

# Back & Forth Between Malleable Finite Models

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### **(0) back & forth in model theory**

- Ehrenfeucht–Fraïssé and Karp thms

### **(I) modal back & forth: bisimulation**

- modal Ehrenfeucht–Fraïssé and Karp thms
- bisimulation invariant FO & compactness phenomena
- controlling edge cycles in finite coverings

### **(II) from graphs to hypergraphs: guarded bisimulation**

- guarded back & forth and guarded logic
- controlling hypergraph cycles in finite coverings
- amalgamation of controlled acyclicity & applications

## classical back & forth

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model-theoretic comparison game, **Ehrenfeucht–Fraïssé game**

two players explore differences/similarities between  
two locally isomorphic configurations  $\mathcal{A}, \mathbf{a}$  and  $\mathcal{B}, \mathbf{b}$

single round: dynamic challenge/response probing of similarity,  
by extension of configuration  $\mathcal{A}, \mathbf{a}; \mathcal{B}, \mathbf{b} \rightsquigarrow \mathcal{A}, \mathbf{aa}; \mathcal{B}, \mathbf{bb}$

**existence of winning strategy** for second player  
extensionally represented by **back & forth system**  
establishes notions of structural equivalence:

$\ell$ rounds	$\simeq^\ell$	$\ell$ -partial isomorphism
any finite no. of rounds	$\simeq^\omega$	finite isomorphism
unbounded play	$\simeq^\infty$	partial isomorphism ( $\simeq_{\text{part}}$ )

**variation:**  $k$  pebbles (to be moved), controlling  
configuration size as a bdd resource

## Ehrenfeucht–Fraïssé and Karp theorems

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for finite relational vocabularies, back & forth equivalences capture logical indistinguishability w.r.t. first-order logic FO or its infinitary variant  $\text{FO}_\infty = \mathcal{L}_{\infty\omega}$

**one round corresponds to probing one level of  $\exists$  assertions !**

$\simeq^{\ell}$	( $\ell$ rounds)	$\equiv_{\text{FO}}^{\ell}$
$\simeq^{\omega}$	(all finite plays)	$\equiv_{\text{FO}}$
$\simeq^{\infty}$	(unbdd game)	$\equiv_{\text{FO}_\infty}$

**useful facts:**

$\simeq^{\ell}$  has finite index with  $\text{FO}_\ell$ -definable classes

$\simeq^{\infty} \rightsquigarrow \simeq$  on countable models

$\simeq^{\infty} \rightsquigarrow \simeq^{\omega}$  on  $\omega$ -saturated models

## (I) modal back & forth

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### bisimulation game and equivalence

challenge/response probing of transition systems or graphs, Kripke structures, ...

**configurations:** single states or vertices, worlds, ...

**single round:** move of single pebble along an edge (transition)  
in one structure needs to be matched in the other

winning strategies/back & forth systems establish equivalences:

$\ell$ rounds	$\sim^\ell$	$\ell$ -bisimulation equivalence
any finite no. of rounds	$\sim^\omega$	
unbounded play	$\sim^\infty$	bisimulation equivalence ( $\sim$ )

$\sim$  is for modal/process/temporal logics

what  $\simeq_{\text{part}}$  is for classical logic, ... and can also be seen as the  
mother of all back & forth equivalences

## modal Ehrenfeucht–Fraïssé and Karp theorems

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**one round corresponds to one level of  $\diamond$  assertions !**

relates to modal logic  $ML \subseteq FO$  with quantification relativised to direct edge-neighbours ( $\diamond_R, \square_R$ )

$$\exists y (Rxy \wedge \varphi(y)) \quad / \quad \forall y (Rxy \rightarrow \varphi(y))$$

over transitions systems (Kripke structures) with finite vocabulary:

$\sim^l$	( $l$ rounds)	$\equiv_{ML}^l$
$\sim^\omega$	(all finite plays)	$\equiv_{ML}$
$\sim = \sim^\infty$	(unbdd game)	$\equiv_{ML^\infty}$

**useful facts:**

$\sim^l$  has finite index with  $ML_l$ -definable classes

$\sim \rightsquigarrow \sim^\omega$  on e.g.  $\omega$ -saturated models

## what is bisimulation invariant FO, $\text{FO}/\sim$ ?

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a question in the spirit of classical “preservation” theorems

