We use the methods developed in [4], [6], [7] to solve the isomorphism problem of unitary forms of infinite split Kac–Moody groups over finite fields of square order.

1 Introduction

The isomorphism problem for Kac–Moody groups has been studied and solved in [4], [6], [7]. In addition, [4] contains a description of the isomorphisms of unitary forms of complex Kac–Moody groups with respect to the compact involution; see also [15].

An adaption of the methods developed in [4], [6], [7] allows us to prove the following theorem. (We refer to Section 2 for definitions.)

Main Result. Let \( q \) and \( r \) be arbitrary prime powers, let \( G \) and \( G' \) be infinite split Kac–Moody groups over \( \mathbb{F}_q \) and \( \mathbb{F}_r \), respectively, and let \( K \) and \( K' \) be their respective unitary forms.

If there exists an isomorphism \( \varphi : K \to K' \), then the following hold.

(i) \( q = r \).

(ii) There exist a bijection \( \pi : S \to S' \), an inner automorphism \( \nu \) of \( K' \) and for each \( i \in S \) a diagonal-by-field automorphism \( \gamma_i \) of \( SU_2(\mathbb{F}_q) \) such that the diagram

\[
\begin{array}{ccc}
SU_2(\mathbb{F}_q) & \xrightarrow{\gamma_i} & SU_2(\mathbb{F}_r) \\
\varphi_i \downarrow & & \varphi'_i \downarrow \\
K & \xrightarrow{\nu \circ \varphi} & K'
\end{array}
\]

commutes for every \( i \in S \).

As a byproduct of our strategy, one may conclude that the Weyl groups of \( K \) and \( K' \) and the underlying Kac–Moody root data (cf. [7, Section 2.3.2]) are isomorphic. Moreover, we have the following.

Corollary (Strong rigidity). For any isomorphism \( \varphi : K \to K' \) there exists a unique isomorphism \( \psi : G \to G' \) satisfying \( \psi|_K \equiv \varphi \).

Throughout the paper we assume that the reader is familiar with Kac–Moody groups and their buildings as described, for instance, in [1], [4, Chapter 1]. Section 2 may serve as a reminder

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about these concepts. In Section 3 we collect information about the structure of maximal finite subgroups, and in Section 4 we prove the Main Result.

This note can be considered as being part of an ongoing project of understanding unitary forms of Kac–Moody groups over finite fields. We are particularly interested in these unitary forms as over sufficiently large finite fields they are lattices in the completions of the ambient Kac–Moody groups with respect to the topology of compact convergence (cf. [12]).

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2 Basics

Let $G$ be a split Kac–Moody group over a finite field (cf. [1, Section 8.11], [4, Chapter 3], [18, Chapter 8], [21]) and let $(G, B_+, B_-, N, S)$ be the associated saturated twin Tits system with Weyl group $W$. For $\varepsilon \in \{+, -\}$ we have the Bruhat, resp. Birkhoff decompositions

$$G = \bigsqcup_{w \in W} B_\varepsilon w B_\varepsilon$$

see [1, Section 6.2 and Proposition 6.81], [4, Chapter 1], [18, Chapter 1].

Conjugates of the fundamental Borel subgroups $B_+$ and $B_-$ are called Borel subgroups of $G$. The intersection $T := B_+ \cap B_-$ is called the fundamental maximal split torus of $G$; each of its conjugates is called a maximal split torus. A fundamental parabolic subgroup $P_\varepsilon$ of $G$ is a subgroup containing a fundamental Borel group $B_\varepsilon$. Any conjugate of a fundamental parabolic subgroup is simply called parabolic subgroup. Given a fundamental parabolic subgroup $P_\varepsilon$ of $G$, there exists $J \subseteq S$ such that

$$P_\varepsilon = \bigsqcup_{w \in W_J} B_\varepsilon w B_\varepsilon$$

for the special subgroup $W_J := \langle J \rangle$ of $W$. The set $J$ is called the type of $P$. Moreover, $P$ and $J$ are called spherical, if $W_J$ is finite.

