A polynomial rate of asymptotic regularity for compositions of projections in Hilbert space

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Abstract

This paper provides an explicit polynomial rate of asymptotic regularity for (in general inconsistent) feasibility problems in Hilbert space. In particular, we give a quantitative version of Bauschke's solution of the minimal displacement problem as well as of various generalizations of this problem. The results in this paper have been obtained by applying a general proof-theoretic method for the extraction of effective bounds from proofs due to the author ('proof mining') to Bauschke's proof.

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1 Introduction

In a remarkable paper [1], Bauschke showed the following result: let H be a Hilbert space, $C_1, \ldots, C_N \subseteq H$ be nonempty closed and convex subsets and $P_{C_1}, \ldots, P_{C_N}$ the corresponding metric projections, then the composition $T = P_{C_N} \circ \ldots \circ P_{C_1}$ is asymptotically regular (in the sense of [6]), i.e.

$$\|T^{n+1}x - T^n x\| \stackrel{n\to\infty}{\to} 0$$

for each $x \in H$.

This is relatively easy to show under the assumption that T possesses a fixed point which e.g. is trivially the case if the convex sets have a nonempty intersection. In the latter case the fixed points of T are in fact precisely the points in that intersection and the problem to find such a point is often referred to as an 'image recovery problem' or as a 'convex feasibility problem'. To show the asymptotic regularity of some appropriate iteration procedure involving T usually is the first step in proving that the iterations at least weakly converge to a fixed point of T with strong convergence in the boundedly compact case or under additional metric regularity assumptions on $C_1, \ldots, C_N$.

The problem of whether the above stated asymptotic regularity holds without assuming at least the existence of a fixed point of T, in fact, the question of whether T has approximate fixed points at all, had remained open (except for the case $N = 2$) until [1] and was referred to as the 'minimal displacement conjecture' (see [2] for this and for general background information on the topic of this paper). In this situation one also speaks of a potentially 'inconsistent feasibility problem'.

While the result proved by Bauschke is very concrete and easy to state, the proof uses a variety of nontrivial results from the abstract theory of maximal monotone operators as well as from the fixed point theory of firmly and strongly nonexpansive mappings and Bauschke does not provide any rate of convergence.

In this paper we construct such a rate of convergence $\varphi(\varepsilon, b, N, K)$ which only depends on the error ε, a norm upper bound $b \geq \|x\|$ on the starting point of the iteration, the number N of sets and a norm upper bound $K \geq \|(c_1, \ldots, c_N)\|$ on **some arbitrary** point $c = (c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N \subseteq H^N$, where H^N is equipped with the induced inner product:

$$\forall \varepsilon > 0 \, \forall x \in H \ (\|x\| \leq b \to \forall n \geq \varphi(\varepsilon, b, N, K) \, (\|T^{n+1}x - T^n x\| \leq \varepsilon)).$$

The bound φ is a simple polynomial in the data ε, b, N, K.

The rate of convergence is easier to state if we add as an additional input an upper bound $D \geq \|x - Tx\|$ on the initial displacement which, however, can be computed in terms of N, b, K, e.g. $D := 2b + NK$ (see remark 2.15), and so is actually redundant.

In [3], the asymptotic regularity result from [1] is extended to arbitrary firmly nonexpansive mappings $T_1, \ldots, T_N : H \to H$ provided that each T_i has approximate fixed points. Our quantitative analysis easily extends to this situation. In the case, where each T_i possesses even a fixed point p_i, the bound from the case for projections actually applies unchanged if we replace $K \geq \|c\|$ by $K \geq \|(p_1, \ldots, p_N)\|$. The general case needs some refinement of our analysis and the bound will depend on norm bounds for approximate fixed points of $T_1, \ldots, T_N$.

The results in this paper are obtained as an instance of a general methodology, called 'proof mining', which uses tools from mathematical logic to extract explicit effective bounds from proofs (see [12]). In fact, general logical so-called metatheorems guarantee the extractability of such bounds in quite general situations and actually provide an algorithm which in principle allows one to carry out such an extraction from a given proof. In practice, however, one will mostly follow this general algorithm as a guideline for the extraction with many optimizations tailored to the specific case at hand. That the general framework of abstract real Hilbert spaces, convex subsets and nonexpansive operators nicely fits into the logical methodology was already established in [11, 8, 12]. That metric projections can be handled was shown in [9].

The central conditions for the extractability of effect bounds (for $x \in X, T : X \to X$ with majorants $b \geq \|x\|$ and $T^* : \mathbb{N} \to \mathbb{N}$ for x and T, see below)

$$\forall n \in \mathbb{N}\ (A_\forall(n, x, T) \to \exists k \leq \Phi(n, b, T^*)\, B_\exists(n, x, T, k))$$

from a proof of

$$\forall n \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X\ \ (A_\forall(n, x, T) \to \exists k \in \mathbb{N}\, B_\exists(n, x, T, k))$$

are that

- X belongs to a suitable uniform class of structures,
- $A_\forall$ can be written (maybe with additional moduli witnessing some property) as a purely universal condition,
- the condition $A_\forall$ guarantees that T is effectively majorizable (in the input data), i.e. one can construct a function $T^*(n) : \mathbb{N} \to \mathbb{N}$ s.t.

$$\forall n \in \mathbb{N}\, \forall x \in X\ \ (n \geq \|x\| \to T^*(n) \geq \|T(x)\|),$$

- the property $B_\exists$ is purely existential.