NB:  $\sim$ -invariance is *not* an elementary notion ( $\sim = \sim^\infty$ )

**theorem (van Benthem 83):**  $\text{FO}/\sim \equiv \text{ML}$

interestingly (& easily) equivalent to this

**compactness phenomenon** for  $\varphi(x) \in \text{FO}$ :

$$\sim\text{-invariance} \Rightarrow \sim^\ell\text{-invariance for some finite } \ell \quad (*)$$

bridging the gap between finite & infinitary equivalence

in the following: **two (almost orthogonal) proofs**  
with rather different perspectives & potential

## (A) classical proof

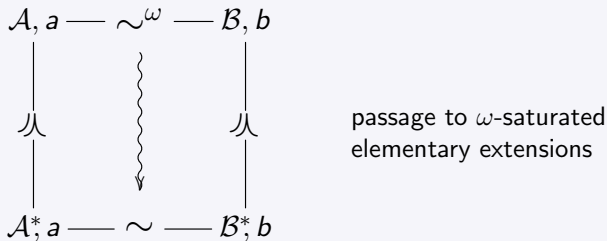
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two applications of FO-compactness in chain

$\sim$ -invariance  $\xRightarrow{(ii)}$   $\sim^\omega$ -invariance  $\xRightarrow{(i)}$   $\sim^\ell$ -invariance  
for suff. large  $\ell$

(i) standard compactness argument for contrapositive

(ii) saturation argument for upgrading  $\sim^\omega \rightsquigarrow \sim$  :



NB: smooth & elegant; lose track of  $\ell = \ell(\varphi)$ ; **no chance for fmt**



## (B) compactness without compactness

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try orthogonal upgrading according to

$$\begin{array}{ccc} \mathcal{A}, a & \text{---} \sim^\ell \text{---} & \mathcal{B}, b \\ \downarrow \sim & \downarrow \text{wavy} & \downarrow \sim \\ \mathcal{A}^*, a & \text{---} \equiv_q^{\text{FO}} \text{---} & \mathcal{B}^*, b \end{array} \quad \text{for suitable } \ell = \ell(q)$$

**possible obstructions to  $\sim^\ell \rightsquigarrow \simeq^q / \equiv_q^{\text{FO}}$  :**

differences w.r.t. (i) **small multiplicities**

(ii) **short cycles**

**need to avoid both** (no level of  $\sim$  controls either feature)

## use structural upgrades in products/coverings

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- (i) to avoid all multiplicities less than  $q$ ,  
pass to direct product with  $q$ -clique  $K_q$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sim^\ell} & \mathcal{B} \\ \downarrow \sim & & \downarrow \sim \\ \mathcal{A} \otimes K_q & \xrightarrow{\sim^\ell} & \mathcal{B} \otimes K_q \end{array}$$

- (ii) to avoid all short cycles, can pass to  
product with Cayley graph of large girth (\*)

both structural upgrades are available in fmt (\*), and support  
van Benthem and Rosen thms:  $\text{FO}/\sim \equiv \text{ML}$  classically and fmt  
slightly different argument yields optimal value  $\ell(q) = 2^q - 1$

- (\*) finite Cayley groups of large girth available from simple combinatorial  
group action on finite coloured trees (Biggs 89)

## variations & extensions

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### argument (B) allows for many variations

dealing with

- enriched modal logics & notions of bisimulation eq.
- other (non-elementary) classes of structures of interest

→ Dawar–O<sub>09</sub> for plenty of examples

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**so far – so good** (but all about graph-like structures)  
**in part II deal with richer relational formats:**

graphs  $\rightsquigarrow$  hypergraphs

edge traversal  $\rightsquigarrow$  non-trivial overlaps/amalgamation

*modal*  $\rightsquigarrow$  *guarded*

## (II) guarded back & forth

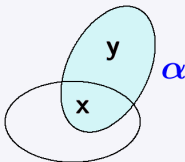
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**guarded fragment**  $\mathbf{GF} \subseteq \mathbf{FO}$  (Andréka–van Benthem–Németi 97)

