A First-Order Framework for Inquisitive Modal Logic

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Abstract

We present a natural standard translation of inquisitive modal logic InqML into first-order logic over the natural two-sorted relational representations of the intended models, which captures the built-in higher-order features of InqML. This translation is based on a graded notion of flatness that ties the inherent second-order, team-semantic features of InqML over information states to subsets or tuples of bounded size. A natural notion of pseudo-models, which relaxes the non-elementary constraints on the intended models, gives rise to an elementary, purely model-theoretic proof of the compactness property for InqML. Moreover, we prove a Hennessy-Milner theorem for InqML, which crucially uses ω-saturated pseudo-models and the new standard translation. As corollaries we also obtain van Benthem style characterisation theorems.

1 Introduction

Inquisitive logics have recently been expounded systematically by Ciardelli in [2], following up on previous work with Roelofsen [6] and earlier sources especially in the work of Groenendijk cited there. The fundamental motivation is to provide logics with expressive means to deal not just with assertions but also with questions. While the general programme can be carried out systematically for various logics, like propositional logic in [6] and first-order logic in [2, 3, 9], it certainly seems particularly natural also at the level of modal logics, as outlined in [2], where Ciardelli gives a first detailed account of inquisitive modal logic in 2016. In its epistemic interpretation, for instance, modal logic offers the natural classical framework for distinctions between different states of affairs (facts, about which basic assertions can be made) and cognitive states (information states, about which more complex assertions, e.g. concerning knowledge, can be made). This is a very natural context in which one may want to give semantics also to questions. The study of questions is more generally well motivated – also at the more foundational philosophical or linguistic level – by considerations about language and logic in all kinds of scenarios that relate facts, knowledge and information. For instance one may want to account for the conceptual difference between '*knowing that*' and '*knowing whether*' something is the case. And indeed, inquisitive modal logic provides connectives and modal operators that neatly capture such distinctions; in particular it also offers, right at the propositional level, a non-trivial disjunction of alternatives p and $\neg p$, whose semantics captures the idea of the question '*whether* p', which is suggestively denoted as $?p$. This novel formula $?p$ is meant to specify, as a kind of $p/\neg p$ alternative, information states that support one of the admissible answers, but crucially without specifying which one. The semantics for inquisitive modal logic is given in terms of satisfaction of formulae in information states (*support semantics* in [6, 2]), i.e. in sets of possible worlds rather than in individual possible worlds. This latter feature also accounts for the conceptual links between the semantics of inquisitive logics and *team semantics* for logics of dependence [12, 16]. Not too surprisingly maybe, the semantic modelling for such phenomena in a modal framework involves not just possible worlds and relations between them (as is the case for basic modal logic) but includes information states as primary objects, together with relationships between (sets of) possible worlds and (sets of) information states.

In this sense, the setting of inquisitive modal logic puts an extra level of (set-theoretic) complexity on top of the familiar modal modelling (cf. [1, 7]). For instance, in the epistemic setting where possible worlds are associated with *states of affairs*: where Kripke models assign to possible worlds sets of alternative possible worlds (*information states*), inquisitive models assign to possible worlds sets of such possible information states (*inquisitive states*), which may be thought of as possible answers, or possible information updates. Correspondingly, natural relational encodings involve both the set of worlds and a set of information states, as two relevant sorts on an equal footing. The way these sorts live in a base set and its power set already suggests a degree of logical complexity that might be more of a challenge for a direct first-order account of the semantics of InqML than the well-known standard translation for basic modal logic ML. Indeed, the relational encodings of the intended inquisitive models form a non-elementary class, due to non-elementary closure conditions on the universe of information states. So, for instance, the straightforward compactness argument for ML, which just applies first-order compactness through standard translation, cannot be used directly.

From a model-theoretic perspective, translating a non-classical logic like InqML into the classical framework of first-order logic FO provides a systematic advantage. With a standard translation we can investigate InqML in the well-known FO setting and explore model-theoretic features of InqML, such as compactness, in this context.

Compared to previous translations as used in [4, 5], the one we present here combines several advantages: it puts minimal requirements on the relational encodings of the inquisitive models; it is more directly defined by natural induction on the full syntax of InqML and is consequently more intuitive. At the technical level, the main novelty is an application of a concept we call *graded flatness*. The same concept was previously considered in (first-order) dependence logic in [13] and in the context of inquisitive first-order logic in [8] under the name of *coherence*, but its application in the modal context seems to represent an innovation.

At a conceptual level, our treatment may also suggest the relaxed class of models, which we call *pseudo-models* and whose relational counterparts form an elementary class, as an alternative to the intended models for InqML, whose relational counterparts form a non-elementary class. Somewhat surprisingly, pseudo-models faithfully reflect some of the most salient logical features. Analogous ideas appear to have been considered in neighbourhood semantics [15], where monotonic models do not translate into an elementary class, while general neighbourhood models do and in some connections can be used in the analysis of the former [10]. But uses of intermediate classes of 'weak' models have also more classically been explored, e.g. in the context of monadic second-order logic or of topological model theory [17]. We exemplify the value of pseudo-models and our standard translation as intermediaries in model-theoretic arguments in two separate applications.

The first application is a new, purely model-theoretic argument to establish compactness for InqML.[1] In the classical case for basic model ML, the standard translation into FO suffices to put ML into the elementary setting of Kripke structures as simple relational structures. The case of InqML needs to appeal to pseudo-models because the relational counterparts of the intended models do not themselves form an elementary class.

The second application yields a Hennessy-Milner theorem for InqML w.r.t. the natural notion of inquisitive bisimulation studied in [4]. Here the intended models, which support the Hennessy-Milner property, are obtained from ω-saturated pseudo-models via a non-elementary closure operation. Incidentally, ω-saturation is also shown to be incompatible with the closure requirements on actual models, which again highlight the crucial rôle of pseudo-models as intermediaries.

Finally we obtain variants of the van Benthem style expressive completeness result for InqML from [4] (for both, the world- and the state-pointed case) as a corollary to the combination of the compactness and Hennessy-Milner assertions. It is different from the results in [4] in referring to a natural strengthening of bisimulation invariance that bridges the gap between relational models and pseudo-models; it is also not applicable in the context of finite model theory.

Our choice of terminology is meant to make contact with team semantic notions to reflect the close relationship between the team semantic treatment of dependence logics [16] and the setting of InqML. Sets of worlds (i.e. information states) can be seen as teams, and in our two-sorted view are treated as first-class objects (of the second sort) along with worlds (as objects of the first sort).

2 Inquisitive modal logic

By $(p_i)_{i \in I}$ we denote a set of propositional variables. Following the terminology in [6, 2] we associate the following two kinds of *states* with a given non-empty set W of *possible worlds*.

Information states: any subset $s \subseteq W$ is called an *information state*; the set of information states over W is $\mathscr{P}(W)$.

Inquisitive states: a non-empty set Π of information states, $\Pi \subseteq \mathscr{P}(W)$, is an *inquisitive state* if it is closed w.r.t. set inclusion: $s \in \Pi$ implies $t \in \Pi$ for all $t \subseteq s$; the set of inquisitive states over W is the set of those non-empty sets of information states in $\mathscr{P}(\mathscr{P}(W))$ that satisfy this characteristic *downward closure* condition.

[1] InqML compactness is known from [2], where it arises as a corollary to a completeness result.

Definition 2.1 ((inquisitive, modal) models).
Let W be a set of possible worlds and $\Sigma : W \to \mathscr{P}(\mathscr{P}(W)) \setminus \{\emptyset\}$ be a function that assigns an inquisitive state $\Sigma(w)$ to every world $w \in W$ (an inquisitive assignment) and $V : (p_i)_{i\in I} \to \mathscr{P}(W)$ a function that assigns a subset of W to every propositional variable (a propositional assignment). Then $\mathbb{M} = (W, \Sigma, V)$ is called an (inquisitive modal) model.
With an inquisitive model $\mathbb{M} = (W, \Sigma, V)$ we associate an induced Kripke model *$\mathcal{K}(\mathbb{M}) = (W, \sigma, V)$, where $\sigma\colon W \to \mathscr{P}(W)$ is defined as $\sigma(w) := \bigcup \Sigma(w)$ (a modal assignment).*

A *state-pointed* (inquisitive modal) model is a pair $\mathbb{M}, s$ which consists of a model $\mathbb{M}$ together with a distinguished information state $s \subseteq W$. If s is a *singleton information state*, i.e. $s = \{w\}$ for some $w \in W$, we also speak of a *world-pointed* inquisitive model $\mathbb{M}, \{w\}$ for which we also write just $\mathbb{M}, w$.

Note that the associated Kripke structure reduces the inquisitive assignment (of inquisitive states $\Sigma(w) \in \mathscr{P}(\mathscr{P}(W))$) to an assignment of single information states $\sigma(w) = \bigcup \Sigma(w) \in \mathscr{P}(W)$ that can be cast as sets of successors w.r.t. a modal accessibility relation. The natural relational encoding of σ is in terms of the accessibility relation

$$R = \{(w, w')\colon w' \in \sigma(w)\} \subseteq W \times W,$$

so that $\sigma(w)$ becomes the set of immediate successors of w w.r.t. R, i.e. the set $\sigma(w) = R[w] = \{w'\colon (w, w') \in R\}$. A corresponding, natural relational encoding of the inquisitive assignment will have to resort to a two-sorted encoding with a second sort of information states (from $\mathscr{P}(W)$) besides the first sort W of worlds (see Section 3.1 below). In this two-sorted scenario, however, the characteristic downward closure condition on the inquisitive states $\Sigma(w)$ would remain non-elementary. This motivates the following relaxation of the notion of models to what we call *pseudo-models*, which may also be cast in the model-theoretic tradition of approximate or weak models that reduce the complexity of higher-order features, similar to, e.g. the use of weak models in topological model theory [17]. As we shall see below, this concept can serve here as a useful tool for the analysis of the intended, proper models.

Compared to Definition 2.1, the following definition of pseudo-models just waives the downward closure requirement on inquisitive assignments.

