# On the Computational Content of the Bolzano-Weierstraß Principle*

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**Abstract**

We will apply the methods developed in the field of 'proof mining' to the Bolzano-Weierstraß theorem BW and calibrate the computational contribution of using this theorem in proofs of combinatorial statements. We provide an explicit solution of the Gödel functional interpretation (combined with negative translation) as well as the monotone functional interpretation of BW for the product space $\prod_{i\in\mathbb{N}}[-k_i, k_i]$ (with the standard product metric). This results in optimal program and bound extraction theorems for proofs based on fixed instances of BW, i.e. for BW applied to fixed sequences in $\prod_{i\in\mathbb{N}}[-k_i, k_i]$.

## Introduction

During the last 15 years, an applied form of proof theory, also called 'proof mining', became more and more prominent that is concerned primarily with the extraction of additional (often computational) information from prima facie ineffective proofs which could not be read of directly (see e.g. [18]). Proof interpretations, most importantly forms of functional interpretation, are the key proof theoretic methods used in this type of proof theory. These methods enjoy a strong modularity in the sense that once the solution of the interpretation of a certain key lemma is found, that solution can be used without change in any other 'unwinding' of proofs using this lemma. Most systematically, proof interpretations have been applied to proofs in (nonlinear) functional analysis. Here sequential compactness principles play an important role in numerous proofs. So far, such uses of sequential compactness could be dealt with using an elimination procedure due to the second author ([14, 15, 13, 17]) which replaces (if the underlying context is elementary enough) applications of fixed instances of these principles by arithmetical counterparts. However, more substantial uses of sequential compactness (e.g. in the context of weak compactness arguments) require to deal with these principles directly by explicitly solving their functional interpretation (see [19, 20]). For the simplest form of sequential compactness, the principle that monotone bounded sequences of real numbers are convergent, this has been done already in [21].

This paper provides an explicit solution of the (negative translation of the) Gödel functional interpretation as well as the monotone functional interpretation of the Bolzano-Weierstraß theorem BW for $[0, 1]$ and other compact metric spaces. Moreover, we argue

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that our solution is of optimal complexity. In fact, we will use it to get optimal program and bound extraction theorems for proofs based on fixed instances $\mathsf{BW}(t)$ of $\mathsf{BW}$, i.e. for $\mathsf{BW}$ applied to fixed bounded sequences in $[0,1]$ given by a term $t$ whose only free variables are the parameters of the theorem to be proved.

As is known from reverse mathematics ([28]) the Bolzano-Weierstraß theorem $\mathsf{BW}$ for compact metric spaces can be proved using arithmetical comprehension $\mathsf{CA}_{ar}$ (also denoted by $\Pi^0_\infty$-$\mathsf{CA}$) and – already for $[0,1]$ – also implies $\mathsf{CA}_{ar}$. All this holds irrespectively of whether $\mathsf{BW}$ is stated to assert the existence of a cluster point or a convergent subsequence and relative to a weak base system $\mathsf{RCA}_0$. In fact, already the special case of $\mathsf{BW}$ stating that every monotone sequence in $[0,1]$ is convergent implies $\mathsf{CA}_{ar}$ (much refined results in this direction can be found in [17]).

From an inspection of Spector's solution of the functional interpretation of classical analysis by his bar recursive functionals $\mathcal{T} + (\mathsf{BR})$ ([29]) it follows that the functional interpretation of $\mathsf{CA}_{ar}$ and hence of $\mathsf{BW}$ can be solved in the fragment $\mathcal{T}_0 + \mathsf{BR}_{0,1}$, where only the primitive recursor $\mathsf{R}_0$ for type 0 and the bar recursor $\mathsf{B}_{0,1}$ for the types $0,1$ are used (see [16]). Here 0 denotes the type of natural numbers and 1 the type of number theoretic functions $f : \mathbb{N} \to \mathbb{N}$. In general, for types $\rho, \tau$ we denote the type of objects that map objects of type $\rho$ to objects of type $\tau$ by $\tau(\rho)$. Pure types are of the form 0 or $0(\rho)$ (where $\rho$ already is a pure type) and can be represented by natural numbers via $n+1 := 0(n)$. The level or degree $deg(\rho)$ of a type $\rho$ is defined as $deg(\tau(\rho)) := \max(deg(\tau), \deg(\rho)+1)$ with $deg(0) := 0$. Up to the type level 2, the functionals definable by closed terms of $\mathcal{T}_0 + \mathsf{BR}_{0,1}$ (and also of $\mathcal{T}_1 + \mathsf{BR}_{0,1}$) coincide with those definable in Gödel's system $\mathcal{T}$ of primitive recursive functionals of finite type ([7, 8]). For type 3 this no longer holds as $\mathsf{B}_{0,1}$ is not definable in $\mathcal{T}$ and already for the type 1 this fails for functionals definable in $\mathcal{T}_2 + \mathsf{BR}_{0,1}$, $\mathcal{T}_0 + \mathsf{BR}_{0,2}$ or $\mathcal{T}_0 + \mathsf{BR}_{1,1}$ (see [16] for all this). Here $T_n$ is the fragment of $T$ that only contains recursors for primitive recursion of type level $\leq n$.
However, for a faithful calibration of the contribution of (single instances of) $\mathsf{BW}$ and the extraction of realizers of optimal complexity level from proofs of $\forall n\, \exists m$-sentences that are based on uses of $\mathsf{BW}$ even the fact that one has a functional interpretation in $\mathcal{T}_0 + \mathsf{BR}_{0,1}$ is too crude. Indeed, it is crucial that the solution of the functional interpretation of $\mathsf{BW}$ uses only minimal number of nested $\mathsf{B}_{0,1}$-applications. In fact, we will show that a single use of $\mathsf{B}_{0,1}$ plus a use of a weak 'binary' form of bar recursion (due to Howard) suffices. Together with results of Howard and Parson this can be used to show that over systems such as

