

On the asymptotic behavior of odd operators

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Abstract

We give quantitative versions of strong convergence results due to Baillon, Bruck and Reich for iterations of nonexpansive odd (and more general) operators in uniformly convex Banach spaces.

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1 Introduction

Let X be a uniformly convex Banach space and $C \subseteq X$ a closed convex subset satisfying $C = -C$. In [2], Baillon, Bruck and Reich showed (among many other things) that the iteration $(T^n x)$ of an odd nonexpansive mapping $T : C \rightarrow C$ that is asymptotically regular at $x \in C$ strongly converges to a fixed point of T . By a famous result due to Ishikawa [4] the averaged mapping $T_\lambda x := (1 - \lambda)x + \lambda T(x)$ with $\lambda \in (0, 1)$ of a nonexpansive mapping $T : C \rightarrow C$ always is asymptotically regular provided that $(T_\lambda^n x)$ is bounded (in fact – by another result from [2] – it suffices that $(\|T_\lambda^n x\|/n)$ converges to 0).

With T also T_λ is nonexpansive and odd and so the sequence (x_n) defined by $x_n := T_\lambda^n x$ (which trivially is bounded) converges strongly towards a fixed point $p \in C$ of T .

We first observe that the condition of T being nonexpansive and odd can be weakened to the condition

$$(W) : \forall x, y \in C \quad (\|Tx + Ty\| \leq \|x + y\|)$$

studied in [14] which also makes the assumption $C = -C$ superfluous.

It is easy to show that there is no computable (in the data at hand) rate of convergence even for $X := \mathbb{R}, C := [0, 1], \lambda := 1/2, x := 1$ in the sense that there is a computable sequence

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(f_l) of odd nonexpansive functions $f_l : [-1, 1] \rightarrow [-1, 1]$ such that there is no computable function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ such that for $x_n^l := (f_l)_{\frac{1}{2}}^n(1)$

$$(+)\ \forall m \geq \delta(l) (|x_m^l - x_{\delta(l)}^l| \leq \frac{1}{2}).$$

Define $f_l(x) := a_l \cdot x$, where $a_l := \sum_{i=0}^{\infty} g(l, i) \cdot 2^{-i-1} \in [0, 1]$ with

$$g(l, n) := \begin{cases} 1, & \text{if } \neg T(l, l, n) \\ 0, & \text{otherwise,} \end{cases}$$

Here T denotes the Kleene T -predicate.

Now observe that

$$(++)\ a_l = 1 \rightarrow x_{\delta(l)}^l = 1 \text{ and } a_l < 1 \rightarrow x_{\delta(l)}^l \in [0, 1/2].$$

While the first implication is immediate from the definition of x_n^l , the second follows using (+) and the fact that (by – an essentially trivial use of – Ishikawa’s theorem [4]) (x_n^l) converges towards the unique fixed point 0 of f_l .

By (++) the computability of δ would allow us to decide whether $a_l = 1$ or $a_l < 1$ and so whether or not $\exists n \in \mathbb{N} T(l, l, n)$ contradicting the undecidability of the (special) Halting problem.

While we do not know whether for single computable operators $T : C \rightarrow C$ in effective uniformly convex spaces, the iteration $x_n := T_{\lambda}^n x$ (for computable $x \in C, \lambda \in (0, 1)$) might have no computable rate of convergence, we show that the rate is computable iff the norm $\|p\|$ of the strong limit p of (x_n) is computable.

Things are much better for a reformulation of the convergence property known in logic as the no-counterexample interpretation of the former ([7, 8], see also [5]) which recently has been popularized under the name of ‘metastability’ by T. Tao (see [11, 12]). Here one considers the statement

$$\forall \varepsilon > 0 \forall g : \mathbb{N} \rightarrow \mathbb{N} \exists k \in \mathbb{N} \forall i, j \in [k; k + g(k)] (\|T^i x - T^j x\| < \varepsilon)$$

which, ineffectively, is equivalent to the strong convergence of $(T^n x)$. Here $[k; k + m] := \{k, k + 1, k + 2, \dots, k + m\}$.

