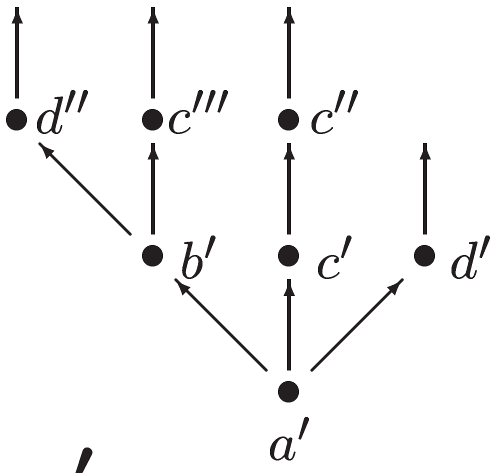
Characterisation Thm (van Benthem '76)

$\varphi \in$ FO equiv. to ML-formula iff
φ preserved under bisimulation

bisimulation & tree models

tree unravelling: every Kripke structure
is bisimilar to a tree



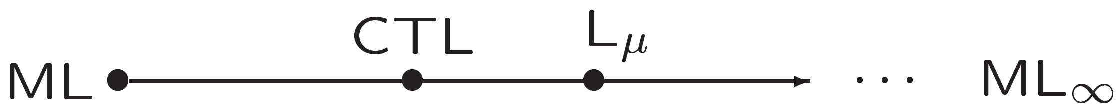


$$\mathfrak{A}, a \quad \sim \quad \hat{\mathfrak{A}}, a'$$

⇒ tree model property

every satisfiable $\varphi \in \mathrm{ML}$ is
satisfied at the root of a tree

shared by all $\sim$-preserving extensions!



tree models $\longrightarrow$ MSO and automata on trees
e.g., Rabin's Thm

more expressive modal logics — beyond FO

L_μ **propositional μ-calculus**
ML + monadic least fixed points

examples

$\mathfrak{A}, a \models \mu_X\big(\Box_R X\big)$ WF: all R-paths terminate

$\mathfrak{A}, a \models \mu_X\big(\Box_{\mathrm{I}}\Diamond_{\mathrm{II}}(X \vee W)\big)$

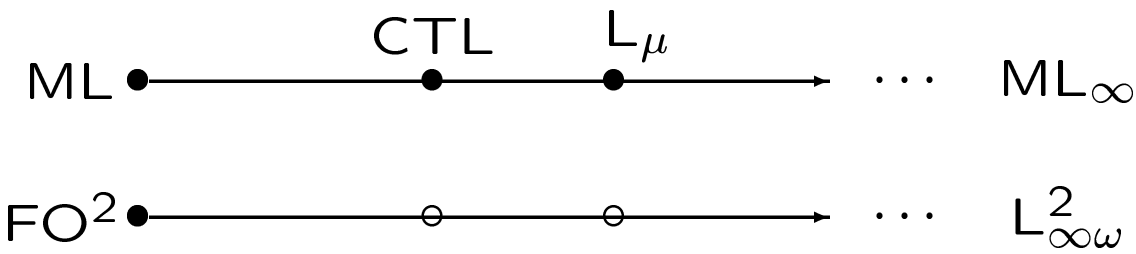
strategy for player II to force game into W

- preserved under bisimulation
  $\longrightarrow$ tree models, automata, ...
- finite model property
  (lost when adding inverses $\Box_{R^-}$)
- SAT(L_μ) decidable (Exptime)
  (Emerson/Jutla'88; with inverses: Vardi'98)

Characterisation Thm (Janin/Walukiewicz'96)

$\varphi \in$ MSO equiv. to L_μ-formula iff
φ preserved under bisimulation

two variables vs. modal



$$FO^2 + \left\{ \begin{array}{l} \text{transitivity} \\ \text{WF conditions} \\ \text{path quant.} \\ \text{monadic l.f.p.} \\ \ \ldots \end{array} \right\}$$