Definition 2.2 (pseudo-models and inquisitive closure).
A pseudo-(inquisitive modal) model *is a structure $\mathbb{M} = (W, \Sigma, V)$ over the set of possible worlds W with propositional assignment $V : (p_i)_{i\in I} \to \mathscr{P}(W)$ and a function $\Sigma : W \to \mathscr{P}(\mathscr{P}(W)) \setminus \{\emptyset\}$, which assigns a non-empty subset $\Sigma(w)$ of $\mathscr{P}(W)$ but not necessarily an inquisitive state to every world $w \in W$.*

With any pseudo-model $\mathbb{M} = (W, \Sigma, V)$ we associate its inquisitive closure *$\mathbb{M}{\downarrow} := (W, \Sigma{\downarrow}, V)$, which is the proper model whose inquisitive assignment $\Sigma{\downarrow}$ is induced by Σ according to*

$$\begin{aligned} \Sigma{\downarrow}\colon W &\longrightarrow \mathscr{P}(\mathscr{P}(W)), \\ w &\longmapsto \Sigma{\downarrow}(w) := \{t \in \mathscr{P}(W)\colon t \subseteq s \text{ for some } s \in \Sigma(w)\}. \end{aligned}$$

We note that the distinction between a pseudo-model and its inquisitive closure is immaterial at the level of the associated Kripke models with their modal assignment $\sigma(w) = \bigcup \Sigma(w) = \bigcup \Sigma{\downarrow}(w)$.

Definition 2.3 (InqML: syntax).
The basic syntax of InqML *is given by the grammar*

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \vee\!\!\!\vee \varphi) \mid \Box\varphi \mid \boxplus \varphi,$$

with negation, disjunction and diamond $\Diamond$ *treated as abbreviations according to* $\neg\varphi := \varphi \to \bot$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, *and* $\Diamond\varphi := \neg\Box\neg\varphi$.

In [2] the symbol $\vee\!\!\!\vee$ is called *intuitionistic disjunction*, $\Box$ the *universal modality* and $\boxplus$ the *inquisitive modality.*

The following extends the standard definition of the semantics of InqML from [6, 2] to pseudo-models in a straightforward manner.

Definition 2.4 (InqML: semantics).
Let $\mathbb{M} = (W, \Sigma, V)$ *be a model or a pseudo-model,* $s \subseteq W$ *an information state. The semantics of* InqML *is defined as follows.*

- $\mathbb{M}, s \models p :\iff s \subseteq V(p)$
- $\mathbb{M}, s \models \bot :\iff s = \emptyset$
- $\mathbb{M}, s \models \varphi \wedge \psi :\iff \mathbb{M}, s \models \varphi$ *and* $\mathbb{M}, s \models \psi$
- $\mathbb{M}, s \models \varphi \to \psi :\iff \forall t \subseteq s : \mathbb{M}, t \models \varphi \Rightarrow \mathbb{M}, t \models \psi$
- $\mathbb{M}, s \models \varphi \vee\!\!\!\vee \psi :\iff \mathbb{M}, s \models \varphi$ *or* $\mathbb{M}, s \models \psi$
- $\mathbb{M}, s \models \Box\varphi :\iff \forall w \in s : \mathbb{M}, \sigma(w) \models \varphi$
- $\mathbb{M}, s \models \boxplus\varphi :\iff \forall w \in s\ \forall t \in \Sigma(w) : \mathbb{M}, t \models \varphi$

We note that the semantic clause for implication in Definition 2.4 refers to *all* subsets $t \subseteq s$, over models or pseudo-models alike. Similarly, the downward closure condition known as *persistency*, as discussed in the following observation, speaks about *all* subsets $t \subseteq s$ of the given information state s, also when interpreted in the non-standard setting of pseudo-models.

Observation 2.5. *Over all models as well as pseudo-models* $\mathbb{M}$, InqML *has the following properties, for all* $\varphi \in$ InqML*:*

(i) $\mathbb{M}, s \models \varphi$ *implies* $\mathbb{M}, t \models \varphi$ *for all* $t \subseteq s$*;*

(ii) $\mathbb{M}, \emptyset \models \varphi$.

Property (i) is called *persistency* in [6, 2] (and usually referred to as *downward closure* in team semantic terminology), while (ii) is called *semantic ex-falso* (reflecting the *empty team property*).

The following gives a further indication that extension of the semantics of InqML beyond the intended inquisitive models is very natural. The proof is by straightforward syntactic induction following the clauses of Definition 2.4; for the $\boxplus$-case one uses persistency, and for the $\Box$-case, which refers to the associated Kripke structure, one uses the fact that the associated modal assignment σ is the same for the pseudo-model and its inquisitive closure.

Proposition 2.6. *Let* $\mathbb{M}$ *be a pseudo-model,* $\mathbb{M}\!\downarrow$ *its inquisitive closure, and* $s \subseteq W$ *any information state over their common universe* W *of possible worlds. Then for* $\varphi \in$ InqML *we have*

$$\mathbb{M}\!\downarrow, s \models \varphi \iff \mathbb{M}, s \models \varphi.$$

This indicates that, as far as e.g. deductive reasoning is concerned, InqML might as well be cast in the extended setting of pseudo-models. The difference is important, though, e.g. in issues concerning what is the natural habitat for the key notion of model equivalence, viz. inquisitive bisimulation equivalence [4] to be discussed in Section 5, or how InqML embeds into classical logics of reference like FO (cf. key results in [4] vs. Corollary 5.8 here).

3 Standard translation

A standard translation serves as a semantically adequate (one could say, truthful) translation between logical frameworks.

Recall the well-known standard translation of modal logic ML into first-order logic FO. It is based on the straightforward transcription of the clauses for the Kripke semantics of ML into their natural first-order analogues over Kripke models viewed as ordinary relational first-order structures. The situation for InqML is different, because inquisitive (pseudo-)models are naturally rendered as two-sorted rather than ordinary single-sorted relational structures.[2] Since an inquisitive assignment is a function from the set W of possible worlds to sets of sets of possible worlds, it is of inherently higher type than a modal assignment. Its natural relational encoding consists of a binary relation not over W itself, but between W (as a first sort) and a set S of information states (as a second sort) where $S \subseteq \mathscr{P}(W)$. In the following we discuss a setting and format for a standard translation of InqML into FO in the natural two-sorted relational framework that is similar in spirit to that in [4] but more liberal and more uniform (cf. Remark 3.5). The technical novelty underpinning this new approach is the application of a graded notion of flatness, *graded flatness*, to InqML, as independently developed by the first author in [14].

We want to associate the semantics of inquisitive modal logic InqML over models (or even pseudo-models) with the semantics of first-order logic over associated two-sorted relational (pseudo-)models. As usual, this task involves two translation levels that need to go hand in hand: transformations linking the underlying (pseudo-)models $\mathbb{M}$ to relational representations $\mathscr{M}$, from which the underlying (pseudo-)models $\mathbb{M}$ can be recovered as $\mathbb{M} = \mathbb{M}(\mathscr{M})$; and a translation of formulae $\varphi \in$ InqML into formulae $\varphi^* \in$ FO such that

$$\mathbb{M}(\mathscr{M}), s \models \varphi \iff \mathscr{M}, s \models \varphi^*.$$

3.1 Relational representations of models

As relational counterparts of inquisitive (pseudo-)models we consider two-sorted relational structures of the form

$$\mathscr{M} = (W, S, \epsilon, E, (P_i)_{i \in I})$$

with some non-empty sets W and S as first and second sorts, linked by two mixed-sorted binary relations $\epsilon, E \subseteq W \times S$, and with unary predicates $P_i \subseteq W$ over the first sort for all p_i, $i \in I$, that encode the propositional assignment as usual. The intended rôles of ϵ and E are as follows: $\epsilon \subseteq W \times S$ encodes

[2] This aspect of two-sortedness is similar to the treatment of neighbourhood models [15], but InqML modelling imposes a different and in some sense tighter link between sorts.

membership of possible worlds in information states, so that $s \in S$ can be associated with $\underline{s} := \{w \in W : (w, s) \in \epsilon\} \in \mathscr{P}(W)$; and $E \subseteq W \times S$ encodes the inquisitive assignment as a relation that associates with each world $w \in W$ the set $E[w] := \{s \in S : (w, s) \in E\} \in \mathscr{P}(S)$.

Definition 3.1 (relational (pseudo-)models)**.**
A structure $\mathscr{M}$ of the type above is a relational (inquisitive modal) model *if the following conditions are satisfied for all $w \in W$, $s, t \in S$ and $a \subseteq S$:*

(i) $\underline{s} = \underline{t} \Rightarrow s = t$ (extensionality);

(ii) $E[w] \neq \emptyset$ (non-emptiness);

(iii) if $a \subseteq \underline{s}$ for $s \in E[w]$, then $a = \underline{t}$ for some $t \in E[w]$ (downward closure).

Correspondingly, a structure $\mathscr{M} = (W, S, \epsilon, E, (P_i)_{i \in I})$ of the same format is a relational (inquisitive modal) pseudo-model *if it satisfies (i) and (ii).*

Relational (pseudo-)models with distinguished information states $s \in S$ are described as *state-pointed* or, in the case of singleton states $\underline{s} = \{w\}$ as *world-pointed*, in analogy with the terminology for models. Due to extensionality (i), we shall identify information states $s \in S$ with sets of worlds $\underline{s} \subseteq W$ and regard the second sort S as a subset of the power set $\mathscr{P}(W)$, with ϵ as the actual membership relation between W and $S \subseteq \mathscr{P}(W)$.

Observation 3.2. *The class $\mathscr{C}$ of all state-pointed (respectively world-pointed) relational pseudo-models is Δ-elementary, i.e. there exists a set of formulae $\Phi \subseteq$ FO such that $\mathscr{C} = Mod(\Phi)$.*[3]

It is fairly easy to see that the class of all relational models cannot be Δ-elementary, as downward closure (condition (iii) in Definition 3.1) cannot be expressed without reference to arbitrary subsets of the first sort. Indeed, if it was Δ-elementary then so would be the class of all *full relational models*, defined by the additional condition that $S = \mathscr{P}(W)$ be the full power set. That, however, is ruled out by the observation that FO does not satisfy compactness over this class of all full relational models: over that class, FO captures the full power of monadic second-order logic MSO over the first sort; so it can, e.g., define the class corresponding to Kripke models that satisfy the well-foundedness condition of Löb frames (cf. [4]).