A split Kac–Moody group $G$ can be defined functorially for any field $F$, cf. [18, Chapter 8], [21]. In particular, for each $s \in S$ there exists a homomorphism $\varphi_s : \text{SL}_2(F) \to G$ with central kernel such that

$$G = \langle \varphi_s(\text{SL}_2(F)) \mid s \in S \rangle.$$ 

Definition 2.1 (Unitary form). Let $G$ be a split Kac–Moody group over the field $F_q$, let $\omega$ be the Chevalley involution of $G$, cf. [4, Chapter 8], [15, Section 2], and let $\theta$ be the composition of $\omega$ and the field involution $x \mapsto x^q$ of $F_q$, called twisted Chevalley involution. The fixed point group $K := \{g \in G \mid \theta(g) = g\}$ is called the unitary form of $G$ with respect to $\theta$.

This unitary form is the finite field analogue of the real group studied in [15, Section 5], where the involved field involution is complex conjugation.

For each $s \in S$, the intersection $K_s := \varphi_s(\text{SL}_2(F_q)) \cap K$ is isomorphic to $\text{SU}_2(F_q)$ and is called a rank one subgroup of $K$. The involution $\theta$ induces an involution of $\varphi_s(\text{SL}_2(F_q))$ which pulls back to the product of the contragredient automorphism of $\text{SL}_2(F_q)$ and the field involution, whose set of fixed elements forms a subgroup isomorphic to $\text{SU}_2(F_q)$.
Remark 2.2. For each $s \in S$, we have $|T \cap \varphi_s(\text{SL}_2(\mathbb{F}_{q^2}))| = q^2$ and $|T_K \cap \varphi_s(\text{SL}_2(\mathbb{F}_{q^2}))| = (q + 1)$. Therefore it is possible for all $q$ to apply [4, Lemma 4.8] to both $T$ and $T_K$ and their conjugates.

Definition 2.3 (Twin building). Let $G$ be a split Kac–Moody group and let $(G, B_+, B_-, N, S)$ be the associated saturated twin Tits system with Weyl group $W$. Let $\Delta_\varepsilon := G/B_\varepsilon$, let

$$
\delta_\varepsilon : \Delta_\varepsilon \times \Delta_\varepsilon \rightarrow W
$$

$$
\delta_\varepsilon(gB_\varepsilon, hB_\varepsilon) := w \text{ if and only if } B_\varepsilon g^{-1}hB_\varepsilon = B_\varepsilon wB_\varepsilon,
$$

and let

$$
\delta_\varepsilon : (\Delta_+ \times \Delta_-) \cup (\Delta_+ \times \Delta_+) \rightarrow W
$$

$$
\delta_\varepsilon(gB_\varepsilon, hB_{-\varepsilon}) := w \text{ if and only if } B_\varepsilon g^{-1}hB_{-\varepsilon} = B_\varepsilon wB_{-\varepsilon}.
$$

The triple $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta_\varepsilon)$ is the twin building associated to $G$, cf. [1, Chapter 8].

Note that the maps $\delta_\varepsilon$ and $\delta_\varepsilon$ are well-defined, because of the Bruhat and Birkhoff decompositions.

Each pair $(\Delta_\varepsilon, \delta_\varepsilon)$ is a building. The Davis realisation of a building, as described in [8], is a CAT(0) space. In particular, the Bruhat-Tits fixed point theorem can be applied to the Davis realisation. For details on CAT(0) spaces, we refer the reader to the book [2].

Definition 2.4 (Flipflop system). Let $G$ be a split Kac–Moody group over $\mathbb{F}_{q^2}$ and let $\theta$ be the twisted Chevalley involution of $G$. The involution $\theta$ induces an involutory automorphism of the twin building associated to $G$, which we also denote by $\theta$. A Phan chamber is a chamber $c \in \Delta$ satisfying $\delta_\varepsilon(c, \theta(c)) = 1_W$. The collection of all Phan chambers contained in $(\Delta_+, \delta_+)$ is called the flipflop system and is denoted by $\Delta_\theta$.