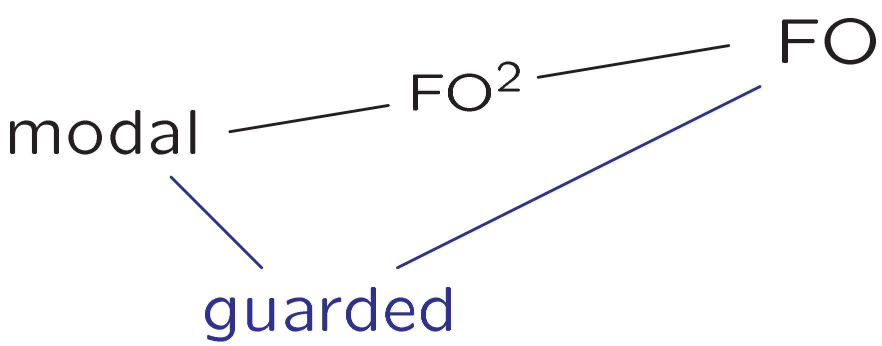
all highly undecidable

(Grädel/O/Rosen'99;O'00)

$\rightarrow$ **robustness of ML** not matched by FO^2

FO^2 also fails interpolation and Beth

(Sain'90;Hodkinson'96)



the Guarded Fragment — guarded logics

Andréka, van Benthem, Németi

generalise modal quantification pattern

$$\forall y\big(Rxy \rightarrow \varphi(y)\big) \;\;/\;\; \exists y\big(Rxy \wedge \varphi(y)\big)$$

to general relational setting

$$\forall \boldsymbol{y}\big(\alpha(\boldsymbol{x},\boldsymbol{y}) \rightarrow \varphi(\boldsymbol{x},\boldsymbol{y})\big) \qquad \forall \boldsymbol{y}.\alpha\;\varphi$$

$$\exists \boldsymbol{y}\big(\alpha(\boldsymbol{x},\boldsymbol{y}) \wedge \varphi(\boldsymbol{x},\boldsymbol{y})\big) \qquad \exists \boldsymbol{y}.\alpha\;\varphi$$

guard: atom α s.t. var(α) $\supseteq$ free(φ)
variations: loosely guarded, packed, ...

GF the **guarded fragment** of FO
FO with guarded quantification

$$\mathrm{ML} \subsetneq \mathrm{GF} \subsetneq \mathrm{FO}$$

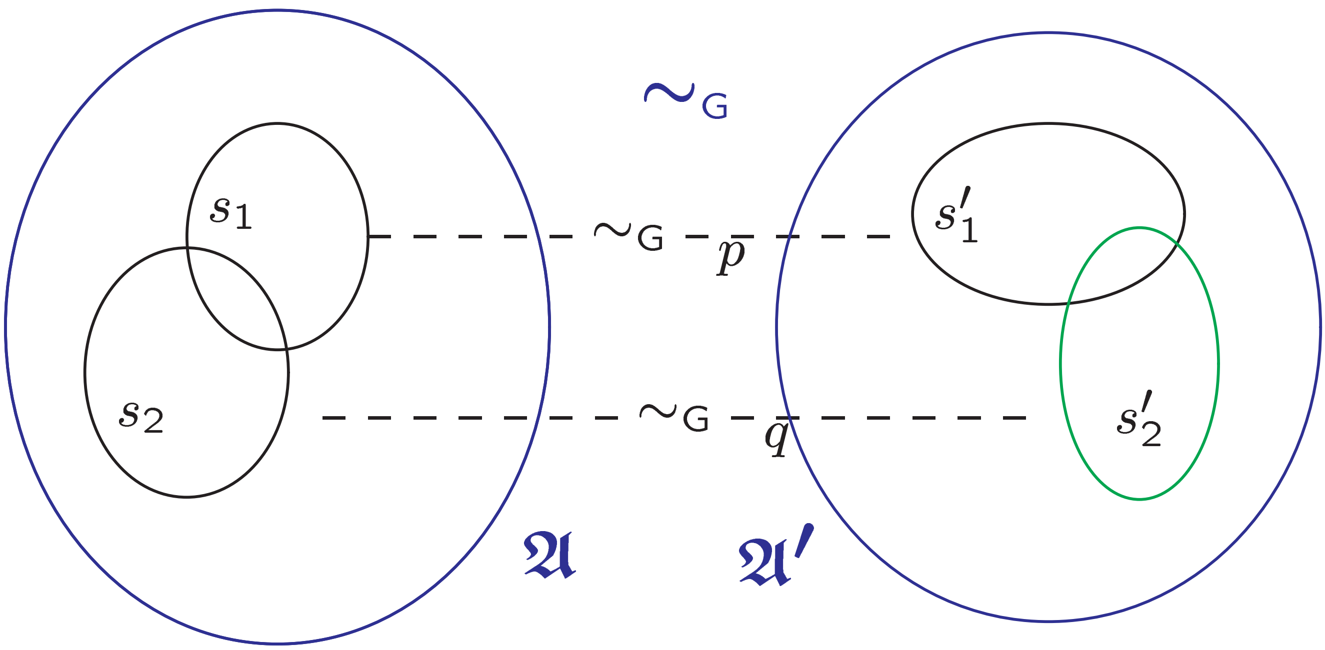
- many parallels with modal logic
- no arity restrictions
- comprises $=$, inverses, universal modalities