$$\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction + \mathsf{QF\text{-}AC} + \Sigma^0_{n+1}\text{-}\mathsf{IA}$$

(which has a functional interpretation in $\mathcal{T}_n$, see [24]) the contribution of a use of $\mathsf{BW}$ in the form

$$\forall n \left( \mathsf{BW}(\xi(n)) \to \exists m\, \varphi_{\mathsf{QF}}(n,m) \right)$$

in a proof of a sentence

$$\forall n \exists m\, \varphi_{\mathsf{QF}}(n,m) \text{ (with quantifier-free } \varphi_{\mathsf{QF}})$$

at most increases the complexity of the extractable algorithm $f$ s.t.

$$\forall n\, \varphi_{\mathsf{QF}}(n, f(n))$$

from $f \in T_n$ to $f \in T_{n+1}$. We will also show that this increase in general is unavoidable, thereby establishing the optimality of our result.
Here $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction$ is the (weakly extensional) extension of Primitive Recursive Arithmetic $\mathsf{PRA}$ to all finite types (i.e. – in contrast to $\mathsf{WE\text{-}PA}^\omega$ – we only include quantifier-free induction and primitive recursion of type $\mathbb{N}$).
Moreover, $\mathsf{QF\text{-}AC}$ is the schema of quantifier-free choice, i.e.

$$\mathsf{QF\text{-}AC} : \quad \forall x^\rho \exists y^\tau \varphi_{\mathsf{QF}}(x,y) \;\to\; \exists g^{\tau(\rho)} \forall x^\rho \varphi_{\mathsf{QF}}(x, g(x))$$

with quantifier-free $\varphi_{\mathsf{QF}}$ (also for tuples of variables $\underline{x}, \underline{y}$ of arbitrary types). Roughly speaking, $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\upharpoonright + \mathsf{QF\text{-}AC}$ is a (conservative) finite type extension of (an appropriate version of) $\mathsf{RCA}_0$.
$\Sigma^0_n$-$\mathsf{IA}$ is the schema of induction for $\Sigma^0_n$-formulas, i.e.

$$\Sigma^0_n\text{-}\mathsf{IA} \quad : \quad \varphi(0) \wedge \forall x^0(\varphi(x) \to \varphi(x+1)) \;\to\; \forall x^0 \varphi(x),$$

where $\varphi \in \Sigma^0_n$ may contain arbitrary parameters.
These results complement the ones obtained in [13, 17] using the aforementioned elimination method. In those papers it is shown that fixed (sequences of) instances of BW, when used over systems containing only Kalmar elementary functionals (but not the recursor $\mathsf{R}_0$), at most contribute by ordinary primitive recursive complexity (i.e. elevate the complexity from being Kalmar elementary to $\mathcal{T}_0$). Again this result is optimal. So far the elimination method has not been developed for systems based on $\mathcal{T}_0$-functionals and stronger ones as above. At the same time, the approach in the present paper does not seem to be fine enough to re-obtain the results based on the elimination method.

Independently of the motivation given so far, our explicit BW-functional seems to be of interest in its own as it exhibits the computational content of BW. The functional dramatically simplifies if we switch to a majorizing functional in the sense of W.A. Howard (and hence to a solution of the monotone functional interpretation of BW). In particular, the use of Howard's 'binary' bar recursion then disappears altogether.