3 On maximal finite subgroups

Our strategy is to show that any isomorphism between two unitary forms induces a bijection between the respective sets of rank one subgroups. This implies that an isomorphism is uniquely determined by its behaviour on the rank one subgroups.

We first investigate the structure of maximal finite subgroups.

Proposition 3.1. Let $q$ be a prime power. Let $G$ be a split Kac–Moody group over $\mathbb{F}_{q^2}$ and let $K = \text{Fix}_G(\theta)$ be the unitary form of $G$ with respect to the twisted Chevalley involution $\theta$.

Let $P_+, P_- \leq G$ be spherical parabolic subgroups of opposite sign of $G$ and set

$$P := P_+ \cap P_- = L(P) \ltimes U(P),$$

where $U(P)$ is the unipotent radical of $P_+ \cap P_-$. If $U(P)$ is non-trivial and $\theta$ normalises $P_+ \cap P_-$, then $K \cap U(P)$ is non-trivial.

Remark 3.2. (i) Let $p = \text{char}(\mathbb{F}_{q^2})$. The above decomposition is the Levi decomposition

$$P = L(P) \ltimes U(P),$$

given in [7, Proposition 3.6], i.e. $L(P)$ is a semisimple Levi subgroup of $P$ and $U := U(P)$ is a $p$-group.

(ii) Note that since $P_+$ and $P_-$ are spherical parabolic subgroups of opposite sign, $P$ is a finite group (cf. [7, Corollary 3.8]).
Proof of Proposition 3.1. Let $R_{P_+}$ and $R_{P_-}$ be the maximal spherical residues of $\Delta_+$, respectively $\Delta_-$. (see Definition 2.3), that are stabilised by $P_+$ and $P_-$ respectively. We may assume that $R_{P_+}$ and $R_{P_-}$ are not opposite, else $U(P)$ is trivial and there is nothing to show. Let $c \in \Sigma \cap R_{P_+}$ such that $w := \delta_*(c, \theta(c))$ has minimal length. Since $w^{-1} = \delta_* (\theta(c), c) = \delta_* (\theta(\theta(c)), \theta(c)) = \delta_* (c, \theta(c)) = w$, the Weyl group element $w$ is an involution. Therefore, by classical Coxeter group theory (see [19]) or by CAT(0) theory (see [17], [9, Theorem 12.3.3]), there exist a non-trivial spherical subset $I$ of $S$ and $s_1, \ldots, s_h \in S$ such that $w = s_1 \cdots s_h w_I s_h \cdots s_1$ with $l(w) = l(w_I) + 2h$, where $w_I$ denotes the longest word of the spherical Coxeter system $(I, I)$. Hence there exists a chamber $d \in \Delta_+$ with $\delta_* (d, \theta(d)) = w_I$ with the property that each twin apartment containing $d$ and $\theta(d)$ also contains $c$ and $\theta(c)$.

Let $R_I(d)$ be the $I$-residue of $\Delta_+$ of $d$. Since $\delta_* (d, \theta(d)) = w_I$, the residue $\theta(R_I(d))$ contains a chamber opposite $d$, so that the restriction of $\delta_*$ to $(R_I(d) \times \theta(R_I(d))) \cup (\theta(R_I(d)) \times R_I(d))$ turns the pair $(R_I(d), \theta(R_I(d)))$ into a spherical twin building on which $\theta$ acts. The product $\theta \circ \text{proj}_{R_I(d)}$ is an involutory automorphism of the spherical building $R_I(d)$, which has $d$ as a fixed point.

