Expressive Completeness

a basic model-theoretic concern
in varied (modal) settings

Martin Otto

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expressive completeness

generic setting:

want **concrete & effective syntax** for

some **class of structural properties**
expressive completeness

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want **concrete & effective syntax** for

some **class of structural properties**

presented in semantic terms
expressive completeness

generic setting:

want **concrete & effective syntax** for some **class of structural properties** presented in semantic terms as a semantic subclass of some given syntactic background class
expressive completeness

generic setting:

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presented in semantic terms

as a semantic subclass of some
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examples:

- FO-properties preserved under extensions;
  corresponding to $\exists^*\text{-FO} \subseteq \text{FO}$ (Łos–Tarski)
expressive completeness

generic setting:

want concrete & effective syntax for some class of structural properties presented in semantic terms as a semantic subclass of some given syntactic background class

examples:

• FO-properties preserved under extensions; corresponding to $\exists^*\text{-FO} \subseteq \text{FO}$ (Łos–Tarski)

• FO-properties preserved under bisimulation; corresponding to $\text{ML} \subseteq \text{FO}$ (van Benthem) $\text{FO}/\sim \equiv \text{ML}$
expressive completeness

generic setting:

want **concrete & effective syntax** for some **class of structural properties** presented in semantic terms as a semantic subclass of some given syntactic background class

remarks:

• not to be confused with deductive completeness as familiar from modal correspondence theory
expressive completeness

generic setting:

want **concrete & effective syntax** for some **class of structural properties** presented in semantic terms as a semantic subclass of some given syntactic background class

remarks:

- undecidability vs. effective syntax (!)
motivation – from classical model theory

- correspondences between semantic and syntactic features
- the non-trivial parts of classical ‘preservation theorems’
- usefull syntactic normal forms
- logical transfer phenomena (→ upgrading, below)
motivation – from classical model theory

• correspondences between semantic and syntactic features
  universal algebra + logic

• the non-trivial parts of classical ‘preservation theorems’

• usefull syntactic normal forms

• logical transfer phenomena (→ upgrading, below)

some classical preservation theorems:

pres. in hom. images — positive FO

pres. under homs — positive-existential FO (Lyndon–Tarski)

pres. in extensions — $\exists^*$-FO (Łos–Tarski)

monotonicity — positivity
motivation – from finite model theory (fmt)

(A) same motivation — fewer positive results

classical expressive completeness proofs invariably fail
motivation – from finite model theory (fmt)

(A) same motivation — fewer positive results

classical expressive completeness results typically fail

some survive – with new proofs that give new insights
motivation – from finite model theory (fmt)

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- Łos–Tarski thm fails in fmt (Tait, Gurevich)
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some survive – with new proofs that give new insights

- Łos–Tarski thm fails in fmt (Tait, Gurevich)
- Lyndon–Tarski thm true in fmt (Rossman’08)
  with new proof & new bounds (!)
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• van Benthem’s thm true in fmt (Rosen’97)
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  with new proofs & new bounds (→ below)

(B) new motivation & ramifications

• other classes of interest besides ‘just finite’

• complexity as another semantic constraint
motivation – from modal model theory

a different sense of correspondence

variation of the underlying class of frames/models familiar from classical modal correspondence theory
motivation – from modal model theory

a different sense of correspondence

variation of the underlying class of frames/models familiar from classical modal correspondence theory

→ a clear sense of natural, restricted classes of models/frames varying the domain of (model-theoretic) discourse
motivation – from modal model theory

a different sense of correspondence

variation of the underlying class of frames/models familiar from classical modal correspondence theory

→ a clear sense of natural, restricted classes of models/frames

varying the domain of (model-theoretic) discourse

rather than sticking with basic modal logic ML as the (syntactic) background logic, can look at

semantic criterion of bisimulation invariance over specific classes of frames/models
motivation – from descriptive complexity

complexity is a semantic constraint

e.g., the class of all Ptime recognisable

$\textit{properties of finite structures}$
motivation – from descriptive complexity

complexity is a semantic constraint

e.g., the class of all Ptime recognisable
properties of finite structures (!)