These first two conditions are satisfied for Hilbert spaces X (see [12]) and metric projections onto closed and convex subsets (see [9]) as well as for firmly nonexpansive mappings (and also for strongly nonexpansive mappings with SNE-modulus, see below and [13]). The majorizability for nonexpansive mappings and so, in particular, for metric projections in Hilbert space and firmly or strongly nonexpansive mappings is easily seen. In the case of a metric projection $P_{C_i} : X \to C_i$ onto a nonempty closed and convex subset $C_i \subseteq X$ one just needs a norm upper bound $K \geq \|c_i\|$ for some arbitrary $c_i \in C_i$ since for all $n \in \mathbb{N}$ and $x \in X$

$$n \geq \|x\| \to n + K \geq \|P_{C_i}x - P_{C_i}0\| + \|P_{C_i}0\| \geq \|P_{C_i}x\|.$$

The existential property $B_\exists(n, T, k)$ is $\|T^k x - T^{k+1}x\| < 2^{-n}$ and, due to the fact that $(\|T^k x - T^{k+1}x\|)_{k\in\mathbb{N}}$ is nonincreasing, $\exists k \leq \varphi(n, b, K)\, (\|T^k x - T^{k+1}x\| < 2^{-n})$ implies that $\forall k \geq \varphi(n, b, K)\, (\|T^k x - T^{k+1}x\| < 2^{-n})$.

These general logical facts already predict (modulo the formalizability of Bauschke's proof in the formal framework to which the aforementioned logical metatheorems apply) that an effective rate of convergence in the above case which only depends on ε, b, K, N must be in principle extractable from Bauschke's proof. In this paper, we present the result of such an extraction and prove the correctness of the rate of convergence thus obtained without any reference to tools from logic.

Notations: $\mathbb{R}_+, \mathbb{R}_+^*$ denote the sets of nonnegative and strictly positive real numbers resp. $\mathbb{N} := \{0, 1, 2, \ldots\}$. For set-valued operator $A : X \to 2^X$ the range of A is denoted by $ran(A)$ and its closure (in norm) by $cl(ran(A))$.

2 Main results

The proof by Bauschke proceeds by using abstract operator theory to show that T has approximate fixed points and then uses results due to Bruck and Reich to conclude from this fact that T - as a strongly nonexpansive mapping - is asymptotically regular. We start by giving a quantitative version of the latter argument. In fact, the asymptotic regularity follows from a result in [14](Proposition 2.1) stating that $T^n x/n$ converges to 0 (if T has approximate fixed points, where here one may have an arbitrary normed space and T can be any nonexpansive function) and a result in [7](Proposition 1.2) which says that for strongly nonexpansive mappings T (again in arbitrary normed spaces) one has $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = \lim_{n\to\infty} \|T^n x/n\|$.

The following lemma is an easy quantitative version of Proposition 2.1 in [14] (for the special case where $c_n = 1$ needed in our paper):

Lemma 2.1. *Let* $(X, \|\cdot\|)$ *be a normed space,* $C \subseteq X$ *an arbitrary nonempty subset and* $T : C \to C$ *be a nonexpansive mapping satisfying* $\inf\{\|x - Tx\| : x \in C\} = 0$. *Let* $\alpha : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ *be such that*

$$\forall \varepsilon > 0\, \exists y \in C\ \ (\|y\| \le \alpha(\varepsilon) \wedge \|y - Ty\| \le \varepsilon)\,.$$

Then for all $b > 0$ *and* $x \in C$ *with* $\|x\| \le b$ *and* $x_k := T^k x$ *one has*

$$\forall \varepsilon > 0\, \forall k \ge \varphi(\varepsilon, b, \alpha)\ \left(\frac{\|x_{k+1}\|}{k+1} \le \varepsilon \right),$$

where

$$\varphi(\varepsilon, b, \alpha) = \left\lceil \frac{6b + 4\alpha(\varepsilon/2)}{\varepsilon} - 1 \right\rceil.$$

Proof: As in the proof of [14](Prop.2.1, for $c_k := 1$) one shows that for all $y \in C, k \in \mathbb{N}$

$$\|x_{k+1} - x\| \le 2\|x - y\| + (k+1)\|y - Ty\|$$

and so

$$\frac{\|x_{k+1}\|}{k+1} \le \frac{\|x\| + 2\|x - y\|}{k+1} + \|y - Ty\|.$$

Applied to $y_\varepsilon \in C$ with $\|y_\varepsilon\| \le \alpha(\varepsilon/2)$ and $\|y_\varepsilon - Ty_\varepsilon\| \le \frac{\varepsilon}{2}$ this gives

$$\frac{\|x_{k+1}\|}{k+1} \le \frac{3b + 2\alpha(\varepsilon/2)}{k+1} + \frac{\varepsilon}{2}$$

and so

$$\forall k \ge \left\lceil \frac{6b + 4\alpha(\varepsilon/2)}{\varepsilon} - 1 \right\rceil \ \left(\frac{\|x_{k+1}\|}{k+1} \le \varepsilon \right).$$

