

Gödel functional interpretation and weak compactness

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Abstract

In recent years, proof theoretic transformations (so-called proof interpretations) that are based on extensions of monotone forms of Gödel's famous functional ('Dialectica') interpretation have been used systematically to extract new content from proofs in abstract nonlinear analysis. This content consists both in effective quantitative bounds as well as in qualitative uniformity results. One of the main ineffective tools in abstract functional analysis is the use of sequential forms of weak compactness. As we recently verified, the sequential form of weak compactness for bounded closed and convex subsets of an abstract (not necessarily separable) Hilbert space can be carried out in suitable formal systems that are covered by existing metatheorems developed in the course of the proof mining program. In particular, it follows that the monotone functional interpretation of this weak compactness principle can be realized by a functional Ω^* definable from bar recursion (in the sense of Spector) of lowest type. While a case study on the analysis of strong convergence results (due to Browder and Wittmann resp.) that are based on weak compactness indicates that the use of the latter seems to be eliminable, things apparently are different for weak convergence theorems such as the famous Baillon nonlinear ergodic theorem. For this theorem we recently extracted an explicit bound on a metastable (in the sense of T. Tao) version of this theorem that is primitive recursive relative to (a somewhat restricted form of) Ω^* .

In the current paper we for the first time give the construction of Ω^* (in the form needed for the unwinding of Baillon's theorem). As a corollary to the fine analysis of the use of bar recursion in this construction we obtain that Ω^* elevates arguments in T_n at most to resulting functionals in T_{n+2} (here T_n is the fragment of Gödel's T with primitive recursion restricted to the type level n). In particular, one can conclude from this that our bound on Baillon's theorem is at least definable in T_4 .

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1. Introduction

In recent years, proof theoretic transformations (so-called proof interpretations) that are based on extensions of monotone forms ([11, 14]) of Gödel’s famous functional (‘Dialectica’) interpretation ([9, 17]) have been used systematically to extract new content from proofs in abstract nonlinear analysis. This content consists both in effective quantitative bounds as well as in qualitative uniformity results (see e.g. [15] for a survey). The latter are a consequence of the fact that the extractable bounds are guaranteed to not depend on parameters from abstract spaces or selfmappings of such spaces but only on certain majorizing data, where majorization is defined in terms of metric distances. Because of this, even in the absence of compactness uniformity can be established as long as certain local metric bounds are given. ‘Guaranteed’ here means that general logical metatheorems allow to infer a-priori the extractability of such bounds provided that the theorem in question has the appropriate logical form and can be proved in the formal systems covered by these metatheorems (see [14] as well as [13, 8]). By ‘abstract spaces’ we mean structures such as general metric, normed, uniformly convex Banach, Hilbert, hyperbolic or CAT(0)-spaces that are axiomatically ‘hard-wired’ into the type structure of our formal systems without any separability assumptions on these spaces. The latter condition is essential as the uniform version of separability (automatically imposed by the monotone functional interpretation) results in total boundedness (e.g. of the unit ball) which would make it impossible to deal with non-compact contexts.

One of the main ineffective tools in abstract functional analysis is the use of sequential forms of weak compactness. Hence the issue arises whether e.g. the weak compactness of bounded closed and convex subsets $C \subset X$ in an abstract Hilbert spaces X can be proved in such a formal context. While the formal systems used in the papers mentioned above are very strong in containing full countable (and dependent) choice and so full impredicative comprehension over numbers, they are mathematically weak in the sense that no comprehension over points in X is possible (and – due to the absence of separability – one per se cannot compensate for this by comprehension over numbers). Nevertheless, this issue was solved in [14] where the formalizability of this weak compactness result in even a fragment of our formal context based on countable choice of arithmetical formulas was shown. Moreover, various very general projection arguments are covered as well.

The next step was to actually analyze some concrete proofs in functional analysis that use weak compactness (and projection arguments). A first case study was carried out recently in [18] where proofs of results due to F.E. Browder [6] and R. Wittmann [26] are analyzed. Both theorems state the strong convergence of some explicit sequences in Hilbert space but use sequential weak compactness in proving this. A surprising outcome of this case study is that in both cases the use of weak compactness in the end can be eliminated resulting in primitive recursive bounds (of rather low complexity) of the so-called metastable versions of these theorems. The term ‘metastability’ is due to T. Tao in [24] and essentially refers to the Kreisel no-counterexample interpretation of the Cauchy property which in turn is equivalent to the Gödel functional interpretation (combined with negative translation) in this case as Cauchyness is a Π_3^0 -property (see [25] for the significance of uniform bounds on metastability in ergodic theory).

While for Browder’s theorem already a more elementary proof due to Halpern did exist before (which is analyzed in [18] as well, again with a resulting primitive recursive bound), this is new for Wittmann’s theorem. The latter can be seen as an important nonlinear generalization of the von Neumann Mean Ergodic Theorem establishing the strong convergence of the so-called Halpern iteration schema which in the linear case coincides with the ergodic averages treated in the Mean

Ergodic Theorem. Surprisingly, the resulting bound is of essentially similar complexity than that obtained for the linear case of the Mean Ergodic Theorem in [20] (which also treats the uniformly convex case) and even better than that from [1] (which gave the first effective bound for the Mean Ergodic Theorem).

Very recently ([19]) we analyzed a proof of Baillon’s famous nonlinear ergodic theorem ([3]) due to Brézis and Browder [5] which states the weak convergence of ergodic averages (rather than of Halpern iterations as in Wittmann’s theorem) for general nonlinear nonexpansive operators in Hilbert space. This time the analysis of the proof crucially uses the (monotone) functional interpretation of the weak compactness principle mentioned above. In that paper we determine the precise form of the monotone functional interpretation of the sequential weak compactness for $C \subset X$ (in the form of the existence of a weak cluster point as it is this formulation that is needed)² and conclude from the provability of this in the aforementioned formal context that it has a solution Ω^* that is given by a closed term in $T_0 + B_{0,1}$. Here T_0 is the fragment of Gödel’s T with primitive recursion for type 0 only and $B_{0,1}$ is Spector’s [23] bar recursor of lowest type (used in the special form of Φ_0 from the appendix). Moreover, in [19] a bound φ on a suitable metastable version of Baillon’s theorem is extracted that is primitive recursive (in the sense of T_0) relative to Ω^* and hence $\varphi \in T_0 + B_{0,1}$. Since φ is of type 2 (while Ω^* is of type 3) it follows from a result in [12] that φ is definable in Gödel’s T .

In the current paper we for the first time carry out the actual construction of Ω^* . As this will turn out to be rather involved we restrict things here to the case of closed bounded balls around 0 (w.l.o.g. the closed unit ball $B_1(0)$) instead of C . This saves us of from having to treat additionally the so-called Mazur lemma that is needed to show that C is weakly closed (which is much easier for balls). Even with this restriction things are so complicated that it is virtually impossible to write down a closed expression for Ω^* . It is rather that the rest of this paper constitutes the description of Ω^* .

Despite of the technical nature of this investigation, we believe that it is of broader significance for the following reasons:

- The construction of Ω^* exhibits the finitary combinatorial content of one of the central infinitary and ineffective existence principles in mathematics.
- In the course of this construction we develop a number of quantitative projection lemmas which are of independent interest.
- The detailed construction of Ω^* reveals that besides ordinary primitive recursive constructions precisely two (nested) instances of $B_{0,1}$ occur. Using Howards’s ordinal analysis of $T_0 + B_{0,1}$ from [10] (see also [21](proof of theorem 4.16)) it follows that Ω^* produces a functional in T_{n+2} when applied to a functional in T_n (where T_n is the fragment of T with primitive recursion restricted to the recursor R_n) to yield a type-2 functional. As in our bound φ on Baillon’s theorem ([19]) we make two nested uses of Ω^* relative to T_0 , we get as a crude estimate that $\varphi \in T_4$. Of course we do not claim this to be optimal. In fact, in the light of the final comments in [19] it is not ruled out that a bound $\varphi \in T_0$ might exist and be extractable from different proofs of Baillon’s theorem.

²Actually, only a slightly weaker form of this is needed and we focus on that but the general case can easily be incorporated as well.

2. A uniform quantitative form of sequential weak compactness

Throughout this paper, X will be a (real) Hilbert space and $B_1(0)$ the closed unit ball in X .

As shown in [16], the well-known fact that every sequence (x_n) in $B_1(0)$ has a weak cluster point in $B_1(0)$ (as well as a weakly convergent subsequence) can be formalized in a fragment of classical analysis \mathcal{A}^ω augmented by an abstract axiomatically formulated Hilbert space X , i.e. – using the notation from [14] – in a fragment \mathcal{T} of $\mathcal{A}^\omega[X, \langle \cdot, \cdot \rangle, \mathcal{C}]$. That fragment is based on Peano arithmetic (in all finite types over \mathbb{N}, X) with restricted induction and primitive recursion plus quantifier-free choice and arithmetical comprehension, i.e. – roughly speaking – a finite type extension of the system ACA_0 from reverse mathematics (see [22]). As shown in [14], \mathcal{T} has (via negative translation) a monotone functional interpretation (called NMD-interpretation in the terminology of [14]) by functionals in $T_0 + B_{0,1}$. Functionals definable in $T_0 + B_{0,1}$ do not define total functionals in the full set-theoretic model \mathcal{S}^ω over \mathbb{N} but only in the model of strongly majorizable functionals \mathcal{M}^ω due to [4] (e.g. see [14] for details). Nevertheless, functionals of type level ≤ 2 (i.e. functionals taking only numbers and number theoretic functions as arguments and numbers as values) in $T_0 + B_{0,1}$ do define total functionals (e.g. of type $\mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}$) and – as shown in [12] – define exactly those functionals (of the respective type) that are primitive recursive in the sense of Gödel’s T . In the next theorem from [19] $\mathcal{M}^{\omega, X}$ denotes the extension of Bezem’s model \mathcal{M}^ω to all finite types (i.e. all function spaces) over \mathbb{N} and X from [8, 14] and $\Omega^* \gtrsim \Omega$ expresses that Ω^* (strongly) majorizes Ω in the sense of [14] (definition 17.50 with $a := 0_X$ as we are in the normed case). For completeness, we include the definition of $\mathcal{M}^{\omega, X}$ here:

Definition 2.1. Let \mathbf{T} resp. \mathbf{T}^X denote the set of all finite types over \mathbb{N} resp. over \mathbb{N} and X . For $\rho \in \mathbf{T}^X$ we define $\widehat{\rho} \in \mathbf{T}$ inductively as follows:

$$\widehat{\mathbb{N}} := \mathbb{N}, \quad \widehat{X} := \mathbb{N}, \quad \widehat{\rho \rightarrow \tau} := \widehat{\rho} \rightarrow \widehat{\tau}.$$

Definition 2.2. Let X be a nontrivial (real) Hilbert space. The extensional type structure $\mathcal{M}^{\omega, X}$ of all hereditarily strongly majorizable set-theoretic functionals of type $\rho \in \mathbf{T}^X$ over \mathbb{N} and X is defined as

$$\left\{ \begin{array}{l} M_{\mathbb{N}} := \mathbb{N}, \quad n \gtrsim_{\mathbb{N}} m := n \geq m \wedge n, m \in \mathbb{N}, \\ M_X := X, \quad n \gtrsim_X x := n \geq \|x\| \wedge n \in M_{\mathbb{N}}, x \in M_X, \\ x^* \gtrsim_{\rho \rightarrow \tau} x := x^* \in M_{\widehat{\rho}}^{M_{\widehat{\rho}}} \wedge x \in M_{\widehat{\tau}}^{M_{\widehat{\rho}}} \\ \quad \wedge \forall y^* \in M_{\widehat{\rho}}, y \in M_{\rho} (y^* \gtrsim_{\rho} y \rightarrow x^* y^* \gtrsim_{\tau} xy) \\ \quad \wedge \forall y^*, y \in M_{\widehat{\rho}} (y^* \gtrsim_{\widehat{\rho}} y \rightarrow x^* y^* \gtrsim_{\widehat{\tau}} x^* y), \\ M_{\rho \rightarrow \tau} := \left\{ x \in M_{\widehat{\tau}}^{M_{\widehat{\rho}}} \mid \exists x^* \in M_{\widehat{\rho}}^{M_{\widehat{\rho}}} : x^* \gtrsim_{\rho \rightarrow \tau} x \right\} \quad (\rho, \tau \in \mathbf{T}^X). \end{array} \right.$$

In the following we consider weak compactness in the form: every sequence (x_n) in the unit ball $B_1(0)$ of a (real) Hilbert space possesses a point $v \in B_1(0)$ such that for every $\varepsilon > 0$ and $w \in X$, we have that $|\langle v - x_n, w \rangle| < \varepsilon$ for arbitrarily large n . In the official definition of v being a weak cluster point of (x_n) one requires that for any finite set w_1, \dots, w_n of vectors one has arbitrarily large indexes n for which this holds simultaneously. However, in our application to Baillon’s theorem, only the former version is needed and, anyhow, our construction of Ω^* can easily be adapted to cover also the more general case.

Theorem 2.3 (Uniform quantitative version of weak sequential compactness [19]). *Applying monotone functional interpretation to the proof of weak sequential compactness of $B_1(0)$ from [16] yields the extractability of a closed term Ω^* in $T_0 + B_{0,1}$ such that the following is true in the model $\mathcal{M}^{\omega, X}$ (for any Hilbert space X)*

$$\left\{ \begin{array}{l} \exists \Omega \lesssim \Omega^* \forall K, W \forall (x_n) \subset B_1(0) \\ \exists v \in X \exists \chi = \Omega(K, W, (x_n)) \exists n \in [K(\tilde{v}, \chi), \chi(W(\tilde{v}, \chi), K(\tilde{v}, \chi))] \\ \quad (|\langle \tilde{v} - x_n, W(\tilde{v}, \chi) \rangle| <_{\mathbb{R}} 2^{-K(\tilde{v}, \chi)}), \end{array} \right.$$

where where

$$\tilde{v} := \frac{v}{\max\{\|v\|, 1\}}.$$

Note that Ω^* does not depend on X .

The statement in theorem 2.3 is equivalent to

$$\left\{ \begin{array}{l} \exists \Omega \lesssim \Omega^* \forall K, W \forall (x_n) \subset B_1(0) \\ \exists v \in B_1(0) \exists \chi = \Omega(K, W, (x_n)) \exists n \in [K(v, \chi), \chi(W(v, \chi), K(v, \chi))] \\ \quad (|\langle v - x_n, W(v, \chi) \rangle| <_{\mathbb{R}} 2^{-K(v, \chi)}). \end{array} \right.$$

The main purpose of this paper is to actually construct the functional Ω^* in this theorem.