We claim that there exist distinct $\theta \circ \text{proj}_{R_I(d)}$-fixed chambers $c$ and $e'$ opposite $d$ in $R_I(d)$. For, if $x \in R_I(d)$ is $(\theta \circ \text{proj}_{R_I(d)})$-fixed, then, for any $s \in I$, the automorphism $(\theta \circ \text{proj}_{R_I(d)}(x)$ acts on the panel $P_s(x)$. Since $\theta$ involves the field automorphism of $\mathbb{F}_{q^2}$, there are exactly $q + 1 \geq 3$ $(\theta \circ \text{proj}_{R_I(d)})$-fixed chambers in $P_s(x)$. Hence the claim follows by induction on the maximal distance in $R_I(d)$ of $(\theta \circ \text{proj}_{R_I(d)})$-fixed chambers from $d$.

There exists a unique twin apartment $\Sigma$ containing $d$, $\theta(d)$, $c$, $\theta(c)$ and there exists a unique twin apartment $\Sigma' \neq \Sigma$ containing $d$, $\theta(d)$, $e'$, $\theta(e')$. By construction both $\Sigma$ and $\Sigma'$ are $\theta$-stable. Moreover, since $c, \theta(c) \in \Sigma \cap \Sigma'$, the intersections $\Sigma \cap R_{P_+}$ and $\Sigma' \cap R_{P_-}$ (resp. $\Sigma \cap R_{P_-}$ and $\Sigma' \cap R_{P_+}$) coincide and are equal to an apartment of $R_{P_+}$ (resp. $R_{P_-}$). As $\Sigma$ and $\Sigma'$ are conjugate by an element $g \in G$ that lies both in $K$ (see [14, Chapter 2] and also Remark 3.3 below) and in $U(P)$ (see [7, Proposition 3.1]), the intersection $\Sigma \cap U(P)$ is necessarily non-trivial.

Remark 3.3. The proof of Proposition 3.1 is strongly inspired by [14]. For the sake of completeness we here also give the argument from [14, Chapter 2] that $\Sigma$ and $\Sigma'$ are conjugate by an element in $K$. The latter argument in turn has been inspired by [13, Lemma 2.4].

Let $x \in \Sigma \cap \Sigma'$, which implies $\theta(x) \in \Sigma \cap \Sigma'$. The unipotent radical $U$ of the Borel subgroup $B$ that stabilizes $x$ acts sharply transitively on the twin apartments containing $x$. Hence there exists a unique $u \in U$ such that $\Sigma' = u \Sigma$. As multiplication with $u$ is a building automorphism fixing $x$, it stabilises the set $\Sigma \cap \Sigma'$ chamberwise. In particular, $u$ fixes $\theta(x)$, whence $\theta(u)$ fixes $x$. Therefore $\theta(u) \in U$. As $\Sigma, \Sigma'$ are $\theta$-stable, $u \Sigma = \Sigma' = \theta(\Sigma') = \theta(u \Sigma) = \theta(u) \Sigma$. Since $u, \theta(u) \in U$ and since $u$ was unique, we conclude that $u = \theta(u) \in K$.

The following result is well known. Since it is not readily available in the literature, we also include a proof.

Lemma 3.4. Let $K$ be the unitary form of a split Kac–Moody group $G$ with respect to $\theta$. Then $K$ acts transitively on the flipflop system $\Delta_\theta$ of $G$.