Observation 3.3. *The class of all relational models is not Δ-elementary.*

From relational (pseudo-)models to (pseudo-)models.
With any relational (pseudo-)model $\mathscr{M} = (W, S, \epsilon, E, (P_i)_{i \in I})$ we associate the (pseudo-)model $\mathbb{M}(\mathscr{M}) = (W, \Sigma, V)$ that decodes the relational information in $\mathscr{M}$ into functional assignments according to

$$\Sigma \colon w \longmapsto \{s \in S : (w, s) \in E\},$$
$$V : p_i \longmapsto \{w \in W : w \in P_i\}.$$

We observe that the actual extension of the second sort $S \subseteq \mathscr{P}(W)$ in $\mathscr{M}$ is immaterial in as far as it may go beyond the range of Σ.

[3] If the set of propositions is finite, then $\mathscr{C}$ is even elementary, i.e. definable by a single FO-formula.

From (pseudo-)models to relational (pseudo-)models.
With a (pseudo-)model $\mathbb{M} = (W, \Sigma, V)$ and a distinguished information state $s \subseteq W$, we associate as a relational representation any relational (pseudo-)model $\mathscr{M} = (W, S, \epsilon, E, (P_i)_{i \in I})$ that encodes Σ and V over sorts W and S, where $S \subseteq \mathscr{P}(W)$ is rich enough to represent the image $\Sigma(w) \subseteq \mathscr{P}(W)$ for all $w \in W$ as well as the distinguished information state s:

$$\begin{aligned} S &\subseteq \mathscr{P}(W) \text{ with } S \supseteq \{s\} \cup \bigcup_{w \in W} \Sigma(w), \\ \epsilon &:= \{(w, s) \in W \times S : w \in s\} = \in\restriction(W \times S), \\ E &:= \{(w, s) \in W \times S : s \in \Sigma(w)\}, \\ P_i &:= \{w \in W : w \in V(p_i)\} \text{ for } i \in I. \end{aligned}$$

Note that $\mathscr{M}$ is fully determined by $\mathbb{M}$ once the actual extension of the second sort S is fixed; that however is naturally only subject to a richness condition. (We argued above that insistence on fullness, i.e. on the maximal extension $S = \mathscr{P}(W)$, may not be advisable.)

Definition 3.4 (relational representations)**.**
A relational (pseudo-)model $\mathscr{M}$ *is a* relational representation *of a given (pseudo-)model* $\mathbb{M}$ *precisely for* $\mathbb{M} = \mathbb{M}(\mathscr{M})$.
A state-pointed relational (pseudo-)model $\mathscr{M}, s$ *is a* relational representation *of the state-pointed (pseudo-)model* $\mathbb{M}, s$ *if in addition the distinguished information state* s *is represented as an element of its second sort* S*.*[4]

It is clear from the above that every state-pointed (pseudo-)model admits relational representations, and that every relational (pseudo-)model $\mathscr{M}$ represents a unique (pseudo-)model, viz. $\mathbb{M}(\mathscr{M})$.

3.2 Graded flatness and the standard translation

Compared to the well-known standard translation for plain modal logic over Kripke models (for which $\mathbb{M}$ and $\mathscr{M}$ are practically identical), InqML involves challenges associated with the semantics of implication and $\Box$. The corresponding clauses in Definition 2.4 involve reference to information states that might not necessarily be directly available in the second sort of $\mathscr{M}$.

There are different suggestions to overcome this problem. An elimination of $\Box$ is possible via so-called *resolutions* [2] or via the $\Box$-free characteristic formulae for finitary bisimulation classes from [4]; then a straightforward translation can be given. That standard translation, however, is based on stronger closure conditions on the universe of information states in the relational encodings of the inquisitive models, which give FO direct access to the relevant information states. Such stronger closure conditions may further interfere with compactness over the required classes of models, as mentioned in § 3.1. A straightforward translation without elimination of $\Box$ is also possible but also requires stronger closure conditions on the class of relational models.

Remark 3.5. *Compared to the previous translations, the present proposal is more general, more uniform and more direct: it works for the natural class*

[4]Note that the distinguished state becomes an element of the second sort of $\mathscr{M}$, hence available as a parameter-definable subset of the first sort, but not as a constant or predicate.

of all pseudo-models and does not require any additional (elementary or non-elementary) closure conditions on the state universe of the intended relational models. Moreover, it is defined directly by induction on the unrestricted standard syntax of InqML*, without appeal to specific syntactic normal forms.*

Our standard translation relies on the following concept of *graded flatness*, which had also been investigated in the team-semantic context of dependence logic (cf. [13]) and in the context of inquisitive first-order logic (cf. [8]) under the name of *coherence*. For our context it was independently (re-)discovered and put to this new use in [14]. Our preferred terminology of *graded flatness* derives from the notion of *flatness* in team semantics. If we think of information states (sets of worlds) as teams, then a formula $\varphi \in$ InqML would be *flat* (in the team semantic sense) if its truth in s is equivalent to truth in $\{w\}$ for all $w \in s$. Graded flatness generalises this idea to quantitative bounds on the size of subsets $s' \subseteq s$ that need to be investigated, rather than singleton subsets. Such a size bound can be obtained as a syntactic parameter as follows.

Definition 3.6 (flatness grade)**.**
The flatness grade $\flat(\varphi) \in \mathbb{N}$ *of* $\varphi \in$ InqML *is defined by syntactic induction, for all* $\psi, \chi \in$ InqML*, according to*

- $\flat(\varphi) := 0$ *for atomic* φ *and all* φ *of the form* $\Box\psi$ *or* $\boxplus\psi$*;*
- $\flat(\psi \wedge \chi) := \max\{\flat(\psi), \flat(\chi)\}$*;*
- $\flat(\psi \rightarrow \chi) := \flat(\chi)$*;*
- $\flat(\psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi) := \flat(\psi) + \flat(\chi) + 1$.

Proposition 3.7 (graded flatness)**.**
Inquisitive modal logic InqML *satisfies the following* graded flatness property*. For all* $\varphi \in$ InqML *and state-pointed (pseudo-)models* $\mathbb{M}, s$*:*

$$\mathbb{M}, s \models \varphi \iff \mathbb{M}, t \models \varphi \text{ for all } t \subseteq s \text{ of size } |t| \leq \flat(\varphi) + 1.$$

Proof. The direction from left to right follows immediately from persistency for InqML. The implication from right to left is shown by syntactic induction on φ. We illustrate the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-case, which is the most interesting.
For $\varphi = \psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi$ let $m := \flat(\psi)$ and $n := \flat(\chi)$ so that $\flat(\varphi) = m + n + 1$, and assume that $\mathbb{M}, t \models \psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi$ for all $t \subseteq s$ of size $|t| \leq m+n+2$. By Definition 2.4 and the induction hypothesis we get that for all $t \subseteq s$ with $|t| \leq m + n + 2$:

$$\begin{array}{rl} & \forall a \subseteq t \text{ with } |a| \leq m+1 : \quad \mathbb{M}, a \models \psi \\ \text{or} & \forall a \subseteq t \text{ with } |a| \leq n+1 : \quad \mathbb{M}, a \models \chi. \end{array}$$

It follows that

$$\begin{array}{rl} & \forall t \subseteq s \text{ with } |t| \leq m+1 : \quad \mathbb{M}, t \models \psi \\ \text{or} & \forall t \subseteq s \text{ with } |t| \leq n+1 : \quad \mathbb{M}, t \models \chi. \end{array}$$

Indeed, if this were false, there would exist information states $t_1 \subseteq s$ with $|t_1| \leq m + 1$ and $t_2 \subseteq s$ with $|t_2| \leq n + 1$ such that $\mathbb{M}, t_1 \not\models \psi$ and $\mathbb{M}, t_2 \not\models \chi$. But then, for $t_0 := t_1 \cup t_2$, the previous statement would be false: we have $|t_0| \leq m + n + 2$ but $t_1 \subseteq t_0$ violates the first disjunct and $t_2 \subseteq t_0$ violates the second disjunct.

With the induction hypothesis we get $\mathbb{M}, s \models \psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi$. ∎

We use the following notation. Generally, we take variable symbols $x, y, \ldots$ to be interpreted over the first sort (worlds), and variable symbols $\lambda, \mu, \ldots$ over the second sort (information states). A tuple of length n is denoted as $\mathbf{x} = (x_1, ..., x_n)$, the set of its components as $\{\mathbf{x}\} = \{x_1, ..., x_n\}$. If the length of a tuple is determined by some flatness grade $\flat(\varphi)$, we write $\mathbf{x}_\varphi$ for the tuple $(x_1, ..., x_{\flat(\varphi)+1})$ and $\{\mathbf{x}_\varphi\}$ for the associated set. For better readability we write $x \in \lambda$ instead of $\epsilon x \lambda$ in the following first-order formulae.

Our standard translation $\varphi \mapsto \varphi^*(\lambda)$ is defined below, by syntactic induction on $\varphi \in$ InqML. The interesting, somewhat non-standard feature involves the necessary passage between sorts: on one hand, φ is translated into the first-order formula $\varphi^*(\lambda)$ in a free variable λ of the second sort, which is to be interpreted as the distinguished information state s; the core induction, on the other hand, deals with auxiliary formulae $\mathsf{ST}(\xi, \mathbf{x})$ in tuples $\mathbf{x}$ of free variables of the first sort that capture the semantics of $\xi^*(\mu)$ for $\mu = \{\mathbf{x}\}$.