We first observe that it suffices to show the special case of theorem 2.3 where W is assumed to satisfy $\|W(w, \chi)\| \leq 1$ for all w, χ . Indeed, suppose we have shown this restricted version. Let now W (and K) be arbitrary and define

$$W_1(w, \chi) := \frac{1}{\max\{\|W(w, \tilde{\chi})\|, 1\}} \cdot W(w, \tilde{\chi})$$

and

$$\tilde{K}_W(w, \chi) := K(w, \tilde{\chi}) + \lceil \log_2(\max\{\|W(w, \tilde{\chi})\|, 1\}) \rceil,$$

where

$$\tilde{\chi}(w, k) := \chi \left(\frac{1}{\max\{\|w\|, 1\}} \cdot w, k + \lceil \log_2(\max\{\|w\|, 1\}) \rceil \right).$$

Now apply the special case of theorem 2.3 to \tilde{K}_W, W_1 . Then for some Ω with $\Omega^* \gtrsim \Omega$ we have

$$\exists v \in X \exists \chi = \Omega(\tilde{K}_W, W_1, (x_n)) \exists n \in [\tilde{K}_W(\tilde{v}, \chi), \chi(W_1(\tilde{v}, \chi), \tilde{K}_W(\tilde{v}, \chi))] \\ (|\langle \tilde{v} - x_n, W_1(\tilde{v}, \chi) \rangle| <_{\mathbb{R}} 2^{-\tilde{K}_W(\tilde{v}, \chi)}).$$

Note that $|\langle \tilde{v} - x_n, W_1(\tilde{v}, \chi) \rangle| <_{\mathbb{R}} 2^{-\tilde{K}_W(\tilde{v}, \chi)}$ implies that

$$|\langle \tilde{v} - x_n, W(\tilde{v}, \tilde{\chi}) \rangle| <_{\mathbb{R}} 2^{-K(\tilde{v}, \tilde{\chi})}.$$

Also

$$n \in [K(\tilde{v}, \tilde{\chi}), \tilde{\chi}(W(\tilde{v}, \tilde{\chi}), K(\tilde{v}, \tilde{\chi}))].$$

Hence $\tilde{\Omega}(K, w, (x_n)) := \Omega(\widetilde{\tilde{K}_W}, W_1, (x_n))$ satisfies the theorem for general W with the majorant $\tilde{\Omega}^*(K^*, W^*) := \Omega^*(\widetilde{\tilde{K}^*}, 1)$, where

$$\tilde{K}^*(n, \chi^*) := K^*(n, \widehat{\chi^*}) + \lceil \log_2(\max\{W^*(n, \widehat{\chi^*}), 1\}) \rceil$$

and

$$\widehat{\chi^*}(n, k) := \chi^*(1, k + \lceil \log_2(\max\{n, 1\}) \rceil).$$

We next show that this special case of theorem 2.3 follows from the (correspondingly special) case where the conclusion on v, χ, n is replaced by

$$(+) \quad \left\{ \begin{array}{l} n \in [K(v, \chi), \chi(W(v, \chi), K(v, \chi))] \wedge \\ |\langle v - x_n, W(v, \chi) \rangle| <_{\mathbb{R}} 2^{-\tilde{K}(v, \chi)-1} \wedge \exists m \in \mathbb{N}(|\langle v - x_m, v \rangle| <_{\mathbb{R}} 2^{-K(v, \chi)-1}). \end{array} \right.$$

Note that here \tilde{v} is replaced by v where the latter is only claimed to be in X (not necessarily in $B_1(0)$).

For any given K, W (with $W(w, \chi) \in B_1(0)$ for all w, χ) we apply (+) to $\tilde{K}(v, \chi) := K(\tilde{v}, \chi)$ and $\tilde{W}(v, \chi) := W(\tilde{v}, \chi)$ and obtain v, χ, n such that

$$(+) \quad \left\{ \begin{array}{l} n \in [\tilde{K}(v, \chi), \chi(\tilde{W}(v, \chi), \tilde{K}(v, \chi))] \wedge \\ |\langle v - x_n, \tilde{W}(v, \chi) \rangle| <_{\mathbb{R}} 2^{-\tilde{K}(v, \chi)-1} \wedge \exists m \in \mathbb{N}(|\langle v - x_m, v \rangle| <_{\mathbb{R}} 2^{-\tilde{K}(v, \chi)-1}). \end{array} \right.$$

first show that the second conjunct in (+)' implies that $\|v\| \leq 1 + 2^{-\tilde{K}(v, \chi)-1}$:

Suppose that $\|v\| > 1 + 2^{-\tilde{K}(v, \chi)-1}$. Then

$$\langle v, v \rangle = \|v\|^2 > \|v\|(1 + 2^{-\tilde{K}(v, \chi)-1}) > \|v\| \geq \|v\| \cdot \|x_m\| \geq \langle v, x_m \rangle$$

and so

$$\begin{aligned} |\langle v, v - x_m \rangle| &= |\langle v, v \rangle - \langle v, x_m \rangle| = \langle v, v \rangle - \langle v, x_m \rangle \\ &> \|v\|(1 + 2^{-\tilde{K}(v, \chi)-1}) - \|v\| = 2^{-\tilde{K}(v, \chi)-1} \|v\| > 2^{-\tilde{K}(v, \chi)-1} \end{aligned}$$

which contradicts the second conjunct in (+)'.

$\|v\| \leq 1 + 2^{-\tilde{K}(v, \chi)-1}$ in turn implies that $\|\tilde{v} - v\| \leq 2^{-\tilde{K}(v, \chi)-1}$:

Case 1: $\|v\| \leq 1$. Then $\tilde{v} = v$ and so we are done.

Case 2: $\|v\| > 1$. Then $1 \leq \max\{\|v\|, 1\} = \|v\| \leq 1 + 2^{-\tilde{K}(v, \chi)-1}$ and so

$$\|\tilde{v} - v\| = \left\| \frac{v}{\|v\|} - \frac{\|v\| \cdot v}{\|v\|} \right\| = \frac{1}{\|v\|} \|(1 - \|v\|)v\| = |1 - \|v\|| \leq 2^{-\tilde{K}(v, \chi)-1}.$$

$\|\tilde{v} - v\| \leq 2^{-\tilde{K}(v, \chi)-1}$ together with (+)' yields that

$$\begin{aligned} |\langle W(\tilde{v}, \chi), \tilde{v} - x_n \rangle| &= |\langle \tilde{W}(v, \chi), \tilde{v} - x_n \rangle| \leq |\langle \tilde{W}(v, \chi), v - x_n \rangle| + |\langle \tilde{W}(v, \chi), \tilde{v} - v \rangle| \\ &< 2^{-\tilde{K}(v, \chi)-1} + \|\tilde{W}(v, \chi)\| \cdot \|\tilde{v} - v\| \leq 2^{-\tilde{K}(v, \chi)-1} + 2^{-\tilde{K}(v, \chi)-1} \leq 2^{-\tilde{K}(v, \chi)} = 2^{-K(\tilde{v}, \chi)}, \end{aligned}$$

which is the conclusion of theorem 2.3 for K, W . □

Finally, in order to show (+), it even suffices to prove (+) with the last conjunct being dropped and then apply that statement to

$$W'(v, \chi) := \begin{cases} W(v, \chi), & \text{if } \neg \exists n \in [K(v, \chi), \chi(W(v, \chi), K(v, \chi))] \\ & (|\langle v - x_n, W(v, \chi) \rangle| < 2^{-K(v, \chi)}) \\ v, & \text{otherwise.} \end{cases}$$

Then we obtain v, χ such that

$$\exists n \in [K(v, \chi), \chi(W'(v, \chi), K(v, \chi))] (|\langle v - x_n, W'(v, \chi) \rangle| < 2^{-K(v, \chi)})$$

which – by the definition of W' – implies

$$\exists n \in [K(v, \chi), \chi(w, K(v, \chi))] (|\langle v - x_n, w \rangle| < 2^{-K(v, \chi)})$$

for both $w = W(v, \chi)$ as well as $w = v$.

So putting everything together (and disregarding the inessential issue of $2^{-K(\tilde{v}, \chi)}$ versus $2^{-K(v, \chi)-1}$) we have shown that it suffices to establish theorem 2.3 with \tilde{v} replaced by v .

Our functional Ω^* will not only be computable but even be primitive recursive in bar recursion (in the sense of Spector [23]) Φ_0 of lowest type which in turn can be seen as a special case of $B_{0,1}$ as defined in [14](p.202-203). See also [21] and the appendix of this paper for details.

Such a use of bar recursion already is needed for the functional interpretation of the Bolzano-Weierstraß principle (for a suitable compact Polish space) on which the proof of weak sequential compactness of $B_1(0)$ is based. Let (x_n) be a sequence in $B_C(0) \subset X$ (for $C > 0$) and consider the following separable closed linear subspace of X :

$$\mathcal{L} := \overline{\text{Lin}_{\mathbb{R}}\{x_n : n \in \mathbb{N}\}}$$

and its countable dense subset

$$\mathcal{L}_{\mathbb{Q}} := \bigcup_{n \in \mathbb{N}} \text{Lin}_{\mathbb{Q}}\{x_0, \dots, x_n\}.$$

Let (y_k) be some (effective in (x_n)) standard enumeration of $\mathcal{L}_{\mathbb{Q}}$. To be more specific we could (relying on the primitive recursive sequence codings $\langle \cdot, \dots, \cdot \rangle$, $(\cdot)_{(\cdot)}$, lth , $*$ and pairings j, j_1, j_2 as well as the encoding of rational numbers and their basic operations, all from [14]) take

$$y_n := r_{j_1((n)_0)} \cdot x_{j_2((n)_0)} + \dots + r_{j_1((n)_{lth(n)-1})} \cdot x_{j_2((n)_{lth(n)-1})}$$

(for $n > 0$ and $y_0 := 0$), where r_n denotes the (unique) rational number encoded by n . Note that there is a simple primitive recursive function $\gamma(n, C)$ with values in \mathbb{N} such that for (x_n) in $B_C(0)$ we have

$$\gamma(n, C) \geq \|y_n\| \quad \text{for all } n \in \mathbb{N}.$$

In the above encoding, we could take e.g.

$$\gamma(n, C) := C \cdot n \cdot lth(n) \geq C \cdot \sum_{i=0}^{lth(n)-1} |j_1((n)_i)|_{\mathbb{Q}} \geq \|y_n\|$$

using basic properties of the sequence coding and representation of rational numbers from [14]. Note that γ is monotone in n , i.e. $\gamma(n+1, C) \geq \gamma(n, C)$.

Linear C -bounded operators $\tilde{L} : \mathcal{L} \rightarrow \mathbb{R}$ can be represented as points (satisfying appropriate conditions)

$$L \in \prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|]$$

via $L(k) := \tilde{L}(y_k)$ (see [7] and [16] for details). Equipped with the product metric

$$d(a, b) := \sum_{i=0}^{\infty} \frac{1}{2^i} \cdot \frac{|a_i - b_i|}{1 + |a_i - b_i|},$$

$\prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|]$ is a compact metric space.

Now consider the sequence (L_n) of C -bounded operators in $\prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|]$ given by

$$L_n(k) := \langle x_n, y_k \rangle.$$

Then this sequence possesses a cluster point in $\prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|]$, i.e. the following holds

$$(+) \exists L \in \prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|] \exists f : \mathbb{N} \rightarrow \mathbb{N} \forall k \in \mathbb{N} \exists j \leq f(k) (j \geq k \wedge d(L_j, L) < 2^{-k}).$$

Note that L again represents a C -bounded linear functional $\tilde{L} : \mathcal{L} \rightarrow \mathbb{R}$.

Functional interpretation (combined with negative translation) of (+) yields functionals realizing (bar recursively in (L_n) and C) ‘ $\exists f_{\tilde{k}}, L_{\tilde{k}}$ ’ in

$$(*) \left\{ \begin{array}{l} \forall \tilde{k} \exists L_{\tilde{k}} \in \prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|] \exists f_{\tilde{k}} : \mathbb{N} \rightarrow \mathbb{N} \exists j \leq f_{\tilde{k}}(\tilde{k}[f_{\tilde{k}}, L_{\tilde{k}}]) \\ (j \geq \tilde{k}[f_{\tilde{k}}, L_{\tilde{k}}] \wedge d(L_j, L_{\tilde{k}}) < 2^{-\tilde{k}[f_{\tilde{k}}, L_{\tilde{k}}]}). \end{array} \right.$$

Here \tilde{k} is a functional that maps arguments $L \in \prod_{n \in \mathbb{N}} [-C\|y_n\|, C\|y_n\|]$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ to natural numbers. Note that in contrast to the ‘real’ cluster point L the approximate cluster point $L_{\tilde{k}}$ (while still C -bounded on $\mathcal{L}_{\mathbb{Q}}$) inherits from being close to L_j only an approximate form of the linearity condition

$$y_k =_X r_n \cdot_X y_i +_X r_m \cdot_X y_j \rightarrow a_k =_{\mathbb{R}} r_n a_i +_{\mathbb{R}} r_m a_j,$$

where $a_n := L_{\tilde{k}}(n)$ and (r_n) some standard enumeration of \mathbb{Q} , and hence also of the continuity condition. This will cause some technical problems further below as we have to make 2^{-k} so small that we get for all the points for which we use the linearity of $L_{\tilde{k}}$ a sufficiently good approximate form of linearity. Based on the primitive recursive enumeration (y_n) of $\mathcal{L}_{\mathbb{Q}}$ one can construct a primitive recursive function $\xi : \mathbb{N}^4 \rightarrow \mathbb{N}$ such that (provably) $y_{\xi(n,m,i,j)} =_X r_n \cdot_X y_i +_X r_m \cdot_X y_j$. By an approximate form of the above linearity we mean

$$|L(y_{\xi(n,m,i,j)}) - (r_n L(y_i) +_{\mathbb{R}} r_m L(y_j))| < 2^{-l}$$

for some $l \in \mathbb{N}$.