In [18], the second author has argued that the solutions provided by monotone functional interpretation of principles $P$ directly correspond to the 'finitary' versions of $P$ as discussed in Tao's program of 'finitary analysis' (see [31]). Following [31], the discussion in [18] focuses on the monotone convergence principle PCM and the infinitary pigeonhole principle IPP:

$$\mathsf{IPP} \quad : \quad \forall n \in \mathbb{N}\, \forall f : \mathbb{N} \mapsto \{0, \ldots, n\}\, \exists i \le n \forall k \in \mathbb{N}\, \exists m \ge k \;\; (f(m) = i).$$

Already for IPP, it is nontrivial to arrive at a 'correct' finitization (see [5] for a thorough discussion). However, IPP is nothing else as the special case of BW for the discrete spaces $\{0, 1, 2, \ldots, n\}$. Hence to carry out the explicit solution of the (monotone) functional interpretation is a step further towards investigating the role of functional interpretations in connection with the program of finitizing analytical principles.

In this paper we also treat the Bolzano-Weierstraß principle for the compact (w.r.t. the product metric) metric space $\Pi_{i \in \mathbb{N}}[-k_i, k_i]$ (for sequences $(k_i)$ in $\mathbb{R}_+$) whose functional interpretation has the same complexity as the one for the case $[0, 1]$. This is of relevance for the logical analysis of proofs that use the weak compactness of closed, bounded convex sets in Hilbert spaces which can be reduced to the sequential compactness of $\Pi_{i \in \mathbb{N}}[-k_i, k_i]$ (see [19, 20]).

For simplicity, let us come back to the case of $[0, 1]$ and the Bolzano-Weierstraß theorem in the form stating that every sequence $(x_n)$ of rational numbers in $[0, 1]$ has a cluster point. In order to obtain a solution of the functional interpretation of optimal complexity one has to start with an appropriate proof of this statement: one standard way is to select one of the subintervals $[0, \frac{1}{2}], [\frac{1}{2}, 1]$ that contains infinitely many elements of the sequence $(x_n)$ (by IPP at least one of the two intervals has this property) and then to continue with that interval. In this way one gets a nested sequence $I_0 \supset I_1 \supset I_2, \ldots$ of intervals $I_k$ of length $2^{-k}$ that converges to a cluster point. In order to decide whether an interval $I_k$ contains infinitely many elements of $(x_n)$ one needs $\Pi^0_2$-comprehension since

$$\forall m \exists n \ge m\, (x_n \in I_k) \in \Pi^0_2.$$

However, in order to get the existence of just **some** sequence $I_0 \supset I_1 \supset I_2, \ldots$ of intervals $I_n$ with the above property (rather than deciding this property which would be necessary only for finding – say – a left-most sequence, i.e. for constructing the limit inferior of $(x_n)$ which indeed is of strictly greater complexity, see [17]) one can use König's lemma for 0/1-trees. Note though, that this is not a use of what is called weak König's lemma (WKL) in reverse mathematics (see [28]) since the tree predicate is not quantifier-free but is $\Pi^0_2$ (and so, in fact, is an instance of what we call $\Pi^0_2$-WKL). Nevertheless, using a single instance of $\Sigma^0_1$-comprehension (short: $\Sigma^0_1$-CA) one can reduce such a $\Pi^0_2$-formula to a $\Pi^0_1$-tree predicate (by absorbing the inner existential quantifier). Now WKL for $\Pi^0_1$-trees (i.e. $\Pi^0_1$-WKL) can easily be reduced to the usual WKL. In this way the use of $\Pi^0_2$-comprehension is replaced by a use of $\Sigma^0_1$-CA plus the use of WKL, where the latter is known not to contribute to the complexity of extractable bounds. In fact, rather than first reducing $\Pi^0_2$-WKL using $\Sigma^0_1$-CA to $\Pi^0_1$-WKL and subsequently to WKL, we work directly with $\Sigma^0_1$-WKL and reduce this via $\Sigma^0_1$-CA in one step to WKL. The functional interpretation of BW, therefore, essentially boils down to solving the functional interpretation of $\Sigma^0_1$-WKL.

## Notation and Common Expressions

By "$\equiv$" we refer to syntactic identity. We will write $\Pi^0_1$ and $\Sigma^0_1$ for the purely universal arithmetic formulas, i.e. $\forall n^0\ \varphi_{\mathsf{QF}}(n)$, and the purely existential arithmetic formulas, i.e. $\exists n^0\ \varphi_{\mathsf{QF}}(n)$, where in general $\varphi_{\mathsf{QF}}$ denotes a quantifier-free formula, which may contain parameters of arbitrary type.

For the encoding of a given finite sequence $s$ of natural numbers we write $\mathrm{lh}(s)$ for the length of $s$ and denote by $[s]$ the type one function defined by

$$[s](i^0) \ :=_0 \begin{cases} s(i) & \text{if } i <_0 \mathrm{lh}(s), \\ 0 & \text{else.} \end{cases}$$

For a type one function $f$ and a natural number $n$ we define the corresponding encoding of the finite sequence $\overline{f}n$ of length $n$ as follows:

$$\overline{f}n \ := \ \langle f(0), f(1), \cdots, f(n-1) \rangle.$$

Given two finite sequences $s$ and $t$ we write $s * t$ for the concatenation of $s$ and $t$. We write shortly $s * \langle 0 \rangle$ and $s * \langle 1 \rangle$ as $s * 0$ and $s * 1$. Following the notation of Avigad and Feferman [1], we use '$s \subseteq t$' to denote $t$ is an extension of $s$ (i.e. the sequence $t$ has the sequence $s$ as an initial segment). We denote the empty sequence by $\varnothing$.