Definition 3.8 (standard translation)**.**
For $\varphi \in$ InqML *define its* standard translation $\varphi^*(\lambda) \in$ FO *in one free state variable* λ *as*

$$\varphi^*(\lambda) := \forall \boldsymbol{x}_\varphi \Big(\bigwedge_{k \leq \flat(\varphi)+1} x_k \in \lambda \;\longrightarrow\; \mathsf{ST}(\varphi, \boldsymbol{x}_\varphi) \Big),$$

where the auxiliary first-order formulae $\mathsf{ST}(\xi, \boldsymbol{x})$*, with free world variables among* $\boldsymbol{x}$*, are defined by syntactic induction according to:*

- $\mathsf{ST}(p, \boldsymbol{x}) := \bigwedge_{k \leq n} P x_k$
- $\mathsf{ST}(\bot, \boldsymbol{x}) := \bigwedge_{k \leq n} \neg x_k = x_k$
- $\mathsf{ST}(\psi \wedge \chi, \boldsymbol{x}) := \mathsf{ST}(\psi, \boldsymbol{x}) \wedge \mathsf{ST}(\chi, \boldsymbol{x})$
- $\mathsf{ST}(\psi \to \chi, \boldsymbol{x}) := \forall \boldsymbol{y} \Big[\Big(\bigwedge_{k \leq n} \bigvee_{l \leq n} y_k = x_l \Big) \longrightarrow \big(\mathsf{ST}(\psi, \boldsymbol{y}) \to \mathsf{ST}(\chi, \boldsymbol{y}) \big) \Big]$
- $\mathsf{ST}(\psi \mathbin{\vee\!\!\!\vee} \chi, \boldsymbol{x}) := \mathsf{ST}(\psi, \boldsymbol{x}) \vee \mathsf{ST}(\chi, \boldsymbol{x})$
- $\mathsf{ST}(\Box \psi, \boldsymbol{x}) := \bigwedge_{k \leq n} \forall \boldsymbol{y}_\psi \forall \boldsymbol{\mu}_\psi \Big(\bigwedge_{l \leq \flat(\psi)+1} \big(E x_k \mu_l \wedge y_l \in \mu_l \big) \;\longrightarrow\; \mathsf{ST}(\psi, \boldsymbol{y}_\psi) \Big)$
- $\mathsf{ST}(\boxplus \psi, \boldsymbol{x}) := \bigwedge_{k \leq n} \forall \mu \, (E x_k \mu \to \psi^*(\mu))$

The intuition for, e.g. the translation of $\Box$ in this definition is the following. We want to suitably mimic the semantics of $\Box$ in our context, hence we want to say that for all worlds $x_k \in \mathbf{x}$, $\psi^*(\lambda)$ is satisfied by the information state $\sigma(x_k)$. For this we need access to the states $\sigma(x_k)$. Via the subformula $E x_k \mu_l \wedge y_l \in \mu_l$ of our translation we can check whether a world y_l is element of $\sigma(x_k)$. Since $\sigma(x_k)$ is in general not necessarily represented as an element of the second sort of the relational pseudo-model that is supposed to satisfy the translation, we cannot access it directly via a state variable. Instead we can fix all its substates of size $\flat(\psi)+1$ in the shape of tuples of worlds via

$$\forall \mathbf{y}_\psi \forall \boldsymbol{\mu}_\psi \Big(\bigwedge_{l \le \flat(\psi)+1} \big(Ex_k\mu_l \wedge y_l \in \mu_l\big) \longrightarrow \mathsf{ST}(\psi, \mathbf{y}_\psi)\Big).$$

By Proposition 3.7 this correctly expresses that $\sigma(x_k)$ satisfies $\psi^*(\lambda)$.

Remark 3.9. *We may analogously define a standard translation $\pi^*(x)$ in the single free world variable x of the first sort, which covers the world-pointed case just as the standard translation $\varphi^*(\lambda)$ covers the state-pointed case. For this we may just use $\varphi^*(x) := \mathsf{ST}(\varphi, x)$.*[5]

The following shows that the proposed standard translation is adequate – in preserving the semantics and turning InqML into a *syntactic fragment* of FO – over the rich class of relational encodings of (pseudo-)models.

Proposition 3.10 (InqML as a fragment of FO)**.**
Let $\varphi \in$ InqML and let $\varphi^*(\lambda) \in$ FO be its standard translation. Let $\mathbb{M}, s$ be a (pseudo-)model and $\mathscr{M}, s$ be a relational representation of $\mathbb{M}, s$. Then

$$\mathbb{M}, s \models \varphi \iff \mathscr{M}, s \models \varphi^*(\lambda),$$

where the state variable λ is interpreted as s.

Proof. We show below that, for $\varphi \in$ InqML,

$$\mathbb{M}, \{\mathbf{w}\} \models \varphi \iff \mathscr{M}, \mathbf{w} \models \mathsf{ST}(\varphi, \mathbf{x}) \qquad (*)$$

for all finite tuples of worlds $\mathbf{w}$ from W and matching tuples $\mathbf{x}$ of variables. From this we obtain the claim of the proposition, that $\mathbb{M}, s \models \varphi \Leftrightarrow \mathscr{M}, s \models \varphi^*(\lambda)$, as follows. By Proposition 3.7, $\mathbb{M}, s \models \varphi$ is equivalent to

$$\forall t \subseteq s \text{ with } |t| \le \flat(\varphi)+1 : \mathbb{M}, t \models \varphi,$$

which is further equivalent to

$$\forall \mathbf{w}_\varphi \in s : \mathbb{M}, \{\mathbf{w}_\varphi\} \models \varphi.$$

(Note that for the direction from right to left we need $\mathbb{M}, \emptyset \models \varphi$ on the right-hand side, but the set of all tuples $\mathbf{w}_\varphi \in s$ of size $1 \le |\mathbf{w}_\varphi| \le \flat(\varphi)+1$ does not contain the empty tuple. This, however, is unproblematic for the equivalence since $\mathbb{M}, \emptyset \models \varphi$ trivially holds by Observation 2.5.)
With $(*)$ we find that $\mathbb{M}, s \models \varphi$ is equivalent to

$$\forall \mathbf{w}_\varphi \in s : \mathscr{M}, \mathbf{w}_\varphi \models \mathsf{ST}(\varphi, \mathbf{x}_\varphi),$$

which translates equivalently into

$$\mathscr{M}, s \models \forall \mathbf{x}_\varphi \Big(\bigwedge_{k \le \flat(\varphi)+1} x_k \in \lambda \longrightarrow \mathsf{ST}(\varphi, \mathbf{x}_\varphi)\Big),$$

which is the same as $\mathscr{M}, s \models \varphi^*(\lambda)$, by Definition 3.8.

[5]This naturally also captures semantics of $\varphi^*(\lambda)$ for $\lambda = \{x\}$, as it should.

It remains to show (∗) by syntactic induction. We explicitly treat the $\Box$- and $\boxplus$-steps, and show the implication from left to right in each case.

For the case of $\varphi = \Box\psi$ assume $\mathbb{M}, \{\mathbf{w}\} \models \Box\psi$, which, by Definition 2.4 and Proposition 3.7, means that

$$\forall w \in \{\mathbf{w}\}\ \forall t \subseteq \sigma(w) \text{ with } |t| \leq \flat(\psi) + 1 : \mathbb{M}, t \models \psi.$$

This is further equivalent to

$$\forall w \in \{\mathbf{w}\}\ \forall \mathbf{u}_\psi \in \sigma(w) : \mathbb{M}, \{\mathbf{u}_\psi\} \models \psi,$$

and by induction hypothesis to

$$\forall w \in \{\mathbf{w}\}\ \forall \mathbf{u}_\psi \in \sigma(w) : \mathscr{M}, \mathbf{u}_\psi \models \mathsf{ST}(\psi, \mathbf{y}_\psi).$$

This condition is correctly rendered in FO as

$$\mathscr{M}, \mathbf{w} \models \mathsf{ST}(\Box\psi, \mathbf{x}) = \bigwedge_{k\leq n} \forall \mathbf{y}_\psi \forall \boldsymbol{\mu}_\psi \Big(\bigwedge_{l \leq \flat(\psi)+1} \big(Ex_k\mu_l \wedge y_l \in \mu_l\big) \longrightarrow \mathsf{ST}(\psi, \mathbf{y}_\psi) \Big) :$$

universal quantification over $w \in \{\mathbf{w}\}$ is expressed by the conjunction over the $x_k \in \mathbf{x}$ enumerating the $w_k \in \mathbf{w}$; the quantification $\forall \mathbf{u}_\psi \in \sigma(w)$ is represented by the quantification $\forall \mathbf{y}_\psi \forall \boldsymbol{\mu}_\psi$, relativised so that the y_l are instantiated by worlds from $\sigma(w_k)$.

For the case $\varphi = \boxplus\psi$ assume $\mathbb{M}, \{\mathbf{w}\} \models \boxplus\psi$. With Definition 2.4 and Proposition 3.7 we get

$$\forall w \in \{\mathbf{w}\}\ \forall t \in \Sigma(w)\ \forall a \subseteq t \text{ with } |a| \leq \flat(\psi) + 1 : \mathbb{M}, a \models \psi,$$

hence

$$\forall w \in \{\mathbf{w}\}\ \forall t \in \Sigma(w)\ \forall \mathbf{u}_\psi \in t : \mathbb{M}, \{\mathbf{u}_\psi\} \models \psi.$$

By induction hypothesis we get

$$\forall w \in \{\mathbf{w}\}\ \forall t \in \Sigma(w)\ \forall \mathbf{u}_\psi \in t : \mathscr{M}, \mathbf{u}_\psi \models \mathsf{ST}(\psi, \mathbf{y}_\psi).$$

Again we can express this in FO according to

$$\mathscr{M}, \mathbf{w} \models \bigwedge_{k \leq n} \forall \mu \Big[Ex_k\mu \longrightarrow \Big(\forall \mathbf{y}_\psi \Big(\bigwedge_{l \leq \flat(\psi)+1} y_l \in \mu \longrightarrow \mathsf{ST}(\psi, \mathbf{y}_\psi) \Big) \Big) \Big],$$

which is $\mathscr{M}, \mathbf{w} \models \mathsf{ST}(\boxplus\psi, \mathbf{x})$ by Definition 3.8. □

Remark 3.11. *The analogue of Proposition 3.10 for world-pointed models and our world-version of the standard translation from Remark 3.9 is easily checked.*

4 Compactness for InqML

It is known that InqML has a sound and strongly complete proof calculus and therefore satisfies compactness (cf. [2]). We use our standard translation to give a new, purely model-theoretic proof, essentially by reduction to first-order compactness. But while the corresponding reduction is totally straightforward for basic modal logic ML over Kripke structures, we here need to deal with the additional complication that the class of relational models is not Δ-elementary. Correspondingly, a detour through pseudo-models plays an essential rôle in our proof. Moreover, the proof also shows that compactness over the class of all pseudo-models works in a straightforward manner. We interpret this as an additional, natural indication that InqML could also be explored over the extended class of all pseudo-models.