□

Remark 2.2. *The above lemma also holds (with the same proof) if* $\inf\{\|x - Tx\|\} = \xi > 0$, *where then* α *is such that*

$$\forall \varepsilon > 0\, \exists y \in C \ \ (\|y\| \leq \alpha(\varepsilon) \wedge \|y - Ty\| \leq \xi + \varepsilon)$$

and the conclusion says

$$\forall \varepsilon > 0\, \forall k \geq \varphi(\varepsilon, b, \alpha) \ \ \left(\frac{\|x_{k+1}\|}{k+1} \leq \xi + \varepsilon\right).$$

Definition 2.3 ([13], Definition 2.4)**.** *Let* $C \subseteq X$ *be an arbitrary set. A function* $T : C \to X$ *is called strongly nonexpansive (SNE) with SNE-modulus* $\omega : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ *if*

$$\forall d \in \mathbb{R}_+\, \forall \varepsilon > 0\, \forall x, y \in C$$
$$(\|x - y\| \leq d \wedge \|x - y\| - \|Tx - Ty\| < \omega(d, \varepsilon) \to \|(x - y) - (Tx - Ty)\| < \varepsilon).$$

Remark 2.4. *As shown in* [13], *a function* $T : C \to X$ *is strongly nonexpansive in the sense introduced in* [7] *iff it possesses an SNE-modulus in the above sense.*

A proof-theoretic analysis of the proof of Proposition 1.2 in [7] together with the quantitative analysis of strongly nonexpansive mappings from [13] results in:

Theorem 2.5. *Under the assumptions of Lemma 2.1, if* T *is additionally strongly nonexpansive with SNE-modulus* ω, *then for* $x \in C$, $x_n := T^n x$ *and* $D > 0$ *such that* $\|x - Tx\| \leq D$ *one has*

$$\forall \varepsilon > 0\, \forall n \geq \psi(\varepsilon, b, D, \alpha, \omega) \ \ (\|x_{n+1} - x_n\| < \varepsilon),$$

where

$$\psi(\varepsilon, b, D, \alpha, \omega) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6)}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

with

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}.$$

Proof: Consider $y_n := x_{n+1} - x_n = T^{n+1}x - T^n x$ and let $k \in \mathbb{N}$. Since T is in particular nonexpansive, the sequence $(\|y_n\|)_{n \in \mathbb{N}}$ is nonincreasing and $\|y_n\| \leq \|y_0\| \leq D$ for all $n \in \mathbb{N}$. Hence by [12](Prop.2.27 and Rem.2.29.1) applied to $g(n) := k$ (and so $\tilde{g}(n) = n + k$) one gets

$$\exists n \leq k \left\lceil \frac{D}{\varepsilon} \right\rceil \ \left(\bigwedge_{i=0}^{k-1} (\|y_{n+i}\| - \|y_{n+i+1}\| < \varepsilon) \right).$$

Applying this to $\omega(D, \tilde{\varepsilon}))$ as ε and using that ω is an SNE-modulus for T and that $y_{n+i} = x_{n+i+1} - x_{n+i}$, $y_{n+i+1} = Tx_{n+i+1} - Tx_{n+i}$ one obtains

$$\forall i \leq k - 1 \ (\|y_{n+i} - y_{n+i+1}\| < \tilde{\varepsilon})$$

and so

$$\forall 1 \leq i \leq k \ \ (\|y_{n+i} - y_n\| < i \cdot \tilde{\varepsilon}).$$

For $\tilde{\varepsilon} \le 2(k+1)\varepsilon/3(k+1)k = 2\varepsilon/3k$ (if $k \ge 1$ and $\tilde{\varepsilon} > 0$ arbitrary if $k = 0$) we then get

$$\exists n \le k \left\lceil \frac{D}{\omega(D,\tilde{\varepsilon})} \right\rceil \left(\sum_{i=1}^{k} \|y_{n+i} - y_n\| < (k+1)\frac{\varepsilon}{3} \right).$$

Note that

$$T^{n+k+1}x - T^n x = \sum_{i=0}^{k} y_{n+i} = \sum_{i=1}^{k} (y_{n+i} - y_n) + (k+1)y_n.$$

Hence

$$\left\| \frac{T^{n+k+1}x - T^n x}{k+1} - (T^{n+1}x - T^n x) \right\| < \frac{\varepsilon}{3}$$

and so (using again that T is nonexpansive)

$$\|T^{n+1}x - T^n x\| < \frac{\|T^{n+k+1}x - T^n x\|}{k+1} + \frac{\varepsilon}{3} \le \frac{\|T^{k+1}x - x\|}{k+1} + \frac{\varepsilon}{3} \le \frac{\|T^{k+1}x\|}{k+1} + \frac{\|x\|}{k+1} + \frac{\varepsilon}{3}.$$