• a priori a semantic class in the sense of complexity theory
why?
motivation – from descriptive complexity

complexity is a semantic constraint

e.g., the class of all Ptime recognisable properties of finite structures (!)

• a priori a *semantic class* in the sense of complexity theory because of the hidden condition of ≃-closure
motivation – from descriptive complexity

complexity is a semantic constraint

e.g., the class of all Ptime recognisable properties of finite structures (!)

• a priori a semantic class in the sense of complexity theory because of the hidden condition of $\sim$-closure

• not known to possess a syntactic characterisation

the long-open logic-for-Ptime issue

finding a logic for Ptime is an expressive completeness issue
motivation – from descriptive complexity

complexity is a semantic constraint

e.g., the class of all Ptime recognisable properties of finite structures (!)

• a priori a semantic class in the sense of complexity theory because of the hidden condition of \(\sim\)-closure

• not known to possess a syntactic characterisation the long-open logic-for-Ptime issue

finding a logic for Ptime is an expressive completeness issue

remark: natural positive solution for Ptime properties of linearly ordered finite structures: least fixed-point logic LFP (Immermann, Vardi)
motivation – from descriptive complexity

Complexity is a semantic constraint

e.g., the class of all Ptime recognisable
properties of finite structures (!)

- a priori a semantic class in the sense of complexity theory because of the hidden condition of \( \sim \)-closure
- not known to possess a syntactic characterisation

The long-open logic-for-Ptime issue

Finding a logic for Ptime is an expressive completeness issue

Remark: natural positive solution for Ptime properties of linearly ordered finite structures:
least fixed-point logic LFP (Immermann, Vardi)
plan

• model-theoretic upgrading & model constructions
• specific constructions/issues in the modal setting
• specific constructions/issues in the guarded setting
• on descriptive complexity in these settings
**a general line**

**classical lemma** (based on compactness)

for fragment $L \subseteq FO$ (closed under $\land$, $\lor$) and $\varphi \in FO$ t.f.a.e.

- $\varphi \equiv \varphi' \in L$
- $\varphi$ preserved under $L$-transfer, $\Rightarrow_L$
a general line

classical lemma (based on compactness)

for fragment $L \subseteq FO$ (closed under $\land, \lor$) and $\varphi \in FO$ t.f.a.e.

• $\varphi \equiv \varphi' \in L$

• $\varphi$ preserved under $L$-transfer, $\Rightarrow_L$

non-classical substitute (based on Ehrenfeucht–Fraïssé)

for natural fragments $L \subseteq FO$

can typically replace $\Rightarrow_L$

by finite index approximants $\Rightarrow^\ell_L$

for some $\ell \in \mathbb{N}$ (which $\ell = \ell(\varphi)$? extra insight: $\varphi' \in L^\ell$)
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example:
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example: Łos–Tarski thm

\[
\begin{align*}
\varphi \text{ pres. under extensions} & \quad \bullet \\
\varphi \text{ pres. under } \exists^*-\text{transfer, } \Rightarrow \exists & \quad \bullet \\
\varphi \text{ formalisable in } \exists^*-\text{FO} & 
\end{align*}
\]

for \( \varphi \in \text{FO} \), equivalence of

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upgrading example:

for $\varphi \in \text{FO}$, equivalence of

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\varphi \text{ pres. under extensions}
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\[
\varphi \text{ pres. under } \exists^*-\text{transfer, } \Rightarrow \exists
\]

\[
\varphi \text{ formalisable in } \exists^*-\text{FO}
\]

Łos–Tarski thm

crux: if $\varphi \in \text{FO}$ is preserved under extensions,

then $A \Rightarrow \exists B$ implies $A \Rightarrow \varphi B$
**model-theoretic upgrading**

*the* technical key to expressive completeness results

**upgrading example:**

\[
\begin{align*}
\varphi \text{ pres. under extensions} & \quad \bullet \\
\varphi \text{ pres. under } \exists^*\text{-transfer}, \Rightarrow_{\exists} & \quad \bullet \\
\varphi \text{ formalisable in } \exists^*\text{-FO} & 
\end{align*}
\]