We consider the satisfiability version of compactness. Of course, by semantic ex-falso (cf. Observation 2.5), any set $\Phi \subseteq$ InqML is trivially satisfied by any model $\mathbb{M}, \emptyset$. For a non-trivial statement we need to exclude the empty state.

Proposition 4.1 (compactness).
InqML *satisfies compactness, i.e. a set of formulae* $\Phi \subseteq$ InqML *is satisfiable by some non-empty state of some (pseudo-)model if, and only if, every finite subset of* Φ *is satisfiable by some non-empty state of some (pseudo-)model.*

Proof. Let $\Phi \subseteq$ InqML be a set of formulae such that any finite subset $\Phi_0 \subseteq \Phi$ is satisfiable by a state-pointed model $\mathbb{M}_{\Phi_0}, s_{\Phi_0}$ such that $s_{\Phi_0} \neq \emptyset$. We want to show that Φ is satisfiable as well.
Let $\Phi^*(\lambda) \subseteq$ FO (respectively $\Phi_0^*(\lambda) \subseteq$ FO) be the set of all standard translated formulae of Φ (respectively Φ_0) and let for each $\Phi_0 \subseteq \Phi$ the relational model $\mathscr{M}_{\Phi_0}, s_{\Phi_0}$ be a relational representation of $\mathbb{M}_{\Phi_0}, s_{\Phi_0}$. Then Proposition 3.10 yields $\mathscr{M}_{\Phi_0}, s_{\Phi_0} \models \Phi_0^*$ for all finite $\Phi_0^* \subseteq \Phi^*$.
We let $\Delta = \Delta(\lambda) \subseteq$ FO be a set of formulae defining the class of all state-pointed relational pseudo-models (see Observation 3.2) with non-empty state ($\lambda \neq \emptyset$) and let $\tilde{\Phi} := \Phi^* \cup \Delta$. Since $\mathscr{M}_{\Phi_0}, s_{\Phi_0} \models \Delta$ for all finite $\Phi_0 \subseteq \Phi$, any finite subset of $\tilde{\Phi}$ is satisfiable. Hence by compactness of FO, $\tilde{\Phi}$ is satisfiable by some relational pseudo-model $\mathscr{M}, s$.
Then Proposition 3.10 yields $\mathbb{M}, s \models \Phi$ for $\mathbb{M}, s := \mathbb{M}(\mathscr{M}), s$ and Proposition 2.6 entails $\mathbb{M}{\downarrow}, s \models \Phi$ for the inquisitive closure $\mathbb{M}{\downarrow}, s$ of $\mathbb{M}, s$. $\square$

With persistency it is easy to see that compactness over the class of all state-pointed (pseudo-)models implies compactness over the class of all world-pointed (pseudo-)models.

5 A Hennessy-Milner class for InqML

The notion of inquisitive bisimulation from [4] plays an essential rôle for InqML as does ordinary bisimulation for plain modal logic ML (for background cf. [7]). In particular, inquisitive n-bisimulation equivalence $\sim_n$ between two inquisitive models represents a finite approximation to full inquisitive bisimulation equivalence; at level $n \in \mathbb{N}$, $\sim_n$ is related to InqML-equivalence up to nesting depth n (of the modalities $\Box$ and $\boxplus$), in the familiar style of an Ehrenfeucht–Fraïssé correspondence for InqML [4] (see Theorem 5.1 below). Full inquisitive bisimulation equivalence $\sim$ between world- or state-pointed inquisitive models is naturally defined in terms of back&forth systems, bisimulation relations, or winning strategies for the defender (player **II**) in the corresponding infinite back&forth game. The finite levels $\sim_n$ for $n \in \mathbb{N}$ are best understood as approximations to $\sim$ in terms of winning strategies for player **II** in the n-round back&forth game; the common refinement $\sim_\omega$ of all the finite levels $\sim_n$ is defined in terms of winning strategies for any finite number of rounds, which in general is characteristically weaker than full $\sim$. The crucial feature of the back&forth games for inquisitive bisimulation is that each round comprises two phases so that the probing of state-pointed positions is interleaved with a probing of intermediate world-pointed positions (for details of the game and a fuller discussion we refer to [4]).

The following summarises the Ehrenfeucht–Fraïssé correspondence derived in [4]. We write $\equiv^n_{\mathrm{InqML}}$ for InqML-equivalence between world- or state-pointed inquisitive models up to nesting depth n, and use the term $\sim_n$*-types* for $\sim_n$-equivalence classes (of world- or state-pointed) inquisitive modal models.

Theorem 5.1 (inquisitive Ehrenfeucht–Fraïssé [4])**.** *For world- or state-pointed inquisitive modal models over a finite signature and* $n \in \mathbb{N}$*:*

$$\begin{aligned} \mathbb{M}, w \sim_n \mathbb{M}', w' \quad &\Leftrightarrow \quad \mathbb{M}, w \equiv^n_{\mathrm{InqML}} \mathbb{M}', w', \\ \mathbb{M}, s \sim_n \mathbb{M}', s' \quad &\Leftrightarrow \quad \mathbb{M}, s \equiv^n_{\mathrm{InqML}} \mathbb{M}', s'. \end{aligned}$$

In particular, InqML *is preserved under* $\sim$*, and* $\sim_\omega$ *coincides with* InqML*-equivalence over finite signatures. Moreover, for world- or state-pointed inquisitive modal models* $\mathbb{M}, w$ *or* $\mathbb{M}, s$ *over a finite signature, there are* characteristic formulae $\chi^n_{\mathbb{M},w}$ *and* $\chi^n_{\mathbb{M},s}$ *of* InqML *that define* $\sim_n$*-types (up to persistency, cf. Observation 2.5) in the sense that*

$$\begin{aligned} \mathbb{M}', w' \models \chi^n_{\mathbb{M},w} \quad &\Leftrightarrow \quad \mathbb{M}', w' \sim_n \mathbb{M}, w, \\ \mathbb{M}', s' \models \chi^n_{\mathbb{M},s} \quad &\Leftrightarrow \quad \mathbb{M}', s' \sim_n \mathbb{M}, s_0 \textit{ for some } s_0 \subseteq s. \end{aligned}$$

We remark that the characteristic formulae $\chi^n_{\mathbb{M},w}$, for $\sim_n$-types of worlds, are *truth-conditional* (in the terminology of [2, 4]) or *flat* (in the terminology of team semantics). This means that

$$\begin{aligned} \mathbb{M}', s' \models \chi^n_{\mathbb{M},w} \quad &\Leftrightarrow \quad \mathbb{M}', \{w'\} \models \chi^n_{\mathbb{M},w} \text{ for all } w' \in s' \\ &\Leftrightarrow \quad \mathbb{M}', w' \models \chi^n_{\mathbb{M},w} \text{ for all } w' \in s'. \end{aligned}$$

For plain modal logic ML and many of its relatives, the significant gap between $\sim$ and $\sim_\omega$, and between $\sim_\omega$ and $\equiv_{\mathrm{ML}}$ for infinite signatures, is bridged by modally saturated and in particular by ω-saturated models. This phenomenon, which for basic modal logic is (a generalisation of) the well-known Hennessy–Milner theorem, is a crucial tool in the model theory of modal logic [7]: ω-saturated Kripke structures are modally saturated and satisfy the Hennessy–Milner correspondence

$$\mathcal{K}, w \sim \mathcal{K}', w' \;\Leftrightarrow\; \mathcal{K}, w \sim_\omega \mathcal{K}', w' \;\Leftrightarrow\; \mathcal{K}, w \equiv_{\mathrm{ML}} \mathcal{K}', w'.$$

We here establish an inquisitive version of this Hennessy-Milner phenomenon, which technically works with ω-saturated relational pseudo-models and our standard translation. It turns out that the passage through pseudo-models is absolutely crucial for the argument (cf. Observation 5.3).

In what follows we could consider signatures of arbitrary infinite cardinalities, i.e. with arbitrary index sets I for the family of basic propositions $(p_i)_{i \in I}$; but to simplify the formal account, and without any essential loss of generality, we stick to at most countably infinite I. Consider InqML-types of worlds of inquisitive (pseudo-)models: for a world $w \in W$ in the (pseudo-)model $\mathbb{M}$, let

$\rho_w := \{\varphi \in \mathrm{InqML} \colon \mathbb{M}, w \models \varphi\}$ be its InqML-type. Even if the overall signature is infinite, any individual InqML- or FO-formula can only refer to finitely many basic propositions (or their relational counterparts). In this sense we have $\mathrm{InqML} = \bigcup_k \mathrm{InqML}_k$ for its fragments InqML_k that use just the first k basic propositions from $I_k \subseteq I$, for $k \in \mathbb{N}$. The InqML-type of a world in a (relational) pseudo-model therefore always is the union of its restrictions to finite sub-signatures, which are types in corresponding reducts.

For bisimulation types, i.e. equivalence classes w.r.t. $\sim$ or $\sim_n$, however, the situation is different. If we denote as $\sim_{n,k}$ the notion of n-bisimilarity of the I_k-reducts of models, then even at the level of $n = 1$, all the $\sim_{1,k}$-types do not determine the $\sim_1$-type of a world w, as the former do e.g. not determine whether w has a Kripke-successor in which infinitely many basic propositions are satisfied simultaneously.