For finite tuples of variables (not necessarily of the same type) $x_1, x_2, \ldots, x_k$ we write $\underline{x}$. By $\underline{x}^{\underline{\rho}}$ we mean $x_1^{\rho_1}, x_2^{\rho_2}, \ldots, x_k^{\rho_k}$.

In most cases we will use the Greek letters $\varphi$,$\psi$, $\chi$ to denote formulas, the lower case Latin letters $f$,$g$,$h$ for functions, the letters $a$,$b$,$i$,$j$,... for natural numbers and encodings, and the capitals $A$, $B$, ... for functionals.

We denote $\lambda n^0.1^0$, $\lambda f^1.1^0$, ... by $\mathbf{1} \equiv \mathbf{1}^1$, $\mathbf{1}^2$, ... and we use bold numbers to denote the type level of a term, e.g. we write $t^{\mathbf{1}}$ for $t^{1(0)}$. We use this superscript as a shortcut for a specific type having the given type level. So, by $\forall X^{\mathbf{2}}$ we mean for all $X$ of an appropriate type, e.g. $2(1(0))$, not for all $X$ which are of any type with level 2.

We distinguish between full schemas and principles and their restrictions to a concrete instance. For a given principle $\varphi$ we write $\varphi(f)$ for the concrete instance of $\varphi$ applied to $f$. We will define the principles in this form in most cases. It is also the preferable form to be analyzed, since most proofs in mathematics use only concrete instances of the common principles. We will write shortly $\varphi$ for the full second-order closure $\forall f \varphi(f)$ of $\varphi(f)$.

# 1 Representations and Interpretations

## 1.1 Fragments and Extensions of $\mathsf{WE\text{-}PA}^\omega$

In the following, $\mathsf{WE\text{-}PA}^\omega$ denotes weakly extensional (see below) Peano arithmetic extended to the language of functionals for all finite types (e.g. see [32] or [18]). The intuitionistic variant of this system (weakly extensional Heyting arithmetic in all finite types) is denoted by $\mathsf{WE\text{-}HA}^\omega$. Furthermore, we define the extensional systems:

**Definition 1.1.** By replacing the weak extensionality rule

$$\mathsf{QF\text{-}ER} \quad : \quad \frac{\varphi_{\mathsf{QF}} \to s =_\rho t}{\varphi_{\mathsf{QF}} \to r[s] =_\tau r[t]},$$

by the *axioms of higher type extensionality*:

$$\mathsf{E}_\rho \quad : \quad \forall z^\rho, x_1^{\rho_1}, y_1^{\rho_1}, \ldots, x_k^{\rho_k}, y_k^{\rho_k} \Big( \bigwedge_{i=1}^{k} (x_i =_{\rho_i} y_i) \to z\underline{x} =_0 z\underline{y} \Big),$$

in the systems $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{WE\text{-}PA}^\omega$, where $\rho = 0\rho_k \ldots \rho_1$, we obtain the systems $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$. Here $s =_\rho t$ is defined extensionally, i.e. as

$$\forall x_1^{\rho_1}, \ldots, x_k^{\rho_k} (s\underline{x} =_0 t\underline{x}).$$

On the other hand, sometimes it is sufficient to have weaker systems. This leads us to the definition of the following fragments:

**Definition 1.2.** For all four systems $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{WE\text{-}PA}^\omega$, $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$ let correspondingly $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$, $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$, $\widehat{\mathsf{E\text{-}HA}}^\omega\!\restriction$, $\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction$ denote the fragments where we only have the recursor $R_0$ for type-0-recursion and the induction schema is restricted to the schema of quantifier-free induction:

$$\mathsf{QF\text{-}IA} \quad : \quad \Big(\varphi_{\mathsf{QF}}(0) \wedge \forall n^0 \big(\varphi_{\mathsf{QF}}(n) \to \varphi_{\mathsf{QF}}(n+1)\big)\Big) \to \forall n^0 \varphi_{\mathsf{QF}}(n),$$

where $\varphi_{\mathsf{QF}}$ is a quantifier-free formula and may contain parameters of arbitrary types.