Now let $k := \left\lceil \frac{3(6b+4\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \ge \frac{3b}{\varepsilon} - 1$, then using Lemma 2.1

$$\frac{\|T^{k+1}x\|}{k+1} \le \frac{\varepsilon}{3} \text{ and } \frac{\|x\|}{k+1} \le \frac{b}{k+1} \le \frac{\varepsilon}{3}$$

and so by the above

$$\exists n_0 \le k \left\lceil \frac{D}{\omega(D,\tilde{\varepsilon})} \right\rceil \left(\|T^{n_0+1}x - T^{n_0}x\| < \frac{\|T^{k+1}x\|}{k+1} + \frac{\|x\|}{k+1} + \frac{\varepsilon}{3} \le \varepsilon \right).$$

Since

$$k \le \frac{18b + 12\alpha(\varepsilon/6)}{\varepsilon},$$

the theorem now follows using that $(\|T^{n+1}x - T^n x\|)_{n\in\mathbb{N}}$ is nonincreasing. $\square$

Remark 2.6. *In the situation of Remark 2.2, Theorem 2.5 holds with* $\|x_{n+1} - x_n\| < \xi + \varepsilon$.

We now come to the main part of Bauschke's proof where the theory of maximally monotone operators is used to show that T has approximate fixed points.

In the following, H is a real Hilbert space and $C_1, \ldots, C_N \subseteq H$ are nonempty closed and convex subsets of H. Let $C := C_1 \times \ldots \times C_N$. Following [1], we consider H^N equipped with the induced inner product

$$\langle x, y \rangle := \sum_{n=1}^{N} \langle x_n, y_n \rangle \text{ for all } x = (x_1, \ldots, x_N),\ y = (y_1, \ldots, y_N) \in H^N.$$

We will need the following notions

$R : H^N \to H^N,\ R(x_1, x_2, \ldots, x_N) := (x_N, x_1, \ldots, x_{N-1})$ (right-shift),
$L : H^N \to H^N,\ L(x_1, x_2, \ldots, x_N) := (x_2, x_3, \ldots, x_N, x_1)$ (left-shift),
$M := I - R$ and $S := \frac{1}{2}M + \frac{1}{2}M^* = I - \frac{1}{2}R - \frac{1}{2}L$
(symmetric part of M, M^* adjoint operator),
$q : H^N \to \mathbb{R},\ q(x) := \frac{1}{2}\langle x, Mx\rangle$,
$q^*(x^*) := \sup\limits_{x \in H^N} \{\langle x^*, x\rangle - q(x)\}$ (conjugate function of q),
$N_C x := \begin{cases} \{u \in H^N\ :\ \sup\langle C - x, u\rangle \le 0\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$ (normal cone of C)

Lemma 2.7 ([1], Fact 2.2(ii))**.** $q^* \circ S = q$.

The next lemma is a quantitative version of the relevant part of Proposition 2.3 in [1]:

Lemma 2.8. *Let $y = (y_1, \ldots, y_N) \in H^N$ be such that $\sum_{n=1}^N y_n = 0$ and $\|y\| \le K$ with $K \ge 1$. Then $q^*(y) \le \frac{N^3(N-1)^2}{4} \cdot K^2$.*

Proof: As in the proof of Proposition 2.3 in [1] one defines for $1 \le n \le N-1$

$$z_n := y_1 + 2y_2 + \ldots n y_n \text{ and } z_N := 0.$$

Then for $n = 1, \ldots, N-1$

$$\|z_n\| \le \left\| \sum_{i=1}^n i y_i \right\| \le \sum_{i=1}^n i\|y_i\| \le K \sum_{i=1}^n i = \frac{Kn(n+1)}{2}.$$

Again as in [1], one now defines by backwards recursion $x = (x_1, \ldots, x_N)$ as

$$x_N := 0 \text{ and } (n+1)x_n - n x_{n+1} = z_n \text{ for } 1 \le n \le N-1.$$

As shown in [1], $y = S(2x)$ and so by Lemma 2.7 $q^*(y) = q(2x)$.
One easily verifies that

$$\|x_n\| \le \frac{\|z_n\|}{n+1} + \|x_{n+1}\| \text{ for } 1 \le n \le N-1$$

and so for $n = 1, \ldots, N$

$$\|x_n\| \le \sum_{i=1}^{N-1} \frac{\|z_i\|}{i+1} \le \frac{1}{2} \sum_{i=1}^{N-1} i \cdot K \le \frac{N(N-1)}{4} \cdot K$$

and in turn

$$\|x\| \le \frac{\sqrt{N} \cdot N(N-1)}{4} \cdot K \text{ and } |q(2x)| \le 2\|x\| \cdot \|M(x)\| \le 4\|x\|^2 \le \frac{N^3(N-1)^2}{4} \cdot K^2.$$

$\Box$

Lemma 2.9. *Let $c = (c_1, \ldots, c_N) \in C$ with $\|c\| \leq K$. Then*

$$\sup_{w \in H^N} \langle w - c, -Mw \rangle \leq \frac{N^3(N-1)^2}{2} \cdot K^2.$$

