Bisimulation Unravellings and Covers

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bisimulation

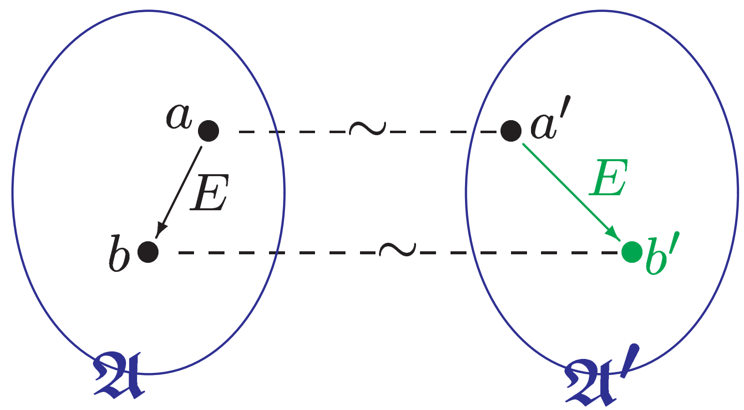
back & forth – games and equivalences between
transition systems / Kripke structures / coloured graphs

$$\mathfrak{A} = (A, P, \ldots, E, \ldots)$$

behavioural equivalence — modal Ehrenfeucht-Fraïssé

$$\boldsymbol{Z : \mathfrak{A} \sim \mathfrak{A}'}$$

- $\emptyset \neq Z \subseteq A \times A'$
- respects unary predicates
- back & forth along edges



forth

modal logics

relativised quantification along edges/transitions $\Box_E\varphi(x) \equiv \forall y\big(Exy \to \varphi(y)\big)$

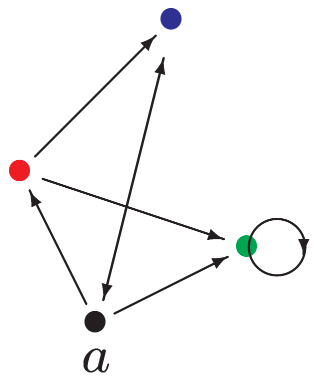
basic modal logic ML atomic props, $\wedge$, $\vee$, $\neg$, $\Box_E$, $\Diamond_E$

bisimulation invariance of $\varphi \in$ ML

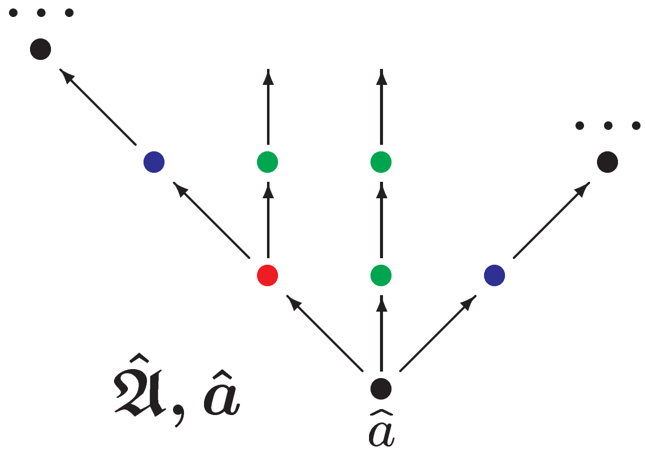
$$\mathfrak{A}, a \sim \mathfrak{A}', a' \;\Rightarrow\; \mathfrak{A}, a \models \varphi \;\Leftrightarrow\; \mathfrak{A}', a' \models \varphi$$

bisimilar companions that are trees

tree unravellings of transition systems / Kripke structures



$$\mathfrak{A}, a \quad \sim \quad \hat{\mathfrak{A}}, \hat{a}$$



tree model property

satisfiability $\Rightarrow$ satisfiability in a tree

tree models $\longrightarrow$ monadic second-order logic on trees
decidability (Rabin's Thm)
tree-automata