We then give an explicit effective (in fact even primitive recursive) and highly uniform rate $\Phi(b, \alpha, \varepsilon, g)$ of metastability of $(T^n x)$

$$\forall \varepsilon \in (0, 2] \forall g : \mathbb{N} \rightarrow \mathbb{N} \forall b \in \mathbb{N}^* \forall x \in C_b \exists n \leq \Phi(b, \alpha, \varepsilon, g) \forall i, j \in [n; n + g(n)] (\|T^i x - T^j x\| < \varepsilon)$$

that (in addition to ε and g) only depends on a norm upper bound $b \geq \|x\|$ of x and a uniform rate α of asymptotic regularity of T on $C_b := \{x \in C : \|x\| \leq b\}$, i.e.

$$\forall \varepsilon > 0 \forall b \in \mathbb{N}^* \forall x \in C_b \forall n \geq \alpha(b, \varepsilon) (\|T^{n+1} x - T^n x\| < \varepsilon).$$

In fact, instead of α a (uniform) rate on the metastable version of asymptotic regularity, i.e. a φ such that

$$\forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall b \in \mathbb{N}^* \forall x \in C_b \exists k \leq \varphi(b, f, \varepsilon) \forall i \in [k; k + f(k)] (\|T^{i+1}x - T^i x\| < \varepsilon),$$

is sufficient.

The bound Φ is independent of X (and C) except for a modulus of uniform convexity η of X (and an upper bound b on $\|x\|$). The extraction of this bound is an instance of a general logical metatheorem which not only guarantees the extractability of such bounds for large classes of proofs but also provides an algorithm for the actual construction of the bound from a given proof. This then results again in an ordinary proof that no longer relies on any facts from logic (see [5], in particular Chapters 17 and 18, for all this).

Using the optimal rate of asymptotic regularity α for T_λ from [1] this gives an effective (and even primitive recursive) rate of metastability for the strong convergence of (x_n) (as defined above) that only depends on ε, g and b .

A primitive recursive rate on the metastability of the **Cesàro means** (i.e. ergodic averages) of operators in Hilbert space satisfying Wittmann's condition was recently extracted from Wittmann's [14] proof of strong convergence of these means by Safarik [10]. For another quantitative strong nonlinear ergodic theorem see [6]. Again, these results have been obtained using the aforementioned proof-theoretic approach.

2 Results

In the following, let X be a uniformly convex Banach space with a modulus of convexity $\eta : (0, 2] \rightarrow (0, 1]$, i.e.

$$\forall x, y \in B_1(0) \forall \varepsilon \in (0, 2] \left(\left\| \frac{x+y}{2} \right\| > 1 - \eta(\varepsilon) \rightarrow \|x - y\| < \varepsilon \right),$$

where $B_d(0)$ denotes the closed ball with center 0 and radius d in X .

Lemma 2.1. *Let $x, y \in B_d(0) \subset X$ with $0 < d \leq b \in \mathbb{N}$. Then*

$$\forall \varepsilon \in (0, 2] \left(\left\| \frac{x+y}{2} \right\| > d(1 - \eta(\varepsilon/b)) \rightarrow \|x - y\| < \varepsilon \right).$$

Proof: Define $\tilde{x} := x/d, \tilde{y} := y/d$ so that $\tilde{x}, \tilde{y} \in B_1(0)$.

Assume that $\left\| \frac{x+y}{2} \right\| > d(1 - \eta(\varepsilon/b))$. Then

$$\left\| \frac{\tilde{x} + \tilde{y}}{2} \right\| = \frac{1}{d} \left\| \frac{x+y}{2} \right\| > 1 - \eta(\varepsilon/b)$$

and so $\frac{1}{d}\|x - y\| = \|\tilde{x} - \tilde{y}\| < \frac{\varepsilon}{b}$. Hence $\|x - y\| < \frac{d\varepsilon}{b} \leq \varepsilon$. □

Notation: For $b \in \mathbb{N}^*$ define $C_b := \{x \in C : \|x\| \leq b\}$.