Proof. By [4, Lemma 4.8] (cf. Remark 2.2 of the present article) there exists a one-to-one correspondence between $G$-conjugates of $T$ and twin apartments of the twin building of $G$. Let $(\Sigma_+, \Sigma_-)$ be the twin apartment in the twin building of $G$ corresponding to the fundamental torus $T$ and let $c \in \Sigma_+$. By definition of $\theta$ this implies $\delta_* (c, \theta(c)) = 1_W$. Let $c'$ be an arbitrary chamber with $\delta_* (c', \theta(c')) = 1_W$ and let $(\Sigma'_+, \Sigma'_-)$ be the unique twin apartment containing both $c'$ and $\theta(c')$. By strong transitivity there is a $g \in G$ such that $g \Sigma_+ = \Sigma'_+$ and $g \Sigma_- = \Sigma'_-$ and there is an $h \in G$ normalizing $(\Sigma'_+, \Sigma'_-)$ such that $h gc = c'$; hence $h \theta(c) = \theta(c')$. Now $\theta(hg) - 1 hl = \theta((hg) - 1 hl) = \theta((hg) - 1 hl c') = c$ and, similarly, $\theta((hg) - 1 hl \theta(c) = \theta(c)$, so $t = \theta(hg) - 1 hl \in T$. Moreover $\theta(t) = (hg) - 1 hl \theta(hg) = t^{-1}$.

Let $\mathbb{F}_{q^2}$ denote the algebraic closure of $\mathbb{F}_{q^2}$ and note that $T(\mathbb{F}_{q^2})$ is a connected linear algebraic group. Let $\sigma$ be the endomorphism of raising elements of $\mathbb{F}_{q^2}$ to the $q$th power; note that the fixed point set of $\mathbb{F}_{q^2}$ under $\sigma^2$ is exactly $\mathbb{F}_r$. The map $\theta : T(\mathbb{F}_{q^2}) \to T(\mathbb{F}_{q^2}) : g \mapsto \sigma^2 (g^{-1})$ is an endomorphism of algebraic groups, which satisfies $\theta^2 (g) = \sigma^2 (g)$, so that $s \in T$ if and only
if \( s = \theta^2(s) \) for \( s \in T(\mathbb{F}_{q^2}) \). Consequently \( T(\mathbb{F}_{q^2})_\theta \subseteq T(\mathbb{F}_{q^2})_{q^2} = T \) is finite and so by Lang’s Theorem [16, Corollary to Theorem 1] there is an \( s \in T(\mathbb{F}_{q^2}) \) such that \( \theta(s)s^{-1} = t \). Now \( \theta^2(s) = \theta(t)s = t^{-1}\theta(s) = s \), so \( s \in T \). Hence the element \( hgs \) lies in \( K \) and it maps \( c \) to \( c' \). Therefore \( K \) acts transitively on the flipflop system \( \Delta_\theta \).

Recall that the \( p \)-core \( O_p(X) \) of a finite group \( X \) is the largest normal \( p \)-subgroup of \( X \).

**Proposition 3.5.** Let \( q \) be a power of the prime \( p \), let \( G \) be an infinite split Kac–Moody group over \( \mathbb{F}_{q^2} \), let \( K \) be its unitary form, and let \( P_+, P_- \leq G \) be opposite maximal spherical parabolic subgroups with the property that \( \theta(P_+) = P_- \).

Then \( P_+ \cap P_- \cap K = \text{Fix}_{P_+\cap P_-}(\theta) \) is a maximal finite subgroup of \( K \) with trivial \( p \)-core. Conversely, any maximal finite subgroup with trivial \( p \)-core is obtained in this fashion.

**Proof.** Let \( R_{P_+} \) and \( R_{P_-} \) be the respective residues of \( \Delta_+ \) and \( \Delta_- \) associated to \( P_+ \) and \( P_− \). As \( P_+ \) and \( P_- \) are opposite, so are \( R_{P_+} \) and \( R_{P_-} \), and the intersection \( P := P_+ \cap P_- \) is finite and semisimple. Hence

\[
(R_{P_+}, R_{P_-}, \delta_{1|R_{P_+} \times R_{P_-}})
\]

is a spherical twin building. By [22], this twin building can be canonically identified with the spherical building \( \Delta(P) \) of the finite semisimple group \( P = P_+ \cap P_- \). The product \( \theta \circ \text{proj}_{R_{P_+}} \) yields an involution on \( \Delta(P) \), which we also denote by \( \theta \). Recall that an element of \( P \) is semisimple if and only if it is \( p \)-regular, and that it is unipotent if and only if it is \( p \)-singular. Hence \( P \), and therefore \( F := P \cap K = \text{Fix}_P(\theta) \), have trivial \( p \)-cores.