**crux:** if \( \varphi \in FO \) is preserved under extensions,

then \( \mathcal{A} \Rightarrow_{\exists} \mathcal{B} \) implies \( \mathcal{A} \Rightarrow_{\varphi} \mathcal{B} \)

compactness argument

yields this upgrading:
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example: van Benthem’s thm

for $\varphi \in \text{FO}$, equivalence of

$\begin{cases}
\varphi \text{ pres. under bisimulation} \\
\varphi \text{ pres. under ML-transfer} \\
\varphi \text{ formalisable in ML}
\end{cases}$
model-theoretic upgrading

*the* technical key to expressive completeness results

**upgrading example:** van Benthem’s thm

for \( \varphi \in \text{FO} \), equivalence of

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\begin{cases}
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**crux:** if \( \varphi \in \text{FO} \) is preserved under bisimulation,

then \( A \equiv_{\text{ML}} B \) implies \( A \equiv_{\varphi} B \)
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example: van Benthem’s thm

for φ ∈ FO, equivalence of

\[\begin{cases} & φ \text{ pres. under bisimulation} \\
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crux: if φ ∈ FO is preserved under bisimulation,
then \( A \equiv_{\text{ML}} B \) implies \( A \equiv_\varphi B \)

compactness argument (e.g. modal saturation) yields this upgrading:
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example: van Benthem–Rosen thm, recast
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example: van Benthem–Rosen thm, recast

for \( \varphi \in \text{FO} \), equivalence of

\[
\begin{align*}
\varphi \text{ pres. under bisimulation} \quad &\implies \varphi \text{ pres. under } \sim^l \text{ for some } l \\
\varphi \text{ expressible in ML} \quad &\implies \varphi \text{ pres. under bisimulation}
\end{align*}
\]
model-theoretic upgrading

the technical key to expressive completeness results

upgrading example: van Benthem–Rosen thm, recast

for $\varphi \in \text{FO}$, equivalence of

\[
\begin{cases}
\varphi \text{ pres. under bisimulation} \\
\varphi \text{ pres. under } \sim^\ell \text{ for } \ell = 2^{\text{qr}(\varphi)} \\
\varphi \text{ expressible in } \text{ML}^\ell \text{ for } \ell = 2^{\text{qr}(\varphi)}
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crux: if \( \varphi \in \text{FO} \) is preserved under bisimulation,
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crux: if $\varphi \in \text{FO}$ is preserved under bisimulation,
then $A \sim^\ell B$ implies $A \equiv^\varphi B$ for $\ell = 2^{qr(\varphi)}$

game argument
and model construction
provides this upgrading,

classically and fmt:

$A \sim^\ell B$
model-theoretic upgrading

The technical key to expressive completeness results

upgrading example: van Benthem–Rosen thm, recast

for $\varphi \in \text{FO}$, equivalence of

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\varphi \text{ pres. under bisimulation} \\
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\]

crux: if $\varphi \in \text{FO}$ is preserved under bisimulation,

then $A \sim^l B$ implies $A \equiv_\varphi B$ for $l = 2^{qr(\varphi)}$

cf. classical case:

via compactness

\[
\begin{array}{c}
A \\ \sim \\
\downarrow \\
\equiv_{\text{ML}} \\
\downarrow \\
A^* \\
\sim \\
\downarrow \\
B^*
\end{array}
\]

\[
\begin{array}{c}
A \\
\equiv_{\text{ML}} \\
\downarrow \\
B \\
\sim \\
\downarrow \\
B^*
\end{array}
\]
the modal and guarded worlds