Let us now define how we precisely extend L from $\mathcal{L}_{\mathbb{Q}}$ to \mathcal{L} : let C be the completion operator from [14] (pp. 432-434) and $(z_n) \mapsto \widehat{(z_n)}$ the construction used in the definition of C (for simplicity we write $\widehat{(z_n)}$ but note that $\widehat{z_n}$ is defined in terms of z_i for all $i \leq n$ and coincides with z_i for a suitable such i). Roughly speaking, this construction transforms a given sequence in a fast converging Cauchy sequence unless it was already such a sequence. We define for a given sequence $(z_n) = (y_{f(n)})_n$ in $\mathcal{L}_{\mathbb{Q}}$ and $z = C((z_n))$

$$L(z) := L(f) := \lim_{k \rightarrow \infty} L(\widehat{(z_{k+3})}).$$

Here the outer $\widehat{(\)}$ -construction is that for real numbers from [14] (p.79) while the inner one is that for sequences in X from [14] (p.433). More precisely, as L needs its argument y_k to be given via the index k , we use the following version of the construction from [14] (p.433):

$$\widehat{f}(n) := \begin{cases} f(n), & \text{if } \forall k < n (d_X(y_{f(k)}, y_{f(k+1)})(k+1) <_{\mathbb{Q}} 6 \cdot 2^{-k-1}) \\ f(k) & \text{for the least counterexample } k < n, \text{ otherwise.} \end{cases}$$

So officially we use $\lim_{k \rightarrow \infty} L(\widehat{(y_{\widehat{f}(k+3)})})$ although we will write $\lim_{k \rightarrow \infty} L(\widehat{(y_{f(k+3)})})$ (note that extensionally $\widehat{y_{f(n)}} =_X y_{\widehat{f}(n)}$). Our construction extends L from \mathbb{N} to $\mathbb{N} \rightarrow \mathbb{N}$ via the embedding $i \mapsto \lambda n.i$.

Now let L be bounded by 1 and hence (if linear) in $Lip(1)$. Then (by [14], p.433) $\|\widehat{z_{k+3}} - \widehat{z_{k+4}}\| < 7 \cdot 2^{-k-4}$ and so

$$|L(\widehat{z_{k+3}}) - L(\widehat{z_{k+4}})| \stackrel{!}{=} |L(\widehat{z_{k+3}} - \widehat{z_{k+4}})| < 7 \cdot 2^{-k-4} < 7 \cdot 2^{-k-4} + 2^{-k-4} = 2^{-k-1}.$$

Hence $(L(\widehat{z_{k+3}}))_k = (L(\widehat{(z_{k+3})}))_k$ for all $k \in \mathbb{N}$ (see [14], p.79) and so $L(z) =_{\mathbb{R}} \lim L(\widehat{z_k})$. Thus if (z_n) is a fast converging Cauchy sequence (with rate 2^{-n} so that $\widehat{z_n} =_X z_n$ for all $n \in \mathbb{N}$), then $L(z) =_{\mathbb{R}} \lim L(z_k)$. Note that for

$$|L(z) - L(\widehat{z_{k+3}})| < 2^{-k-1}$$

it suffices to replace the used linearity conditions (for all $i \leq k+3$)

$$L(z_i) - L(z_{i+1}) =_{\mathbb{R}} L(z_i - z_{i+1})$$

(to get ‘!’) by their approximate form

$$|(L(z_i) - L(z_{i+1})) - L(z_i - z_{i+1})| < 2^{-k-4}.$$

In the rest of this paper we will mark (approximate) uses of linearity by ‘!’ to finally address this issue at the end of section 4.

By monotone functional interpretation one can extract majorants L^* for the functional $\tilde{k} \mapsto L_{\tilde{k}}$ (this is essentially trivial) and (which is highly nontrivial) f^* for $\tilde{k} \mapsto f_{\tilde{k}}$ that are definable primitive recursively in Φ_0 . These majorants no longer depend on (L_n) (but only on C instead). An explicit construction is given in [21] (see also the appendix to the present paper). For the rest of the paper the radius C will be 1.

3. Quantitative projection lemmas

Further below we will need a number of quantitative projection lemmas:

Lemma 3.1 ([16]). *The following holds:*

$$\left\{ \begin{array}{l} \forall \varepsilon > 0 \forall K \geq 1 \forall x, y, z \in X \\ (\|z\| \leq K \wedge \|x - y\|^2 \leq \|x - (y + \alpha z)\|^2 + \frac{\varepsilon^2}{K^2} \rightarrow |\langle x - y, z \rangle| \leq \varepsilon), \end{array} \right.$$

where

$$\alpha := \frac{\langle x - y, z \rangle}{\max\left(\left(\varepsilon / (2 \max(\|x\|, \|y\|, 1))\right)^2, \|z\|^2\right)}.$$

In particular, if $V \subseteq X$ is a linear subspace, then:

$$\forall x \in X \forall y \in V (\forall z \in V (\|x - y\| \leq \|x - z\|) \rightarrow \forall z \in V (\langle x - y, z \rangle = 0)).$$

Lemma 3.2. *Let $S \subseteq X$ be any subset of X with $0 \in S$ (e.g. a linear subspace), $x \in X$, Φ be a selfmapping $S \rightarrow S$ and $\varepsilon > 0$. Then*

$$\exists y \in S (\|y\| \leq 2\|x\| \wedge \|x - y\|^2 \leq \|x - \Phi(y)\|^2 + \varepsilon).$$

Moreover, for $\|x\| \leq N \in \mathbb{N}$ one can find an $i < \left\lceil \frac{N^2}{\varepsilon} \right\rceil$ such that y can be taken as $y := \Phi^{(i)}(0)$ and $\|\Phi^{(j)}(0)\| \leq 2\|x\|$ for all $j \leq i$.

Proof: Suppose that for all $i < \left\lceil \frac{N^2}{\varepsilon} \right\rceil =: K$ we would have that

$$\|x - \Phi^{(i)}(0)\|^2 > \|x - \Phi^{(i+1)}(0)\|^2 + \varepsilon,$$

where $N \geq \|x\|$. Then

$$N^2 \geq \|x - 0\|^2 > \|x - \Phi^{(K)}(0)\|^2 + K \cdot \varepsilon \geq \|x - \Phi^{(K)}(0)\|^2 + N^2 \geq N^2$$

which is a contradiction.

Now let $i_0 < \left\lceil \frac{N^2}{\varepsilon} \right\rceil$ be minimal with

$$\|x - \Phi^{(i_0)}(0)\|^2 \leq \|x - \Phi^{(i_0+1)}(0)\|^2 + \varepsilon$$

and $y := \Phi^{(i_0)}(0)$.

Case 1: $i_0 = 0$. Then $y = 0$ and so $\|y\| \leq 2\|x\|$.

Case 2: $i_0 > 0$: Then for all $l \in \{0, 1, 2, \dots, i_0 - 1\}$ we have

$$\|x - \Phi^{(l)}(0)\|^2 > \|x - \Phi^{(l+1)}(0)\|^2 + \varepsilon > \|x - \Phi^{(l+1)}(0)\|^2.$$

Hence

$$\|x\|^2 = \|x - \Phi^{(0)}(0)\|^2 \geq \|x - \Phi^{(j)}(0)\|^2$$

for all $j \leq i_0$ and so $\|x\| \geq \|x - \Phi^{(j)}(0)\|$ which yields $\|\Phi^{(j)}(0)\| \leq 2\|x\|$.

In particular: $\|y\| \leq 2\|x\|$. □

Let $\Phi : S \rightarrow S$ be a selfmapping of a subset $S \subseteq X$, then we say that $\Phi^* : \mathbb{N} \rightarrow \mathbb{N}$ majorizes Φ (short: $\Phi^* \gtrsim \Phi$) if

$$\forall x \in S, n \in \mathbb{N} (n \geq \|x\| \rightarrow \Phi^*(n) \geq \|\Phi(x)\|).$$

Lemma 3.3. *Let V be any linear subspace of X . Then the following holds:*

$$\left\{ \begin{array}{l} \forall \varepsilon > 0 \forall N \in \mathbb{N}^* \forall x \in X \forall \Phi : V \rightarrow V \forall \Phi^* : \mathbb{N} \rightarrow \mathbb{N} \\ (\|x\| \leq N \wedge \Phi^* \gtrsim \Phi \rightarrow \exists \hat{y} \in V (\|\hat{y}\| \leq 2N \wedge |\langle x - \hat{y}, \Phi(\hat{y}) \rangle| \leq \varepsilon)), \end{array} \right.$$

where, moreover, \hat{y} can be constructed as $\hat{y} := \Psi^{(i)}(0)$ for a suitable $i \leq \left\lceil \frac{N^2(\Phi^*(2N))^2}{\varepsilon^2} \right\rceil$ with $\|\Psi^{(j)}(0)\| \leq 2N$ for all $j \leq i$, where

$$\Psi : V \rightarrow V, \quad \Psi(y) := y + \alpha_{x,y,\varepsilon}^\Phi \cdot \Phi(y)$$

with

$$\alpha_{x,y,\varepsilon}^\Phi := \frac{\langle x - y, \Phi(y) \rangle}{\max\left(\left(\varepsilon/(2 \max(\|x\|, \|y\|, 1))\right)^2, \|\Phi(y)\|^2\right)}.$$

Finally

$$|\alpha_{x,\hat{y},\varepsilon}^\Phi| \leq \left\lceil \frac{48N^3\Phi^*(2N)}{\varepsilon^2} \right\rceil =: D_{N,\Phi^*,\varepsilon}.$$

Proof: Lemma 3.2 applied to Ψ and $\varepsilon' := \frac{\varepsilon^2}{(\Phi^*(2N))^2}$ yields a $\hat{y} \in V$ with

$$\|\hat{y}\| \leq 2\|x\| \leq 2N \wedge \|x - \hat{y}\|^2 \leq \|x - \Psi(\hat{y})\|^2 + \frac{\varepsilon^2}{(\Phi^*(2N))^2},$$

where \hat{y} has the form $\Psi^{(i)}(0)$ for a suitable $i \leq \left\lceil \frac{N^2(\Phi^*(2N))^2}{\varepsilon^2} \right\rceil$ and $\|\Psi^{(j)}(0)\| \leq 2N$ for all $j \leq i$.

Since $\|\hat{y}\| \leq 2N$ we have $\Phi^*(2N) \geq \|\Phi(\hat{y})\|$. Hence lemma 3.1 implies (using the definition of Ψ) that $|\langle x - \hat{y}, \Phi(\hat{y}) \rangle| \leq \varepsilon$.

Finally

$$\begin{aligned} |\alpha_{x,\hat{y},\varepsilon}^\Phi| &\leq \left\lceil \frac{16N^2\|x - \hat{y}\| \cdot \|\Phi(\hat{y})\|}{\varepsilon^2} \right\rceil \\ &\leq \left\lceil \frac{48N^3\Phi^*(2N)}{\varepsilon^2} \right\rceil. \end{aligned}$$

□

Definition 3.4. *Let $\delta \in \mathbb{Q}_+^*$. By the canonical rational δ -approximation to a real number x we mean $i \cdot \delta$ (resp. $-i \cdot \delta$) for the least $i \leq |x|/\delta$ such that $i \cdot \delta$ is closest to x if $x \geq 0$ (resp. if $x < 0$).*

Remark 3.5. *In the following we sometimes use canonical rational approximations. Things could also be made constructive (relative to the use of the constants of the language) using the rational*

approximations provided already by e.g. the functional $\|\cdot\|_X$ (of type $X \rightarrow 1$) which outputs a type-1 object encoding a fast converging sequence of rational numbers. The latter, however, is itself provided by the official interpretation of the constant $\|\cdot\|_X$ in the model $\mathcal{M}^{\omega, X}$ via the – again ineffective – selection $(\|x\|)_\circ$ of a canonical representation of $\|x\|$ (see [14](def.17.7 and p.429)). The crucial thing is that \circ is trivially majorizable.

Definition 3.6. We say that a function $f^* : \mathbb{N} \rightarrow \mathbb{N}$ majorizes a function $f : \mathbb{N} \rightarrow \mathbb{N}$ (short: $f^* \succcurlyeq f$), if

$$\forall n, m \in \mathbb{N} (m \geq n \rightarrow f^*(m) \geq f^*(n), f(n)).$$

Lemma 3.7. Let $\mathcal{L}_\mathbb{Q}$ be as before and $\Phi : \mathcal{L}_\mathbb{Q} \rightarrow \mathcal{L}_\mathbb{Q}$ be a function that – on the codes k of elements $y_k \in \mathcal{L}_\mathbb{Q}$ – is given by $\widehat{\Phi} : \mathbb{N} \rightarrow \mathbb{N}$ (i.e. $y_{\widehat{\Phi}(k)} = \Phi(y_k)$). Let $\varepsilon \in \mathbb{Q}_+^*$, $N \in \mathbb{N}^*$ and $x \in X$ be with $\|x\| \leq N$. Furthermore, let $\Phi^*, \widehat{\Phi}^* : \mathbb{N} \rightarrow \mathbb{N}$ be such that $\Phi^* \succcurlyeq \Phi$ (in the above defined sense) and $\widehat{\Phi}^* \succcurlyeq \widehat{\Phi}$ as in the previous definition. Then the following holds:

$$(+) \exists \widehat{k} \leq \chi(N, \Phi^*, \widehat{\Phi}^*, \varepsilon) (\|y_{\widehat{k}}\| \leq 2N \wedge |\langle x - y_{\widehat{k}}, \Phi(y_{\widehat{k}}) \rangle| \leq \varepsilon),$$

where $y_{\widehat{k}} = \Psi^{(i)}(0)$ for a suitable $i \leq \left\lceil \frac{3N^2(\Phi^*(2N))^2}{\varepsilon^2} \right\rceil =: I$ with $\|\Psi^{(j)}(0)\| \leq 2N$ for all $j \leq i$ and

$$\Psi : \mathcal{L}_\mathbb{Q} \rightarrow \mathcal{L}_\mathbb{Q}, \quad \Psi(y) := y + r_{x, y, \varepsilon}^{\delta, \Phi} \cdot \Phi(y),$$

where $r_{x, y, \varepsilon}^{\delta, \Phi}$ is the canonical rational δ -approximation to $\alpha_{x, y, \varepsilon}^\Phi$ (as defined in lemma 3.3) with (using the definition of $D_{N, \Phi^*, \varepsilon}$ from lemma 3.3)

$$\mathbb{Q}_+^* \ni \delta \leq \min \left(\frac{\varepsilon}{\sqrt{3}(\Phi^*(2N))^2}, \frac{\varepsilon^2}{6(\Phi^*(2N))^3(3N + D_{N, \Phi^*, \varepsilon} \Phi^*(2N))}, 1 \right).$$

Finally:

$$\chi(N, \Phi^*, \widehat{\Phi}^*, \varepsilon) := (\Psi^*)^{(I)}(\langle 0 \rangle),$$

where $\Psi^*(k) := \max\{\Psi'(0), \dots, \Psi'(k), k\}$ with

$$\Psi'(k) := \max \left\{ \langle y_k + (-1)^l(i \cdot \delta) \cdot y_j \rangle : i \leq \left\lceil \frac{D_{N, \Phi^*, \varepsilon}}{\delta} \right\rceil ; j \leq \widehat{\Phi}^*(k); l \in \{0, 1\} \right\}$$

and $\langle 0 \rangle$ is some code of $0 \in \mathcal{L}_\mathbb{Q}$ (here $\langle y_k + (-1)^l(i \cdot \delta) \cdot y_j \rangle$ is some canonical code built up primitive recursively from k, j and some standard code of the rational number $(-1)^l(i \cdot \delta)$).