For $n, m \in \mathbb{N}$ we define $n \dot{-} m := n - m$ if $n \geq m$ and $:= 0$, otherwise.

Theorem 2.2. *Let $C \subseteq X$ be any nonempty subset of X and $T : C \rightarrow C$ a selfmapping of C that satisfies Wittmann's [14] condition*

$$(W) : \forall x, y \in C \left(\|Tx + Ty\| \leq \|x + y\| \right).$$

Moreover, assume that for each $0 < b \in \mathbb{N}$ the mapping T is (uniformly on C_b) asymptotically regular with a rate $\alpha : \mathbb{N} \times \mathbb{R}_+^ \rightarrow \mathbb{N}$, i.e.*

$$\forall \varepsilon > 0 \forall b \in \mathbb{N}^* \forall x \in C_b \forall n \geq \alpha(b, \varepsilon) \left(\|T^{n+1}x - T^n x\| < \varepsilon \right).$$

Then $(T^n x)_{n \in \mathbb{N}}$ converges strongly with the following rate of metastability

$$\forall \varepsilon \in (0, 2] \forall g : \mathbb{N} \rightarrow \mathbb{N} \forall b \in \mathbb{N}^* \forall x \in C_b \exists n \leq \Phi(b, \alpha, \varepsilon, g) \\ \forall i, j \in [n; n + g(n)] \left(\|T^i x - T^j x\| < \varepsilon \right),$$

where

$$\Phi(b, \alpha, \varepsilon, g) := \Psi(b, h_{b, \alpha, \varepsilon, g}, \frac{\delta_b(\varepsilon)}{2}) \text{ with} \\ h_{b, \alpha, \varepsilon, g}(n) := h(n) := \max \left\{ \alpha \left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n), 1\}} \right) \div n, g(n) \right\} \text{ and} \\ \Psi(b, f, \delta) := \tilde{f}^{(\lceil b/\delta \rceil)}(0) \text{ with } \tilde{f}(n) := n + f(n) \text{ for } f : \mathbb{N} \rightarrow \mathbb{N}, \\ \delta_b(\varepsilon) := \frac{\varepsilon}{2} \cdot \eta(\varepsilon/b).$$

If T is continuous and C closed, then the strong limit of $(T^n x)_{n \in \mathbb{N}}$ is a fixed point of T . For the metastability statement the completeness of X is not needed.

Proof: It suffices to prove the metastability statement which (ineffectively) implies the strong Cauchy property of the sequence (and so using the completeness of X its convergence). That for continuous T (and closed C) the limit is a fixed point of T then trivially follows from the asymptotic regularity of T .

Let $\varepsilon \in (0, 2]$, $b \in \mathbb{N}^*$, $g : \mathbb{N} \rightarrow \mathbb{N}$ and C, T, x be as in the theorem. By the condition (W) the sequence $(\|T^n x\|)_{n \in \mathbb{N}}$ is nonincreasing and hence convergent. By [5] (Proposition 2.27, Remark 2.29) it follows that Ψ is a rate of metastability for this sequence, i.e.

$$\forall \delta > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \Psi(b, f, \delta) \forall i, j \in [n; n + f(n)] \left(\left| \|T^i x\| - \|T^j x\| \right| < \delta \right).$$

For $\delta := \frac{\delta_b(\varepsilon)}{2}$ and $f := h := h_{b, \alpha, \varepsilon, g}$ let $n \in \mathbb{N}$ be such a number.