FO-quantification relativised to *guarded* subsets/tuples contained in some  $[a] = \{a : a \text{ in } \mathbf{a}\}$  for  $\mathbf{a} \in R^A$

$$\exists \mathbf{y}(\alpha(\mathbf{xy}) \wedge \varphi(\mathbf{xy})) \quad / \quad \forall \mathbf{y}(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy}))$$

with *guard atom*  $\alpha$  covering all the free variables of  $\varphi$



associated back & forth game limits configurations to guarded subsets/tuples, challenge/response must preserve local isomorphism type and respect overlaps

graph game  $\rightsquigarrow$  hypergraph game

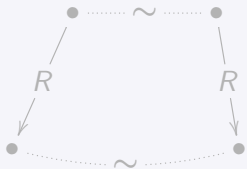
transitions  $\rightsquigarrow$  overlap/amalgamation

## challenges in hypergraph games/coverings

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challenge/response



compare: modal

### **crucial difference: moves no longer forgetful!**

some “history” persists in elements carried through chains of overlaps/amalgamation steps

### **look to control (short) hypergraph cycles:**

establish degrees of qualified hypergraph acyclicity in finite coverings that guarantee acyclicity (tree-decomposability) in small induced sub-configurations

## a notion of coset acyclicity

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**want:** local acyclicity in natural (reduced) products with  $\mathbb{G}$

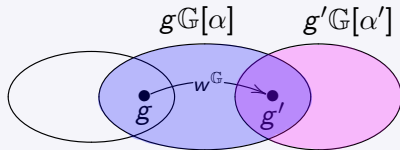
**need:** more than large girth (no short generator cycles) in  $\mathbb{G}$

no short generator cycles  
large girth

$\rightsquigarrow$

no short coset cycles  
(coset)  $N$ -acyclicity

links in a coset cycle:

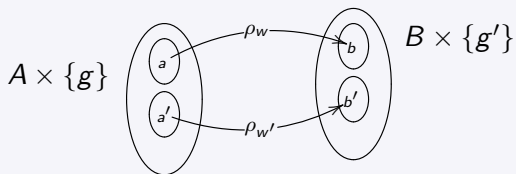


**useful facts:**

- (coset!) 2-acyclicity guarantees relational consistency in amalgamation chains based on  $\mathbb{G}$
- higher levels of acyclicity guarantee “local tree-likeness” of amalgams, i.e., local freeness and universality w.r.t. homs

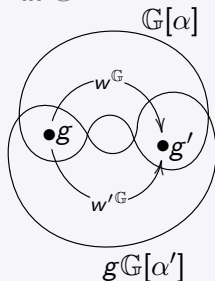
## why coset 2-acyclicity matters: a sketch

in reduced product



$$w^G = g^{-1}g = w'^G$$

in  $\mathbb{G}$



NB: the disjoint union of two partial isomorphisms  
need not be a partial isomorphism

if  $\mathbb{G}[\alpha]$  carries  $a$  and  $\mathbb{G}[\alpha']$  carries  $a'$ ,  $w^G = w'^G$  may constitute a  
bad cycle in  $\mathbb{G}$  unless also  $w = w' \in \mathbb{G}[\alpha \cap \alpha']$  (carrying both)

— i.e., unless the 2-cycle degenerates

## core results

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### **theorem** (JACM12/arXiv15)

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for every finite generator set and  $N$  can find finite coset  $N$ -acyclic groups/groupoids  $\mathbb{G}$

idea: in an inductive construction generate  $\mathbb{G}$  from (semi)group action on amalgamation chains that unfold short coset cycles

cf. constructions of acyclic Cayley graphs (Alon, Biggs)  
here lifted to more intricate adaptation for coset cycles  
(and for groupoids!)

natural (reduced) products with these yield

(\*) finite  $N$ -acyclic coverings of hypergraphs/relational structures

(\*\*) finite  $N$ -acyclic realisations of any amalgamation pattern over finite families of finite structures & partial isos



## some applications/uses

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(reduced) products with suitable acyclic groups/groupoids yield:

(\*) finite  $N$ -acyclic coverings of hypergraphs/relational structures

→ local tree-decomposability is key to upgrading  $\sim_g^{\ell} \rightsquigarrow \simeq^q$   
to show  $\text{FO}/\sim_g \equiv \text{GF}$  (fmt and classically)

(\*\*) finite  $N$ -acyclic realisations of any amalgamation pattern  
over finite families of finite structures & partial isos

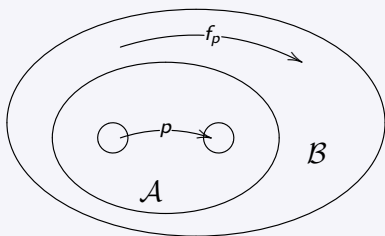
→ generic finite  $N$ -acyclic hypergraph coverings