Proof: As in 'Step 3' in the proof of Theorem 3.1 in [1] one shows (for the relevant case where $x := c$ and $y := 0$ and so $My = 0$) that

$$\sup_{w \in H^N} \langle w - c, -Mw \rangle \leq 2q^*(\frac{1}{2}M^*c),$$

where $M^*c = z = (z_1, \ldots, z_N) = (c_1 - c_2, c_2 - c_3, \ldots, c_{N-1} - c_N, c_N - c_1)$. Since $\sum_{n=1}^N z_n = 0$ and $\|\frac{1}{2}z\| \leq K$, Lemma 2.8 implies that

$$q^*((1/2)z) \leq \frac{N^3(N-1)^2}{4} \cdot K^2$$

and so the lemma follows. □

Lemma 2.10. *With c, K as before, if*

$$\sup_{w \in H^N} \langle w - c, -Mw \rangle \leq L,$$

then

$$\forall \varepsilon \in (0,1)\, \exists b \in H^N, x \in C\ (b \in N_C(x) + M(x) \wedge \|x\| \leq 4(K^2 + 2L)/\varepsilon \wedge \|b\| \leq \varepsilon).$$

Proof: As in the proof of Theorem 3.1 in [1] (Steps 1 and 2) one shows that $N_C + M$ is maximal monotone. Clearly, $0 = 0 + 0 \in N_C(c) + M(0)$. We now follow the reasoning from the proof of Theorem 4 in [4] (adapted to the special case at hand): By the monotonicity of N_C is follows that

$$\langle h_1, z - c \rangle \geq 0,\ \forall (z, h_1) \in G(N_C).$$

By the assumption in the lemma we have

$$\langle Mw, w - c \rangle \geq -L,\ \forall w \in H^N$$

and so adding these inequalities

$$(*)\ \langle h, w - c \rangle \geq -L,\ \forall (w, h) \in G(N_C + M).$$

We now follow quantitatively the proof of Lemma 1 in [4]: Since $N_C + M$ is maximal monotone, $\varepsilon I + (N_C + M)$ has full range (by Minty's theorem) and so, in particular, $0 \in ran(\varepsilon I + (N_C + M))$, i.e.

$$\forall \varepsilon > 0\, \exists u_\varepsilon \in H^N\, (0 \in \varepsilon u_\varepsilon + (N_C + M)(u_\varepsilon)).$$

In fact, $u_\varepsilon \in C$. By $(*)$ we have

$$\forall (w,h) \in G(N_C + M)\ (\langle -h, w - c\rangle \leq L).$$

Applied to $w := u_\varepsilon, h := -\varepsilon u_\varepsilon$, this gives us $\langle \varepsilon u_\varepsilon, u_\varepsilon - c\rangle \leq L$ and so

$$\frac{1}{2}\varepsilon \|u_\varepsilon - c\|^2 \leq \frac{1}{2}\varepsilon \|c\|^2 + L \leq \frac{1}{2}\varepsilon K^2 + L.$$

Hence (using that $\varepsilon \in (0,1)$)

$$\sqrt{\varepsilon} \cdot \|u_\varepsilon\| \leq \sqrt{\varepsilon} \cdot \|c\| + \sqrt{\varepsilon \cdot K^2 + 2L} \leq \sqrt{\varepsilon K^2} + \sqrt{\varepsilon K^2 + 2L} \leq 2\sqrt{K^2 + 2L}.$$

Now take $\tilde{\varepsilon} := \varepsilon^2/(4(K^2 + 2L))$. Then $\|\tilde{\varepsilon} \cdot u_{\tilde{\varepsilon}}\| \leq \varepsilon$ and

$$\|u_{\tilde{\varepsilon}}\| \leq \frac{2\sqrt{K^2 + 2L}}{\sqrt{\tilde{\varepsilon}}} = \frac{4(K^2 + 2L)}{\varepsilon}.$$

The lemma is now satisfied with $b := -\tilde{\varepsilon} u_{\tilde{\varepsilon}}$ and $x := u_{\tilde{\varepsilon}}$. $\square$

The next theorem is a quantitative version of the minimal displacement conjecture (proved in [1]):

Theorem 2.11. *Let H be a real Hilbert space and $C_1, \ldots, C_N \subseteq H$ be nonempty closed and convex subsets and P_{C_i} the metric projections onto C_i for $i = 1, \ldots, N$. Let $c = (c_1, \ldots, c_N)$ be an arbitrary element of $C := C_1 \times \ldots \times C_N$ and $K \geq \|c\|$ (with $K \geq 1$). Let $T := P_{C_N} \circ \ldots \circ P_{C_1}$. Then for every $\varepsilon \in (0,1)$ there exists a point $y \in C_N$ with*

$$\|y\| \leq \alpha_K(\varepsilon) \text{ and } \|Ty - y\| \leq \varepsilon,$$

where

$$\alpha_K(\varepsilon) := \frac{4(K^2 + N^3(N-1)^2K^2)N^2}{\varepsilon}.$$

Proof: For given $\varepsilon \in (0,1)$, let x be as in Lemma 2.10. Inspecting the proofs of 'Steps 6-9' in the proof of Theorem 3.1 in [1] shows that for the N-th component x_N of x one has that

$$\|x_N - Tx_N\| \leq N^2 \varepsilon.$$

The theorem now follows from Lemma 2.9 and Lemma 2.10. $\square$

Definition 2.12 ([5])**.** *A mapping $T : H \to H$ is called firmly nonexpansive if*

$$\forall x, y \in H\ (\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle).$$

Lemma 2.13. *Let T be as in Theorem 2.11. Then T is strongly nonexpansive with SNE-modulus*

$$\omega_T(d, \varepsilon) := \frac{1}{16d}\left(\frac{\varepsilon}{N}\right)^2.$$

This modulus also holds for $T = T_N \circ \ldots \circ T_1$ for any firmly nonexpansive mappings $T_1, \ldots, T_N : H \to H$.