Proof: By lemma 3.2 applied to Ψ (and using that $\Psi'(k)$ is an upper bound for some code m s.t. $y_m = \Psi(y_k)$) there exists a $\widehat{k} \leq (\Psi^*)^{(I)}(\langle 0 \rangle)$ such that

$$\|y_{\widehat{k}}\| \leq 2\|x\| \leq 2N \wedge \|x - y_{\widehat{k}}\|^2 \leq \|x - \Psi(y_{\widehat{k}})\|^2 + \frac{\varepsilon^2}{3(\Phi^*(2N))^2}$$

and $y_{\widehat{k}} = \Psi^{(i)}(0)$ for some $i \leq I$.

The fact that $r_{x, y_{\widehat{k}}, \varepsilon}^{\delta, \Phi}$ is δ -close to $\alpha_{x, y_{\widehat{k}}, \varepsilon}^\Phi$ implies that

$$\|x - \Psi(y_{\widehat{k}})\| \leq \|x - (y_{\widehat{k}} + \alpha_{x, y_{\widehat{k}}, \varepsilon}^\Phi \cdot \Phi(y_{\widehat{k}}))\| + \underbrace{\delta \cdot \Phi^*(2N)}_{\delta :=}$$

Since

$$\|x - (y_{\widehat{k}} + \alpha_{x, y_{\widehat{k}}, \varepsilon}^{\Phi} \cdot \Phi(y_{\widehat{k}}))\| \leq \|x\| + \|y_{\widehat{k}}\| + |\alpha_{x, y_{\widehat{k}}, \varepsilon}^{\Phi}| \cdot \Phi^*(2N) \leq 3N + D_{N, \Phi^*, \varepsilon} \cdot \Phi^*(2N)$$

we get

$$\begin{aligned} \|x - \Psi(y_{\widehat{k}})\|^2 &\leq \|x - (y_{\widehat{k}} + \alpha_{x, y_{\widehat{k}}, \varepsilon}^{\Phi} \cdot \Phi(y_{\widehat{k}}))\|^2 + 2\tilde{\delta} \cdot (3N + D_{N, \Phi^*, \varepsilon} \cdot \Phi^*(2N)) + \tilde{\delta}^2 \\ &\leq \|x - (y_{\widehat{k}} + \alpha_{x, y_{\widehat{k}}, \varepsilon}^{\Phi} \cdot \Phi(y_{\widehat{k}}))\|^2 + \frac{2\varepsilon^2}{3(\Phi^*(2N))^2}. \end{aligned}$$

Hence

$$\|x - y_{\widehat{k}}\|^2 \leq \|x - (y_{\widehat{k}} + \alpha_{x, y_{\widehat{k}}, \varepsilon}^{\Phi} \cdot \Phi(y_{\widehat{k}}))\|^2 + \frac{\varepsilon^2}{(\Phi^*(2N))^2}.$$

Since $\|\Phi(y_{\widehat{k}})\| \leq \Phi^*(2N)$, lemma 3.1 (applied to $V := \mathcal{L}$) now yields that

$$|\langle x - y_{\widehat{k}}, \Phi(y_{\widehat{k}}) \rangle| \leq \varepsilon.$$

□

Corollary to the proof of lemma 3.7: Lemma 3.7 also holds (with $y_{\widehat{\Phi}(k)}$ instead of $\Phi(y_{\widehat{k}})$ in (+)) for arbitrary functions $\widehat{\Phi} : \mathbb{N} \rightarrow \mathbb{N}$ that might not be extensional in the sense of $y_{k_1} = y_{k_2} \rightarrow y_{\widehat{\Phi}(k_1)} = y_{\widehat{\Phi}(k_2)}$ as long as we still have that

$$n \geq \|y_k\| \rightarrow \Phi^*(n) \geq y_{\widehat{\Phi}(k)}.$$

Then $\widehat{\Phi}$ will not come from any **function** $\Phi : \mathcal{L}_{\mathbb{Q}} \rightarrow \mathcal{L}_{\mathbb{Q}}$ but only defines an intensional operation $\mathcal{L}_{\mathbb{Q}} \rightarrow \mathcal{L}_{\mathbb{Q}}$.

Lemma 3.8. *Let $k_0 \in \mathbb{N}$ be such that $|L(y_{k_0})| > 0$. With $z := \frac{y_{k_0}}{L(y_{k_0})}$ define $w_k := y_k - L(y_k) \cdot z \in \text{Kern}(L)$ for all $k \in \mathbb{N}$. Let $\widehat{\Phi} : \mathbb{N} \rightarrow \mathbb{N}$ be a function and $x \in X$ with $\|x\| \leq N \in \mathbb{N}$. Furthermore, let $\Phi^*, \widehat{\Phi}^* : \mathbb{N} \rightarrow \mathbb{N}$ be such that*

$$\forall n, k \in \mathbb{N} (n \geq \|w_k\| \rightarrow \Phi^*(n) \geq \|w_{\widehat{\Phi}(k)}\|)$$

and $\widehat{\Phi}^* \succeq \widehat{\Phi}$ as in the previous definition. Then the following holds:

$$(+) \forall \varepsilon \in \mathbb{Q}_+^* \exists \widehat{k} \leq \chi(N, \Phi^*, \widehat{\Phi}^*, \varepsilon) (\|w_{\widehat{k}}\| \leq 2N \wedge |\langle x - w_{\widehat{k}}, w_{\widehat{\Phi}(\widehat{k})} \rangle| \leq \varepsilon),$$

where $\widehat{k} = \widehat{\Psi}^{(i)}(0)$ for a suitable $i \leq \left\lceil \frac{3N^2(\Phi^*(2N))^2}{\varepsilon^2} \right\rceil =: I$ with $\|w_{\widehat{\Psi}^{(j)}(0)}\| \leq 2N$ for all $j \leq i$ and

$$\widehat{\Psi} : \mathbb{N} \rightarrow \mathbb{N} \quad \widehat{\Psi}(k) := \xi(k, \widehat{\Phi}(k), r_{x, k, \varepsilon}^{\delta, \widehat{\Phi}}) \stackrel{!}{=} w_k + r_{x, k, \varepsilon}^{\delta, \widehat{\Phi}} \cdot w_{\widehat{\Phi}(k)},$$

where ξ is a primitive recursive function (operating on the code of r) such that

$$\forall k, j \in \mathbb{N} \forall r \in \mathbb{Q} (w_k + r \cdot w_j = w_{\xi(k, j, r)})$$

and $r_{x,k,\varepsilon}^{\delta,\widehat{\Phi}}$ is the canonical rational δ -approximation to

$$\alpha_{x,k,\varepsilon}^{\widehat{\Phi}} := \frac{\langle x - w_k, w_{\widehat{\Phi}(k)} \rangle}{\max\left(\left(\varepsilon/(2\max(\|x\|, \|w_k\|, 1))\right)^2, \|w_{\widehat{\Phi}(k)}\|^2\right)}$$

with (using the definition of $D_{N,\Phi^*,\varepsilon}$ from lemma 3.3)

$$\mathbb{Q}_+^* \ni \delta \leq \min\left(\frac{\varepsilon}{\sqrt{3}(\Phi^*(2N))^2}, \frac{\varepsilon^2}{6(\Phi^*(2N))^3(3N + D_{N,\Phi^*,\varepsilon}\Phi^*(2N))}, 1\right).$$

Finally:

$$\chi(N, \Phi^*, \widehat{\Phi}^*, \varepsilon) := (\Psi^*)^{(I)}(0),$$

where $\Psi^*(k) := \max\{\Psi'(0), \dots, \Psi'(k), k\}$ with

$$\Psi'(k) := \max\left\{\xi^*(k, \widehat{\Phi}^*(k), (-1)^l(i \cdot \delta)) : i \leq \left\lceil \frac{D_{N,\Phi^*,\varepsilon}}{\delta} \right\rceil \text{ and } l \in \{0, 1\}\right\},$$

where ξ^* is some primitive recursive majorant of ξ .

Proof: By (the proof of) lemma 3.2 applied to $\widehat{\Psi}$ there exists a \widehat{k} such that

$$\|w_{\widehat{k}}\| \leq 2\|x\| \leq 2N \wedge \|x - w_{\widehat{k}}\|^2 \leq \|x - w_{\widehat{\Psi}(\widehat{k})}\|^2 + \frac{\varepsilon^2}{3(\Phi^*(2N))^2}$$

and $\widehat{k} = \widehat{\Psi}^{(i)}(0)$ for some $i \leq I$.

The fact that $r_{x,\widehat{k},\varepsilon}^{\delta,\widehat{\Phi}}$ is δ -close to $\alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}}$ implies that

$$\|x - w_{\widehat{\Psi}(\widehat{k})}\| \leq \|x - (w_{\widehat{k}} + \alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}} \cdot w_{\widehat{\Phi}(\widehat{k})})\| + \underbrace{\delta \cdot \Phi^*(2N)}_{\delta:=}$$

Since

$$\|x - (w_{\widehat{k}} + \alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}} \cdot w_{\widehat{\Phi}(\widehat{k})})\| \leq \|x\| + \|w_{\widehat{k}}\| + |\alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}}| \cdot \Phi^*(2N) \leq 3N + D_{N,\Phi^*,\varepsilon} \cdot \Phi^*(2N)$$

we get

$$\begin{aligned} \|x - w_{\widehat{\Psi}(\widehat{k})}\|^2 &\leq \|x - (w_{\widehat{k}} + \alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}} \cdot w_{\widehat{\Phi}(\widehat{k})})\|^2 + 2\delta \cdot (3N + D_{N,\Phi^*,\varepsilon} \cdot \Phi^*(2N)) + \delta^2 \\ &\leq \|x - (w_{\widehat{k}} + \alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}} \cdot w_{\widehat{\Phi}(\widehat{k})})\|^2 + \frac{2\varepsilon^2}{3(\Phi^*(2N))^2}. \end{aligned}$$

Hence

$$\|x - w_{\widehat{k}}\|^2 \leq \|x - (w_{\widehat{k}} + \alpha_{x,\widehat{k},\varepsilon}^{\widehat{\Phi}} \cdot w_{\widehat{\Phi}(\widehat{k})})\|^2 + \frac{\varepsilon^2}{(\Phi^*(2N))^2}.$$

Since $\|w_{\widehat{\Phi}(\widehat{k})}\| \leq \Phi^*(2N)$, lemma 3.1 now yields that

$$|\langle x - w_{\widehat{k}}, w_{\widehat{\Phi}(\widehat{k})} \rangle| \leq \varepsilon.$$

□

Remark 3.9. The construction of (w_k) is taken from [2] where it is shown that this sequence is dense in $\text{Kern}(L)$ which we will need further below.

Lemma 3.10. Let Φ_0^* be the standard majorant of the Spector bar recursor functional Φ_0 (see the appendix as well as [14]). For

$$\chi^* := \chi^*(N, d) := \lambda k, \widehat{\Phi}^* . \chi(N, \lambda n.d, \widehat{\Phi}^*, 2^{-k})$$

(with χ as in lemma 3.8) define

$$\varphi^* := \varphi^*(N) := \lambda K, H. \Phi_0^*(K, \lambda n, v. \chi^*(n, \lambda D.H(v(D))), 0, 0).$$

Then

$$\exists \varphi \lesssim \varphi^* \forall K, H (\|w_{g(K(g))}\| \leq 2N \wedge |\langle x - w_{g(K(g))}, \check{w}_{H(g)} \rangle| \leq 2^{-K(g)})$$

where $g = \varphi(K, H), \check{w}_l := \begin{cases} w_l, & \text{if } \|w\| \leq d \\ 0(=w_0), & \text{otherwise} \end{cases}$ and with $\mathbb{N} \ni d \geq 4N$ and $x \in X, (w_k), N$ as before. φ^* is selfmajorizing even in the hidden arguments N, d .

Proof: By lemma 3.8, χ^- defined as (using the construction $f^M(n) := \max\{f(i) : i \leq n\}$)

$$\begin{aligned} \chi_{k_0, x, L}^-(k, \widehat{\Phi}) &:= \chi_{k_0, x, L}^-(N, d, k, \widehat{\Phi}) := \\ \min n \leq \chi(N, \lambda n.d, \widehat{\Phi}^M, 2^{-k}) &(\|w_n\| \leq 2N \wedge |\langle x - w_n, \check{w}_{\widehat{\Phi}} \rangle| \leq \varepsilon) \end{aligned}$$

resp. χ^* solves the functional (D)-interpretation resp. the monotone functional (MD)-interpretation of

$$\forall k \in \mathbb{N} \neg \neg \exists n \in \mathbb{N} \forall l \in \mathbb{N} (\|w_n\| \leq 2N \wedge |\langle x - w_n, \check{w}_l \rangle| \leq 2^{-k})$$

(Note that $\check{w}_{\widehat{\Phi}(n)} = w_{\widehat{\Phi}^-(n)}$, where $\widehat{\Phi}^-(n) := \widehat{\Phi}(n)$ if $\|w_{\widehat{\Phi}(n)}\| \leq d$ and $:= 0$, otherwise. Furthermore, $\widehat{\Phi}^* \gtrsim \widehat{\Phi}$ implies that $\widehat{\Phi}^* \gtrsim \widehat{\Phi}^-$ and so, in particular, $\widehat{\Phi}^M \gtrsim \widehat{\Phi}^-$).

From [14] (pp.200-205) it then follows that φ resp. φ^* satisfies the functional resp. monotone functional interpretation of

$$\neg \neg \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall l \in \mathbb{N} (\|w_n\| \leq 2N \wedge |\langle x - w_n, \check{w}_l \rangle| \leq 2^{-k}),$$

i.e. $g = \varphi(K, H)$ realizes

$$\forall K, H \exists g (\|w_{g(K(g))}\| \leq 2N \wedge |\langle x - w_{g(K(g))}, \check{w}_{H(g)} \rangle| \leq 2^{-K(g)})$$

while $\varphi^* \gtrsim \varphi$. Here φ is defined as φ^* but with Φ_0 instead of Φ_0^* and the solution of the D-interpretation of the $\forall \neg \neg \exists \forall$ -statement χ^- instead of χ^* . \square

Lemma 3.11. With the same assumptions as in lemma 3.10 we have

$$\exists \psi \lesssim \psi^* \forall K, H \forall i \leq K(g) (\|w_{g(i)}\| \leq 2N \wedge |\langle x - w_{g(i)}, \check{w}_{H(i,g)} \rangle| \leq 2^{-2i-1}),$$

where $g = \psi(K, H)$ and $\psi^* := \lambda K, H. \varphi^*(\widetilde{K}, \widetilde{H}_K)$ with $\widetilde{K}(g) := 2K(\tilde{g}) + 1, \widetilde{H}_K(g) := H(K(\tilde{g}), \tilde{g})$, where $\tilde{g}(k) := g(2k + 1)$ with φ^* as in lemma 3.10.