→ local-to-global lifting arguments for (partial) symmetries  
in finite structures (aka EPPA results)

plus finite model properties and characterisation theorems or finite  
controllability results for extensions like guarded negation fragment

## EPPA results: from local to global symmetries

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**given:** finite relational  $\mathcal{A} \in \mathcal{C}$  and  $P \subseteq \text{PartIso}(\mathcal{A})$

**want:** finite extension  $\mathcal{B} \in \mathcal{C}$  extending  $p \in P$  to  $f_p \in \text{Aut}(\mathcal{B})$   
(provided there is at least an infinite such  $\mathcal{B}$  in  $\mathcal{C}$ )

Hrushovski 92      for finite graphs

Herwig 95      for finite relational structures

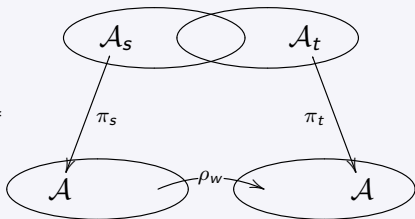
Herwig–Lascar 00      for finite relational structures that  
omit finitely many finite homomorphisms

## EPPA as an application of (\*\*)

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get finite EPPA extension structure  $\mathcal{B} \supseteq \mathcal{A}$  for  $(\mathcal{A}, P)$ ,  
with hypergraph structure  $(\mathcal{B}, \mathcal{S})$  and projections  $(\pi_s)_{s \in \mathcal{S}}$  s.t.:

- $\mathcal{B} = \bigcup_{s \in \mathcal{S}} \mathcal{A}_s$  where  $\mathcal{A}_s := \mathcal{B} \upharpoonright s$
- $(\pi_s: \mathcal{A}_s \simeq \mathcal{A})_{s \in \mathcal{S}}$  an atlas for  $\mathcal{B}$
- all overlaps between charts induced by compositions  $w \in P^*$



- up to any desired size bound, every small substructure of  $\mathcal{B}$  is acyclic and covered by  $\mathcal{A}$ -charts that form a free amalgam
- hence “ $N$ -locally free” and universal w.r.t. bdd size homs

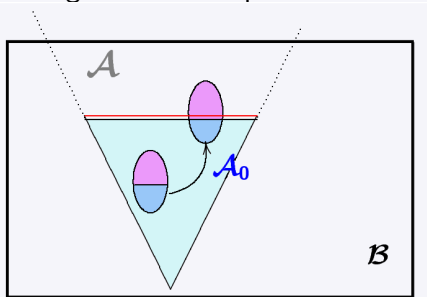
## back to guarded logics

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### finite model property for GF

based on Herwig's EPPA (Grädel 99)

after relational Skolemisation use EPPA to obtain finite model  
as finite closure of suitable finite substructure of infinite model  
w.r.t. guarded  $\forall\exists$ -requirements



$\mathcal{B} \supseteq \mathcal{A}_0$  EPPA extension

## ... and more applications/extensions

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Grädel's argument for fmp for GF extends, via Herwig–Lascar EPPA and acyclic coverings, to

- fmp relative to classes with forbidden homomorphisms
- finite controllability of UCQ w.r.t. guarded constraints
- similar results for the richer guarded negation fragment  $\text{GNF} \supseteq \text{GF}, \exists^* \text{posFO}$  (Bárány–ten Cate–Segoufin 11)

$N$ -acyclic Cayley groups can be used for characterisations of

- $\text{GNF} \subseteq \text{FO}$ , and
- the common knowledge extension of modal logic:

$$\text{ML}[\text{CK}] \equiv \text{FO}^*/\sim \quad (\text{classically and fmp})$$

work in progress with Felix Canavoi

## some related references

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**Bárány–Gottlob–O**\_\_(LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

**Bárány–ten Cate–O**\_\_(VLDB 2012, arXiv:1203.0077): Queries with guarded negation

**Grädel–O**\_\_(2014): The freedoms of (guarded) bisimulation

**Hodkinson–O**\_\_(BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

**Herwig–Lascar**(Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

**O**\_\_(Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

**O**\_\_(arXiv:1404.4599): Finite groupoids, finite coverings and symmetries in finite structures

( $\rightarrow$  <http://www.mathematik.tu-darmstadt.de/~otto/>)