guarded bisimulation $\sim_{\mathsf{G}}$

Ehrenfeucht-Fraïssé equivalence
associated to guarded quant. $\forall \boldsymbol{y}.\alpha$, $\exists \boldsymbol{y}.\alpha$

$s \subseteq A$ guarded in $\mathfrak{A}$:
$s = \left\{ a \,\middle|\, a \text{ in } \boldsymbol{a} \right\}$, $\boldsymbol{a} \in R$ for some R in $\mathfrak{A}$

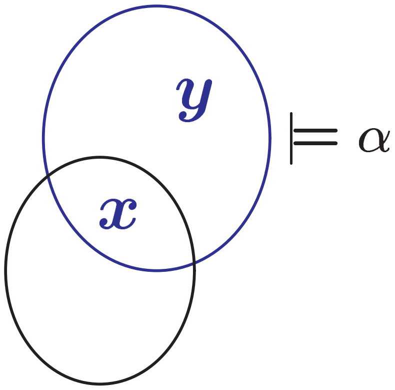
back & forth:



p, q partial isomorphisms
agree on common domain $s_1 \cap s_2$

GF as a fragment of FO

GF preserved under guarded bisimulation



$\exists \boldsymbol{y}.\alpha(\boldsymbol{x},\boldsymbol{y})$__

carries over via
forth property

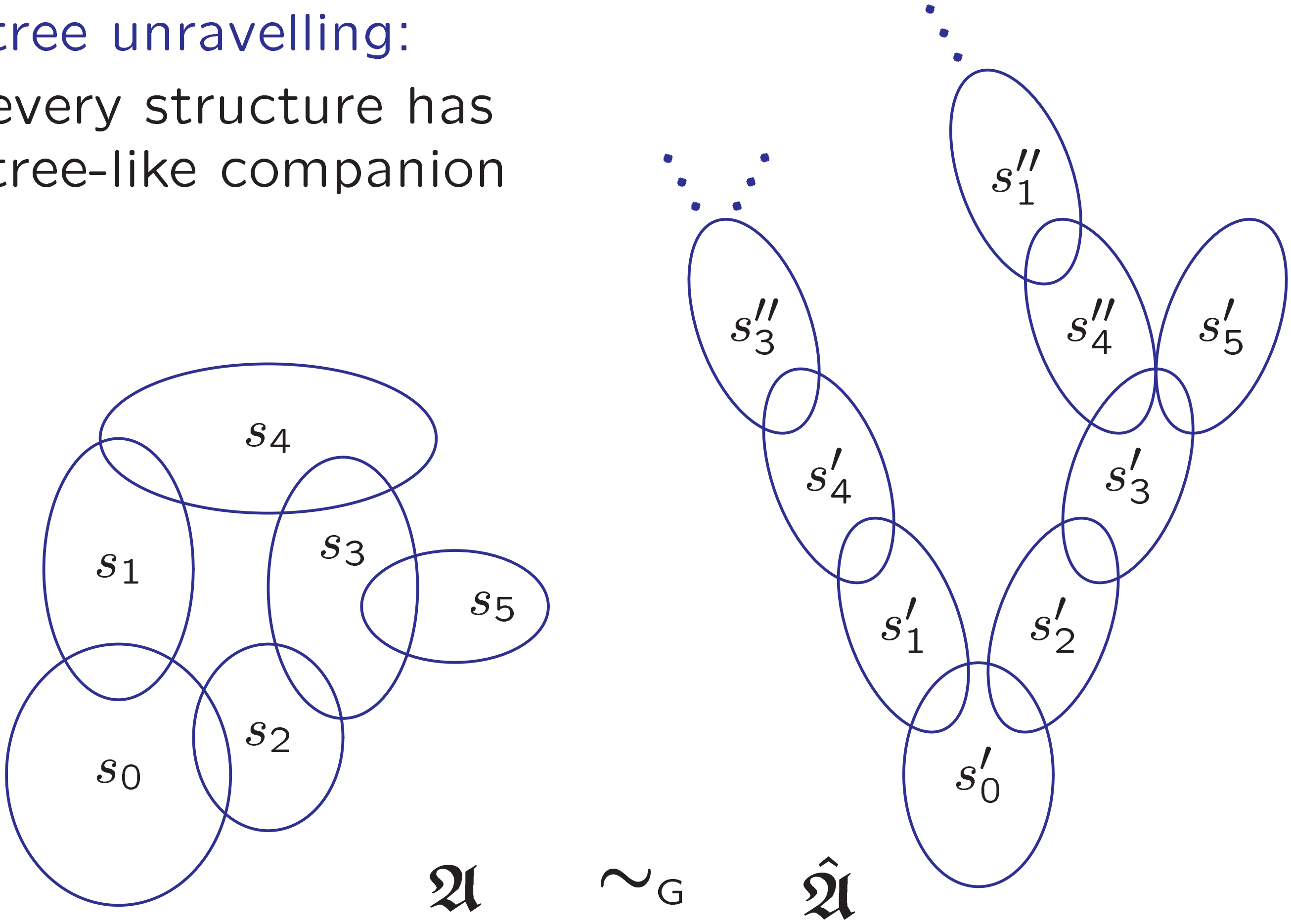
Characterisation Thm (Andreka, van Benthem, Nemeti'98)

$\varphi \in$ FO equiv. to GF-formula iff
φ preserved under guarded bisimulation

guarded bisimulation & tree-like models

tree unravelling:

every structure has
tree-like companion



$$\mathfrak{A} \quad \sim_{\mathrm{G}} \quad \hat{\mathfrak{A}}$$

$T := \big(V, E\big)$ tree of all **guarded paths**

$$\hat{\mathfrak{A}} = \dot{\bigcup_{v \in V}} \mathfrak{A}{\restriction} s_v \Big/ \approx$$

$\approx$ induced by $=$ on overlappings along E

tree decompositions

tree $T=(V,E)$ together with $\lambda\colon V \longrightarrow \mathcal{P}(A)$

a **tree decomposition** $\mathfrak{A}$ if

- $s \subseteq A$ guarded $\Rightarrow$ $s \subseteq \lambda(v)$ for some $v \in V$
- $\left\{ v \in V \;\middle|\; a \in \lambda(v) \right\}$ connected for all $a \in A$

width: $\max_{v\in V}|\lambda(v)| - 1$

unravellings have tree decomp. of width $\max |s| - 1$

$\Rightarrow$ **bounded tree width model property**

every satisfiable $\varphi \in$ GF in vocabulary of width d has a model of tree-width $d-1$

(Grädel 99)

shared by all $\sim_{\mathsf{G}}$-preserving extensions!

