A note on the monotone functional interpretation

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Abstract

We prove a result relating the author's monotone functional interpretation to the bounded functional interpretation due to Ferreira and Oliva. More precisely we show that (over model of majorizable functionals) largely a solution for the bounded interpretation also is a solution for monotone functional interpretation although the latter uses the existence of an underlying precise witness. This makes it possible to focus on the extraction of bounds (as in the bounded interpretation) while using the conceptual benefit of having precise realizers at the same time without having to construct them.

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1 Introduction

Monotone functional interpretation, which was first introduced in [7], is the most frequently used proof-theoretic tool in the ongoing program of extracting new information (such as effective uniform bounds) from proofs in analysis (see e.g. [8]). Here instead of realizers for the well-known Gödel functional ('Dialectica') interpretation, majorizing functionals are extracted while precise realizers – which may not be effective and usually are not uniform – are not constructed but only have to exist.

A functional interpretation that differs more radically from Gödel's interpretation is the bounded functional interpretation introduced in [2]. Here the whole interpretation is changed in a way which requires classically false 'nonstandard' principles to prove the equivalence between the interpretation of a formula and the original formula. This has applications in establishing conservation results for these new principles but w.r.t. bound extractions one has to restrict the matrix of the formula in the theorem treated to get a correct bound for the original formula rather than the interpreted one. When applied to proofs based on classical logic – which requires a negative translation as a processing step – both interpretations

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become more similar as in this case one has to restrict things essentially to $\forall\exists$-theorems. Experience from proof mining indicates that using the monotone interpretation has the conceptual benefit that one can reason at the level of precise realizers while running in parallel to this the construction of the majorants. However, one then has – in addition to the construction of the majorant – also to show the existence of a precise realizer (below the majorant) that does not have any access to majorants of its arguments but only to the arguments themselves. This, sometimes, causes extra complications (see e.g. the discussion of WKL below or the need to define the functional J_ε in the proof of Theorem 3.3. in [9] by unbounded search although it has a trivial majorant).

In this note we show how to overcome this by proving that – ineffectively – the existence of such a precise realizer can be guaranteed without any extra work (i.e., in particular, without having to construct such a realizer first by the Dialectica interpretation) and without having to modify the majorant.

Consider a sentence of the form $S :\equiv \forall x^\rho \exists y^\tau\, A(x,y)$. Provided that A is quantifier-free (or – in the case of the bounded interpretation – at least bounded) the (combination of negative translation and the) bounded interpretation would extract from a proof of such a theorem a self-majorizing term φ^* such that

$$(1)\ \forall x, x^*\, (x^* \gtrsim x \to \exists y \lesssim \varphi^*(x^*)\, A(x,y)),$$

whereas the monotone interpretation extracts a φ^* such that

$$(2)\ \exists \varphi \lesssim \varphi^*\, \forall x\, A(x, \varphi(x)).$$

Here $\gtrsim$ denotes Bezem's [1] strong majorization relation (which is a variant of Howard's [4] majorization).[1]

Obviously, (2) implies (1) (take $y := \varphi(x)$) but the other direction in general does not hold for the simple reason that (1) only claims something for majorizable functionals x while (2) talks about all x. E.g. consider the sentence (x_1, x_2 can be contracted)

$$S :\equiv \forall x_1^2\, \forall x_2^{0\to 1} \exists y^0\, (y \geq x_1(\min(x_2(y), 1))).$$

Here '1' is the constant-1 function and 'min' the pointwise minimum. Then (1) trivially holds for $\varphi^*(x_1, x_2) := x_1(1)$ while e.g. in the model $\mathcal{S}^\omega$ of all set-theoretic functionals there is no functional φ (even if not required to be majorized by some φ^*) satisfying (2) since this would imply S and hence (arguing – using extensionality – by contraposition and applying quantifier-free countable choice to $\forall y^0 \exists z^1 (y < x(\min_1(z,1)))$

$$\forall x^2\, \exists y^0\, \forall z \leq_1 1\, (y \geq x(z))$$

which, however, fails to hold in $\mathcal{S}^\omega$.

Another example (pointed out to us by P. Oliva) comes from Spector's [11] bar condition

$$S :\equiv \forall x_1^2\, \forall x_2^1\, \exists y^0\, (x_1(\overline{x_2, y}) < y),$$

[1] The monotone interpretation also works with Howard's original notion.

where $\overline{x_2, y}(k) := x_2(k)$, for $k < y$, and $:= 0$, otherwise. Again, S (and hence (2)) does not hold in $\mathcal{S}^\omega$ while (1) does: take $\varphi^*(x_1, x_2) := x_1(x_2) + 1$.