Define $d := \|T^n x\| = \max\{\|T^k x\| : k \in [n; n + h(n)]\} \leq b$. Then

$$(1) \forall k \in [n; n + h(n)] \left(d - \frac{\delta_b(\varepsilon)}{2} < \|T^k x\| \leq d \right).$$

From the assumption on α we get

$$(2) \forall i \in \mathbb{N}^* \forall \varepsilon > 0 \forall k \geq \alpha(b, \varepsilon/i) \forall j \leq i \left(\|T^k x - T^{k+j} x\| < \varepsilon \right),$$

since

$$\|T^k x - T^{k+j} x\| \leq \sum_{l=0}^{j-1} \|T^{k+l} x - T^{k+l+1} x\| < \sum_{l=0}^{j-1} \frac{\varepsilon}{i} \leq \varepsilon$$

for all $0 < j \leq i$ and $k \geq \alpha(b, \varepsilon/i)$.

For $k := n + h(n) \geq \alpha\left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n), 1\}}\right)$ we get from (1) and (2) that for all $i \in [n; n + g(n)] \subseteq [n; k]$:

$$\begin{aligned} \forall j \leq g(n) \quad \left(2(d - \frac{\delta_b(\varepsilon)}{2})\right) &\leq 2\|T^k x\| \leq \|T^{k+j} x + T^k x\| + \|T^k x - T^{k+j} x\| \\ &< \|T^{k+j} x + T^k x\| + \delta_b(\varepsilon) \\ &\stackrel{(W)}{\leq} \|T^{i+j} x + T^i x\| + \delta_b(\varepsilon). \end{aligned}$$

Hence

$$(3) \quad \forall i, j \in [n; n + g(n)] \quad \left(d - \delta_b(\varepsilon) < \left\| \frac{T^i x + T^j x}{2} \right\| \right).$$

Case 1: $d := \|T^n x\| < \frac{\varepsilon}{2}$. Then

$$\forall i, j \in [n; n + g(n)] \quad (\|T^i x - T^j x\| \leq \|T^i x\| + \|T^j x\| \leq 2\|T^n x\| < \varepsilon)$$

and so we are done.

Case 2: $d \geq \frac{\varepsilon}{2}$. Then by the definition of $\delta_b(\varepsilon)$ and (3) we have

$$(4) \quad \forall i, j \in [n; n + g(n)] \quad \left(d(1 - \eta(\varepsilon/b)) < \left\| \frac{T^i x + T^j x}{2} \right\| \right).$$

Using (1), (4) and lemma 2.1 yields that

$$\forall i, j \in [n; n + g(n)] \quad (\|T^i x - T^j x\| < \varepsilon).$$

□

Remark 2.3. If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $0 < \varepsilon_1 \leq \varepsilon_2 \rightarrow \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2)$, then we can replace $\delta_b(\varepsilon)$ in the bound in theorem 2.2 by $\delta_b(\varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b)$. In particular, in the case of a Hilbert space X (where one can take $\eta(\varepsilon) := \varepsilon^2/8$, see e.g. [6]), this yields $\delta_b(\varepsilon) := \frac{\varepsilon^2}{8b}$.

Proof: With $\delta_b(\varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b)$ one gets instead of (4) in the proof of theorem 2.2

$$(4)' \quad \left\{ \begin{array}{l} \forall i, j \in [n; n + g(n)] \\ \left(d(1 - \eta(\varepsilon/d)) = d(1 - \frac{\varepsilon}{d} \cdot \tilde{\eta}(\varepsilon/d)) \leq d(1 - \frac{\varepsilon}{d} \cdot \tilde{\eta}(\varepsilon/b)) < \left\| \frac{T^i x + T^j x}{2} \right\| \right) \end{array} \right.$$

The claim now follows using lemma 2.1 since $T^i x, T^j x \in B_d(0)$ for $i, j \in [n; n + g(n)]$. □

The above extraction of the rate of metastability Φ from the proof given in [2] (and also the fact that Φ only depends on the arguments $b, \alpha, \varepsilon, g$) is an instance of a general logical metatheorem (see [3] Theorem 6.3.2 or [5] Theorem 17.69.2 and note that the condition (W) is purely universal and implies that T is majorized by the identity function). In fact, that metatheorem even guarantees such a bound when the rate of asymptotic regularity α is replaced by a weaker rate of metastability φ instead, i.e.