tree represented models

$\longrightarrow$ MSO and automata on trees
e.g., Rabin's Thm
modulo coding

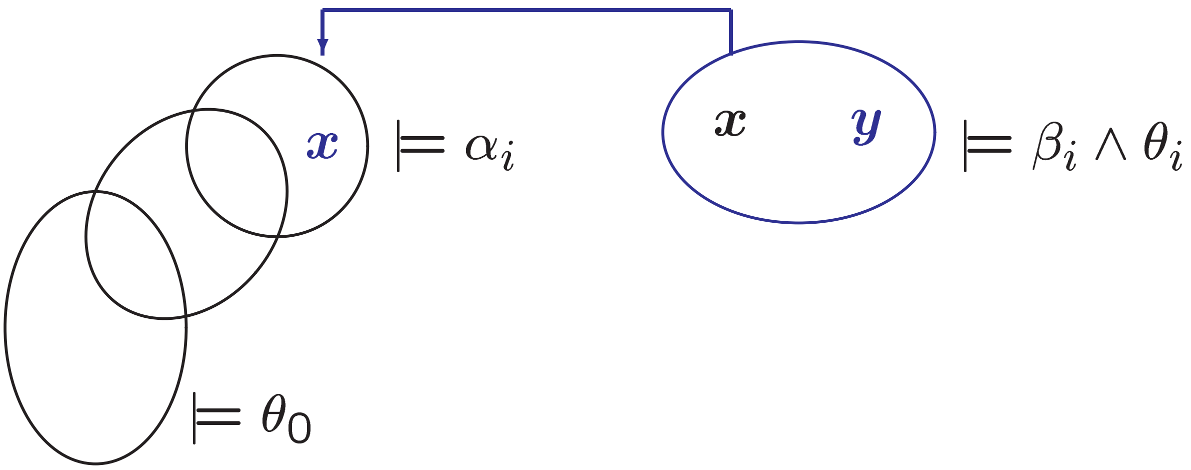
SAT and finite model property

following Grädel'99

after Skolemisation for SAT, w.l.o.g.

$$\varphi(\boldsymbol{x}) = \theta_0(\boldsymbol{x}) \wedge \bigwedge_i \underbrace{\forall \boldsymbol{x}.\alpha_i\, \exists \boldsymbol{y}.\beta_i\, \theta_i(\boldsymbol{x}, \boldsymbol{y})}_{\text{witness requirements}}$$

θ_i quantifier-free



φ satisfiable $\Leftrightarrow$

φ has **uniform** tree-like model

constructed over finite collection of guarded $\simeq$-types
with a uniform witnessing function

decidability

bounded size certificates for satisfiability

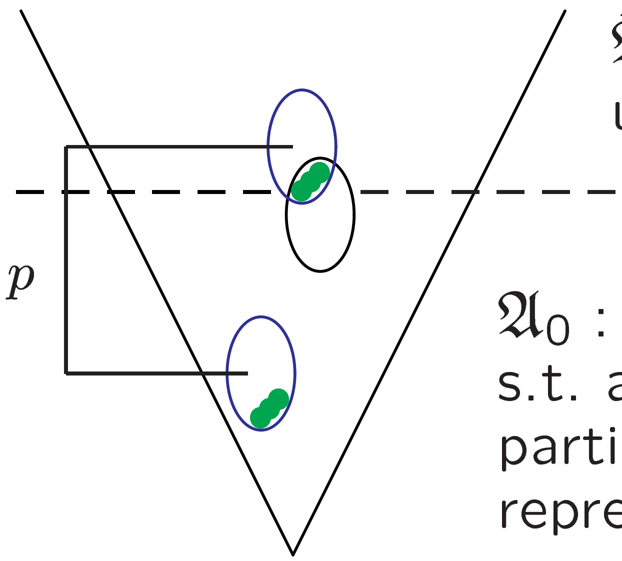
$\Rightarrow$ SAT(GF) decidable

analysis via alternating game protocols
to establish existence of witness function:

$$\text{SAT(GF) is} \begin{cases} \text{2-Exptime} & \text{general vocabulary} \\ \text{Exptime} & \text{bdd. width} \end{cases}$$

(Grädel'99)

finite model property



$\hat{\mathfrak{A}} \models \varphi$
uniform tree-like

$\mathfrak{A}_0$: a large initial segement s.t. all missing witnesses have partially isomorphic representatives in $\mathfrak{A}_0$

Finite Completion Thm (Herwig'95)

for finite $\mathfrak{A}_0$ find finite $\mathfrak{A} \supseteq \mathfrak{A}_0$ s.t.