In the following we want to consider standard translations of InqML-formulae on world-pointed relational inquisitive models and therefore use the world-version of the standard translation from Remark 3.9 whenever this is appropriate. With a world w in some relational pseudo-model we associate its complete InqML-type ρ_w with a partial first-order type $\rho^*_w(x)$ consisting of the standard translations of all formulae in ρ_w. So ρ_w is determined by (the standard translations of) the characteristic formulae $\chi^{n,k}_w \in \mathrm{InqML}_k$ for the $\sim_n$-types of w in the I_k-reducts, i.e. the $\sim_{n,k}$-types, for all $n, k \in \mathbb{N}$, by Theorem 5.1.

From now on, for reasons of simplicity, we identify InqML-formulae like $\chi^{n,k}_v$ with their standard translation in FO, which is expressed as $\chi^{n,k}_v(x)$ in a first-order variable of the first sort, or as $\chi^{n,k}_v(\lambda)$ in a first-order variable of the second sort in case we consider state-pointed relational inquisitive models.

Recall from classical model theory (see e.g. [11]) that a first-order structure is *ω-saturated* if it realises every first-order type with finitely many parameters. Generally, a (partial) *first-order type* of a σ-structure $\mathcal{A}$ is a set of $\mathrm{FO}(\sigma)$-formulae Φ in some tuple of free variables that is consistent with the complete $\mathrm{FO}(\sigma)$-theory of $\mathcal{A}$; a tuple of elements of A *realises* Φ if all $\varphi \in \Phi$ are satisfied by this tuple. A (partial) type with parameters $\mathbf{a}$ from A is a (partial) type of the expansion of $\mathcal{A}$ with new constant symbols for the $a \in \mathbf{a}$, so that $\Phi_{\mathbf{a}}$ can specify first-order properties in relation to these parameters in $(\mathcal{A}, \mathbf{a})$.[6]

Classical chain constructions based on FO compactness establish that any first-order structure admits elementary extensions that are ω-saturated. Compactness is also at the heart of the straightforward argument for the following fact, to be used in the proof of our main result of this section.

In the following, we say that an element $s \in \mathscr{P}(W)$ is *represented* in the second sort of a relational pseudo-model if $s \in S \subseteq \mathscr{P}(W)$.

Lemma 5.2. *Let $s \in S$ be represented in the second sort of an ω-saturated relational pseudo-model $\mathscr{M}$, $\rho_v(x)$ as above (not necessarily realised in $\mathscr{M}$). Then $\rho_v(x)$ is realised by a world in s in $\mathscr{M}$ iff*

$$\mathscr{M}, s \models \exists x \left(x \in s \wedge \chi^{n,k}_v(x)\right) \quad \textit{for all } k, n \in \mathbb{N}.$$

[6] We here look at partial FO-types in the two-sorted setting of relational pseudo-models. These may specify properties of worlds (in free variables of the first sort) as well as of information states (in free variables of the second sort), and could involve parameters from both sorts. As it turns out, we actually just need to consider partial types of individual elements (of either sort) with single parameters from the other sort for our purposes.

This lemma implies in particular that ω-saturation is at odds with full downward closure. This is the reason why our inquisitive version of the Hennessy–Milner correspondence must be based on ω-saturated relational pseudo-models rather than relational encodings of proper models.

Observation 5.3. *In general, ω-saturation is incompatible with the $\downarrow$-closure condition on information states. It requires* all *information states s that are represented in the second sort to be closed under "limits" of partial $\sim$-types of worlds $w \in s$. For instance, one cannot have the analogue of the information state "all Kripke-paths from worlds $w \in s$ are finite" without imposing a uniform finite bound on the lengths of all these paths.*

Indeed, an information state whose worlds admit arbitrarily long but just finite Kripke-paths (Löb condition) cannot be represented in an ω-saturated pseudo-model, because any such state would also have to contain a world with an infinite Kripke-path. This follows as ω-saturation would inductively always provide a next Kripke-successor that still has Kripke-paths of unbounded finite lengths in front of it.

We define an auxiliary equivalence relation between state-pointed pseudo-models as follows: $\mathbb{M}, s$ and $\mathbb{M}', s'$ are *bulk-equivalent*,

$$(\dagger) \quad \mathbb{M}, s \equiv^{bulk}_{\mathrm{InqML}} \mathbb{M}', s',$$

if for every $w \in s$ there is some $w' \in s'$ such that $\mathbb{M}, w \equiv_{\mathrm{InqML}} \mathbb{M}', w'$, and vice versa. In general, $\equiv^{bulk}_{\mathrm{InqML}}$ is a proper strengthening of $\equiv_{\mathrm{InqML}}$; the two do coincide, however, for ω-saturated pseudo-models and for information states that are represented in the second sort. In that situation, ordinary InqML-equivalence reflects the natural relationship between state- and world-based inquisitive bisimulation: state-equivalence $\mathbb{M}, s \equiv_{\mathrm{InqML}} \mathbb{M}', s'$ as a flat lifting of world-equivalence, in the spirit of our bulk equivalence.

Remark 5.4. *For state-pointed pseudo-models $\mathbb{M}, s$ and $\mathbb{M}', s'$ that stem from ω-saturated relational state-pointed pseudo-models $\mathscr{M}, s$ and $\mathscr{M}', s'$ with s and s' being represented in the second sort of $\mathscr{M}$ and $\mathscr{M}'$, respectively, the following are equivalent:*

(i) $\mathbb{M}\!\downarrow, s \equiv_{\mathrm{InqML}} \mathbb{M}'\!\downarrow, s'$*;*

(ii) $\mathbb{M}\!\downarrow, s \equiv^{bulk}_{\mathrm{InqML}} \mathbb{M}'\!\downarrow, s'$*, i.e., for every $w \in s$ there is some $w' \in s'$ such that $\mathbb{M}, w \equiv_{\mathrm{InqML}} \mathbb{M}', w'$, and vice versa.*

Proof. Note that references to inquisitive closures $\mathbb{M}\!\downarrow, s$ could be replaced by $\mathbb{M}, s$ in both (i) and (ii), as generally $\mathbb{M}, s \equiv_{\mathrm{InqML}} \mathbb{M}\!\downarrow, s$ by Proposition 2.6.

The implication (ii) $\Rightarrow$ (i) is obvious in light of the definition of the semantics of InqML (over pseudo-models) in combination with its graded flatness. This implication does not require ω-saturation.

For (i) $\Rightarrow$ (ii), let $\mathbb{M}, s \equiv_{\mathrm{InqML}} \mathbb{M}', s'$ and consider the claim in (ii) e.g. for $w \in s$. On the basis of ω-saturation and Lemma 5.2, it suffices to show that for all $n, k \in \mathbb{N}$, there is some $w' \in s'$ s.t. $\mathbb{M}', w' \models \chi^{n,k}_w$. If this were not the case for some n, k, then $\mathbb{M}', s' \models \neg\chi^{n,k}_w$, which is impossible since obviously not $\mathbb{M}, s \models \neg\chi^{n,k}_w$ (cf. remarks on the truth-conditional, flat nature of $\chi^{n,k}_w$ in connection with Theorem 5.1 above). □

Lemma 5.5. *For pairs of state-pointed pseudo-models* $\mathbb{M}, s$ *and* $\mathbb{M}', s'$ *stemming from* ω*-saturated relational pseudo-models* $\mathscr{M}$ *and* $\mathscr{M}'$*:*

$$\mathbb{M}{\downarrow}, s \equiv^{bulk}_{\text{InqML}} \mathbb{M}'{\downarrow}, s' \;\Rightarrow\; \mathbb{M}{\downarrow}, s \sim \mathbb{M}'{\downarrow}, s'.$$

Note that the information states $s \subseteq W$ and $s' \subseteq W'$ in the premise of the claim of the lemma are arbitrary information states and need not themselves be represented in the second sort of the underlying relational pseudo-models $\mathscr{M}$ and $\mathscr{M}'$. However, by Remark 5.4, Lemma 5.5 implies the same conclusion for states s and s' that are represented in the second sort of $\mathscr{M}$ and $\mathscr{M}'$, respectively, under the weaker assumption that $\mathbb{M}{\downarrow}, s \equiv_{\text{InqML}} \mathbb{M}'{\downarrow}, s'$.

Proof. The idea is to represent or simulate the infinite bisimulation game from [4] on $\mathbb{M}{\downarrow}; \mathbb{M}'{\downarrow}$ in terms of the underlying relational pseudo-models $\mathscr{M}; \mathscr{M}'$. The challenge lies in dealing with those information states that may essentially occur in the bisimulation game without being represented in the second sort of the underlying relational pseudo-models.

For the claim of the lemma it suffices to show that player **II** can maintain $\equiv^{bulk}_{\text{InqML}}$ through a single round in the following sense: starting from a state position $\big(\mathbb{M}{\downarrow}, s; \mathbb{M}'{\downarrow}, s'\big)$ with $\mathbb{M}{\downarrow}, s \equiv^{bulk}_{\text{InqML}} \mathbb{M}'{\downarrow}, s'$, player **II** can respond to any world challenge from player **I** to get to a world position $\big(\mathbb{M}{\downarrow}, w; \mathbb{M}'{\downarrow}, w'\big)$ such that $\mathbb{M}{\downarrow}, w \equiv_{\text{InqML}} \mathbb{M}'{\downarrow}, w'$, and from there respond to any state challenge from player **I** to get again to a state position $\big(\mathbb{M}{\downarrow}, t; \mathbb{M}'{\downarrow}, t'\big)$ such that $\mathbb{M}{\downarrow}, t \equiv^{bulk}_{\text{InqML}} \mathbb{M}'{\downarrow}, t'$.

Let $\mathbb{M}{\downarrow}, s \equiv^{bulk}_{\text{InqML}} \mathbb{M}'{\downarrow}, s'$ and assume w.l.o.g. that player **I** picks an element $w \in s$; then by the definition of $\equiv^{bulk}_{\text{InqML}}$ at (†), player **II** can respond with some $w' \in s'$ with $\mathbb{M}{\downarrow}, w \equiv_{\text{InqML}} \mathbb{M}'{\downarrow}, w'$. If player **I** now picks (again w.l.o.g.) an information state $t \in \Sigma{\downarrow}(w)$ in $\mathbb{M}{\downarrow}$, we need player **II** to respond with some $t' \in \Sigma'{\downarrow}(w')$ in $\mathbb{M}'{\downarrow}$ such that $\mathbb{M}{\downarrow}, t \equiv^{bulk}_{\text{InqML}} \mathbb{M}'{\downarrow}, t'$.