<table>
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<th>guarded logic</th>
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<td>Kripke structures:</td>
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<td>coloured graphs</td>
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the modal and guarded worlds

modal logic
Kripke structures:
coloured **graphs**
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→ classically:
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guarded logic
relational structures:
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guarded bisimulation:
**hypergraph bisimulation**
→ classically:
guarded tree unfolding
**acyclic hypergraph models**

modal model theory = model theory of
bisimulation invariance

guarded model theory = model theory of
guarded bisimulation invariance
the modal and guarded worlds

modal logic | guarded logic
---|---
Kripke structures: coloured **graphs** | relational structures: coloured **hypergraphs**
modal bisimulation: **graph bisimulation** | guarded bisimulation: **hypergraph bisimulation**
→ classically: tree unfolding, **tree models** | → classically: guarded tree unfolding, **acyclic hypergraph models**

modal model theory = model theory of bisimulation invariance
guarded model theory = model theory of guarded bisimulation invariance
specific model constructions for upgrading

the classical modal example

for van Benthem–Rosen, it suffices to show:

$$\varphi(x) \in \text{FO } \sim\text{-inv. } \Rightarrow \varphi \text{ } \ell\text{-local for } \ell = 2^{qr(\varphi)} \text{ (hence } \sim^\ell\text{-inv.)}$$
specific model constructions for upgrading

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\[ \varphi(x) \in \text{FO} \sim\text{-inv.} \Rightarrow \varphi \text{ } \ell\text{-local for } \ell = 2^{qr(\varphi)} \text{ (hence } \sim\ell\text{-inv.)} \]

\[ \rightarrow \text{ analysis of } q\text{-round Ehrenfeucht–Fraïssé game for } q = qr(\varphi) \text{ on } \]

\[ a \]

versus
specific model constructions for upgrading

the classical modal example

for van Benthem–Rosen, it suffices to show:

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\[ \rightarrow \text{ analysis of } q\text{-round Ehrenfeucht–Fraïssé game for } q = qr(\varphi) \text{ on } \]

\[
\begin{array}{ccccccc}
\circ & \cdots & \circ & \bullet & \circ & \cdots & \circ \\
& & & A & & & \\
\end{array}
\sim \quad \begin{array}{c}
\circ \\
q \text{ copies of } A
\end{array}
\sim \quad \begin{array}{c}
\circ \\
q \text{ copies of } A \upharpoonright N^\ell(a)
\end{array}

versus

\[
\begin{array}{ccccccc}
\circ & \cdots & \circ & \bullet & \circ & \cdots & \circ \\
& & & a & & & \\
\end{array}
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the modal and guarded worlds

modal logic

Kripke structures:
coloured graphs

modal bisimulation:
graph bisimulation

→ classically:
tree unfolding,
tree models

guarded logic

relational structures:
coloured hypergraphs

guarded bisimulation:
hypergraph bisimulation

→ classically:
guarded tree unfolding
acyclic hypergraph models
acyclicity in (graph) covers

for upgrading $\sim^\ell$ (and its variants) to $\equiv_q$

more generally need \{ \begin{align*}
\text{uniform degree of local acyclicity} \\
& \text{& finite saturation w.r.t. multiplicities}
\end{align*} \}

modularity of FO Ehrenfeucht–Fraïssé game (locality of FO)
then guarantees upgrading
acyclicity in (graph) covers

for upgrading $\sim^\ell$ (and its variants) to $\equiv_q$

more generally need

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\begin{align*}
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\]

modularity of FO Ehrenfeucht–Fraïssé game (locality of FO)
then guarantees upgrading

local acyclicity = ‘local uncluttering’

local normalisation up to $\sim$
replacing (infinite) tree unfolding

question: does every finite Kripke structure possess a finite bisimilar companion without any short undirected cycles?
acyclicity in finite bisimilar graph covers

**bisimilar cover** \( \pi : \hat{A} \sim \to A : \)

- homomorphism with the *back*-property
- = bisimulation induced by a function/projection
- = bounded morphism

- bisimilar tree-unfoldings provide acyclic covers
- if \( A \) has cycles, then any *acyclic cover* is infinite
acyclicity in finite bisimilar graph covers

bisimilar cover $\pi : \hat{A} \sim \to A$:

- homomorphism with the back-property
- = bisimulation induced by a function/projection
- = bounded morphism

- bisimilar tree-unfoldings provide acyclic covers
- if $A$ has cycles, then any acyclic cover is infinite

thm O’04
every finite Kripke structure/frame admits bisimilar covers by finite $\ell$-locally acyclic structures/frames