(*) $\forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall b \in \mathbb{N}^* \forall x \in C_b \exists k \leq \varphi(b, f, \varepsilon) \forall i \in [k; k + f(k)] (\|T^{i+1}x - T^i x\| < \varepsilon)$.

We will briefly demonstrate this now. In fact, one only needs φ for constant- c functions ($c \in \mathbb{N}$) that we also denote by c . Modifying φ to $\varphi'(b, c, l, \varepsilon) := \varphi(b, c + l, \varepsilon) + l$ one gets for each $l \in \mathbb{N}$

$$(**) \exists k \leq \varphi'(b, c, l, \varepsilon) \forall i \in [k; k + c] (k \geq l \wedge \|T^{i+1}x - T^i x\| < \varepsilon).$$

Now define (using (**)) $\alpha_{n,g}(b, \varepsilon)$ as the least $k \leq \varphi'(b, g(n), n + g(n), \varepsilon)$ such that

$$\forall i \in [k; k + g(n)] (k \geq n + g(n) \wedge \|T^{i+1}x - T^i x\| < \varepsilon).$$

Then theorem 2.2 holds with α and $h_{b,\alpha,\varepsilon,g}$ being replaced by $\alpha_{n,g}$ and $h_{b,\varphi,\varepsilon,g}(n) := \alpha_{n,g}\left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n), 1\}}\right) - n$ respectively. Replacing $h_{b,\varphi,\varepsilon,g}$ by the monotone upper bound

$$h_{b,\varphi,\varepsilon,g}^*(n) := \max\{\varphi'(b, g(m), m + g(m), \delta_b(\varepsilon)/\max\{g(m), 1\}) - m : m \leq n\}$$

yields an upper bound

$$\Phi(b, \varphi, \varepsilon, g) := \Psi(b, h_{b,\varphi,\varepsilon,g}^*, \frac{\delta_b(\varepsilon)}{2}) \geq \Psi(b, h_{b,\varphi,\varepsilon,g}, \frac{\delta_b(\varepsilon)}{2})$$

satisfying theorem 2.2. This yields the following qualitative improvement of theorem 2.2

Corollary 2.4. *For the strong convergence of $(T^n x)$ in theorem 2.2 one can weaken the asymptotic regularity assumption to*

$$\forall \varepsilon > 0 \forall c \in \mathbb{N} \forall x \in C \exists k \in \mathbb{N} \forall i \in [k; k + c] (\|T^{i+1}x - T^i x\| < \varepsilon).$$

If T is continuous and C is closed, then the limit of $(T^n x)$ is a fixed point of T .

Proof: By the reasoning above, the sequence $(T^n x)$ is metastable (note that for metastability in the point x we also only need the above weak form of asymptotic regularity in x) and hence is strongly Cauchy. For closed C the limit is in C and – for continuous T – a fixed point of T as the condition in the corollary implies that

$$\forall \varepsilon > 0 \forall n \in \mathbb{N} \exists k \geq n (\|T^{k+1}x - T^k x\| < \varepsilon).$$

□

In the following, we apply theorem 2.2 to averages mappings for which effective (full) rates of asymptotical regularity are known (here ‘ π ’ denotes the constant π):

Theorem 2.5. *Let X be a uniformly convex Banach space and $C \subseteq X$ a closed and convex subset. Assume that $T : C \rightarrow C$ satisfies (W) and is nonexpansive. Let $\lambda \in (0, 1)$ and define $T_\lambda x := (1 - \lambda)x + \lambda T x$, $x_n := T_\lambda^n x$ for $x \in C$. Then $(T_\lambda^n)_{n \in \mathbb{N}}$ strongly converges to a fixed point $p \in C$ of T and the following rate of metastability holds:*

$$\forall \varepsilon \in (0, 2] \forall g : \mathbb{N} \rightarrow \mathbb{N} \forall b \in \mathbb{N}^* \forall x \in C_b \exists n \leq \Phi(b, \alpha, \varepsilon, g) \forall i, j \in [n; n + g(n)] (\|x_i - x_j\| < \varepsilon),$$

where Φ is as in theorem 2.2 and $\alpha(b, \varepsilon) := \left\lceil \frac{b \cdot \lambda}{\pi(1-\lambda)\varepsilon^2} \right\rceil$.