Proof: Metric projections in Hilbert space are firmly nonexpansive and so by Corollary 2.18 in [13] (applied to $\lambda := 1/2$) have $\frac{1}{16d}\varepsilon^2$ as SNE-modulus. The lemma now follows from [13](Theorem 2.10). $\square$

Corollary 2.14. *Under the conditions of Theorem 2.11, the sequence* $(x_n) := (T^n x)$ *is asymptotically regular with rate of convergence* $\psi(\varepsilon, b, D, \alpha_K, \omega_T)$ *with* ψ *as in Theorem 2.5,* α_K *as in Theorem 2.11 and* ω_T *as in Lemma 2.13, i.e.*

$$\forall \varepsilon \in (0,1)\, \forall n \geq \psi(\varepsilon, b, D, \alpha_K, \omega_T)\ (\|x_{n+1} - x_n\| < \varepsilon),$$

where $b, D > 0$ *with* $b \geq \|x\|$ *and* $D \geq \|Tx - x\|$.

Remark 2.15.

1. *The input* D *in the Corollary 2.14 actually is redundant as such a* D *can be computed in terms of* b, K, N*:* $\|c\| \leq K$ *implies that* $\|c_i\| \leq K$ *for* $1 = 1, \ldots, N$ *and so*

$$\|P_{C_i} 0\| \leq \|c_i\| \leq K$$

and -using that P_{C_i} *is nonexpansive - in turn*

$$\forall y \in H\, (\|P_{C_i} y\| \leq \|P_{C_i} y - P_{C_i} 0\| + \|P_{C_i} 0\| \leq \|y\| + K.$$

Inductively, it now follows that for all $i = 1, \ldots, N$

$$\|(P_{C_i} \circ \ldots \circ P_{C_1})x\| \leq b + iK$$

and so, in particular, $\|Tx\| \leq b + NK$ *and, consequently,*

$$\|Tx - x\| \leq \|Tx\| + \|x\| \leq 2b + NK.$$

So we may always take $D := 2b + NK$.

2. *By an affine shift one can always reduce the general situation to the case where* $x = 0$*. Hence the bound in Corollary 2.14 also holds if we take* $b := 0$ *but then require that* $K \geq \|(c_1, \ldots, c_N) - (x, \ldots, x)\|$*, i.e. the bound then only depends on the relative distances between each* c_i *and* x.

In [3], it is observed that the approach in [1] extends to arbitrary firmly nonexpansive mappings $T_1, \ldots, T_N : H \to H$ as long as each T_i possesses arbitrary good approximate fixed points. One then uses instead of N_C the maximal monotone operator

$$A(x) := (A_1 x_1, \ldots, A_N x_N), \text{ where } A_i := T_i^{-1} - I.$$

The quantitative analysis given above is largely independent of whether one has N_C or A except the issue that we now only have $0 \in cl(ran(A))$ instead of $0 \in ran(A)$. Let us first consider the case where each T_i actually possesses a fixed point $p_i \in H$. Then $0 \in A(p_1, \ldots, p_N)$ and the proofs of Theorem 2.11 and Corollary 2.14 go through with the only change that we now need a norm bound K on $p = (p_1, \ldots, p_N)$ rather than on $c \in C_1 \times \ldots \times C_N$. In particular we have

Theorem 2.16. *Let H be a real Hilbert space and $T_1, \ldots, T_N : H \to H$ be firmly nonexpansive mappings which posses fixed points $p_1, \ldots, p_N \in H$ resp. Let $K \geq \|p = (p_1, \ldots, p_N)\|$ (with $K \geq 1$). Then for $T := T_N \circ \ldots \circ T_1$ and $x_n := T^n x$ for $x \in H$ with $b \geq \|x\|$ and $D \geq \|Tx - x\|$*

$$\forall \varepsilon \in (0,1)\, \forall n \geq \psi(\varepsilon, b, D, \alpha_K, \omega_T)\ (\|x_{n+1} - x_n\| \leq \varepsilon)$$

with ψ as in Theorem 2.5, α_K as in Theorem 2.11 and ω_T as in Lemma 2.13.

If we only have the existence of ε-approximate fixed points $p_{1,\varepsilon}, \ldots, p_{N,\varepsilon} \in H$ for $T_1, \ldots, T_N$ resp. for every $\varepsilon > 0$, .i.e.

$$\|T_i p_{i,\varepsilon} - p_{i,\varepsilon}\| < \varepsilon \text{ for } i = 1, \ldots, N,$$

then we have to refine Lemma 2.9 and Lemma 2.10:

Lemma 2.17. *Let $x = (x_1, \ldots, x_N) \in H^N$ with $\|x\| \leq K$ (with $K \geq 1$) and $f = (f_1, \ldots, f_N) \in H^N$ with $\|f\| \leq 1$. Then*

$$\sup_{w \in H^N} \langle w - x, Mf - Mw \rangle \leq \frac{N^3(N-1)^2}{2} \cdot (K+1)^2 + 2K.$$