The ordered field of *rational Numbers* within $\mathsf{WE\text{-}HA}^\omega$ is represented by codes $j(n,m)$ of pairs $(n,m)$ of natural numbers (in this paper let $j(n,m)$ define the rational number $\frac{\frac{n}{2}}{m+1}$ if $n$ is even and $-\frac{\frac{n+1}{2}}{m+1}$ otherwise, for $j$ we use the Cantor pairing function). The standard relations and operations (like e.g. $+_\mathbb{Q}$, $|\cdot|_\mathbb{Q}$, $<_\mathbb{Q}$) are defined in a natural way (see e.g. [18]). By $\langle r \rangle$ we mean the smallest code of the rational number $r$.
The Archimedean ordered Field of *real Numbers* within $\mathsf{WE\text{-}HA}^\omega$ is represented by sequences of rational numbers with a fixed rate of convergence $2^{-n}$.

**Definition 1.3.** *Representation of a real number*
A function $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$\forall n \left( |fn -_\mathbb{Q} f(n+1)|_\mathbb{Q} <_\mathbb{Q} \langle 2^{-(n+1)} \rangle \right),$$

represents a real number.

We arrange for each function $f^1$ to code a unique real number in this way.

**Definition 1.4.** Let $f : \mathbb{N} \mapsto \mathbb{N}$ be a function. Define $\widehat{f}$ by

$$\widehat{f}n := \begin{cases} fn & \text{if } \forall k < n \left( |fk -_\mathbb{Q} f(k+1)|_\mathbb{Q} <_\mathbb{Q} \langle 2^{-k-1} \rangle \right) \\ fk & \text{else} \end{cases},$$

where $k$ is the least number such that $k < n$ and

$$|fk -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} \geq_{\mathbb{Q}} \langle 2^{-k-1} \rangle.$$

The function $\widehat{f}$ defines a uniquely determined real number, so we say also that $f$ defines a uniquely determined real number, namely the one represented by $\widehat{f}$.

The functional which maps $f$ to $\widehat{f}$ can be defined primitive recursively in $\mathsf{WE\text{-}HA}^\omega$. I.e. we can reduce quantifiers ranging over $\mathbb{R}$ to quantifiers ranging over type 1 objects. The usual operations and relations on $\mathbb{R}$ can be defined in $\mathsf{WE\text{-}HA}^\omega$ in a an intuitive way (see e.g. [18]).
For the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$, we define for the coding $n = \langle r \rangle$ of a rational number $r$ its coding $n_{\mathbb{R}} := \lambda k.n_{\mathbb{Q}}$ for the real number corresponding to $r$. However, usually we omit the intermediate encoding to a rational number and write shortly $r_{\mathbb{R}}$ instead of $(\langle r \rangle)_{\mathbb{R}}$. We do not introduce $\mathbb{R}$ as the set of equivalence classes of representatives, but consider only the representatives themselves. The structure

$$(\mathbb{N}^{\mathbb{N}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, <_{\mathbb{R}})$$

represents the Archimedean ordered field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, <)$ in $\mathsf{WE\text{-}HA}^\omega$.

## 1.2 Gödel's Functional Interpretation

In [7], K. Gödel introduced his famous functional 'Dialectica' interpretation for Heyting arithmetic HA and – via some negative translation as a pre-processing step – also Peano arithmetic PA in his quantifier-free system $\mathcal{T}$ of primitive recursive functionals of all finite types. This interpretation was extended to classical analysis (obtained from a finite type extension of PA by the addition of the schema of countable choice) by C. Spector in [29] using his bar recursive functionals $\mathcal{T} + \mathsf{BR}$, where for the treatment of countable choice for numbers and arithmetical formulas (sufficient to derive arithmetical comprehension) bar recursion of lowest types is enough (see e.g. [18]). In this paper, it is always the combination of negative translation and functional interpretation that is used. This combination becomes particularly convenient to formulate if one uses a negative translation due to Krivine, as it then coincides with the so-called Shoenfield variant [27] (for the fragment $\{\forall, \vee, \neg\}$) as was shown in [30], where this interpretation is given for the full language (i.e. $\{\forall, \exists, \rightarrow, \vee, \wedge, \neg\}$ with $A \leftrightarrow B :\equiv (A \rightarrow B) \wedge (B \rightarrow A)$).
We now give the definition of this Shoenfield interpretation:

**Definition 1.5** ([27, 30])**.** To each formula $\varphi(\underline{a})$ in $\mathcal{L}(\mathsf{WE\text{-}PA}^\omega)$ with the tuple of free variables $\underline{a}$ we associate its *Sh-interpretation* $\varphi^{Sh}(\underline{a})$, which is a formula of the form

$$\varphi^{Sh}(\underline{a}) :\equiv \forall \underline{u}\, \exists \underline{x}\, \varphi_{Sh}(\underline{u}, \underline{x}, \underline{a}),$$