Again we denote by \( \Delta_\theta \) the flipflop system of \( \Delta(P) \). Let \( c \in \Delta_\theta \). Then \( \delta(c, \theta(c)) = w_0 \in W_P \), where \( w_0 \) is the longest word in the Coxeter group \( W_P \) associated to \( P \). Hence \( \Sigma := \text{conv}(c, \theta(c)) \) is a \( \theta \)-stable apartment containing two opposite Phan chambers, which implies that \( \Sigma \) consists of Phan chambers only. By Lemma 3.4 the group \( F \) acts transitively on the set of Phan chambers. Hence the orbit of \( c \) under the action of \( F \) on \( \Delta(P) \) meets every double coset \( BwB \), where \( w \in W_P \) and \( B := P_+ \). Hence any parabolic subgroup of \( P \) containing \( F \) necessarily contains all double cosets of \( P \) modulo \( B \). We conclude that \( F \) cannot stabilise a proper residue of the building \( \Delta(P) \).

The Davis realisation (see [8]) of each half \( \Delta_\pm \) of the twin building \( \Delta(G) \) admits one obvious fixed point of \( F \), namely \( R_{P_+} \). We claim that these are unique. To show this, suppose there is some other spherical residue \( R_{Q_\pm} \) in \( \Delta_\pm(G) \) which is stabilised by \( F \). By maximality of \( R_{P_+} \), the residue \( R_{Q_\pm} \) cannot contain \( R_{P_+} \) properly. Moreover, the residues \( R_{P_+} \) and \( R_{Q_\pm} \) must be disjoint, else their intersection would yield a proper residue of \( R_{P_+} \) stabilised by \( F \), a contradiction to what we established above. Now consider the projection of \( R_{Q_\pm} \) onto \( R_{P_+} \). If \( F \) stabilises \( R_{Q_\pm} \), then it also stabilises the image of the projection. Hence the projection must be surjective. In view of [7, Corollary 2.8], the residues \( R_{P_+} \) and \( R_{Q_\pm} \) are therefore opposite, which is absurd as \( G \) is infinite, whence \( \Delta_\pm(G) \) non-spherical. Hence \( R_{P_+} \) is the unique fixed point of \( F \) in the Davis realisation of \( \Delta(G) \).

However, the stabiliser of \( R_{P_+} \) in \( G \) is \( P_+ \), whence the stabiliser of the residues \( R_{P_\pm} \) in \( K \) is equal to \( P_+ \cap P_- \cap K = P \cap K = F \). Since by the Bruhat–Tits fixed point theorem every finite subgroup fixes a point in the Davis realisation (as it is \( \text{CAT}(0) \)), we conclude that \( F \) must be maximal among finite subgroups of \( K \) with trivial \( p \)-core.

Conversely, let \( F \leq K \) be maximal finite with trivial \( p \)-core. Since \( F \) is finite, it has a bounded orbit on \( \Delta_\pm \), whence the Bruhat–Tits fixed point theorem again implies that there exists some positive spherical parabolic subgroup \( P_+ \leq G \) with \( F \leq P_+ \). Without loss of generality, we may choose \( P_+ \) with this property and of minimal rank. Since \( F \) is pointwise fixed by \( \theta \), it follows from the construction that \( F \) is also contained in \( \theta(P_+) = P_- \). Now \( P_+ \) and \( P_- \) are spherical, thus \( P := P_+ \cap P_- \) is a finite group. In particular, we see that \( F \leq P \cap K \) and \( P \cap K \) is finite. By maximality of \( F \), we get that \( F = P \cap K \).