guarded logics

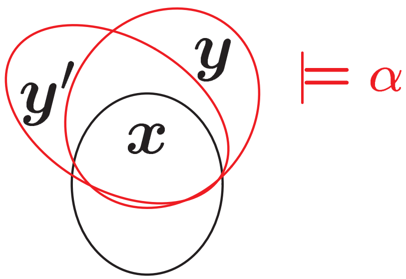
Andréka, van Benthem, Németi

quantification relativised
to relational (hyper-)edges

$$\forall \boldsymbol{y}\big(\alpha(\boldsymbol{x},\boldsymbol{y}) \ \rightarrow \varphi(\boldsymbol{x},\boldsymbol{y})\big)$$

guard atom: free(φ) $\subseteq$ var(α)

to **guarded subsets**
$(a_1, \dots, a_k) \in R \ \rightsquigarrow \ \{a_1, \dots, a_k\}$



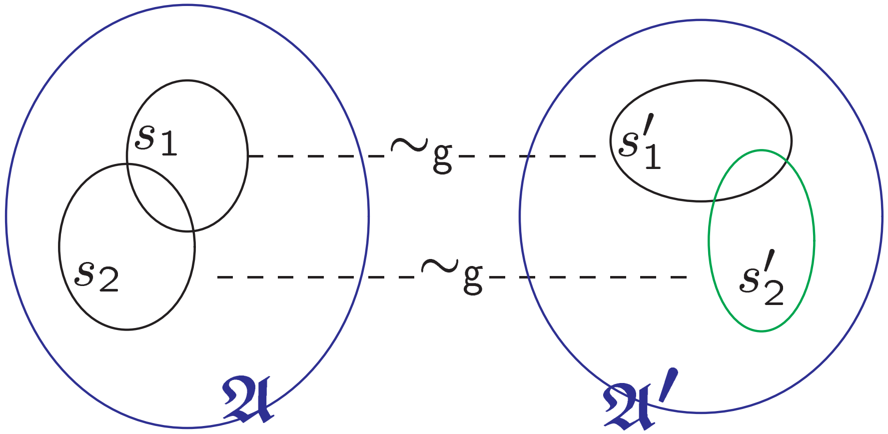
Guarded Fragment GF $\subseteq$ FO

subsumes ML(and extensions)
mirrors modal model theory
good algorithmic features

guarded bisimulation – back & forth

between general relational structures

$$Z : \mathfrak{A} \sim_{\mathrm{g}} \mathfrak{A}'$$



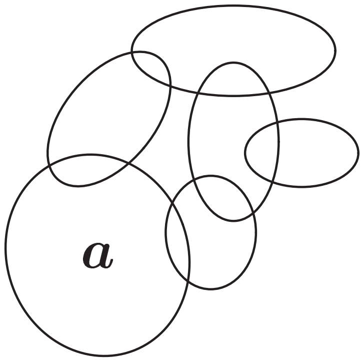
- $\emptyset \neq Z \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{A}')$
- local isomomorphisms between guarded patches
- guarded back & forth

forth

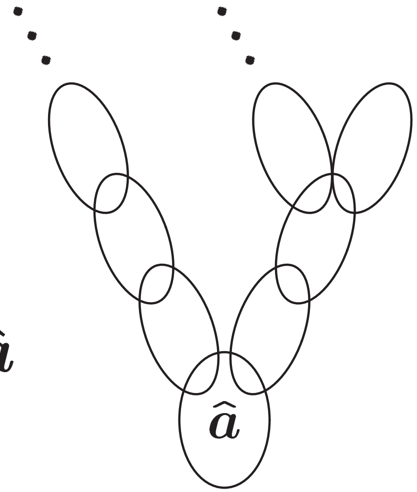
any $\varphi \in \mathrm{GF}$ is $\sim_{\mathrm{g}}$-invariant