For the last statement no completeness of X or closedness of C is needed.

Proof: For $x \in C_b$ it follows from a deep result due to Baillon and Bruck [1] (and using that $\|x_n - T(x_n)\| = \lambda \|T_\lambda^{n+1}x - T_\lambda^n x\|$) that α is a rate of asymptotic regularity for T_λ (this result even holds in arbitrary normed spaces).¹ With T also T_λ satisfies (W) since

$$\begin{aligned} \|T_\lambda x + T_\lambda y\| &= \|(1-\lambda)x + \lambda T x + (1-\lambda)y + \lambda T y\| \\ &\leq (1-\lambda)\|x + y\| + \lambda\|T x + T y\| \\ &\leq (1-\lambda)\|x + y\| + \lambda\|x + y\| = \|x + y\|. \end{aligned}$$

Hence the corollary follows from theorem 2.2 applied to T_λ (note that the proof for the metastability statement did not use the completeness of X nor the closedness of C). \square

Remark 2.6. For nonexpansive T the condition (W), in particular, holds when $C = -C$ and T is odd, i.e. $T(-x) = -T(x)$.

The proof of theorem 2.2 (and theorem 2.5) immediately yields an effective rate of convergence of $(T^n x)_{n \in \mathbb{N}}$ (instead of a rate of metastability only) provided one has a rate $\Psi_{x,T}$ of convergence for $(\|T^n x\|)_{n \in \mathbb{N}}$ given, i.e. for $d := \lim_{n \rightarrow \infty} \|T^n x\|$

$$\forall \varepsilon > 0 \forall n \geq \Psi_{x,T}(\varepsilon) (\|T^n x\| - d < \varepsilon).$$

Then $\Psi_{x,T} \left(\frac{\delta_b(\varepsilon)}{2} \right)$ is a rate of convergence of $(T^n x)_{n \in \mathbb{N}}$. This leads to the following (using the notion of computability for Banach spaces and mappings between Banach spaces from [9] and [13]).

Corollary 2.7. Let X be a computable uniformly convex Banach space with a computable modulus of uniform convexity η and C be a closed and convex subset. Let $T : C \rightarrow C$ be a computable nonexpansive mapping satisfying condition (W) and $x \in C$ be a computable point. Finally, let $\lambda \in (0, 1)$ be computable. Then $(T_\lambda^n x)_{n \in \mathbb{N}}$ converges effectively (i.e. with a computable rate of convergence) to its limit $p := \lim_{n \rightarrow \infty} T_\lambda^n x$ if and only if $\|p\|$ is computable.

Proof: The assumptions yields that (x_n) with $x_n := T_\lambda^n x$ is a computable sequence in X . If (x_n) converges effectively, then also p and hence $\|p\|$ is computable. Conversely, suppose that $\|p\|$ is computable. Then there is a computable function $\rho : \mathbb{Q}_+^* \rightarrow \mathbb{N}$ such that

$$\forall q \in \mathbb{Q}_+^* (\|T_\lambda^{\rho(q)} x\| - \|p\| < q)$$

since ' $\|T_\lambda^n x\| - \|p\| < q$ ' is computably enumerable in n, q . Since $(\|T^n x\|)_{n \in \mathbb{N}}$ is nonincreasing, ρ in fact is a rate of convergence. The comments preceding this corollary now yield a computable rate of convergence for $(T_\lambda^n x)_{n \in \mathbb{N}}$. \square

¹The bound in [1] is stated for sequences in $B_1(0)$ but can easily be adapted to $B_b(0)$ by switching to the norm $\|x\|_b := \frac{1}{b} \cdot \|x\|$. The α in our theorem results from this adaptation.

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