where $\varphi_{Sh}(\underline{a})$ is a quantifier-free formula. Each of $\underline{x}$ and $\underline{y}$ is a tuple of variables whose types, as well as the length of each tuple, depend only on the logical structure of $\varphi$. We also write $\varphi_{Sh}(\underline{x}, \underline{y}, \underline{a})$ for $\varphi_{Sh}(\underline{a})$. If some variables $\underline{z}$ of $\varphi$ are exhibited, as $\varphi(\underline{z}, \underline{a})$, then we write $\varphi_{Sh}(\underline{x}, \underline{y}, \underline{z}, \underline{a})$ for $\varphi_{Sh}(\underline{a})$.
We define the construction of $\varphi^{Sh}$ inductively as follows (with $\underline{x}\,\underline{y}$ denoting $x_1\underline{y}, \ldots, x_n\underline{y}$ for $\underline{x} = x_1, \ldots, x_n$). In the inductive steps we assume that

$$\varphi^{Sh}(\underline{a}) :\equiv \forall \underline{u}\, \exists \underline{x}\, \varphi_{Sh}(\underline{u}, \underline{x}, \underline{a}) \text{ and } \psi^{Sh}(\underline{b}) :\equiv \forall \underline{v}\, \exists \underline{y}\, \varphi_{Sh}(\underline{v}, \underline{y}, \underline{b})$$

are already defined.

(S1) $\varphi^{Sh}(\underline{a}) \equiv: \varphi_{Sh}(\underline{a})$ for atomic $\varphi(\underline{a})$,

(S2) $(\neg\varphi)^{Sh} \equiv \forall \underline{f} \exists \underline{u}\, \neg\varphi_{Sh}(\underline{u}, \underline{f}\,\underline{u})$, (S3) $(\varphi \vee \psi)^{Sh} \equiv \forall \underline{u}, \underline{v} \exists \underline{x}, \underline{y}\, \big(\varphi_{Sh}(\underline{u}, \underline{x}) \vee \psi_{Sh}(\underline{v}, \underline{y})\big)$,

(S4) $(\forall z\, \varphi)^{Sh} \equiv \forall z, \underline{u} \exists \underline{x}\, \varphi_{Sh}(z, \underline{u}, \underline{x})$,

(S5) $(\varphi \to \psi)^{Sh} \equiv \forall \underline{f}, \underline{v} \exists \underline{u}, \underline{y}\, \big(\varphi_{Sh}(\underline{u}, \underline{f}\, \underline{u}) \to \psi_{Sh}(\underline{v}, \underline{y})\big)$,

(S6) $(\exists z\, \varphi)^{Sh} \equiv \forall \underline{U} \exists z, \underline{f}\, \varphi_{Sh}(z, \underline{U}\, z\, \underline{f}, \underline{f}(\underline{U}\, z\, \underline{f}))$,

(S7) $(\varphi \wedge \psi)^{Sh} \equiv \forall n^0, \underline{u}, \underline{v}\, \exists \underline{x}, \underline{y}\, \big((n{=}0 \to \varphi_{Sh}(\underline{u}, \underline{x})) \wedge (n{\neq}0 \to \psi_{Sh}(\underline{v}, \underline{y}))\big)$.

Note that, even inuitionistically, $(\varphi \wedge \psi)^{Sh} \leftrightarrow \forall \underline{u}, \underline{v} \exists \underline{x}, \underline{y}\, \big(\varphi_{Sh}(\underline{u}, \underline{x}) \wedge \psi_{Sh}(\underline{v}, \underline{y})\big)$. We denote this equivalence by (S7*).

Sometimes, a partial (weaker) interpretation of the implication is sufficient when the witnessing data from the premise are not needed for further use of the Sh-interpretation of a given formula. E.g. if the premise can be proved directly. Typically, we would analyze in such cases an implication as:

$$\forall \underline{f} \exists \underline{y}\, \big(\forall \underline{u}\, \varphi_{Sh}(\underline{u}, \underline{f}\, \underline{u}) \to \psi_{Sh}(\underline{v}, \underline{y})\big).$$

*Remark.* The Shoenfield version of the functional interpretation is often – for obvious reasons – called $\forall\exists$-form, whereas the Dialectica interpretation (and hence also the combination ND of some negative translation N with the Dialectica interpretation D) always is of the form $\exists\forall$. If the Krivine negative translation is used (see [30]), the latter is nothing else but the result of a final application of the axiom schema of quantifier-free choice QF-AC to the Shoenfield interpretation. One should stress though that this passage from the $\forall\exists$-form to the $\exists\forall$-version which also is implicitly present in the soundness theorem of the Shoenfield interpretation (stating the extractability of suitable terms realizing the $\forall\exists$-form) is necessary for the interpretation to be sound for the modus ponens rule.

## 1.3 Bar Recursion

We give the definition in the form presented e.g. in [22] or [18]. Implicitly we assume that tuples of variables are contracted to single variables. Alternatively, one could use a simultaneous form of bar recursion (see [18]).