Proof: As in 'Step 3' in the proof of Theorem 3.1 in [1] one shows that

$$\sup_{w \in H^N} \langle w - x, Mf - Mw \rangle \leq -\langle x, Mf \rangle + 2q^* \left(\frac{1}{2} Mf + \frac{1}{2} M^* x \right),$$

where

$$\frac{1}{2} Mf + \frac{1}{2} M^* x = \frac{1}{2} z := \frac{1}{2} (x_1 - x_2 + (f_1 - f_N), x_2 - x_1 + (f_2 - f_1), \ldots, x_N - x_1 + (f_N - f_{N-1}).$$

Since $\sum_{n=1}^N z_n = 0$ and $\frac{1}{2}\|z\| \leq K + 1$, Lemma 2.8 implies that

$$q^*((1/2)z) \leq \frac{N^3(N-1)^2}{4} \cdot (K+1)^2.$$

Also

$$-\langle x, Mf \rangle \leq \|x\| \cdot \|Mf\| \leq 2\|x\| \leq 2K$$

and so the lemma follows. □

Lemma 2.18. *Let $\varepsilon \in (0,1)$, x be as in the previous lemma and assume that $\tilde{T} := (T_1, \ldots, T_N)$ possesses ε-approximate fixed points $p_\varepsilon = (p_{1,\varepsilon}, \ldots, p_{N,\varepsilon}) \in H^N$ for every $\varepsilon \in (0,1)$ with $\|\tilde{T} p_\varepsilon\| \leq K(\varepsilon)$. If L is such that $\forall g \in H$ with $\|g\| \leq 1$*

$$\sup_{w \in H^N} \langle w - \tilde{T} p_{\varepsilon/4}, Mg - Mw \rangle \leq L,$$

then

$$\forall \varepsilon \in (0,1)\, \exists b, x \in H^N\ (b \in A(x) + M(x) \wedge \|x\| \leq 16((K(\varepsilon/4) + 1)^2 + 2L)/\varepsilon \wedge \|b\| \leq \varepsilon).$$

Proof: By Corollary 2.6 in [3] $A + M$ is maximal monotone. By the existence of ε-fixed points of $\tilde{T}$ one has, taking an $\varepsilon/4$-fixed point p with $\|p\| \leq K(\varepsilon/4)$, the existence of $q \in A(\tilde{T}p)$ with $\|q\| \leq \varepsilon/4 \leq 1$, namely $q = p - \tilde{T}p$. Hence $f := q + Mq \in A(\tilde{T}p) + Mq$ with $\|f\| \leq 3\|q\| \leq 3\varepsilon/4$. We now follow again the reasoning from the proof of Theorem 4 in [4]: By the monotonicity of A it follows that

$$\langle h_1 - q, z - \tilde{T}p\rangle \geq 0, \ \forall (z, h_1) \in G(A).$$

By the assumption in the lemma we have

$$\langle Mw - Mq, w - \tilde{T}p\rangle \geq -L, \ \forall w \in H^N$$

and so

$$(*) \ \langle h - f, w - \tilde{T}p\rangle \geq -L, \ \forall (w, h) \in G(A + M).$$

We now follow quantitatively the proof of Lemma 1 in [4]: Since $A+M$ is maximal monotone, $\tilde{\varepsilon}I + (A + M)$ has full range for every $\tilde{\varepsilon} > 0$ by Minty's theorem and so, in particular, $f \in ran(\tilde{\varepsilon}I + (A + M))$, i.e.

$$\forall \tilde{\varepsilon} > 0 \, \exists u_{\tilde{\varepsilon}} \in H^N \, (f \in \tilde{\varepsilon}u_{\tilde{\varepsilon}} + (A + M)(u_{\tilde{\varepsilon}})).$$

By $(*)$ we have

$$\forall (w, h) \in G(A + M) \ (\langle f - h, w - \tilde{T}p\rangle \leq L).$$

Applied to $w := u_{\tilde{\varepsilon}}, h := f - \tilde{\varepsilon}u_{\tilde{\varepsilon}}$, this gives us $\langle \tilde{\varepsilon}u_{\tilde{\varepsilon}}, u_{\tilde{\varepsilon}} - \tilde{T}p\rangle \leq L$ and so

$$\frac{1}{2}\tilde{\varepsilon}\|u_{\tilde{\varepsilon}} - \tilde{T}p\|^2 \leq \frac{1}{2}\tilde{\varepsilon}\|\tilde{T}p\|^2 + L \leq \frac{1}{2}\tilde{\varepsilon}K(\varepsilon/4)^2 + L.$$

Hence (for $\tilde{\varepsilon} \in (0, 1)$)

$$\sqrt{\tilde{\varepsilon}} \cdot \|u_{\tilde{\varepsilon}}\| \leq \sqrt{\tilde{\varepsilon}K(\varepsilon/4)^2} + \sqrt{\tilde{\varepsilon} \cdot K(\varepsilon/4)^2 + 2L} \leq 2\sqrt{K(\varepsilon/4)^2 + 2L}.$$