It remains to show that the groups \( P_+ \) and \( P_- \) constructed above are opposite and maximal spherical. The group \( F \) has trivial \( p \)-core by assumption, hence we may apply Proposition 3.1 and see that \( P \) has trivial unipotent radical. Comparing this with [7, Proposition 3.1 and Proposition 3.6] and the fact that \( P_+ \) and \( P_- \) are of minimal rank, we see that \( P_+ \) and \( P_- \) must be opposite,
Let $q$ be a maximal spherical Levi subgroup of $K$. By Proposition 3.5, the tori $T$ of the present article). By the above discussion and the fact that the fundamental tori stabilises a unique twin apartment $\Sigma$ of the twin building of $K$. In particular, contained in $P$ and $F$ embedded in $K$ for some opposite maximal spherical parabolic subgroups of $L$ and $L'$, and their respective unitary forms by $K$ and $L$. We conclude this section by recording a first structural property of isomorphisms of unitary forms.

**Corollary 3.6.** Let $\varphi : K \to K'$ be an isomorphism of unitary forms. Then $\varphi$ maps maximal spherical Levi subgroups of $K$ to maximal spherical Levi subgroups of $K'$.

**Proof.** By Proposition 3.5, a maximal spherical Levi subgroup $L$ of $K$ is associated to some reductive group $P$ such that $L = P \cap K$, where $P$ is the intersection of two opposite maximal spherical parabolic subgroups. As the image of $L$ under $\varphi$ is maximal finite in $K'$, it holds (again by Proposition 3.5) that $\varphi(L) = L' \cap K' = P' \cap K'$, where $P'$ is the intersection of some opposite maximal spherical parabolic subgroups of $G'$ and $L'$ is the Levi subgroup of their intersection. Hence the image of a maximal spherical Levi subgroup is again a maximal spherical Levi subgroup. □

## 4 Isomorphisms

We continue to denote by $G$ and $G'$ split Kac–Moody groups over the fields $\mathbb{F}_q$ and $\mathbb{F}_{r^2}$, respectively, and their respective unitary forms by $K$ and $K'$.

Firstly, we record that we may recognise the characteristic of the ground field.

**Proposition 4.1.** Let $K$ be an infinite unitary form and let $p$ be a prime. Then the set of orders of finite $p$-subgroups of $K$ is unbounded if and only if $p = \text{char}(\mathbb{F}_q)$.

**Proof.** If $p \neq \text{char}(\mathbb{F}_q)$, then the set of orders of finite $p$-subgroups of $K$ is bounded by [7, Proposition 6.2]. For $p = \text{char}(\mathbb{F}_q)$ it follows from the proof of [12, Theorem 1] that the sets of orders of the stabilisers $\text{Stab}_K(c)$ and of their $p$-Sylow subgroups, respectively, are unbounded. □

The following result is an adaption of [7, Theorem 5.1] to unitary forms.

**Proposition 4.2.** Let $K$ and $K'$ be unitary forms of infinite split Kac–Moody groups over fields $\mathbb{F}_q$ and $\mathbb{F}_{r^2}$. Let $\varphi : K \to K'$ be an isomorphism. Then there exists $g \in K'$ such that

1. $q = r$,
2. the tori $\varphi(T_K)$ and $T_{K'}$, are conjugate under $g$, and
3. $\{g\varphi(K_\alpha)g^{-1} | \alpha \in \Phi\} = \{K'_\alpha | \alpha \in \Phi\}$.