For this consider the partial FO-type describing the essential properties of t (in variable λ of the second sort) in $\mathscr{M}$ (with parameter w):[7]

$$\Phi_w(\lambda) := \{Ew\lambda\} \cup \big\{\exists x \big(x \in \lambda \wedge \chi^{n,k}_v(x)\big) : n, k \in \mathbb{N}, v \in t\,\big\},$$

where $\chi^{n,k}_v(x)$ is (the standard translation of) the InqML-formula that characterises the $\sim_{n,k}$-type of $\mathbb{M}{\downarrow}, v$. We want to show that $\Phi_{w'}(\lambda)$ (with parameter w') is finitely satisfiable in $\mathscr{M}'$ and hence a partial type of $\mathscr{M}'$. It suffices to show that $\mathscr{M}', w'$ satisfies every formula of the form

$$\varphi(x) = \exists\lambda\Big(Ex\lambda \wedge \bigwedge_{i \le m} \exists y \big(y \in \lambda \wedge \chi^{n,k}_{v_i}(y)\big)\Big),$$

for $n, m, k \in \mathbb{N}$ and $v_1, \ldots, v_m \in t$. For this claim we can use the fact that $\mathbb{M}, w \equiv_{\text{InqML}} \mathbb{M}', w'$, which implies that $\mathbb{M}{\downarrow}, w \sim_{n+1,k} \mathbb{M}'{\downarrow}, w'$. So for the state $t \in \Sigma{\downarrow}(w)$ there is some $t^* \in \Sigma'{\downarrow}(w')$ in $\mathbb{M}'{\downarrow}$ with $\mathbb{M}{\downarrow}, t \sim_{n,k} \mathbb{M}'{\downarrow}, t^*$. Let $\hat{t}' \in \Sigma'(w')$ be an information state in the pseudo-model $\mathbb{M}'$ that is represented in the second sort of $\mathscr{M}'$, such that $t^* \subseteq \hat{t}'$. Then for every $v \in t$ there is some $v' \in t^* \subseteq \hat{t}'$ with $\mathbb{M}{\downarrow}, v \sim_{n,k} \mathbb{M}'{\downarrow}, v'$, hence with $\mathbb{M}'{\downarrow}, v' \models \chi^{n,k}_v(y)$. So

[7] This partial type is an approximation of the bisimulation type of t, in two senses: it only stipulates positive assertions about bisimulation types of worlds v in t; and each such $\sim$-type of a world v can only be represented by its $\sim_{n,k}$-approximants.

$\mathscr{M}', w' \models \varphi(x)$, with $\hat{t}'$ as an existential witness for λ. Then $\Phi_{w'}(\lambda)$ is consistent with the FO-theory of $\mathscr{M}', w'$ (i.e. a partial type with parameter w' in $\mathscr{M}'$).

By ω-saturation of $\mathscr{M}'$ there is a realisation $\tilde{t}'$ of $\Phi_{w'}(\lambda)$. So (by Lemma 5.2 and using ω-saturation again) there is, for every world $v \in t$, some world $v' \in \tilde{t}'$ that realises its InqML-type ρ_v, i.e. such that $\mathbb{M}{\downarrow}, v \equiv_{\mathrm{InqML}} \mathbb{M}'{\downarrow}, v'$. Now

$$t' := \{v' \in \tilde{t}' : \exists v \in t \text{ with } \mathbb{M}{\downarrow}, v \equiv_{\mathrm{InqML}} \mathbb{M}'{\downarrow}, v'\} \subseteq \tilde{t}'$$

is an information state in $\Sigma'{\downarrow}(w')$, as $\tilde{t}' \in \Sigma'(w')$ in $\mathbb{M}'$ ($\tilde{t}' \in E[w']$ in $\mathscr{M}'$). It follows that $t' \in \Sigma'{\downarrow}(w')$ is an appropriate response to match $t \in \Sigma{\downarrow}(w)$. Indeed, for every world $v \in t$ there is a world $v' \in t'$ and vice versa, with $\mathbb{M}{\downarrow}, v \equiv_{\mathrm{InqML}} \mathbb{M}'{\downarrow}, v'$. So we have $\mathbb{M}{\downarrow}, t \equiv^{bulk}_{\mathrm{InqML}} \mathbb{M}'{\downarrow}, t'$, and t' is as desired. □

Lemma 5.5 does give rise to a nice class of models with a Hennessy–Milner property for InqML.

Corollary 5.6 (a Hennessy–Milner class for InqML)**.**
If $\mathbb{M} = \mathbb{M}(\mathscr{M})$ and $\mathbb{M}' = \mathbb{M}(\mathscr{M}')$ arise as pseudo-models from ω-saturated relational pseudo-models $\mathscr{M}$ and $\mathscr{M}'$, then their ${\downarrow}$-closures, which are proper models, satisfy the following Hennessy–Milner property for bulk-equivalent[8] states:

$$\mathbb{M}{\downarrow}, s \equiv^{bulk}_{\mathrm{InqML}} \mathbb{M}'{\downarrow}, s' \;\;\Leftrightarrow\;\; \mathbb{M}{\downarrow}, s \;\sim\; \mathbb{M}'{\downarrow}, s'.$$

Similarly, for worlds:

$$\mathbb{M}{\downarrow}, w \equiv_{\mathrm{InqML}} \mathbb{M}'{\downarrow}, w' \;\;\Leftrightarrow\;\; \mathbb{M}{\downarrow}, w \;\sim\; \mathbb{M}'{\downarrow}, w'.$$

Proof. The left-to-right implication is from Lemma 5.5; the converse follows from preservation of InqML under inquisitive bisimulation. □

As further corollaries we obtain characterisation results for InqML as a fragment of FO, in the style of a van Benthem theorem. The relevant notions of inquisitive bisimulation equivalence can be naturally transferred from models to their relational encodings as in [4].

Definition 5.7. *Inquisitive bisimulation equivalence for world-pointed relational models, $\mathscr{M}, w \sim \mathscr{M}', w'$, is defined by direct transfer from the underlying inquisitive models, as $\mathscr{M}, w \sim \mathscr{M}', w' \;:\Leftrightarrow\; \mathbb{M}(\mathscr{M}), w \sim \mathbb{M}(\mathscr{M}'), w'$; and analogously for the state-pointed case.*

The following corollary crucially differs from the characterisation obtained along very different routes in [5]. On the one hand, the underlying elementary class of pseudo-models, which our new characterisation refers to, is considerably wider than the non-elementary class of relational encodings of proper models. On the other hand, it therefore significantly involves preservation assumptions that bridge the gap between pseudo-models and proper models. Involving the natural extension of the semantics of InqML to this wider class of pseudo-models, it also enriches the analysis of the relation between InqML and FO in another direction.

We first state and sketch the proof for the simpler version of this characterisation for world properties; the more general characterisation of state properties,

[8] See (†) for the definition of $\equiv^{bulk}_{\mathrm{InqML}}$ as the flat lifting of $\equiv_{\mathrm{InqML}}$ and see also Remark 5.4.

with persistency as an additional semantic constraint, is then treated in Corollary 5.11.

Compare Definition 2.2 for the notion of *inquisitive closure*, which involves downward closure w.r.t. information states in inquisitive assignments, in the passage from Σ to $\Sigma\!\downarrow$ or the analogous passage in the relational encoding of Σ in terms of E.

Corollary 5.8. *The following are equivalent for $\varphi = \varphi(x) \in$ FO (in the vocabulary of two-sorted relational (pseudo-)models and in a single free variable x of the first sort):*

(i) – *φ and $\neg\varphi$ are preserved under passage to inquisitive closures over the elementary class of all world-pointed relational pseudo-models, and*
– *φ is preserved under inquisitive bisimulation $\sim$ over the (non-elementary) class of all relational encodings of world-pointed models;*

(ii) φ is logically equivalent over the class of all world-pointed relational pseudo-models to (the standard translation of) some formula $\psi \in$ InqML.

Proof. For (i) $\Rightarrow$ (ii) we argue for the contrapositive. If φ is not equivalent to any InqML-formula, then by compactness, there are ω-saturated relational pseudo-models $\mathscr{M}, w$ and $\mathscr{M}', w'$ such that $\mathscr{M}, w \equiv_{\mathrm{InqML}} \mathscr{M}', w'$ but $\mathscr{M}, w \models \varphi$ while $\mathscr{M}', w' \models \neg\varphi$. By the previous corollary $\mathscr{M}\!\downarrow, w \sim \mathscr{M}'\!\downarrow, w'$, and as $\mathscr{M}, w \models \varphi$ while $\mathscr{M}', w' \models \neg\varphi$, φ and $\neg\varphi$ cannot be preserved under $\sim$ and under passage to inquisitive closures as stipulated in (i). □

Remark 5.9. *The combined preservation condition in (i) above can equivalently be replaced by a single preservation condition w.r.t. the natural extension of inquisitive bisimulation to (relational encodings of) pseudo-models. For this one could just define, in extension of Definition 5.7 and, e.g. for world-pointed pseudo-models, $\mathscr{M}, w \sim \mathscr{M}', w' \;:\Leftrightarrow\; \mathbb{M}(\mathscr{M})\!\downarrow, w \sim \mathbb{M}(\mathscr{M}')\!\downarrow, w'$.*

For the following we adapt the notion of inquisitive closure (from Definition 2.2) to state-pointed relational pseudo-models so as to include all subsets of the distinguished information state in the second sort:[9] the inquisitive closure $(\mathscr{M}, s)\!\downarrow$ augments the second sort of the relational encoding of $\mathbb{M}(\mathscr{M})\!\downarrow$ by representations of all subsets of s in the second sort in as far as they are not already covered as subsets of information states in inquisitive assignments.