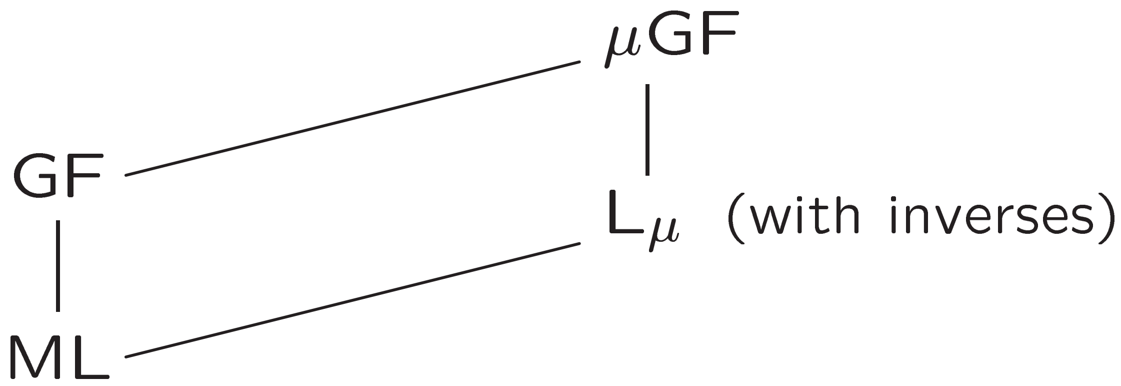
- all partial automorphisms of $\mathfrak{A}_0$ extend to automorphisms of $\mathfrak{A}$
- $\mathfrak{A}$ has no "new" guarded subsets

$\Rightarrow$ **finite model property**

$\rightarrow$ Hodkinson's generalisation (2000) covers relaxed notions of "guarded"

other guarded logics — beyond FO

μGF **guarded fixed-point logic**
no arity restrictions



- preserved under guarded bisimulation
- not the finite model property
- decidable for SAT (same complexity as GF)
(Grädel/Walukiewicz'99)

decidability can be proved using
tree-like models interpreted in S2S (Grädel'99)

one more characterisation thm

recall

$$\mathrm{ML} \equiv \mathrm{FO}\,\big/\!\sim$$

$$\mathrm{L}_\mu \equiv \mathrm{MSO}\,\big/\!\sim$$

compare

$$\mathrm{GF} \equiv \mathrm{FO}\,\big/\!\sim_{\mathrm{G}}$$

$$\mu\mathrm{GF} \equiv \;?\; \big/\!\sim_{\mathrm{G}}$$

not MSO: higher arity fixed points

not SO: too strong over words (finite strings)

need guarded variant of SO

SO-quantifiers range over
guarded relations

guarded relations: sets of guarded tuples
guarded tuples: tuples within guarded sets

N.B.: not preserved under $\sim_{\mathrm{G}}$

guarded second-order logic GSO

GSO SO with SO-quantifiers ranging over guarded relations

examples

$(V, E, S) \models$

$$
\begin{aligned}
& \forall xy.Sxy\; Exy \;\wedge\; \forall x\,(\exists y.Sxy \;\wedge\; \exists y.Syx) \\
\wedge\;\; & \forall X\; \forall x\, \neg\big(\exists y.Sxy\; Xy \;\wedge\; \exists y.Sxy\; \neg Xy\big) \\
\wedge\;\; & \forall X\; \forall x\, \neg\big(\exists y.Syx\; Xy \;\wedge\; \exists y.Syx\; \neg Xy\big)
\end{aligned}
$$

$(*)$ $S \subseteq E$ selects unique E-succ./pred.

$(V, S) \models$

$$\forall X\; \big[\big(\exists x.Xx \wedge \forall xy.Sxy\; (Xx \leftrightarrow Xy)\big) \rightarrow \forall x\; Xx\big]$$

$(**)$ (V, S) connected

$(V, E) \models \exists S\,[\;(*) \;\wedge\; (**)\;]$ Hamiltonicity

semantic robustness

$$
\begin{aligned}
\text{GSO} \quad &\equiv \quad \text{FO} + \text{guarded SO qfrs} \\
&\equiv \quad \text{GF} + \text{guarded SO qfrs} \\
&\equiv \quad \text{GF} + \text{ordinary SO qfrs} \\
&\equiv \quad \text{MSO on } \textbf{incidence structures}
\end{aligned}
$$

$$
\begin{array}{ccccc}
\textbf{MSO} & \text{———} & \textbf{GSO} & \text{———} & \textbf{SO} \\
| & & | & & \\
\mathbf{L}_\mu & & \mu\textbf{GF} & &
\end{array}
$$