tree-like companions up to $\sim_{\mathbf{g}}$

guarded unravellings



$$\mathfrak{A}, a \;\sim_{\mathbf{g}}\; \hat{\mathfrak{A}}, \hat{a}$$



based on tree of all guarded paths in $\mathfrak{A}$

generalised tree model property

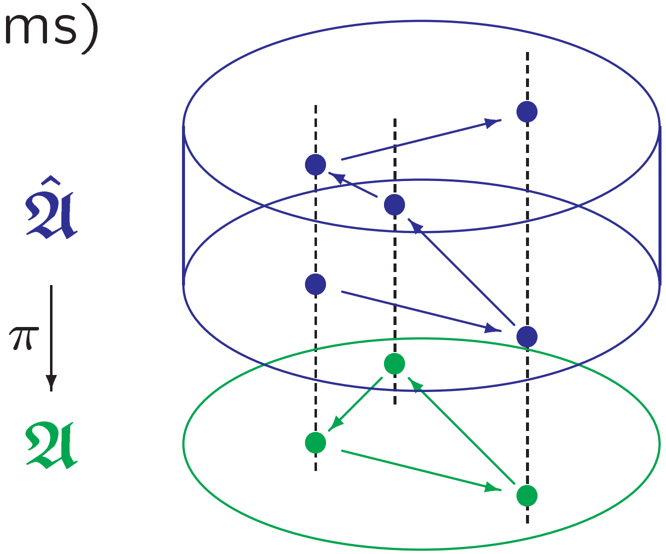
satisfiability $\Rightarrow$ tree-decomposable models

$\longrightarrow$ interpretable in trees, tree automata,...

from unravellings to covers

bisimilar covers (of transition systems)

homomorphism $\pi \colon \hat{\mathfrak{A}} \rightarrow \mathfrak{A}$
whose graph induces a
two-way global bisimulation



unravellings are (infinite) covers, acyclic

finite locally acyclic covers

theorem O'01

for every $k \in \mathbb{N}$ and finite transition system $\mathfrak{A}$
there is a **finite bisimilar cover** $\boldsymbol{\pi : \hat{\mathfrak{A}} \to \mathfrak{A}}$
where $\hat{\mathfrak{A}}$ **k-locally acyclic**
π **degree-preserving**

analogous: guarded covers of finite transition systems

group operations on underlying tournament

characterisation thms

Barwise, van Benthem, Andréka, Németi; Rosen

$$\mathbf{ML} \equiv \mathbf{FO}\,/\!\sim$$

for first-order φ: $\quad \varphi$ $\sim$-invariant $\Leftrightarrow$ φ expressible in ML

analogously

$$\mathbf{GF} \equiv \mathbf{FO}\,/\!\sim_{\mathbf{g}}$$

effective syntax for undecidable invariances

characterisation theorems in finite model theory

alternative, constructive proofs in classical case

theorem over finite transition systems

$\mathbf{ML} \equiv \mathbf{FO}/\!\sim$ (Rosen) $\qquad$ $\mathbf{GF} \equiv \mathbf{FO}/\!\sim_{\mathbf{g}}$

e.g., $\varphi \in$ FO $\sim$-invariant over finite transition systems $\Leftrightarrow$ φ expressible in ML over finite transition systems

methodological shift

classical logic $\rightarrow$ combinatorics

FO model constructions $\rightarrow$ local control over FO in bisimilar covers

$\rightarrow$ quantitative bounds

analysis of games provides

finite approximations $\approx^\ell$ for full equivalence $\approx$

$$\boxed{\begin{array}{l}\text{crucial for characterisation thms:}\\ \varphi\ \approx\text{-invariant} \;\Rightarrow\; \varphi\ \approx^\ell\text{-invariant for some finite } \ell\end{array}}$$

classically

$$\begin{array}{ccc} \mathfrak{A} & \approx^\ell & \mathfrak{A}' \\ \models \varphi & & \models \neg\varphi \\ & \text{all } \ell & \end{array}$$

upgrade via
compactness
ω-saturation

$$\begin{array}{ccc} \mathfrak{A} & \approx & \mathfrak{A}' \\ \models \varphi & & \models \neg\varphi \end{array}$$

alternative, constructive proof

$$\begin{array}{ccc} \mathfrak{A} & \approx^\ell & \mathfrak{A}' \\ \models \varphi & & \models \neg\varphi \\ & \ell = \ell(\varphi) & \end{array}$$