In the following, we invoke the set of all *boolean combinations of (standard translations) of* InqML*-formulae*, meaning the closure of all standard translations of InqML-formulae under the classical FO-connectives $\neg$, $\wedge$ and $\vee$.

Lemma 5.10. *Over the class of inquisitive closures of state-pointed relational pseudo-models with non-empty distinguished states: if a boolean combination of (standard translations of)* InqML*-formulae is persistent, then it is logically equivalent to (the standard translation of) a single* InqML*-formula.*

Proof. For first-order $\psi(\lambda)$ in a free variable λ of the second sort, let

$$\psi^{\downarrow}(\lambda) := \forall\mu\big(\mu \subseteq \lambda \rightarrow \psi(\mu)\big),$$

where $\mu \subseteq \lambda$ stands for the natural rendering in our 2-sorted first-order framework, which is literally adequate in inquisitive closures of state-pointed relational

[9] Representation of all subsets of the distinguished information state in the second sort is essential if we want to capture the idea of persistency in the relational setting.

pseudo-models. Then ψ is persistent if, and only if, $\psi \equiv \psi^{\downarrow}$ over this class. It is easy to check that $\downarrow$ commutes with conjunction in the sense that

$$(\psi_1 \wedge \psi_2)^{\downarrow}(\lambda) \equiv \psi_1^{\downarrow}(\lambda) \wedge \psi_2^{\downarrow}(\lambda).$$

Consider now a persistent formula $\psi(\lambda)$ that is a boolean combination of standard translations, w.l.o.g. in conjunctive normal form. As $\downarrow$ commutes with conjunction, it suffices to show that the $\downarrow$-version of each conjunct of the conjunctive normal form satisfies the claim of the lemma. And as the standard translation commutes with conjunction and disjunction (i.e. $\vee\!\!\!\vee$ in InqML vs. $\vee$ in FO), it suffices to prove expressibility of $\psi^{\downarrow}$ as a standard translation of an InqML-formula for a simple disjunction of the form

$$\psi(\lambda) = \neg\varphi_1^*(\lambda) \vee \varphi_2^*(\lambda)$$

for $\varphi_1, \varphi_2 \in$ InqML with standard translations $\varphi_i^*(\lambda)$. We here think of φ_1 as a (possibly empty) conjunction and of φ_2 as a (possibly empty) disjunction. The degenerate cases, in which ψ has trivial positive or negative contributions, correspond to (classical) $\bot$ or $\top$ in the places of φ_2^* or φ_1^*, respectively. And as classical $\bot$ agrees with the standard translation of inquisitive $\bot$ only for non-empty states, it is important that we aim for equivalence w.r.t. non-empty states only. But in all those scenarios we find that $\psi^{\downarrow}(\lambda)$ is equivalent to the standard translation of the inquisitive implication $\varphi_1 \rightarrow \varphi_2 \in$ InqML:

$$\psi^{\downarrow}(\lambda) \equiv (\varphi_1 \rightarrow \varphi_2)^*(\lambda)$$

over the class of inquisitive closures of state-pointed relational pseudo-models with non-empty distinguished states. □

Corollary 5.11. *The following are equivalent for* $\varphi = \varphi(\lambda) \in$ FO *(in the vocabulary of two-sorted relational (pseudo-)models and in a single free variable* λ *of the second sort):*

(i)
- *φ and $\neg\varphi$ are preserved under passage to the inquisitive closure of state-pointed relational pseudo-models,*
- *φ is preserved under inquisitive bisimulation $\sim$ over the (non-elementary) class of all relational encodings of state-pointed models,*
- *φ is persistent over the (non-elementary) class of inquisitive closures of state-pointed relational pseudo-models;*

(ii) φ is logically equivalent to (the standard translation of) some formula $\psi \in$ InqML *over the class of all relational pseudo-models with non-empty distinguished information state.*

Proof. As in the proof of Corollary 5.8, the implication (ii) $\Rightarrow$ (i) is obvious, and we argue indirectly for (i) $\Rightarrow$ (ii): we show that, if φ is *not* equivalent, w.r.t. non-empty information states, to any *boolean combination* of InqML-formulae then it cannot satisfy the preservation and invariance conditions in (i). This then implies (ii) with Lemma 5.10. The crucial claim for that is the following. If $\varphi(\lambda)$ is not equivalent to any boolean combination of standard translations of InqML-formulae, then there is a set $\Psi = \Psi(\lambda)$ of first-order formulae in the single free variable λ of the second sort, such that

(a) for all standard translations $\psi(\lambda)$ of InqML-formulae, either $\psi \in \Psi$ or $\neg\psi \in \Psi$,

(b) $\Psi \cup \{\varphi\}$ and $\Psi \cup \{\neg\varphi\}$ are both satisfiable by non-empty information states in relational pseudo-models.

Condition (a) says that Ψ is "complete" w.r.t. the set of all standard translations of InqML-formulae; (b) guarantees that there are ω-saturated relational pseudo-models $\mathscr{M}, s$ and $\mathscr{M}', s'$ (models of Ψ with states s, s' that are represented) such that $\mathscr{M}, s \equiv_{\mathrm{InqML}} \mathscr{M}', s'$, whence also $(\mathscr{M}, s)\downarrow, s \equiv_{\mathrm{InqML}} (\mathscr{M}', s')\downarrow, s'$, but $\mathscr{M}, s \models \varphi$ while $\mathscr{M}', s' \models \neg\varphi$. This contradicts the preservation and invariance as stated in (i): preservation of φ and $\neg\varphi$ under passage to closures implies $(\mathscr{M}, s)\downarrow, s \models \varphi$ and $(\mathscr{M}', s')\downarrow, s' \models \neg\varphi$. Moreover, $\mathscr{M}, s \equiv_{\mathrm{InqML}} \mathscr{M}', s'$ implies $(\mathscr{M}, s)\downarrow, s \sim (\mathscr{M}', s')\downarrow, s'$ with Remark 5.4 and Corollary 5.6; but this contradicts preservation of φ under inquisitive bisimulation.

The set Ψ is obtained by a compactness argument, which is more involved than the analogue used in the world-pointed case of Corollary 5.8. The present argument crucially involves the additional persistency assumption in (i). Assuming w.l.o.g. a finite signature we may enumerate all standard translations of InqML-formulae as $(\psi_i(\lambda))_{i\in\mathbb{N}}$, and obtain the desired Ψ as the union of a chain of finite sets Ψ_i as follows. Starting from $\Psi_0 := \emptyset$ we inductively augment Ψ_i by either ψ_i or $\neg\psi_i$ to obtain Ψ_{i+1}. In the inductive process we maintain, as an invariant, the condition that for all boolean combinations of standard translations $\psi(\lambda)$ of InqML-formulae, and over the elementary class of all state-pointed relational pseudo-models with a non-empty distinguished information state,[10]

$$(*) \quad \Psi_i(\lambda) \not\models \varphi(\lambda) \leftrightarrow \psi(\lambda),$$

i.e. that φ remains inequivalent to all boolean combinations of standard translations over the class of all state-pointed relational pseudo-models of Ψ_i with non-empty distinguished state. Then $\Psi = \bigcup_i \Psi_i$ meets requirements (a) and (b); indeed, with compactness, (b) is covered by $(*)$ for (the standard translations of) $\bot$ or $\top$ in the rôle of ψ. If $\Psi \cup \{\varphi\}$ were not satisfiable in a non-empty information state, then $\Psi(\lambda) \models \varphi(\lambda) \leftrightarrow \bot$ over the class of relational pseudo-models with non-empty information states (and over this class, classical $\bot$ is equivalent to the standard translation of inquisitive $\bot$); similarly for $\Psi \cup \{\neg\varphi\}$ and equivalence with $\top$.

The initial set $\Psi_0 = \emptyset$ satisfies condition $(*)$ by assumption. It remains to argue that $(*)$ can be maintained in at least one of the augmentations $\Psi_i \cup \{\psi_i\}$ or $\Psi_i \cup \{\neg\psi_i\}$. Assume this were not the case, i.e. that, for some boolean combinations ξ_0, ξ_1 of standard translations of InqML-formulae,

$$\Psi_i \cup \{\psi_i\} \models \varphi \leftrightarrow \xi_0 \quad \text{and} \quad \Psi_i \cup \{\neg\psi_i\} \models \varphi \leftrightarrow \xi_1.$$

But then

$$\Psi_i \models \varphi \leftrightarrow \big((\psi_i \wedge \xi_0) \vee (\neg\psi_i \wedge \xi_1)\big)$$

contradicts the assumption that $(*)$ was satisfied in the previous step. □

6 Conclusion

Our new standard translation combines several advantages: it is defined directly and inductively, and it works with minimal requirements on the corresponding

[10] In the remainder of this argument we always assume to be working over this elementary class: even where this is not stated explicitly, all first-order consequence relations are to be read in this context.

class of relational models. The basic idea of the standard translation and its use in a model-theoretic compactness proof requires a non-trivial adaptation of the well-known treatment of basic modal logic ML, due to the inherent two-sortedness of the relational representations of the intended models for inquisitive modal logic InqML; the notion of graded flatness plays a key rôle in taming the salient second-order features. Other than for ML, the relational counterparts of the intended models for InqML do not form an elementary class. Nevertheless, with our standard translation we could give a purely model-theoretic compactness proof for InqML over the class of all relational inquisitive models, as well as over the class of all relational pseudo-models. The usefulness of model-theoretic constructions based on pseudo-models has further been exemplified with a novel Hennessy–Milner theorem for InqML, which crucially involves the passage through pseudo-models and again relies on our standard translation. As corollaries we obtained a world-pointed and a state-pointed variant of a van Benthem style theorem that characterises InqML in relation to FO over the elementary class of all pseudo-models. Overall our findings seem to recommend the class of pseudo-models as a useful model-theoretic backdrop for the study of inquisitive modal logic, if not even as a suitable alternative for the class of proper models for some specific aspects. Further investigations in this vein could focus on model-theoretic approaches, via pseudo-models, to special classes of inquisitive frames or, for instance, a Lindström characterisation of inquisitive modal logic.

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