Now take $\tilde{\varepsilon} := (\varepsilon/4)^2/(4(K(\varepsilon/4)^2 + 2L))$. Then $\|\tilde{\varepsilon}u_{\tilde{\varepsilon}}\| \leq \varepsilon/4$ and so $\|f - \tilde{\varepsilon}u_{\tilde{\varepsilon}}\| \leq \varepsilon$ and

$$\|u_{\tilde{\varepsilon}}\| \leq \frac{2\sqrt{K(\varepsilon/4)^2 + 2L}}{\sqrt{\tilde{\varepsilon}}} = \frac{16(K(\varepsilon/4)^2 + 2L)}{\varepsilon}.$$

The lemma is now satisfied with $b := f - \tilde{\varepsilon}u_{\tilde{\varepsilon}}$ and $x := u_{\tilde{\varepsilon}}$. □

Theorem 2.19. *Let H be a real Hilbert space, $K : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ be a function and $T_1, \ldots, T_N : H \to H$ be firmly nonexpansive mappings s.t. for each $\varepsilon > 0$ the mapping $(T_1, \ldots, T_N) : H^N \to H^N$ has an ε-fixed point p_ε with $\|p_\varepsilon\| \leq K(\varepsilon)$. Let $T := T_N \circ \ldots \circ T_1$. Then for every $\varepsilon \in (0, 1)$ there exists a point $y \in H$ with*

$$\|y\| \leq \alpha_K(\varepsilon) \text{ and } \|Ty - y\| \leq \varepsilon,$$

where

$$\alpha_K(\varepsilon) := 16((K(\varepsilon/4) + 2)^2 + N^3(N - 1)^2(K(\varepsilon/4) + 2)^2 + 4K(\varepsilon/4) + 2)N^2/\varepsilon.$$

Proof: For given $\varepsilon \in (0,1)$, let x be as in Lemma 2.18. Inspecting the proofs of Theorem 3.1(ii)-(v) in [3] one shows that for the N-th component x_N of x one has that

$$\|x_N - Tx_N\| \le N^2\varepsilon.$$

The theorem now follows from Lemma 2.17 and Lemma 2.18 noticing that
$\|\tilde{T}p_{\varepsilon/4}\| \le \|p_{\varepsilon/4}\| + \varepsilon/4 \le K(\varepsilon/4) + 1$. □

Corollary 2.20. *Under the conditions of Theorem 2.19, the sequence* $(x_n) := (T^n x)$ *is asymptotically regular with rate of convergence* $\psi(\varepsilon, b, D, \alpha_K, \omega_T)$ *with* ψ *as in Theorem 2.5,* α_K *as in Theorem 2.19 and* ω_T *as in Lemma 2.13, i.e. for* $b, D > 0$ *with* $b \ge \|x\|$ *and* $D \ge \|Tx - x\|$

$$\forall \varepsilon \in (0,1)\, \forall n \ge \psi(\varepsilon, b, D, \alpha_K, \omega_T)\ (\|x_{n+1} - x_n\| < \varepsilon).$$

Remark 2.21. *Also in Theorem 2.16 and in Corollary 2.20 one can compute the upper bound* D *already in terms of the other data: let* $p = (p_1, \ldots, p_N)$ *be a* 1*-approximate fixed point of* $(T_1, \ldots, T_N)$ *with* $\|p\| \le K(1)$*. Then, in particular,* $\|T_i p_i - p_i\| \le 1$ *and so* $\|T_i p_i\| \le \|p_i\| + 1 \le K(1) + 1$ *for* $i = 1, \ldots, N$*. Hence*

$$\|T_i y\| \le \|T_i y - T_i p_i\| + \|T_i p_i\| \le \|y - p_i\| + K(1) + 1 \le \|y\| + 2K(1) + 1$$

and so $\|Tx\| \le b + N(2K(1)+1)$ *and, finally,* $\|Tx - x\| \le 2b + N(2K(1)+1)$*.*

As follows from [13], the asymptotic regularity of SNE-functions T (in arbitrary Banach spaces) and hence of compositions of firmly nonexpansive functions (in uniformly convex Banach spaces) is much easier when T is assumed to have a fixed point. We state here this only for the case of Hilbert space H :

Theorem 2.22. *Let* $C \subseteq H$ *be any subset of a real Hilbert space* H *and let* $T_1, \ldots, T_N : C \to C$ *be firmly nonexpansive mappings. Let* $T := T_N \circ \ldots \circ T_1$ *possess a fixed point* $p \in C$ *and, for* $x \in C$*, let* $b \ge \|x - p\|$*. Then for* $x_n := T^n x$:

$$\forall \varepsilon > 0\, \forall n \ge \lceil b/\omega_T(b,\varepsilon) \rceil\ (\|x_{n+1} - x_n\| < \varepsilon),$$

where

$$\omega_T(b,\varepsilon) := \frac{1}{16b}(\varepsilon/N)^2.$$

Proof: The theorem is immediate from Theorems 2.8, 2.10 and Corollary 2.18 in [13]. □

If $T_1, \ldots, T_N$ have common fixed points then one has explicit bounds on the number of T-iterations needed to obtain a common ε-fixed point of $T_1, \ldots, T_N$: see [13] and - for the case of convex combinations of projections in Hilbert space - [10].

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