$\ell$-local acyclicity: no (undirected) cycles in $\ell$-neighbourhoods, of length $\leq 2\ell + 1$
generic construction in the modal world (graphs)

simple idea: natural product with Cayley group of large girth

given such $G$ with generators $e \in E^\mathcal{A}$:

lift edge $e = (a_1, a_2)$ in $\mathcal{A}$
to edges $\hat{e} = ((a_1, g), (a_2, g \circ e))$
in cover with vertex set $A \times G$
generic construction in the modal world (graphs)

simple idea: natural product with Cayley group of large girth

given such $G$ with generators $e \in E^{|A|}$:

lift edge $e = (a_1, a_2)$ in $\mathcal{A}$
to edges $\hat{e} = ((a_1, g), (a_2, g \circ e))$
in cover with vertex set $A \times G$

a combinatorial group construction (Biggs)

find finite Cayley groups of large girth
for any given finite set $E$ of generators,
generated by group action on $E$-coloured trees
aside: Cayley groups of large girth

given: set $E$ of involutive generators,

bound $N$ on girth (length of shortest cycles)
aside: Cayley groups of large girth

given: set $E$ of involutive generators,
   bound $N$ on girth (length of shortest cycles)

on regularly $E$-coloured tree $T$ of depth $N$,

let $e \in E$ operate through
   swaps of nodes in $e$-edges:
aside: Cayley groups of large girth

given: set $E$ of involutive generators, bound $N$ on girth (length of shortest cycles)

on regularly $E$-coloured tree $T$ of depth $N$,

let $e \in E$ operate through swaps of nodes in $e$-edges:

\[
\begin{array}{c}
\bullet & \xrightarrow{e} & \bullet \\
\bullet & \leftarrow & \bullet
\end{array}
\]

$G := \langle E \rangle^{\text{Sym}(T)} \subseteq \text{Sym}(T)$

subgroup generated by the permutations $e \in E$

no short cycles: $e_1 \circ e_2 \circ \cdots \circ e_k \neq 1$ for $k \leq N$
## Sample results for $\text{FO}/\sim$ and $\text{FO}/\sim \forall$, $\text{FO}/\sim _{-}, \forall$

Based on locally acyclic covers

<table>
<thead>
<tr>
<th>Expression</th>
<th>Equivalence</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\text{FO}/\sim _{*}$</td>
<td>$\equiv \text{ML}[\ast]$</td>
<td>all (finite) frames</td>
</tr>
<tr>
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O'_04, Dawar–O'_09
**sample results for FO/∼ and FO/∼∀, FO/∼−,∀**

**based on locally acyclic covers**  
O‘04, Dawar–O‘09

<table>
<thead>
<tr>
<th>FO/∼*</th>
<th>≡</th>
<th>ML[*]</th>
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<td>(finite) equivalence frames</td>
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**based on tree interpretations**  
Dawar–O‘09

| FO/∼ | ≡ | ML \{ all transitive trees, finite irreflexive transitive trees \} |

\[
\begin{align*}
\text{FO/∼} & \equiv \text{ML} \\
\text{FO/∼*} & \equiv \text{ML}^{\ast}
\end{align*}
\]
sample results for $\forall \neq$ and $\forall, \forall$

based on locally acyclic covers

$\forall \neq^* \equiv ML[*]$ all (finite) frames
$\forall \equiv ML[\forall]$ (finite) rooted frames
$\forall^* \equiv ML[*]$ (finite) equivalence frames

O_'04, Dawar–O_'09

based on tree interpretations

$\forall \equiv ML$ \{all transitive trees, finite irreflexive transitive trees\}

Dawar–O_'09

$\forall \equiv MSO/\equiv \equiv ML[\diamond^*]$ \{finite transitive frames, transitive path-finite frames\}

not generally $\sim$-safe

referring to types within E-clusters

non-trivial in non-irreflexive case
the modal and guarded worlds

modal logic

Kripke structures: coloured graphs
modal bisimulation: graph bisimulation
→ classically:
tree unfolding,
tree models
→ for fmt:
locally acyclic covers

guarded logic

relational structures: coloured hypergraphs
guarded bisimulation: hypergraph bisimulation
→ classically:
guarded tree unfolding
acyclic hypergraph models