**Proof.** Let $p := \text{char}(\mathbb{F}_q) = \text{char}(\mathbb{F}_{r^2})$ (Proposition 4.1) and let $F$ be a maximal finite subgroup of $K$ with trivial $p$-core. Proposition 3.5 implies that $F = P_+ \cap P_- \cap K$ for some opposite maximal spherical parabolic subgroups $P_+$ and $P_-$ of $G$ satisfying $\theta(P_+) = P_-$. For the same reason we can write $\varphi(F)$ as $P'_+ \cap P'_- \cap K'$. Define $P := P_+ \cap P_-$ and $P' := P'_+ \cap P'_-$. By the Levi decomposition (cf. [7, Proposition 3.6]; see also Remark 3.2 of this article) the groups $P$ and $P'$ are semisimple finite groups of Lie type. Since $F$ and $\varphi(F)$ are isomorphic as abstract groups and since, moreover, $F$ and $\varphi(F)$ are twisted finite groups of Lie type in identical characteristics (again Proposition 4.1) embedded in $P$ and $P'$, respectively (Proposition 3.5), we conclude from [11, Table 2.2 in Section 2.2], [3, Table 6 in Chapter 4] that $q = r$ and that the buildings and the diagrams of $P$ and $P'$ coincide. Hence $\varphi$ induces an isomorphism $\psi$ between $P$ and $P'$. Therefore, by [7, Theorem 7.1], the map $\psi$ induces an isomorphism of the twin root datum of $P$ onto the twin root datum of $P'$. In particular, $\psi$ maps rank one subgroups of $P$ to rank one subgroups of $P'$.

Let $H_K := H \cap K$ be a $K$-conjugate of the fundamental maximal split torus $T_K$ of $K$ that is contained in $P$, which is possible by Lemma 3.4, because $P_+$ and $P_-$ are opposite. The torus $H_K$ stabilises a unique twin apartment $\Sigma$ of the twin building of $G$ by [4, Lemma 4.8] (cf. Remark 2.2 of the present article). By the above discussion and the fact that the fundamental tori $T_K$ and
$T'_K$, have the same cardinality, there exists a unique twin apartment $\Sigma'_0$ of the twin building of $G'$ such that $\varphi(H_K) = \text{Fix}_K(\Sigma'_0) \leq P'$. By Lemma 3.4 the twin apartment $\Sigma'_0$ is in the $K'$-orbit of the fundamental twin apartment $\Sigma'$, so that $\varphi(H_K)$ is a $K'$-conjugate of the fundamental torus $T'_K$, of $K'$. Hence there exists $g \in K'$ such that $(c_g \circ \varphi)(T_K) = T'_K$.

By the same arguments together with the facts that $T_K$ and $T'_K$ fix a unique apartment in the respective buildings and each rank one subgroup is contained in a maximal spherical parabolic subgroup of $K$ or $K'$, respectively, we conclude that

$$\{g \varphi(K_\alpha) g^{-1} \mid \alpha \in \Phi\} = \{K'_\alpha \mid \alpha \in \Phi'\}.$$

We now have everything at hand to prove the main result.

**Proof of the Main Result.** Assertion (i) coincides with assertion (i) of Proposition 4.2.

The inner automorphism and the bijection of the index sets in (ii) are provided by assertions (ii) and (iii) of Proposition 4.2. It thus remains to analyse the isomorphism $\varphi$ restricted to the rank one subgroups $K_\alpha \cong \text{SU}_2(\mathbb{F}_q) \cong \text{SL}_2(\mathbb{F}_q)$. By [20] the outer automorphisms of $K_\alpha$ are diagonal-by-field and the result follows.

**Proof of the Strong Rigidity.** Let $g \in G$ be an element normalising $K$. Since $K = \text{Fix}_G(\theta)$, it follows that $g$ centralises $\theta$ and hence $g \in Z(G)K$. Thus, $N_G(K) = Z(G)K$. Since $Z(K) = Z(G) \cap K$, each inner automorphism of $K$ therefore extends to a unique inner automorphism of $G$. As the outer automorphisms of $G$ and $K$ coincide by the Main Result and by [4], it follows that any isomorphism of unitary forms can be lifted uniquely to the ambient Kac–Moody groups.

**Remark 4.3.** For sufficiently large $q$ one can deduce from [5, Corollary 2.7(ii)] and [12] that, in fact, the commensurator $\{g \in G \mid gKg^{-1} \cap K \text{ has finite index in both } K \text{ and } gKg^{-1} \}$ of $K$ in $G$ is equal to $Z(G)K$.

**References**


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