Proof: Apply lemma 3.10 to

$$K'_H(g) := \begin{cases} 2i + 1 \text{ for the least } i \leq K(\tilde{g}) \text{ s.t.} \\ \quad \|w_{g(2i+1)}\| > 2N \vee (|\langle x - w_{g(2i+1)}, \check{w}_{H(i,\tilde{g})} \rangle| > 2^{-2i-1}) \\ \quad \text{if existent,} \\ = 1, \text{ otherwise} \end{cases}$$

and $H'_K(g) := H(\frac{K'_H(g)-1}{2}, \tilde{g})$. Then for $g := \varphi(K'_H, H'_K)$

$$\|w_{g(K'_H(g))}\| \leq 2N \wedge |\langle x - w_{g(K'_H(g))}, \check{w}_{H'_K(g)} \rangle| \leq 2^{-K'_H(g)}$$

and so (by the K'_H -definition)

$$\forall i \leq K(\tilde{g}) (\|w_{\tilde{g}(i)}\| \leq 2N \wedge |\langle x - w_{\tilde{g}(i)}, \check{w}_{H(i,\tilde{g})} \rangle| \leq 2^{-2i-1}).$$

Finally, note that $\psi^* \gtrsim \psi$, where $\psi(K, H) := \tilde{g}$, since $K^* \gtrsim K$ and $H^* \gtrsim H$ implies that $\tilde{K}^* \gtrsim K'_H$ and $\tilde{H}^*_{K^*} \gtrsim H'_K$ and so by $\varphi^* \gtrsim \varphi$

$$\psi^*(K^*, H^*) = \varphi^*(\widetilde{K^*}, \widetilde{H^*_{K^*}}) \gtrsim \varphi(\widetilde{K'_H}, \widetilde{H'_K}) = \psi(K, H).$$

□

Lemma 3.12. *With the same assumptions as in lemma 3.10 we have*

$$\left\{ \begin{array}{l} \exists \theta \lesssim \theta^* \forall K, H (\|w_{g(K(g))}\| \leq 2N \\ \wedge \forall i < K(g) (\|w_{g(i)} - w_{g(i+1)}\| < 2^{-i}) \wedge |\langle x - w_{g(K(g))}, \check{w}_{H(g)} \rangle| \leq 2^{-K(g)}), \end{array} \right.$$

where $g = \theta(K, H)$ and $\theta^* := \lambda K, H. \psi^*(K, H'_K)$ with $H'_K(i, g) := \max(\{\eta^*(g(n), g(n+1)) : n < K(g)\} \cup \{H(g)\})$, where η^* is some majorant of a primitive recursive coding function η with $y_{\eta(i,j)} = y_i - y_j$ for all $i, j \in \mathbb{N}$ and ψ^* as in lemma 3.11.

Proof: Let η be a primitive recursive functions such that $y_{\eta(i,j)} = y_i - y_j$ so that also $w_{\eta(i,j)} \stackrel{!}{=} w_i - w_j$. Consider the set

$$A := \{w_{g(n)} - w_{g(n+1)} : n < K(g)\} \cup \{w_{H(g)}\}$$

and define

$$H'_K(i, g) := \left\{ \begin{array}{l} \min j \leq H_K(g) := \max(\{\eta(g(n), g(n+1)) : n < K(g)\} \cup \{H(g)\}) \text{ s.t.} \\ |\langle x - w_{g(i)}, \check{w}_j \rangle| \text{ is maximal.} \end{array} \right.$$

Now apply lemma 3.11 to K and H'_K . Then for $g := \theta(K, H) := \psi(K, H'_K)$ one has

$$\forall i \leq K(g) (\|w_{g(i)}\| \leq 2N \wedge \forall z \in A (|\langle x - w_{g(i)}, \check{z} \rangle| \leq 2^{-2i-1})).$$

For $i := K(g)$ and $z := w_{H(g)} \in A$ we, in particular, get

$$\|w_{g(K(g))}\| \leq 2N \wedge |\langle x - w_{g(K(g))}, \check{w}_{H(g)} \rangle| \leq 2^{-2K(g)-1} < 2^{-K(g)}.$$

Moreover, for all $i < K(g)$ one has (since $\|w_{g(i)} - w_{g(i+1)}\| \leq 4N \leq d$ for $i \leq K(g)$)

$$|\langle x - w_{g(i)}, w_{g(i)} - w_{g(i+1)} \rangle| \leq 2^{-2i-1}$$

and for all $0 < i \leq K(g)$

$$|\langle x - w_{g(i)}, w_{g(i-1)} - w_{g(i)} \rangle| \leq 2^{-2i-1}.$$

Hence

$$\begin{aligned} \|w_{g(i)} - w_{g(i+1)}\|^2 &= |\langle -(w_{g(i)} - w_{g(i+1)}), w_{g(i)} - w_{g(i+1)} \rangle| \\ &= |\langle x - w_{g(i)} - (x - w_{g(i+1)}), w_{g(i)} - w_{g(i+1)} \rangle| \\ &= |\langle x - w_{g(i)}, w_{g(i)} - w_{g(i+1)} \rangle - \langle x - w_{g(i+1)}, w_{g(i)} - w_{g(i+1)} \rangle| \leq 2^{-2i} \end{aligned}$$

for all $i < K(g)$. Thus $\|w_{g(i)} - w_{g(i+1)}\| \leq 2^{-i}$ for all $i < K(g)$. The proof is concluded by noticing that $\theta^* \gtrsim \theta$ since $K^* \gtrsim K$ and $H^* \gtrsim H$ implies that $(H^*)_{K^*}^+ \gtrsim H'_K$ and so

$$\theta^*(K^*, H^*) = \psi^*(K^*, (H^*)_{K^*}^+) \gtrsim \psi(K, H'_K) = \theta(K, H).$$

□

Lemma 3.13. *With the same assumptions as in lemma 3.10 we have*

$$\exists \xi \lesssim \xi^* \forall K, H \left(\|C((w_{g(k)})_{k \in \mathbb{N}})\| \leq 2N + 1 \wedge |\langle x - C((w_{g(k)})_k), \check{w}_{H(g)} \rangle| \leq 2^{-K(g)} \right)$$

where $g = \xi(K, H)$ and $\xi^* := \lambda K, H. \theta^*(K', H)$ with θ^* as in lemma 3.12 and $K'(g) := K(g) + 4 + \lceil \log_2 d \rceil$. Here C refers to the completion operator from [14] (pp. 432-434).

Proof: By the definition of the operation $(w_{g(k)}) \mapsto \widehat{(w_{g(k)})}$ used in the defining axiom (C) for the completion operator C (see [14], p.433) it follows from lemma 3.12 applied to K', H that $\|w_{g(K'(g))}\| \leq 2N$ and (by (C)) that for $g := \xi(K, H) := \theta(K', H)$

$$\|w_{g(K'(g))} - C((w_{g(k)})_k)\| \leq 2^{-K'(g)+3} \wedge |\langle x - w_{g(K'(g))}, \check{w}_{H(g)} \rangle| \leq 2^{-K'(g)} \leq 2^{-K(g)-1}.$$

Since $\|\check{w}_{H(g)}\| \leq d$, the conclusion follows. □

Lemma 3.14. *With the same assumptions as in lemma 3.10 plus $|L(y_{k_0})| \geq 2^{-b}$ and $\|y_{k_0}\| \leq B$ for $b, B \in \mathbb{N}^*$ we have*

$$\exists \zeta \lesssim \zeta_{b,B}^* \forall K, H \left(\|y\| \leq 2N + 1 \wedge y \in \text{Kern}(L) \wedge |\langle x - y, \check{w}_{H(\alpha)} \rangle| \leq 2^{-K(\alpha)} \right)$$

where $\alpha = \zeta(K, H), y := C((y_{\alpha(k)})_{k \in \mathbb{N}})$ and $\lambda b. \zeta_{b,B}^*$ is selfmajorizing (for fixed B) and defined primitive recursively in ξ^* from lemma 3.13 in the proof below.

Proof: Define $a_k := w_{g(k)}$ and let \widehat{a}_k denote the k -th element of the sequence $\widehat{(a_k)}$ resulting from the transformation $\widehat{(\)}$ used in the defining axiom (C) for the completion operator C (see [14], p.433). For $k \in \mathbb{N}$ let $\alpha(k)$ be the least index such that (for $g = \xi(K, H)$ as in lemma 3.13)

$$\|y_{\alpha(k)} - \widehat{a}_{k+1}\| < 2^{-k-1}.$$

Note that such an index exists since (y_n) is dense in \mathcal{L} . Let $\zeta(K, H)$ be defined as this α .

Then

$$\begin{aligned} \|y_{\alpha(k)} - y_{\alpha(k+1)}\| &\leq \|y_{\alpha(k)} - \widehat{a}_{k+1}\| + \|\widehat{a}_{k+1} - \widehat{a}_{k+2}\| + \|\widehat{a}_{k+2} - y_{\alpha(k+1)}\| \\ &< 2^{-k-1} + 7 \cdot 2^{-k-2} + 2^{-k-2} = 5 \cdot 2^{-k-1} \end{aligned}$$

and so

$$\|y_{\alpha(k)} - \widehat{a}_{k+1}\|(k+1) < 6 \cdot 2^{-k-1}.$$

Hence $\widehat{y_{\alpha(k)}} = y_{\alpha(k)}$ for all $k \in \mathbb{N}$ and so

$$C((y_{\alpha(k)})_k) = \lim_{k \rightarrow \infty} y_{\alpha(k)} = \lim_{k \rightarrow \infty} \widehat{w_{g(k)}} = C((w_{g(k)})_k) \stackrel{!}{\in} \text{Kern}(L).$$

We now construct a majorant for α . First note that

$$w_{g(k+1)} = y_{g(k+1)} - \frac{L(y_{g(k+1)})}{L(y_{k_0})} \cdot y_{k_0},$$

where

$$\left| \frac{L(y_{g(k+1)})}{L(y_{k_0})} \right| \leq 2^b \cdot \gamma(g(k+1), 1) =: C_{b,g,k}$$

with γ as defined in section 2 since $|L(y_{k_0})| \geq 2^{-b}$ and $|L(y_{g(k+1)})| \leq \|y_{g(k+1)}\| \leq \gamma(g(k+1), 1)$.

Hence (using that $\widehat{w_{g(k)}} = w_{g(i)}$ for some $i \leq k$)

$$\begin{aligned} \alpha(k) &\leq \alpha_{b,g,B}^*(k) := \\ &\max \{ \langle y_i + (-1)^q \cdot (j \cdot 2^{-l-1}/B) \cdot y_{k_0} \rangle : l \leq k; j \leq 2^{k+1} \cdot C_{b,g,k} \cdot B; i \leq g(k+1); q \in \{0, 1\} \}. \end{aligned}$$

Here $\langle \rangle$ is some canonical code built up primitive recursively from i, k_0 and the code of the rational number $(-1)^q \cdot (j \cdot 2^{-l-1}/B)$.

Since $\alpha_{b,g,B}^*$ is self-majorizing (also in g) it follows that $\zeta_{b,B}^*(K^*, H^*) := \alpha_{b,\xi^*(K^*, H^*),B}^*$ (with ξ^* as in lemma 3.13) satisfies the lemma. \square

Remark 3.15. As B in the previous lemma we may take e.g. $\gamma(k_0, 1)$ with γ as in section 2.

Corollary to the proof of lemma 3.14: For the weakened version of lemma 3.14 with $y \in \text{Kern}(L)$ being replaced by $|L(y)| < 2^{-l}$ for $l \in \mathbb{N}$ one only needs that

$$\forall i \leq l + 5 \quad (|L(w_{g(i)})| \stackrel{!}{<} 2^{-l-5})$$

instead of $L(w_{g(i)}) = 0$ for all i . This is the case because of (for $a_k := w_{g(k)}$ and \widehat{a}_k as before)

$$|L(y_{\alpha(l+4)}) - L(\widehat{a}_{l+5})| \stackrel{!}{<} 2^{-l-5}$$

with $\widehat{a}_{l+5} = w_{g(i)}$ for some $i \leq l + 5$ and (by (\mathcal{C}))

$$|L(y_{\alpha(l+4)}) - L(y)| \stackrel{!}{<} 2^{-l-1}$$

so that

$$\begin{aligned} |L(y)| &\leq \\ |L(y) - L(y_{\alpha(l+4)})| + |L(y_{\alpha(l+4)}) - L(w_{g(i)})| + |L(w_{g(i)})| &< 2^{-l-1} + 2^{-l-5} + 2^{-l-2} \leq 2^{-l}. \end{aligned}$$

4. Construction of the weak sequential compactness functional

We now start to give the construction of the functional Ω^* in theorem 2.3 leaving that construction for the time being somewhat implicit.

Let K, W and $(x_n) \subseteq B_1(0)$ be given (with majorants K^*, W^*) and $(L_n), L$, be as in (+) in section 2, i.e. $L \in \prod_{n \in \mathbb{N}} [-\|y_n\|, \|y_n\|]$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ where

$$(1) \forall k \in \mathbb{N} \exists n \leq f(k) (n \geq k \wedge d(L_n, L) < 2^{-k}).$$

Now – for $j \in \mathbb{N}$ and $w \in X$ – let $k_{w,j} \in \mathbb{N}$ be such that

$$(2) \|y_{k_{w,j}}\| \leq 2\|w\| \wedge \forall z \in \mathcal{L} \cap B_1(0) (|\langle z, w - y_{k_{w,j}} \rangle| < 2^{-j-1})$$

and define

$$\chi_0(w, n) := f(n + k_{w,n} + 3) \text{ and } k := K(0, \chi_0) + k_{W(0, \chi_0), K(0, \chi_0)} + 3.$$

(1) applied to k yields

$$(3) \left\{ \begin{array}{l} \exists n \in [K(0, \chi_0) + k_{W(0, \chi_0), K(0, \chi_0)} + 3, f(K(0, \chi_0) + k_{W(0, \chi_0), K(0, \chi_0)} + 3)] \\ (d(L_n, L) < 2^{-K(0, \chi_0) - k_{W(0, \chi_0), K(0, \chi_0)} - 3}). \end{array} \right.$$

Hence

$$(4) |L_n(y_{k_{W(0, \chi_0), K(0, \chi_0)}}) - L(y_{k_{W(0, \chi_0), K(0, \chi_0)}})| < 2^{-K(0, \chi_0) - 2},$$

i.e.

$$(5) |\langle x_n, y_{k_{W(0, \chi_0), K(0, \chi_0)}} \rangle - L(y_{k_{W(0, \chi_0), K(0, \chi_0)}})| < 2^{-K(0, \chi_0) - 2}.$$

Case 1: $|L(y_{k_{W(0, \chi_0), K(0, \chi_0)}})| < 2^{-K(0, \chi_0) - 2} =: K_1$. Together with (5) this gives

$$(6) |\langle x_n, y_{k_{W(0, \chi_0), K(0, \chi_0)}} \rangle| < 2^{-K(0, \chi_0) - 1}.$$

(2) applied to $j := K(0, \chi_0), w := W(0, \chi_0)$ and $y := x_n$ yields

$$(7) \|y_{k_{W(0, \chi_0), K(0, \chi_0)}}\| \leq 2\|W(0, \chi_0)\| \wedge |\langle x_n, W(0, \chi_0) - y_{k_{W(0, \chi_0), K(0, \chi_0)}} \rangle| < 2^{-K(0, \chi_0) - 1}.$$

(6) and (7) imply

$$(8) |\langle x_n, W(0, \chi_0) \rangle| < 2^{-K(0, \chi_0)}.$$

Since

$$(9) f(K(0, \chi_0) + k_{W(0, \chi_0), K(0, \chi_0)} + 3) = \chi_0(W(0, \chi_0), K(0, \chi_0))$$

it follows that $n \in [K(0, \chi_0), \chi_0(W(0, \chi_0), K(0, \chi_0))]$ and so the claim of theorem 2.3 is satisfied with $v := 0$ and the above defined χ_0 .