**Definition 1.6.** The *bar recursor* $\mathsf{B}_{\rho,\tau}$ is defined by:

$$(\mathsf{BR}_{\rho,\tau}) \ : \mathsf{B}_{\rho,\tau} yzunx :=_\tau \begin{cases} zn(\overline{x,n}) & \text{if } y(\overline{x,n}) <_0 n \\ u\big(\lambda D^\rho.\mathsf{B}_{\rho,\tau} yzu(n+1)(\overline{x,n} * D)\big)n(\overline{x,n}) & \text{otherwise,} \end{cases}$$

where $(\overline{x,n})(k^0) =_\rho \begin{cases} x(k) & \text{if } k <_0 n \\ 0^\rho & \text{otherwise} \end{cases}$ and $(\overline{x,n} * D)(k^0) =_\rho \begin{cases} x(k) & \text{if } k <_0 n \\ D & \text{if } k =_0 n \\ 0^\rho & \text{otherwise.} \end{cases}$

**Remark.** *Note that for $\rho = 0$, $\overline{f,n}$ is an object of type* 1 *and so is not the same as $\bar{f}n$ which has type* 0.

The form of bar recursion we actually need is the following special case of (BR) that Spector presented in [29] as an operator $\Phi$ such that

$$\Phi_\rho yunxm :=_\rho \begin{cases} xm, \text{ if } m <_0 n \\ 0^\rho, \text{ if } m \geq_0 n \wedge y(\overline{x,n}) < n \\ \Phi_\rho yun'(\overline{x,n} * D_0)m, \text{ otherwise,} \end{cases}$$

where

$$D_0 =_\rho un(\lambda D^\rho.\Phi_\rho yun'(\overline{x,n} * D)).$$

As in this paper we deal only with arithmetical comprehension over numbers we don't need bar recursion for all types. Indeed, $\Phi_0$ and hence $\mathsf{B}_{0,1}$ (from which $\Phi_0$ can be defined via type-0 primitive recursion, see [25] or [18]) is sufficient:

$$\Phi_0 yunxm :=_0 \begin{cases} xm, \text{ if } m <_0 n \\ 0^0, \text{ if } m \geq_0 n \wedge y(\overline{x,n}) < n \\ \Phi_0 yun'(\overline{x,n} * D_0)m, \text{ otherwise,} \end{cases}$$

where $D_0 =_0 un\big(\lambda D^0.\Phi_0 yun'(\overline{x,n} * D)\big)$.

Furthermore, to be able to properly analyze the complexity of the witnessing functionals in later sections, we introduce Howard's schemas of restricted bar recursion as given in [10].

**Definition 1.7.** The *restricted bar recursor* for *Scheme 1*, $\mathsf{B'}_{\mathbf{1}}$, is defined by:

$$\mathsf{B'}_{\mathbf{1}} y^2 z^{(2)0} u n^0 x^1 :=_0$$
$$\begin{cases} zn(\overline{x,n}) & \text{if } y(\overline{x,n}) <_0 n \\ u\big(\mathsf{B'}_{\mathbf{1}} yzu(n+1)(\overline{x,n}*0)\big)\big(\mathsf{B'}_{\mathbf{1}} yzu(n+1)(\overline{x,n}*1)\big)n(\overline{x,n}) & \text{otherwise,} \end{cases}$$

and for *Scheme 2*, $\mathsf{B'}_{\mathbf{2}}$, by:

$$\mathsf{B'}_{\mathbf{2}} y^2 u^1 n^0 x^1 :=_0 \begin{cases} 0 & \text{if } y(\overline{x,n}) <_0 n \\ 1 + \mathsf{B'}_{\mathbf{2}} yu(n+1)\big(\overline{x,n} * (un)\big) & \text{otherwise.} \end{cases}$$

**Remark.** *Note that,* $\mathsf{B'}_{\mathbf{1}}$ *and* $\mathsf{B'}_{\mathbf{2}}$ *are just special forms of* $\mathsf{B}_{0,0}$*. Furthermore applied only to 0,1-sequences (i.e. type one functions* $x : \mathbb{N} \mapsto \{0,1\}$*), as is always the case in this paper,* $\mathsf{B'}_{\mathbf{2}}$ *is a special form of* $\mathsf{B'}_{\mathbf{1}}$*. See [25] for proofs of both statements.*

## 1.4 Majorizability

The following important structural property of the closed terms of all systems used in this papers is due to W.A. Howard [9] and (with a modification incorporated below) M. Bezem [2]:

**Definition 1.8.** The relation $x^*$ s-maj$_\rho$ $x$ ($x^*$ *strongly majorizes* $x$) between functionals of type $\rho$ is defined by induction on $\rho$:

$$x^* \text{ s-maj}_0\, x :\equiv x^* \geq_0 x,$$
$$x^* \text{ s-maj}_{\tau\rho}\, x :\equiv \forall y^*, y(y^* \text{ s-maj}_\rho\, y \to x^* y^* \text{ s-maj}_\tau\, x^* y, xy).$$

Naturally, this definition extends to tuples in the expected way.