Characterisation Thm (Grädel/Hirsch/O'00)

$\varphi \in$ GSO equiv. to μGF-formula iff
φ preserved under guarded bisimulation

methodology: exploit two-way parallels

guarded world $\longleftrightarrow$ **modal world**

here: import Janin/Walukiewicz

proof setting for characterisation thm

guarded		**modal**
$\mathfrak{A}$	$\xrightarrow[\text{tree repr. of unravelling}]{}$	$T(\hat{\mathfrak{A}})$
$\mathfrak{A}(T)$	$\xleftarrow[\text{induced structure}]{}$	T
$\sim_{\mathsf{G}}$	$\longleftrightarrow$	$\sim$
GSO	$\longrightarrow$	**MSO**
μ**GF**	$\longleftarrow$	$\mathbf{L}_\mu$

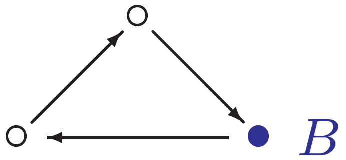
technical crux:

- find adequate class of trees T
  $\sim$-closed, MSO definable

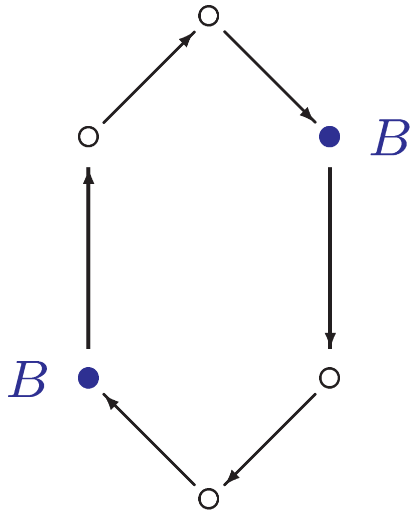
- overcome arity mismatch
  coding, especially: free SO-variables

other issues: failure of interpolation

Hoogland/Marx'99 (simplified)

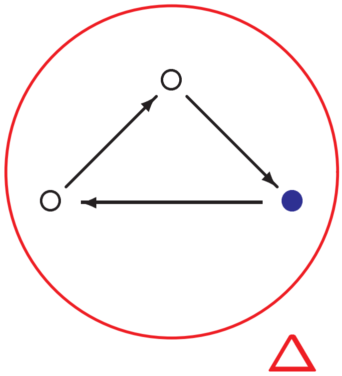


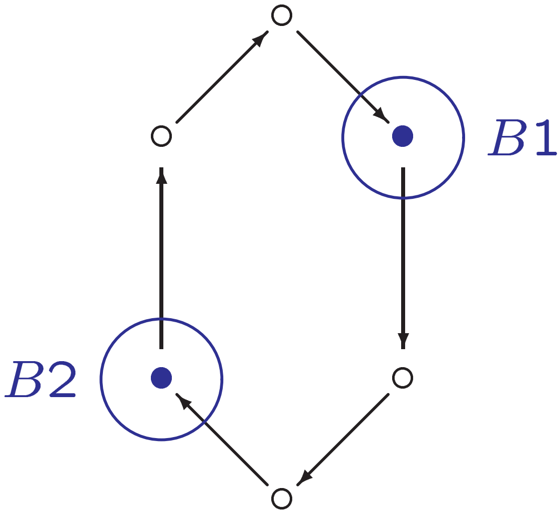
$\sim_G$



expand these with

$\triangle$ (ternary) / $B1$, $B2$ (unary)





failure of interpolation retaining Beth

$$\underbrace{\exists xyz.\Delta xyz \ \ldots}_{\varphi[R,B,\Delta]} \qquad \underbrace{\forall \begin{pmatrix} B \leftrightarrow (B1 \vee B2) \\ \wedge\, B1 \rightarrow \neg\Diamond\Diamond\Diamond B1 \\ \wedge\, B2 \rightarrow \neg\Diamond\Diamond\Diamond B2 \end{pmatrix}}_{\psi[R,B,B1,B2]}$$

$$\varphi[R,B,\Delta] \quad \models \quad \neg\psi[R,B,B1,B2]$$

admits no interpolant in GF$[R,B]$

weaker forms of interpolation survive

GF has Beth's Thm (Hoogland/Marx/O'99)

main observation: no implicit definition of R in GF can require the use of R as a guard ($\sim_G$-preservation)