Even if one only considers $\mathcal{S}^\omega$-true sentences S, (2) in general needs stronger principles for its proof than (1). E.g. consider the weak König's lemma WKL which can be stated as $\forall f^1 \exists b \leq_1 1 \forall k^0 (T(f))(\bar{b}k)$, where $T(f)$ is a primitive recursive transformation which transforms f into (the characteristic function of) an infinite 0/1-tree, unless f was already such a tree (see [8]). Clearly, WKL implies (and – using quantifier-free choice is equivalent to –)

$$S := \forall f^1 \, \forall K^2 \, \exists b \leq 1 \, (T(f))(\bar{b}(K(b)).$$

(1) has a trivially verifiable solution: $\varphi^*(f, K) := 1$ since, given $K^* \gtrsim K$, one only has to produce a **finite** branch of length $K^*(1)$ in $T(f)$ which can be found by primitive recursive bounded search. To show (2) (for this or any φ^*), however, requires the use of a weak binary form of bar recursion (see [5] and [10]).

(1) and (2) get more close once we interpret (1), (2) in the model $\mathcal{M}^\omega$ of all (strongly) majorizable functionals as defined in [1]. Then for the type $\tau = 0$, one can infer (2) from (1) (with the same φ^*) by just defining $\varphi(x) := \min y \, A(x, y)$. This solves (2) (with elementary verifications) in both of the examples above (note that finite branches can be encoded by numbers; also note that to avoid unbounded search in the 2nd example still requires binary bar recursion as φ only has access to K but not to a majorant K^* of K).
In fact, in this way sometimes the existence of precise realizers is established in current applications of proof mining (e.g. in [9], see the proof of theorem 2.15). In this note, we show that this also extends to arbitrary types τ, i.e.

$$\mathcal{M}^\omega \models \forall \varphi^* \big(\varphi^* \gtrsim \varphi^* \wedge (1) \to (2) \big).$$

In fact, we show this even for the extension from finite types over the natural numbers $\mathbb{N}$ to all finite types over $\mathbb{N}$ and an abstract metric, hyperbolic, normed etc. space X with the extended notion of majorization (relative to a reference point $a \in X$) as defined in [3]. This is of relevance as all of the current applications of proof mining in nonlinear functional analysis are based on theories formulated for such extensions (e.g. see [8]).

2 Result

Consider the model $\mathcal{M}^{\omega,X} = \langle M_\rho \rangle_{\rho \in \mathbf{T}^X}$ of all strongly majorizable functionals over the types $0, X$ from [3, 8]. Here $\mathbf{T}^X$ is the set of all finite types over $0, X$ (where '0' represents the type of natural numbers) and $\mathbf{T}$ is the subset of all finite types over 0 alone. For $\rho \in \mathbf{T}^X$, $\widehat{\rho} \in \mathbf{T}$ results from replacing every occurrence of X in ρ by 0.
For $a \in X$ let $x^* \gtrsim^a_\rho x$ be the strong majorizability relation (relative to a) between $x \in M_\rho$ and $x^* \in M_{\widehat{\rho}}$ as defined in [3]. $x \lesssim^a_\rho x^* :\equiv x^* \gtrsim^a_\rho x$. For $\rho \in \mathbf{T}$, the relation $\gtrsim^a_\rho$ does not depend on 'a' and coincides with Bezem's strong majorizability (see [8] for details on all this).

Definition 2.1. *By induction on the type $\rho \in \mathbf{T}$ we define for $A_\rho \subseteq M_\rho$ a functional* $\min A_\rho$ *as follows:*

1) *For $\rho = 0$, we have $A_0 \subseteq \mathbb{N}$ and $\min A_0$ is the usual minimum.*

2) *For $\rho \to \tau$ define $\min A_{\rho\to\tau} := \lambda y \in M_\rho. \min_\tau\{xy : x \in A_{\rho\to\tau}\}$.*

Definition 2.2. *For $\rho \in \mathbf{T}^X$ define $\widehat{x} := \min_{\widehat{\rho}}\{x^* \in M_{\widehat{\rho}} : x^* \gtrsim^a_\rho x\}$.*

Lemma 2.3. *For any $\rho \in \mathbf{T}^X$, the following holds:*

1) $\forall x \in M_\rho\, (\widehat{x} \in M_{\widehat{\rho}} \wedge \widehat{x} \gtrsim^a_\rho x)$,

2) $\forall x \in M_\rho \forall x^* \in M_{\widehat{\rho}}\, (x^* \gtrsim^a_\rho x \to x^* \gtrsim^a_{\widehat{\rho}} \widehat{x})$.