?? fmt ??

from graphs to hypergraphs
**hypergraph bisimulation & covers**

**guarded bisimulation** $\sim_g$ ([hypergraph bisimulation])

the game equivalence for guarded fragment $\text{GF}$

**thm**

Andreka–van Benthem–Nemeti’98

$\text{FO}/\sim_g \equiv \text{GF}$

had been open in fmt since!
hypergraph bisimulation & covers

guarded bisimulation $\sim_g$ (hypergraph bisimulation)
the game equivalence for guarded fragment GF

**thm**
Andreka–van Benthem–Nemeti’98

$\text{FO}/\sim_g \equiv \text{GF}$

had been open in fmt since!

hypergraph cover $\pi : \hat{A} \sim \rightarrow A$

cover of relational structures (hypergraphs)
w.r.t. guarded bisimulation (hypergraph bisimulation)
$= \text{homomorphism with the } \text{back}-\text{property}$
$= \text{guarded bisimulation induced by a function/projection}$
acyclicity in finite bisimilar hypergraph covers

element: $H^3_4$
the full width 3 hypergraph on 4 nodes;
$\equiv$ tetrahedron with faces as hyperedges
acyclicity in finite bisimilar hypergraph covers

example: $H^3_4$
the full width 3 hypergraph on 4 nodes;
= tetrahedron with faces as hyperedges

unfolds into acyclic hypergraph, with typical 1-neighbourhood

even 1-locally *infinite*,

\[ \cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \]
acyclic hypergraph covers

example: $H_4^3$
The full width 3 hypergraph on 4 nodes; 
= tetrahedron with faces as hyperedges

Unfolds into acyclic hypergraph, with typical 1-neighbourhood

Or into *locally finite* hypergraph without short chordless cycles

Even 1-locally *infinite,*
how much acyclicity in finite hypergraph covers?

hypergraph acyclicity  =  chordality  +  conformality

no

no
how much acyclicity in finite hypergraph covers?

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thm Hodkinson–O’03
every finite hypergraph admits a finite conformal cover
applications: reductions from CGF to GF for fmp
Herwig–Lascar–Hrushovski results
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thm

Hodkinson–O’03

every finite hypergraph admits a finite conformal cover

applications: reductions from CGF to GF for fmp

Herwig–Lascar–Hrushovski results

even 1-local acyclic covers may necessarily be infinite: $H_4^3$

N-acyclicity: no small cyclic sub-configurations

relativisation to size $N$ configurations

rather than localisation
**N-acyclic guarded covers**

**thm**

Every finite hypergraph admits covers by finite $N$-acyclic hypergraphs.

Applications:

FMP for GF on classes with forbidden cyclic configurations.
**N-acyclic guarded covers**

**thm**

every finite hypergraph admits covers by finite \(N\)-acyclic hypergraphs

applications:

fmp for GF on classes with forbidden cyclic configurations
fmt version of Andreka–van Benthem–Nemeti:

**thm**

\(\text{FO}/\sim_g \equiv \text{GF} \) over all finite structures
**hypergraph covers and upgrading** $\sim^\ell_g$ to $\equiv_q$

*using more highly acyclic groups*

- to unclutter hyperedges up to $\sim_g$
- for finitary saturation & freeness

stronger form of acyclicity necessary due to unavoidability of local cycles
motivation  model-theoretic upgrading  acyclic covers  from graphs to hypergraphs  canonisation  summary

**hypergraph covers and upgrading** $\sim^\ell_g$ to $\equiv_q$

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stronger form of acyclicity necessary due to unavoidability of local cycles

hyperedge transitions may or may not contribute to progress along a cycle

short chordless cycles may correspond to long generator sequences
other new results in the guarded world

weakly N-acyclic covers  
Barany–Gottlob–O’10

a weaker notion of acyclic covers  
allowing for polynomial size covers to unclutter hyperedges just “projectively”
other new results in the guarded world

weakly N-acyclic covers

Barany–Gottlob–O’10

a weaker notion of acyclic covers
allowing for polynomial size covers to unclutter hyperedges just “projectively”

yield

• near-optimal small models for GF and CGF
• fmp for GF and CGF over classes with forbidden homomorphic embeddings
  → finite control over conjunctive queries/GF constraints
• Ptime reconstruction of canonical finite models from abstract specification of their \( \sim_g \)-class
  → canonisation & capturing (next)
**descriptive complexity: capturing modal/guarded Ptime**