Remark 4.1. Although, of course, the much more difficult

‘Case 2: $|L(y_{k_{W(0, \chi_0), K(0, \chi_0)}})| \geq 2^{-K(0, \chi_0) - 2}$ ’, is still to come, let us indicate already now how an

argument as the one given above can be converted into the construction of Ω^* : first notice that we did not use (2) for **all** $y \in \mathcal{L} \cap B_C(0)$ but only for

$$\{x_n : n \in [j + k_{w,j} + 3, f(j + k_{w,j} + 3)]\}.$$

In this restricted form, a majorant $\lambda w^*, j.k_{w^*,j}^*$ of $\lambda w, j.k_{w,j}$ can be constructed by lemma 3.7. Using K^*, W^* and a majorant for f one can easily compute the majorants $\Phi^*, \widehat{\Phi}^*$ for the function(al)s $\Phi, \widehat{\Phi}$ in question as well as the bound N . We omit the details here as they are similar (but simpler) than that for the case $k_{\alpha,w,j}$ to be treated further below. Observe, moreover, that also (1) has not been used for all k but only for the ‘counter-functional’ $\tilde{k}(f, L) := K(0, \chi_0) + k_{W(0, \chi_0), K(0, \chi_0)} + 3$ (with χ_0 defined in terms of f as above). Hence we only need a solution for the monotone functional interpretation of (1) – i.e. (**) in section 2 – (provided in the appendix) applied to $\tilde{k}^* := \lambda f, L.K^*(0, \chi_0^* f) + k_{1, K^*(0, \chi_0^* f)}^* + 3$, where $(\chi_0^* f)(w^*, j) := \chi_0^*(w^*, j) := f(j + k_{w^*, j}^* + 3)$, to get a majorant for f and consequently for χ_0 as a functional Ω^* in K^*, W^* , where W^* may simply be taken as the constant-1 functional since we assume that W is norm-bounded by 1 (see section 2).

Case 2: $|L(y_{k_{W(0, \chi_0), K(0, \chi_0)}})| \geq 2^{-K(0, \chi_0)-2}$. Put $k_0 := k_{W(0, \chi_0), K(0, \chi_0)}$.

Then $\|y_{k_0}\| \leq 2\|W(0, \chi_0)\| \leq 2$.

Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be such that for $y := \lim \widehat{y_{\alpha(n)}} (= C((y_{\alpha(k)}))_k)$

$$(10) \quad \left\{ \begin{array}{l} \|y\| \leq 5 \wedge |L(y)| \leq 2^{-K(0, \chi_0)-3} \wedge \\ \forall z \in \text{Kern}(L) \cap B_{K_3}(0) \quad (|\langle z, y_{k_0} - y \rangle| \leq 2^{-K(\tilde{x}, \chi)-3}/K_4), \end{array} \right.$$

where K_3, K_4 will be defined below and

$$\tilde{x} := \frac{L(\widehat{x})}{\langle \widehat{x}, \widehat{x} \rangle} \cdot \widehat{x} \text{ with } \widehat{x} := y_{k_0} - y$$

and χ is defined as in (13) below.

Since – by (10) – $\|y\| \leq 5$ we have that

$$(11) \quad \|\widehat{x}\| \leq \|y_{k_0}\| + \|y\| \leq 7.$$

By the case and (10) we obtain that (using in defining $L(\widehat{x})$ the approximating Cauchy sequence $y_{k_0} + \widehat{y_{\alpha(n+3)}}$ for $y_{k_0} - y$ which has the rate of convergence 2^{-n} so that it remains unchanged under the $\widehat{(\)}$ -construction)

$$\begin{aligned} |L(\widehat{x})| &\stackrel{!}{=} |L(y_{k_0}) - L(y)| \geq |L(y_{k_0})| - 2^{-K(0, \chi_0)-3} \\ &\geq K_1 - 2^{-K(0, \chi_0)-3} = 2^{-K(0, \chi_0)-3} \geq 2^{-K(0, \chi_0)-4}. \end{aligned}$$

Using $\|L\| \leq 1$ (!) this implies that

$$(12) \quad \|\widehat{x}\| \geq 2^{-K(0, \chi_0)-4} \geq 2^{-K(0, \chi_0)-5} \text{ and so } |\langle \widehat{x}, \widehat{x} \rangle| = \|\widehat{x}\|^2 \geq 2^{-2K(0, \chi_0)-10}.$$

Now define

$$(13) \quad \chi(w, n) := f(n + k_{\alpha, w, n+3} + 4),$$

where for all $w \in X, j \in \mathbb{N}$ $k_{\alpha, w, j} \in \mathbb{N}$ is such that

$$(14) \|y_{k_{\alpha, w, j}}\| \leq 2\|w\| \wedge \forall z \in \{x_n : n \in \mathbb{N}\} \cup \{\tilde{x}\} (|\langle z, w - y_{k_{\alpha, w, j}} \rangle| \leq 2^{-j}).$$

Note that the condition on $k_{\alpha, w, j}$ does indeed depend on α since \tilde{x} does.

We have (using that $|L(\hat{x})| \stackrel{!}{\leq} \|\hat{x}\| \leq 7 \leq 8$)

$$K_2 \geq \frac{|L(\hat{x})|}{\|\hat{x}\|} = \frac{|L(\hat{x})|}{|\langle \hat{x}, \hat{x} \rangle|} \cdot \|\hat{x}\| = \|\tilde{x}\|,$$

where $K_2 := 8 \cdot 2^{K(0, \chi_0)+5} \leq 8 \cdot 2^{K^*(0, \chi_0^*)+5} =: K_2^*$.

Note also that $1 \geq \|W(\tilde{x}, \chi)\|$ and $\left| \frac{L(\tilde{x})}{\langle \tilde{x}, \tilde{x} \rangle} \right| \leq K_2 \cdot 2^{K(0, \chi_0)+5} =: K_4 \leq K_4^*$

$:= K_2^* \cdot 2^{K^*(0, \chi_0^*)+5}$ for majorants K^*, χ_0^* of K, χ_0 .

Now put $w_1 := W(\tilde{x}, \chi), \hat{j} := K(\tilde{x}, \chi) + 3$ and $k_{\alpha} := k_{\alpha, w_1, \hat{j}}$.

Let

$$n \in [\hat{j} + k_{\alpha} + 1, f(\hat{j} + k_{\alpha} + 1)] \subseteq [K(\tilde{x}, \chi), \chi(W(\tilde{x}, \chi), K(\tilde{x}, \chi))]$$

be such that

$$d(L_n, L) < 2^{-\hat{j}-k_{\alpha}-1}.$$

Then

$$(15) |L_n(y_{k_{\alpha}}) - L(y_{k_{\alpha}})| < 2^{-\hat{j}}.$$

Now define $u := y_{k_{\alpha}} - \xi(y_{k_0} - y) \stackrel{!}{\in} \text{Kern}(L)$, where $\xi := \frac{L(y_{k_{\alpha}})}{L(y_{k_0} - y)} \in \mathbb{R}$. Note that

$$(16) y_{k_{\alpha}} = \xi \cdot \hat{x} + u$$

with

$$(17) \|y_{k_{\alpha}}\| \leq 2\|W(\tilde{x}, \chi)\| \leq 2.$$

Hence $\|L(y_{k_{\alpha}})\| \leq 2$ and, therefore,

$$(18) |\xi| \leq 2 \cdot 2^{K(0, \chi_0)+4}.$$

Thus

$$(19) \|u\| \stackrel{(17), (11)}{\leq} 2 + |\xi| \cdot 7 \stackrel{(18)}{\leq} 2 + 14 \cdot 2^{K(0, \chi_0)+4} =: K_3 \leq K_3^*,$$

where $K_3^* := 2 + 14 \cdot 2^{K^*(0, \chi_0^*)+4}$.

By (14) (applied to $w := W(\tilde{x}, \chi), j := \hat{j}, z := \tilde{x}$) and (16) we get that

$$(20) |\langle \tilde{x}, W(\tilde{x}, \chi) - (\xi \hat{x} + u) \rangle| \leq 2^{-\hat{j}}.$$

By (10) (applied to $z := u$) we have

$$(21) |\langle u, \tilde{x} \rangle| = \left| \frac{L(\hat{x})}{\langle \hat{x}, \hat{x} \rangle} \right| \cdot |\langle u, \hat{x} \rangle| = \left| \frac{L(\hat{x})}{\langle \hat{x}, \hat{x} \rangle} \right| \cdot |\langle u, y_{k_0} - y \rangle| \leq 2^{-\hat{j}}.$$

Hence

$$(22) \quad |\langle \xi \widehat{x} + u, \tilde{x} \rangle - \xi \langle \widehat{x}, \tilde{x} \rangle| \leq 2^{-\widehat{j}}.$$

Also

$$(23) \quad \xi \langle \widehat{x}, \tilde{x} \rangle = \xi \langle \widehat{x}, \frac{L(\widehat{x})}{\langle \widehat{x}, \widehat{x} \rangle} \cdot \widehat{x} \rangle = \xi L(\widehat{x}) = L(y_{k_\alpha}).$$

(22) and (23) imply that

$$(24) \quad |\langle \xi \widehat{x} + u, \tilde{x} \rangle - L(y_{k_\alpha})| \leq 2^{-\widehat{j}}.$$

(20) yields that

$$(25) \quad |\langle W(\tilde{x}, \chi), \tilde{x} \rangle - \langle \xi \widehat{x} + u, \tilde{x} \rangle| \leq 2^{-\widehat{j}}$$

and so using (24)

$$(26) \quad |\langle W(\tilde{x}, \chi), \tilde{x} \rangle - L(y_{k_\alpha})| \leq 2^{-\widehat{j}+1}.$$

Together with (15) this implies

$$(27) \quad |\langle W(\tilde{x}, \chi), \tilde{x} \rangle - L_n(y_{k_\alpha})| \leq 2^{-\widehat{j}+2},$$

i.e.

$$(28) \quad |\langle W(\tilde{x}, \chi), \tilde{x} \rangle - \langle y_{k_\alpha}, x_n \rangle| \leq 2^{-\widehat{j}+2}.$$

By (14) (applied to $z := x_n$) we have

$$(29) \quad |\langle W(\tilde{x}, \chi) - y_{k_\alpha}, x_n \rangle| \leq 2^{-\widehat{j}}.$$

By (28) and (29) we finally obtain

$$(30) \quad |\langle W(\tilde{x}, \chi), \tilde{x} \rangle - \langle W(\tilde{x}, \chi), x_n \rangle| < 2^{-\widehat{j}+3} = 2^{-K(\tilde{x}, \chi)},$$

i.e. $v := \tilde{x}$ and χ satisfy our claim.

In remark 4.1 we showed how to compute a primitive recursive (in K^*, W^* and a majorant f^* of f) selfmajorizing bound $\lambda w^*, j.k_{w^*, j}^*$ for $\lambda k, j.k_{w, j}$ and so – based on this – a primitive recursive majorant k_0^* on k_0 :

$$k_0^* := k_{W^*(0, \chi_0^* f), K^*(0, \chi_0^* f)}^*, \text{ where } \chi_0^*(f, w^*, j) := f(j + k_{w^*, j}^* + 3)$$

and we may take $W^*(0, \chi_0^*) := 1$ as we may assume that $\|W(w, \chi)\| \leq 1$ for all arguments w, χ by the reasoning given in section 2.

Using $\lambda w^*, j.k_{w^*, j}^*$ we below will compute a primitive recursive selfmajorizing bound $\lambda \alpha^*, w^*, j.k_{\alpha^*, w^*, j}^*$ on $\lambda \alpha, w, j.k_{\alpha, w, j}$.

Then (using $\lambda w^*, j.k_{w^*, j}^*$ and $\lambda \alpha^*, w^*, j.k_{\alpha^*, w^*, j}^*$) we compute a bar recursive selfmajorizing bound $\alpha^*(f^*, K^*, W^*)$ (for majorants f^*, K^*, W^* of f, K, W) for $\alpha(f, K, W)$ (viewed as a functional of the arguments shown, i.e. $\lambda f^*, K^*, W^*. \alpha^*(f^*, K^*, W^*)$ majorizes $\lambda f, K, W. \alpha(f, K, W)$).

Now let (for given K, W)

$$\tilde{k}(f, L) := \begin{cases} K(0, \chi_0) + k_0 + 3, & \text{in case 1} \\ K(\tilde{x}, \chi) + k_\alpha + 4, & \text{in case 2,} \end{cases}$$

where $k_0 := k_{W(0,\chi_0),K(0,\chi_0)}$ and $k_\alpha := k_{\alpha,W(\tilde{x},\chi),K(\tilde{x},\chi)+3}$.