Proof: 1) Let $\rho = \rho_1 \to \rho_2 \to \ldots \to \rho_k \to \tau$ with $\tau = 0$ or $\tau = X$. For $\underline{\rho} = \rho_1, \ldots, \rho_k$ let $\underline{y}^*, \underline{y}$ be k-tuples of functionals in $\mathcal{M}^{\omega,X}$ with $y_i^* \gtrsim^a_{\rho_i} y_i$ for $1 \le i \le k$. By the definition of $\widehat{x}$ there exists an $x^* \gtrsim^a_\rho x$ such that $x^*\underline{y}^* =_0 \widehat{x}\underline{y}^*$. Hence $\widehat{x}\underline{y}^* =_0 x^*\underline{y}^* \gtrsim^a_\tau x\underline{y}$. That $\widehat{x}$ is selfmajorizing follows similarly: let $\underline{y}^*, \underline{y}$ be k-tuples of functionals in $\mathcal{M}^{\omega,X}$ with $y_i^* \gtrsim^a_{\widehat{\rho}_i} y_i$ and let x^* be a majorant of x in $\mathcal{M}^{\omega,X}$ with $x^*\underline{y}^* =_0 \widehat{x}\underline{y}^*$. Then (using that x^* is selfmajorizing; [8], lemma 17.75)

$$\widehat{x}\underline{y}^* =_0 x^*\underline{y}^* \ge_0 x^*\underline{y} \ge_0 \widehat{x}\underline{y}.$$

Together with [8] (lemma 17.80) it now follows that $\widehat{x} \gtrsim^a_\rho x$. In particular, $\widehat{x} \in M_{\widehat{\rho}}$ (see [8], lemma 17.75).
2) $x^* \gtrsim^a_\rho x$ implies that $x^* \gtrsim^a_{\widehat{\rho}} x^*$. Also $x^* \ge_{\widehat{\rho}} \widehat{x}$, where $\ge_{\widehat{\rho}}$ is the usual pointwise $\ge$-relation. From these two facts one derives (induction on $\widehat{\rho} \in \mathbf{T}$) that $x^* \gtrsim^a_{\widehat{\rho}} \widehat{x}$. □

Remark 2.4. *For $\rho \in \mathbf{T}$, the definitions and the lemma above are already in [6].*

Let $\mathcal{A}^\omega[X, \ldots]$ be one of the theories from [3, 8] that extends full classical analysis over the types $0, X$ by an abstract (metric, hyperbolic, normed etc.) space X.

Theorem 2.5. *Let $A(x^\rho, y^\tau)$ be a formula in $\mathcal{L}(\mathcal{A}^\omega[X, \ldots])$, where $\rho, \tau \in \mathbf{T}^X$ are arbitrary. Let $a \in X$ be arbitrary as well. Then the following holds in $\mathcal{M}^{\omega,X}$:*

$$\forall \varphi^{*\widehat{\rho}\to\widehat{\tau}} \Big[\varphi^* \gtrsim^a_{\widehat{\rho}\to\widehat{\tau}} \varphi^* \wedge \forall x^\rho, x^{*\widehat{\rho}}(x^* \gtrsim^a_\rho x \to \exists y \lesssim^a_\tau \varphi^*(x^*)\, A(x,y)) \\ \to \exists \varphi^{\rho\to\tau}(\varphi^* \gtrsim^a_{\rho\to\tau} \varphi \wedge \forall x^\rho A(x, \varphi(x)))\Big].$$

Proof: Assume that for a selfmajorizing $\varphi^* \in M_{\widehat{\rho}}$ we have in $\mathcal{M}^{\omega,X}$:

$$\forall x^\rho, x^{*\widehat{\rho}}(x^* \gtrsim^a_\rho x \to \exists y \lesssim^a_\tau \varphi^*(x^*)\, A(x,y)).$$

By lemma 2.3.1) this yields

$$\forall x \in M_\rho \exists y \lesssim^a_\tau \varphi^*(\widehat{x})\, A(x,y).$$

Then by the axiom of choice AC we get

$$\exists \varphi \in M_\rho \to M_\tau \, \forall x \in M_\rho \, (\varphi^*(\widehat{x}) \gtrsim^a_\tau \varphi(x) \wedge A(x, \varphi(x))).$$

It remains to show that $\varphi^* \gtrsim^a_{\rho\to\tau} \varphi \in M_{\rho\to\tau}$: Let $x^* \gtrsim^a_\rho x$ and $\tau = \tau_1 \to \tau_2 \to \ldots \to \tau_k \to \delta$ with $\delta \in \{0, X\}$. Let $\underline{v}^*, \underline{v}$ be such that $v_i^* \gtrsim^a_{\tau_i} v_i$ for $1 \leq i \leq k$. Then (using that φ^* – as well as $\underline{v}^*$ – is selfmajorizing and both parts of lemma 2.3)

$$\varphi^* x^* \underline{v}^* \geq_0 \varphi^* \widehat{x} \underline{v}^* \geq_0 \varphi x \underline{v}, \text{ if } \delta = 0,$$

and

$$(\varphi^* x^* \underline{v}^*)_{\mathbb{R}} \geq_{\mathbb{R}} (\varphi^* \widehat{x} \underline{v}^*)_{\mathbb{R}} \geq_{\mathbb{R}} d_X(a, \varphi x \underline{v}), \text{ if } \delta = X.$$

Thus $\varphi^* x^* \underline{v}^* \gtrsim^a_\delta \varphi x \underline{v}$. Hence by [8] (lemma 17.80) and using again the selfmajorization of φ^* we get $\varphi^* \gtrsim^a_{\rho\to\tau} \varphi \in M_{\rho\to\tau}$. □

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