ℓ-acyclic
$\approx$ covers
(finite)

$$\begin{array}{ccc} \mathfrak{A} & \equiv_{\boldsymbol{q}}^{(\ell)} & \mathfrak{A}' \\ \models \varphi & & \models \neg\varphi \\ & q = q(\varphi) & \end{array}$$

why local control suffices

$\mathrm{G}(\mathfrak{A}) = \Big(A, \big\{(a,b) \colon a, b \in s \text{ for some guarded } s \subseteq A\big\}\Big)$

Gaifman graph

Gaifman's locality theorem

FO can only express Gaifman local properties

finite guarded covers

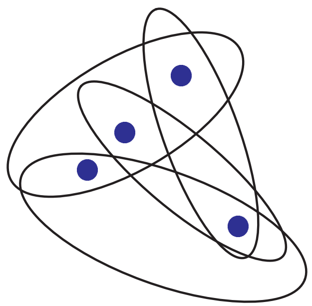
guarded unravellings $\longrightarrow$ **acyclic hypergraphs**

hypergraphs that are

conformal	and	**chordal**
no bad cliques		no bad cycles
+		?

bad cliques: unguarded Gaifman cliques
bad cycles: chordless

unguarded Gaifman cliques
are broken up in unravellings



finite conformal covers

combinatorial construction for hypergraphs

theorem Hodkinson, O'01

for every finite relational $\mathfrak{A}$
there is a **guarded cover** $\pi \colon \hat{\mathfrak{A}} \to \mathfrak{A}$
by a **finite conformal** $\hat{\mathfrak{A}}$

with applications to
- extension properties for partial maps
- finite model theory of guarded logics

the conformal hypergraph cover $\qquad \pi \colon (\hat{\mathbf{A}}, \hat{\mathbf{S}}) \to (\mathbf{A}, \mathbf{S})$

for guarded covers work with associated hypergraph of guarded subsets

geometric intuition: local "generic" cross-sections in fibre bundle over A

critical subsets: $\quad U := \{ u \subseteq A \colon \text{ for all } s \in S,\ u \not\subseteq s \}$

$$\hat{\mathbf{A}} \colon \begin{cases} (a, \chi_a) \ \in \ A \times {}^{U}\mathbb{N} \\ \chi_a(u) < |u| \ , \ \chi_a(u) \neq 0 \ \Leftrightarrow \ a \in u \end{cases}$$

generic subsets of $\hat{A}$

$$\boxed{\begin{array}{l} \pi \text{ is 1–1 and above any } u\text{:} \\ \chi_a(u) \neq \chi_{a'}(u) \text{ for } (a, \chi_a) \neq (a', \chi_{a'}) \end{array}}$$

$\hat{s}$ generic $\Rightarrow \ \pi(\hat{s}) \notin U$
for cardinality reasons

$$\hat{\mathbf{S}} := \{ \hat{s} \subseteq \hat{A} \colon \hat{s} \text{ generic}, \pi(\hat{s}) \in S \}$$

properties

- $\pi\colon (\hat{A}, \hat{S}) \to (A, S)$ is a **conformal bisimilar cover**
- for relational $\mathfrak{A}$, $\pi\colon (\hat{A}, \hat{S}) \to (A, S(\mathfrak{A}))$ induces a **conformal guarded cover** $\pi\colon \hat{\mathfrak{A}} \to \mathfrak{A}$

canonical and homogeneous:

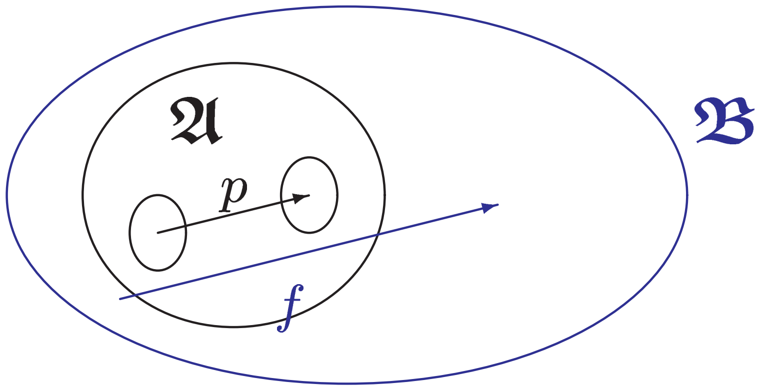
- $\mathrm{Aut}(A, S) \hookrightarrow \mathrm{Aut}(\hat{A}, \hat{S})$: $\quad f$ lifts to $\hat{f}$ s.t. $\pi \circ \hat{f} = f \circ \pi$
- generic "vertical" automorphisms g s.t. $\pi \circ g = \pi$ act transitively on generic sets above every $s \in S$

extension properties for partial maps

Hrushovski/Herwig/Lascar extension thms

profinite topology of free groups (Hall; Ribes, Zalesskii)
automorphism groups of countable structures

for finite relational $\mathfrak{A}$ find finite extension $\mathfrak{B}$
extending every $\boldsymbol{p} \in \mathbf{Part}(\mathfrak{A})$ to $\boldsymbol{f} \in \mathbf{Aut}(\mathfrak{B})$



#-faithful extension: no “new” # in $\mathfrak{B}$

new: clique faithful extension thm Hodkinson, O'01

finite extension $\mathfrak{B} \supseteq \mathfrak{A}$ without **"new" Gaifman cliques**

uses finite conformal covers on top of HHL construction

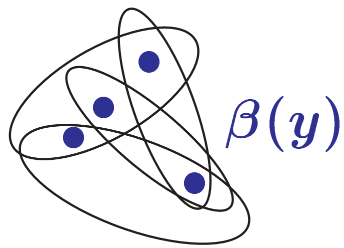
corollary (Herwig/Lascar)

extension property for the class of finite K_n-free graphs

corollary (Hodkinson)

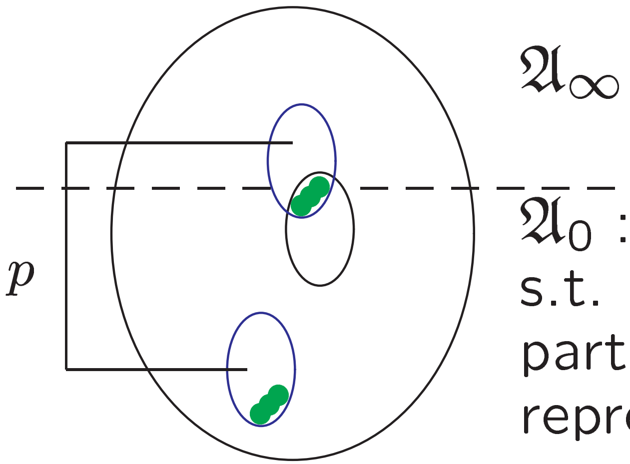
finite model property for the clique guarded fragment CGF

based on Gaifman cliques as visible patches



extension properties and finite models

finite closures of partial model



$\mathfrak{A}_\infty \models \varphi$

$\mathfrak{A}_0$: a large finite segment
s.t. all missing witnesses have
partially isomorphic
representatives in $\mathfrak{A}_0$

φ holds in $\#$-faithfull EPPA extension of $\mathfrak{A}_0$

guarded-set faithful for GF
clique faithful for CGF