Note that χ_0, χ do depend on f and that the case distinction, moreover, depends on L . Applying (*) from section 2 to \tilde{k} yields a function f such that χ_0 (in case 1) resp. χ (in case 2) built with **this** f satisfy the claim as a functional $\Omega(K, W)$ in K, W with either $v := 0$ (in case 1) or $v := \hat{x}$ (in case 2). Using the above (selfmajorizing bounds) one can construct a majorant \tilde{k}^* of \tilde{k} as follows

$$\tilde{k}^* := \lambda f, L. \max\{K^*(0, \chi_0^* f) + k_0^* + 3, K^*(K_2^*, \chi^* f) + k_{\alpha^*}^* + 4\},$$

where $(\chi^* f)(k, j) := \chi^*(k, j) := f(j + k_{\alpha^*}^*, k, j+3 + 4)$ and $k_{\alpha^*}^* := k_{\alpha^*, 1, K^*(1, \chi^* f)+3}$. Now the bar recursive functionals $f_{(\cdot)}^*, L_{(\cdot)}^*$ (mentioned in (**)) in section 2) applied to \tilde{k}^* yield a majorant $\chi^* = \Omega^*(K^*)$ for a χ as a functional $\Omega^* \gtrsim \Omega$ in K^* satisfying the claim.

One obvious problem is that we will not be able to compute $k_{w,j}, k_{\alpha,w,j}$ satisfying respectively (2) and (14) for (modulo some norm bounds) **all** $z \in \mathcal{L}$ resp. for **all** $z \in \{x_n : n \in \mathbb{N}\} \cup \{\tilde{x}\}$. Nor will we be able to compute α such that (10) holds for **all** $z \in \text{Kern}(L)$. However, this is actually not necessary: for $k_{w,j}$ we discussed this already in remark 4.1. Consider next $k_{\alpha,w,j}$: from (14) we only use (via (20)) that

$$|\langle w - y_{k_{\alpha,w,j}}, \tilde{x} \rangle| < 2^{-j}$$

(to establish (25)) as well as

$$|\langle w - y_{k_{\alpha,w,j}}, x_n \rangle| < 2^{-j}$$

for all $n \leq f(j + k_{\alpha,w,j} + 4)$ (to establish (29)). For this to achieve we consider the following operation $\mathcal{L}_{\mathbb{Q}} \rightarrow \mathcal{L}_{\mathbb{Q}}$ given on the codes of elements of $\mathcal{L}_{\mathbb{Q}}$:

$$\widehat{\Phi}_{L,W,f,j,K,\chi_0,\alpha,w}(k) := \widehat{\Phi}(k) := \begin{cases} \langle x_l \rangle, & \text{for the least } l \leq f(j + k + 4) \text{ s.t. } |\langle w - y_k, x_l \rangle| \text{ is} \\ & \text{maximal, if } |\langle w - y_k, x_l \rangle| \geq |\langle w - y_k, \tilde{x} \rangle| \text{ for this } l, \\ \langle r_{\tilde{x},j} \cdot \hat{x}_j \rangle, & \text{otherwise,} \end{cases}$$

where $r_{\tilde{x},j}$ is the canonical $2^{-j-1}/(42\lceil\|w\rceil\rceil)$ -good rational approximation to $\frac{L(\tilde{x})}{\langle \tilde{x}, \tilde{x} \rangle}$ provided that ‘Case 2’ holds and $|L(y)| \leq 2^{-K(0,\chi_0)-3}$ so that $\frac{L(\tilde{x})}{\langle \tilde{x}, \tilde{x} \rangle}$ is defined with $\left| \frac{L(\tilde{x})}{\langle \tilde{x}, \tilde{x} \rangle} \right| \leq K_4 \leq K_4^*$ and $\|\hat{x}\| \leq 7$ (note that these conditions hold for \hat{x} defined in terms of the actual α) and we put $r_{\tilde{x},j} := 0$, otherwise.

Moreover, $\hat{x}_j := y_{k_0} - \widehat{y}_{\alpha(j+4+\lceil\log_2(6K_4\lceil\|w\rceil\rceil)}$ so that (under the above condition) $\|\tilde{x} - r_{\tilde{x},j} \cdot \hat{x}_j\| \leq 2^{-j-1}/(3\lceil\|w\rceil\rceil)$ and consequently

$$|\langle w - y_{k_{\alpha,w,j}}, \tilde{x} \rangle - \langle w - y_{k_{\alpha,w,j}}, r_{\tilde{x},j} \cdot \hat{x}_j \rangle| \leq 2^{-j-1}$$

provided that (as will be the case in lemma 3.7) $\|w - y_{k_{\alpha,w,j}}\| \leq 3\|w\|$.

For all $k \in \mathbb{N}$ we have that

$$\|y_{\widehat{\Phi}(k)}\| \leq K_2 + 1 \leq K_2^* + 1$$

and so $\Phi_{K^*, \chi_0^*}^* := \Phi^*(n) := K_2^* + 1$ satisfies the condition in the corollary to the proof of lemma 3.7.

As $\widehat{\Phi}^*$ in lemma 3.7 we can take (using that W is majorized by 1)

$$\begin{aligned} \widehat{\Phi}_{f,j,K^*,\chi_0^*,\alpha^*,w^*}^*(n) &:= \widehat{\Phi}^*(n) := \\ \max \{ \langle x_l \rangle, \langle (-1)^q i \cdot 2^{-j-1} / (42w^*) \cdot (y_k - y_m) \rangle : q \in \{0, 1\}; \\ l \leq f(j+n+4); i \leq 2^{j+1} \cdot 42w^* \cdot K_4^*; k \leq k_0^*; m \leq \alpha^*(j+4 + \lceil \log_2(6K_4^*w^*) \rceil) \}. \end{aligned}$$

We freely can assume here that the coding $\langle \cdot \rangle$ of elements in $\mathcal{L}_{\mathbb{Q}}$ is such that $\widehat{\Phi}_{f,j,K^*,\chi_0^*,\alpha^*,w^*}^*$ is also selfmajorizing in w^* for, otherwise, we could simply have taken $\max_{a \leq w^*} \{ \widehat{\Phi}_{f,j,K^*,\chi_0^*,\alpha^*,a}^* \}$ instead (similarly for H^* w.r.t. M^*, K^* defined below).

The corollary to the proof of lemma 3.7 applied to $\widehat{\Phi}$ now yields (for all $w \in X$ with $\|w\| \leq w^* \in \mathbb{N}^*$) an upper bound $\chi(w^*, \Phi^*, \widehat{\Phi}^*, 2^{-j-1})$ for a code $k_{\alpha,w,j}$ such that $y_{k_{\alpha,w,j}}$ satisfies the conditions we need. This upper bound is primitive recursive in $f, j, K^*, \chi_0^*, \alpha^*, w^*$ (note that K_4^* is primitive recursive in K^*, χ_0^*), where χ_0^* – using $\lambda w^*, j.k_{w^*,j}^*$ – in turn is primitive recursive in K^*, f . As this bound is selfmajorizing it can be taken as the majorant $\lambda \alpha^*, w^*, j.k_{\alpha^*,w^*,j}^*$ we have been looking for.

For α we first observe that (10) is used in the proof above only for

$$z := u := y_{k_\alpha} - \frac{L(y_{k_\alpha})}{L(y_{k_0} - y)} \cdot (y_{k_0} - y).$$

Hence we can use lemma 3.14³ (taking $N := B := 2$, $d := K_3$ and $b := K(0, \chi_0) + 2$) applied to $K'_f(\alpha) := K(\tilde{x}, \chi f) + \lceil \log_2(K_4) \rceil + 4$, provided that ‘Case 2’ holds and $|L(y)| \leq 2^{-K(0, \chi_0) - 3}$ as well as $\|\widehat{x}\| \leq 7$ (as will be the case for the actual α being constructed) and define $K'_f(\alpha) := 0$, otherwise. $K'_f(\alpha)$ is majorized by $(K'_{f^*})^*(\alpha^*) := K^*(K_2^*, \chi^* f^*) + \lceil \log_2(K_4^*) \rceil + 4$ (in any majorants K^* of K , f^* of f and α^* of α) and $H(\alpha)$ being defined as the least index l such that

$$\|\tilde{w}_l - (y_{k_\alpha} - \frac{L(y_{k_\alpha})}{L(y_{k_0} - y)} \cdot (y_{k_0} - y))\| \leq 2^{-K(\tilde{x}, \chi) - 3} / 2K_3K_4 = 2^{-K(\tilde{x}, \chi) - 4} / K_3K_4 =: M,$$

where $y := C((y_{\alpha(k)})_{k \in \mathbb{N}})$ provided that the above assumptions hold, and $:= 0$, otherwise. Then with $\zeta_{b,B}^*$ and ζ as in lemma 3.14, $\alpha := \zeta(K'_f, H)$ satisfies the claim.

Note that (using the primitive recursive in K^*, f^* constructions of χ_0^* and $\lambda \alpha^*, w^*, j.k_{\alpha^*,w^*,j}^*$) that $(K'_{f^*})^*$ is primitive recursive in K^*, f^* .

H can be majorized by a functional H^* that is primitive recursive in K^*, f^* (where K^*, f^* majorize K, f) using the majorants $\lambda w^*, j.k_{w^*,j}^*$ and $\lambda \alpha^*, w^*, j.k_{\alpha^*,w^*,j}^*$ for $\lambda w, j.k_{w,j}$ and $\lambda \alpha, w, j.k_{\alpha,w,j}$ constructed above. Hence by lemma 3.14 (with the hidden parameter $d^* := K_3^*$ instead of $d := K_3$ and $b^* := K^*(0, \chi_0^*) + 2$ instead of $b := K(0, \chi_0) + 2$) $\zeta_{b^*,2}^*((K'_{f^*})^*, H^*)$ is a majorant for α that is bar recursive (in the sense of Φ_0) in K^*, f^* and selfmajorizing.

³In fact, the statement in the corollary to the proof of this lemma (with $l := K(0, \chi_0) + 3 \leq K^*(0, \chi_0^*) + 3$) is sufficient.

We now give the construction of H^* (writing for simplicity χ_0^*, χ^* instead of $\chi_0^* f^*, \chi^* f^*$): First put $M^* := 2^{-K^*(1, \chi^*)-4} / K_3^* K_4^* \leq M$. By (18) we have

$$\left| \frac{L(y_{k_\alpha})}{L(k_0 - y)} \right| \leq 2^{K^*(0, \chi_0^*)+5}.$$

In view of (7), we may also use $\|y_{k_0} - y\| \leq 7$. Hence (for $\alpha^* \gtrsim \alpha$) and

$$H^*(\alpha^*) := \max \left\{ \langle y_i + (-1)^q \cdot j \cdot \frac{M^*}{56 \cdot 2^{K^*(0, \chi_0^*)+3}} (y_k - y_l) \rangle : q \in \{0, 1\}; i \leq k_{\alpha^*}^*; \right. \\ \left. j \leq 2^{K^*(0, \chi_0^*)+3} 2^{K^*(0, \chi_0^*)+5} \cdot \frac{56}{M^*}; k \leq k_0^*; l \leq \alpha^* (\lceil \log_2 \frac{8 \cdot 2^{K^*(0, \chi_0^*)+3} \cdot 2^{K^*(0, \chi_0^*)+5}}{M^*} \rceil + 3) \right\}$$

one has

$$\exists q \leq H^*(\alpha^*) (\|y_q - (y_{k_\alpha} - \frac{L(y_{k_\alpha})}{L(y_{k_0} - y)} (y_{k_0} - y))\| \leq \frac{M^*}{4 \cdot 2^{K^*(0, \chi_0^*)+3}} \leq \frac{M^*}{4}).$$

Because of $w_q = y_q - L(y_q) \cdot \frac{y_{k_0}}{L(y_{k_0})}$ with $\left\| \frac{y_{k_0}}{L(y_{k_0})} \right\| \leq 2 \cdot 2^{K^*(0, \chi_0^*)+3}$ it follows from

$$|L(y_q)| \leq |L(y_q) - L(y_{k_\alpha} - \frac{L(y_{k_\alpha})}{L(y_{k_0} - y)} (y_{k_0} - y))| + |L(y_{k_\alpha} - \frac{L(y_{k_\alpha})}{L(y_{k_0} - y)} (y_{k_0} - y))| \\ \leq \frac{M^*}{4 \cdot 2^{K^*(0, \chi_0^*)+3}} + 0 \leq \frac{M^*}{3 \cdot 2^{K^*(0, \chi_0^*)+3}}$$

that $\|w_q\| \leq \|y_q\| + 1 \leq K_3 + 2$ (and so $w_q = \check{w}_q$) and $\|\check{w}_q - y_q\| \leq \frac{2M^*}{3}$. Hence

$$\left\| \check{w}_q - (y_{k_\alpha} - \frac{L(y_{k_\alpha})}{L(y_{k_0} - y)} (y_{k_0} - y)) \right\| \leq M^* \leq M.$$

Hence we can take $H^*(\alpha^*)$ satisfies the claim. \square

One problem that we have not yet addressed is that $L := L_{\bar{k}}$ is not a linear functional. The linearity is used on several occasions: firstly, in the proof above (below (10)) we used that $L(y_{k_0} - y) = L(y_{k_0}) - L(y)$. Note that we accommodated space for another error of $2^{-K(0, \chi_0)-4}$ in this line which allows us to replace $L(y_{k_0} - y) = L(y_{k_0}) - L(y)$ by $|L(y_{k_0} - y) - (L(y_{k_0}) - L(y))| \leq 2^{-K(0, \chi_0)-4}$ which follows from

$$(+) \forall i \leq \alpha(K(0, \chi_0) + 12) (|L(y_{k_0} - y_i) - (L(y_{k_0}) - L(y_i))| \leq 2^{-K(0, \chi_0)-5})$$

since (using the completion axiom (C))

$$|L(y) - L(y_{\alpha(\widehat{K(0, \chi_0)+12})})|, |L(y_{k_0} - y) - L(y_{k_0} - y_{\alpha(\widehat{K(0, \chi_0)+12})})| \leq 2^{-K(0, \chi_0)-6}$$

which in turn can be proved based on approximate instances of linearity only (see the discussion at the end of section 2). Here $y_{\alpha(\widehat{K(0, \chi_0)+12})}$ denotes the $(K(0, \chi_0) + 12)$ -th point of the sequence $(\widehat{v_k})$ that results from the sequence (v_k) defined by $v_k := y_{\alpha(k)}$ by applying the operator $\widehat{(\cdot)}$ from the completion axiom (C) in [14] (pp. 433).