**crux of capturing:** semantic constraint on (Ptime) machines

\[ Ptime \rightarrow Ptime/\sim \]
**Descriptive complexity: capturing modal/guarded Ptime**

**Crux of capturing:**

\[ \text{semantic constraint on (Ptime) machines} \]

\[ \sim \text{-invariance: } \text{Ptime} \rightarrow \text{Ptime}/\sim \]

**Here look at**

\[ \left\{ \begin{array}{ll}
\text{Ptime}/\sim & \text{modal Ptime} \\
\text{Ptime}/\sim_g & \text{guarded Ptime}
\end{array} \right. \]

Ptime in the modal and guarded worlds

**How to enforce this (rougther) granularity?**
capturing modal/guarded Ptime

generic pre-processing idea: Ptime canonisation as a filter

\[
\begin{align*}
\mathcal{A} \xrightarrow{I} I(\mathcal{A}) &= I([\mathcal{A}]_\sim) \\
&\xrightarrow{F} F(I(\mathcal{A})) \in [\mathcal{A}]_\sim
\end{align*}
\]
capturing modal/guarded Ptime

generic pre-processing idea: Ptime canonisation as a filter

\[ A \xrightarrow{I} I(A) = I([A]_\sim) \xrightarrow{F} F(I(A)) \in [A]_\sim \]

structure complete invariant/\sim canonical representative/\sim

if in Ptime: \( H := F \circ I \) provides Ptime canonisation & filter

pre-processing with \( H \) enforces \( \sim \)-invariance

trivial for \( \sim \), but not for \( \sim_g \)
Ptime canonisation and Ptime/$\sim$ and Ptime/$\sim_g$

in both cases, natural complete invariant: bisimulation quotient of associated game graph
Ptime canonisation and Ptime/$\sim$ and Ptime/$\sim_g$

in both cases, natural complete invariant: bisimulation quotient of associated game graph

canonisation through reconstruction

in the modal case: bisimulation quotient is canonical representative

$\rightarrow$ capturing Ptime/$\sim$ (O’99)

in the guarded case: non-trivial Ptime construction of a model from this quotient

$\rightarrow$ capturing Ptime/$\sim_g$ (Barany–Gottlob–O’10)

yet another asset of the guarded world
**summary & remarks**

effectively capturing semantic phenomena over interesting classes of structures

e.g., modal/guarded preservation properties
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challenges for (finite) model theory:
model constructions and transformations

new techniques can yield new insights
also into classical results
summary & remarks

effectively capturing semantic phenomena over interesting classes of structures

e.g., modal/guarded preservation properties

challenges for (finite) model theory: model constructions and transformations
new techniques can yield new insights also into classical results

interesting, non-trivial finite model theory of modal and guarded logics with many further worthwhile variations
summary & remarks

many open problems remain,
e.g., the status of the Janin–Walukiewicz thm

\[ \text{MSO}/\sim \equiv L_\mu (\text{fmt?}) \]
**summary & remarks**

many open problems remain, 
e.g., the status of the Janin–Walukiewicz thm

\[
\text{MSO}/\sim \equiv L_\mu \text{ (fmt?)}
\]

e.g., modal Lindström theorems . . . (even in fmt?)

**ML/GF max. expressive \sim/\sim_g\text{-inv. logics with }[\ldots?]**
many open problems remain, e.g., the status of the Janin–Walukiewicz thm

\[ \text{MSO}/\sim \equiv \text{L}_\mu (\text{fmt}?) \]

e.g., modal Lindström theorems . . . (even in fmt?)

\[ \text{ML/GF max. expressive } \sim/\sim_g \text{-inv. logics with [ . . . ? ]} \]

The End