$\rightarrow$ weaker form of interpolation sufficient

other issues: FMT & descriptive complexity

(A) Characterisation Thms

$$\mathbf{ML} = \mathbf{FO} \,/\!\sim$$

survives as a thm of FMT (Rosen'97)

Question: what about GF, L_μ, μGF ?

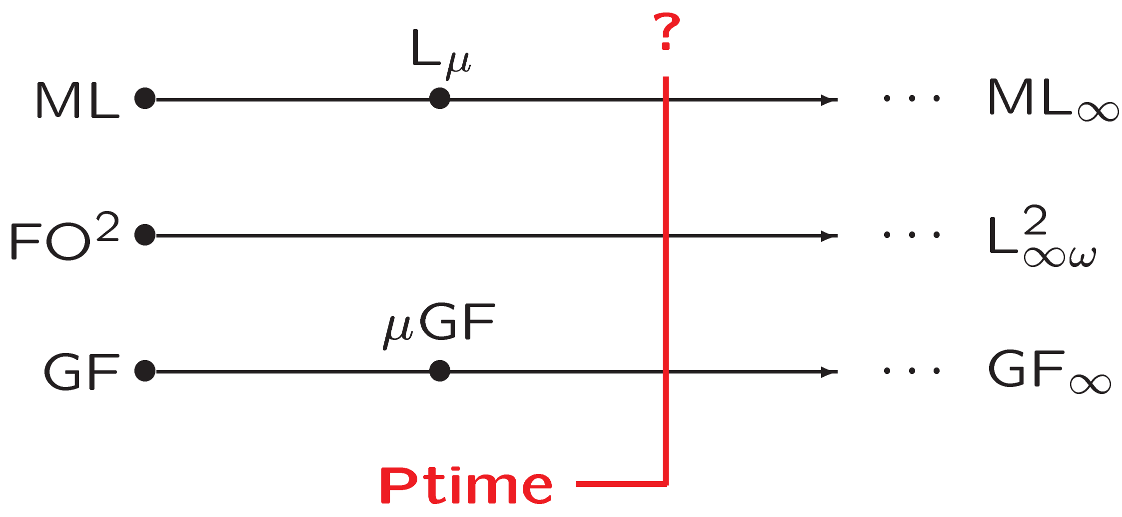
N.B.: classical proof necessitates "detour" through infinite structures

example of failure: $\mathrm{FO}^2 = \mathrm{FO} \,/\!\simeq^2$

not a thm of FMT

other issues:
FMT & descriptive complexity

(B) Complexity Levels Capturing Ptime



effective syntax for the Ptime fragments:

- in the modal world (O'99)
- in the two-variable world (O'96)

Question: what about the guarded world?

$\longrightarrow$ reconstruction problem:
succinct repr. of GF-types $\longmapsto$ canonical realisation
in Ptime?

(C) Combined MC Complexity Efficient Query Evaluation

Model Checking: given φ, $\mathfrak{A}$ decide $\mathfrak{A} \models \varphi$

Query Evaluation: given φ, $\mathfrak{A}$ compute $\varphi[\mathfrak{A}]$

efficient algorithms known for

tree-like $\mathfrak{A}$ arbitrary $\varphi \in$ MSO	arbitrary $\mathfrak{A}$ tree-like $\varphi \in$ FO $(*)$

$(*)$ syntactically tree-like FO formulae:

- bounded tree width $\rightleftarrows$ FO^k
- acyclicity $\rightleftarrows$ **GF**

(Flum/Frick/Grohe'00)

based on earlier results for conjunctive FO queries:
Yannakakis;Chekuri/Rajaraman;Kolaitis/Vardi;Gottlob et al.