From this also another use of linearity gets replaced by approximate linearity, namely the use of

$\|L\| \leq 1$ to derive that $\|\widehat{x}\| \geq 2^{-K(0,\chi_0)-5}$ in (12) (while $|L(y_n)| \leq \|y_n\|$ for all n follows from $L \in \prod[-\|y_n\|, \|y_n\|]$, to extend this to general points in \mathcal{L} needs linearity): suppose that

$$\|\widehat{x}\| = \|y_{k_0} - y\| < 2^{-K(0,\chi_0)-5}.$$

Then $\|y_{k_0} - y_{\alpha(\widehat{K(0,\chi_0)+9})}\| < 2^{-K(0,\chi_0)-5} + 2^{-K(0,\chi_0)-6}$ and so

$$|L(y_{k_0} - y_{\alpha(\widehat{K(0,\chi_0)+9})})| < 2^{-K(0,\chi_0)-5} + 2^{-K(0,\chi_0)-6}$$

which by the above yields

$$|L(y_{k_0} - y)| < 2^{-K(0,\chi_0)-5} + 2^{-K(0,\chi_0)-6} + 2^{-K(0,\chi_0)-6} = 2^{-K(0,\chi_0)-4}$$

contradicting the fact that by (case 2 and) construction $|L(y_{k_0} - y)| \geq 2^{-K(0,\chi_0)-4}$. The other use of $\|L\| \leq 1$ (after (14)) does not need linearity to prove $|L(\widehat{x})| \leq \|\widehat{x}\| + 1 \leq 8$ (while to get this without ‘+1’ would need linearity).

So by making \tilde{k}^* in the proof above possibly somewhat bigger to capture approximate linearity condition (+) above for the points y_{k_0}, y_i with $i \leq \alpha(K(0, \chi_0) + 12)$ one can resolve this issue. Similarly, the use of (an approximate) linearity in the construction of H^* above (again marked by ‘!’) can be handled. Finally, further uses of linearity are made in lemma 3.8 and the proof of lemma 3.12 at the equalities marked with ‘!’. This can be avoided by replacing the sequence (w_n) by a suitable enumeration (v_n) of $\text{Lin}_{\mathbb{Q}}\{w_n : n \in \mathbb{N}\}$. Then, the problem only pops up in the corollary to the proof of lemma 3.14 (applied to $l := K(0, \chi_0) + 3 \leq K^*(0, \chi_0^*) + 3$ in view of (10)) where we need linearity to transform $v_{g(i)}$ back into some w_j and to get from there that $|L(v_{g(i)})| < 2^{-l-2}$. However, from the code $i \leq g(K'(\alpha))$ one can easily compute (primitive recursively) the instances of linearity which – by allowing $|L(v_i)| \leq 2^{-l}$ instead of $L(v_i) = 0$ (thereby replacing l by $l + 1$) – again are only used in an approximate form. We omit the somewhat messy details here.

Corollary to the construction of Ω^* : For $n \geq 0$ let K^* be given by a term $t^*[\underline{h}]$ in the fragment T_n of Gödel’s T (i.e. the fragment where the type degree of ρ in the recursor R_ρ is bounded by n), whose only free variables \underline{h} have type degree ≤ 1 . Then the functional $\lambda \underline{h}. \Omega^*(t^*[\underline{h}])$ is definable in T_{n+2} .

In particular, the type-2 bound φ on Baillon’s theorem in [19] is definable (at least) in T_4 .

Proof: We use the fact (spelled out in detail in [21][proof of corollary 4.18] which in turn uses the ordinal analysis of $B_{0,1}$ from [10]) that a single use of $B_{0,1}$ (and hence of Φ_0) to arguments of the form $t^*[\underline{h}] \in T_n$ results in a type-2 functional that is definable in T_{n+1} . Our majorant \tilde{k}^* (which is of type 2) is primitive recursive (in the sense of T_0) in K^* relative to α^* which in turn is defined by a single use of Φ_0 from K^* (as a functional in f^*) via the functional ζ^* from lemma 3.14, where ζ^* is applied to functionals that are primitive recursive in K^* (and hence definable as well in T_n) only. Hence for $K^* := t^*[\underline{h}] \in T_n$, we get that $\lambda \underline{h}. \tilde{k}^*$ can be defined as $\lambda \underline{h}. s^*[\underline{h}] \in T_{n+1}$ for such a K^* . This functional is then plugged into the functional solving the monotone functional interpretation of the Bolzano-Weierstraß principle from [21] (see the appendix below) which – again by [21] – results in a majorant f^* for f (and hence in $\chi^* f^*$) which is definable in T_{n+2} as a functional in \underline{h} .

In our bound φ on Baillon’s theorem from [19] we first applied (disregarding the dummy variable u

as well as the number parameter b) Ω^* to a functional \tilde{K}_χ that in the function variable χ is definable in T_0 resulting in a type-2 functional $\lambda\chi.\psi_\chi^* \in T_2$ and then apply Ω^* again to a functional K^* that is primitive recursive in the former functional resulting in a type-2 functional $\chi_K^* \in T_4$. \square

Final Remark: Some further work needs to be done to explicitly construct a solution of the functional interpretation of weak compactness in the form of the existence of weakly convergent subsequences (which, however, also follows in our formal framework, see [16]) rather than the existence of a weak cluster point v as treated in this paper. The former has the additional strength that it implies that for different vectors w_1, w_2 (also finitely many) and an error $\varepsilon > 0$ one can get arbitrarily large **common** indices $n \in \mathbb{N}$ for which both $|\langle x_n - v, w_1 \rangle| < \varepsilon$ and $|\langle x_n - v, w_2 \rangle| < \varepsilon$. However, this fact – which is needed in the proof of Mazur’s lemma on the weak closedness of closed, bounded convex subsets C (see [16, 19]) and so for the extension of the construction given in this paper from closed balls around 0 to general C – can be also integrated in our construction of Ω^* and so we expect the adaption of the construction in this paper to the case of general C to be rather straightforward.

5. Appendix

In this appendix we briefly recall from [21] the solution of the monotone functional interpretation of the Bolzano-Weierstraß principle for $\prod_{n \in \mathbb{N}} [-k_n, k_n]$, where (k_n) is a sequence in \mathbb{R}_+ . It is well-known e.g. from reverse mathematics (see [22]), that the Bolzano-Weierstraß principle (already for $[0, 1]$) requires the schema of so-called arithmetical comprehension which, however, is known (see [14]) to have a (monotone) functional interpretation by (in addition to primitive recursion) a principle of bar recursion Φ_0 (used by C. Spector in his seminal paper [23] to give a functional interpretation of full second order arithmetic), though only of lowest type:

$$\Phi_0(y, u, n, x, k) = \begin{cases} x(k), & \text{if } k < n \\ 0, & \text{if } k \geq n \wedge y(\overline{x, n}) < n \\ \Phi_0(y, u, n + 1, (\overline{x, n} * D_0), k), & \text{otherwise,} \end{cases}$$

where

$$D_0 = u(n, \lambda D \in \mathbb{N} \lambda k \in \mathbb{N} . \Phi_0(y, u, n + 1, (\overline{x, n} * D), k)).$$

Here we use the following notation:

$$(\overline{x, n})(k) = \begin{cases} x(k), & \text{if } k < n \\ 0, & \text{otherwise} \end{cases}$$

and

$$(\overline{x, n} * D)(k) = \begin{cases} x(k), & \text{if } k < n \\ D, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Φ_0 is easily definable from the more common definition $B_{0,1}$ of bar recursion of which it is a special case (see [14], pp. 202-203, for all this).

x is a function $\mathbb{N} \rightarrow \mathbb{N}$, while $y : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ and $u : \mathbb{N} \times \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbb{N}$. Bar recursion is a principle of defining a function by recursion over a well-founded tree that corresponds to the proof principle

of bar induction that was considered first by L.E.J. Brouwer in the course of his development of intuitionistic mathematics and which – classically – is a form of dependent choice (for the case at hand, where only numbers are selected this, is not a genuine form of choice though).

It is well-known that Φ_0 is not always defined in the model of all set theoretic functionals as the necessary condition

$$\forall x \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} (y(\overline{x, n}) < n)$$

does not hold for all y . However, it does hold for all continuous (w.r.t. the product topology) y and even for all so-called majorizable y (see [14] for all this). Hence in our application below, where y is a primitive recursive and hence continuous functional, this does not create any problem.

In the following we also need a majorizing functional Φ_0^* for Φ_0 which is given by a slight modification of Φ_0 :

$$\Phi_0^*(y, u, n, x, k) := \max_{i \leq n} \{ \Phi_0(y^m, u_x, i, x, k), x^M(k) \},$$

where

$$y^m(x) := y(x^M) \text{ and } u_x(n, v) := \max\{x^M, v(unv)\}$$

with $x^M(n) := \max\{x(i) : i \leq n\}$.

One of the main results from [21] is that as the functionals L^* and f^* (referred to in (*) in section 2) we can take (denoting by $\mathbf{0}, \mathbf{1}$ the constant-0 resp. 1 function, and, by $\overline{\mathbf{1}}(m)$ the number code under some standard sequence coding of the finite constant-1 sequence of length m):

$$\begin{aligned} f_k^*(n) &:= \Phi_0^*(X^*(\tilde{k}), u_{(Z^*(\tilde{k}))}^*, 0, \mathbf{0}, \overline{\mathbf{1}}(lv^*(n))), \\ L_k^*(n) &:= N_{(K_n)}, \end{aligned}$$

where $N_m(n) := j(m2^{n+3} + 1, 2^{n+2} - 1)$ for some pairing function j and $(K_n)_{n \in \mathbb{N}}$ being a sequence of natural numbers with $K_n \geq k_n$ for all n , with $X^*(\tilde{k})$ and $Z^*(\tilde{k})$ defined primitive recursively in \tilde{k} as follows:

$$\begin{aligned} X^*(\tilde{k}) &:= \lambda g \in \mathbb{N}^{\mathbb{N}} . \overline{\mathbf{1}}(\tilde{k}'(g)), \\ Z^*(\tilde{k}) &:= \lambda g \in \mathbb{N}^{\mathbb{N}} . \max(\tilde{k}'(g), g(\overline{\mathbf{1}}(\tilde{k}'(g)))) . \end{aligned}$$

The functional \tilde{k}' is a simple primitive recursive modification of \tilde{k} , namely

$$\tilde{k}'(g) := lv^* \left(\tilde{k} (\lambda n \in \mathbb{N}. g(\overline{\mathbf{1}}(lv^*(n))), L_k^*) \right).$$

Finally

$$u_Z^*(n, v) := \max\{1, Z(v(1))\}, \text{ and } lv^*(n) := \left(\max_{i \leq n+1} \{K_i, n\} + 2 \right)^4.$$

In our application of this result where $k_i := C\|y_i\|$ we can use as K_i the bound $\gamma(i, C) = C \cdot i \cdot lth(i)$ from section 2.

References

- [1] Avigad, J., Gerhardy, P., Towsner, H., Local stability of ergodic averages. Trans. Amer. Math. Soc. **362**, pp. 261-288 (2010).

- [2] Avigad, J., Simic, K., Fundamental notions of analysis in subsystems of second-order arithmetic. *Ann. Pure Appl. Logic* **139**, pp. 138-184 (2006).
- [3] Baillon, J.B., Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert. *C.R. Acad. Sci. Paris Sér. A-B* **280**, pp. 1511-1514 (1975).
- [4] Bezem, M., Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. *J. Symbolic Logic* **50** pp. 652–660 (1985).
- [5] Brézis, H., Browder, F.E., Nonlinear ergodic theorems. *Bull. Amer. Math. Soc.* **82**, pp. 959-961 (1976).
- [6] Browder, F.E., Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. *Arch. Rational Mech. Anal.* **24**, pp. 82-90 (1967).
- [7] Brown, D.K., Functional analysis in weak subsystems of second order arithmetic. PhD Thesis. The Pennsylvania State University. University Park, PA (1987).
- [8] Gerhardy, P., Kohlenbach, U., General logical metatheorems for functional analysis. *Trans. Amer. Math. Soc.* **360**, pp. 2615-2660 (2008).
- [9] Gödel, K., Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica* **12**, pp. 280–287 (1958).
- [10] Howard, W.A., Ordinal analysis of simple cases of bar recursion. *J. Symb. Logic* **46**, pp. 17-31 (1981).
- [11] Kohlenbach, U., Analysing proofs in analysis. In: W. Hodges, M. Hyland, C. Steinhorn, J. Truss, editors, *Logic: from Foundations to Applications. European Logic Colloquium* (Keele, 1993), pp. 225–260, Oxford University Press (1996).
- [12] Kohlenbach, U., On the no-counterexample interpretation. *J. Symbolic Logic* **64**, pp. 1491-1511 (1999).
- [13] Kohlenbach, U., Some logical metatheorems with applications in functional analysis. *Trans. Amer. Math. Soc.* **357**, no. 1, pp. 89-128 (2005).
- [14] Kohlenbach, U., *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.
- [15] Kohlenbach, U., Effective uniform bounds from proofs in abstract functional analysis. In: Cooper, B., Loewe, B., Sorbi, A. (eds.), *New Computational Paradigms: Changing Conceptions of What is Computable*. Springer Publisher, pp. 223-258 (2008).
- [16] Kohlenbach, U., On the logical analysis of proofs based on nonseparable Hilbert space theory. In: Feferman, S., Sieg, W. (eds.), *‘Proofs, Categories and Computations. Essays in Honor of Grigori Mints*’. College Publications, pp. 131-143 (2010).

- [17] Kohlenbach, U., Gödel's functional interpretation and its use in current mathematics. In: Baaz, M. et al. (eds.), Kurt Gödel and the Foundations of Mathematics. Horizons of Truth. Cambridge University Press, pp. 361-398, 2011.
- [18] Kohlenbach, U., On quantitative versions of theorems due to F.E. Browder and R. Wittmann. *Advances in Mathematics* **226**, pp. 2764-2795 (2011).
- [19] Kohlenbach, U., A uniform quantitative form of sequential weak compactness and Baillon's nonlinear ergodic theorem. To appear in: *Communications in Contemporary Mathematics*.
- [20] Kohlenbach, U., Leuştean, L., A quantitative mean ergodic theorem for uniformly convex Banach spaces. *Ergodic Theory and Dynamical Systems* **29**, pp. 1907-1915 (2009).
- [21] Safarik, P., Kohlenbach, U., On the computational content of the Bolzano-Weierstraß principle. *Math. Logic Quart.* **56**, pp. 508-532 (2010).
- [22] Simpson, S.G., *Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic.* Springer-Verlag. xiv+445 pp. 1999.
- [23] Spector, C., Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. In: *Recursive function theory, Proceedings of Symposia in Pure Mathematics*, vol. 5 (J.C.E. Dekker (ed.)), AMS, Providence, R.I., pp. 1-27 (1962).
- [24] Tao, T., Soft analysis, hard analysis, and the finite convergence principle. Essay posted May 23, 2007. Appeared in: 'T. Tao, *Structure and Randomness: Pages from Year One of a Mathematical Blog.* AMS, 298pp., 2008'.
- [25] Tao, T., Norm convergence of multiple ergodic averages for commuting transformations. *Ergodic Theory and Dynamical Systems* **28**, pp. 657-688 (2008).
- [26] Wittmann, R., Approximation of fixed points of nonexpansive mappings. *Arch. Math.* **58**, pp. 486-491 (1992).