Moreover, Howard and Bezem showed in [9, 2] the following:

**Theorem 1.9** ([9, 2], see also [18])**.** For each closed term $t^\rho$ of $\mathsf{WE\text{-}HA}^\omega + \mathsf{BR}$ one can construct a closed term $t^*$ in $\mathsf{WE\text{-}HA}^\omega + \mathsf{BR}$ of the same type, such that:

$$\mathsf{WE\text{-}HA}^\omega + \mathsf{BR} + \mathsf{BI} \vdash t^* \text{ s-maj}_\rho\, t,$$

where $\mathsf{BI}$ is a suitable principle of bar induction (classically: dependent choice).
$t^*$ only contains recursors or bar recursors of a certain type if already $t$ contains these constants. In particular, $\mathsf{B}_{0,1}$ (and hence $\Phi_0$) can be majorized by a term using in addition to simple primitive recursive constructions only $\mathsf{B}_{0,1}$.

Using monotone functional interpretation (introduced by the 2nd author in [12], see also [18]), one can extract terms which majorize some functionals realizing the usual functional interpretation directly. In this paper we give the *Sh*-interpretation first and then majorize the realizers, still calling this 'monotone *Sh*-interpretation':

**Definition 1.10.** Suppose we have the following *Sh*-interpretation of a formula $\varphi$:

$$\forall \underline{u}\, \exists \underline{x}\, \varphi_{Sh}(\underline{u}, \underline{x}, \underline{a}),$$

where $\underline{a}$ comprises all the free variables of $\varphi$.
Then we say that the terms $\underline{t}^*$ satisfy the monotone *Sh*-interpretation if

$$\exists \underline{X}\, \big(\underline{t}^* \text{ s-maj}\, \underline{X} \;\wedge\; \forall \underline{u}\, \forall \underline{a}\, \varphi_{Sh}(\underline{u}, \underline{X}\; \underline{a}\; \underline{u}, \underline{a})\big).$$

**Remark.** *While the type structures of all strongly majorizable functionals $\mathcal{M}^\omega$ ([2]) and of all continuous functionals $\mathcal{C}^\omega$ ([26]) are models of $\mathsf{E\text{-}PA}^\omega$ with bar recursion (see Definition 1.6) the full set theoretic type structure $\mathcal{S}^\omega$ is not. However, the first two models start to differ only from type 2 on from the third model, where we still have: $\mathcal{C}_2 \subset \mathcal{M}_2 \subset \mathcal{S}_2$. So, if we use bar recursion to define a functional F of type level 2, we know that it is a well defined functional in $\mathcal{C}^\omega$ and in $\mathcal{M}^\omega$ and defines a total (computable) functional: $\mathbb{N}^\mathbb{N} \mapsto \mathbb{N}$ (see [18] for details on this).*

## 1.5 Arithmetical Comprehension

The *Schema of Comprehension* is known in several forms (see e.g. [32]). For us, the following very restricted form is sufficient:

**Definition 1.11** (*Arithmetical Comprehension over numbers for purely existential formulas)***.**

$$\Sigma^0_1\text{-}\mathsf{CA} \quad : \quad \forall f^{1(0)}\, \underbrace{\exists g^1 \forall x^0 \big(\, \big(\exists y^0\, f(x,y) =_0 0\big) \leftrightarrow gx =_0 0\, \big)}_{\equiv:\Sigma^0_1\text{-}\mathsf{CA}(f)}.$$

We define $\Pi^0_1$-$\mathsf{CA}$ and $\Pi^0_1$-$\mathsf{CA}(f)$ for purely universal formulas analogously.

**Remark.** *In $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \Sigma^0_1$-$\mathsf{CA}$, we can derive any instance of arithmetical comprehension*

$$\mathsf{CA}^0_{ar} : \; \exists f \forall x^0\, (f(x) =_0 0 \leftrightarrow A(x)),$$

*where A only contains quantifiers over variables of type 0, by iterated application of $\Sigma^0_1$-$\mathsf{CA}$. However, this is only the case for the full second-order closure $\Sigma^0_1$-$\mathsf{CA}$ of $\Sigma^0_1$-$\mathsf{CA}(f)$ and not for individual instances $\Sigma^0_1$-$\mathsf{CA}(f)$.*

As mentioned already above, the Shoenfield interpretation of $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} +$ $\Sigma^0_1$-$\mathsf{CA}$ can be carried out in $\mathcal{T}_0 + \mathsf{BR}_{0,1}$ (see [16] or [18]).

Below we will need the explicit solution of the Shoenfield interpretation of $\Sigma^0_1$-$\mathsf{CA}$ which we compute now:

Using $\mathsf{QF\text{-}AC}^{0,0}$ (already in $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$) $\Sigma^0_1$-$\mathsf{CA}(f)$ is equivalent to the following modified form

$$\Sigma^0_1\text{-}\widehat{\mathsf{CA}}(f) \quad : \quad \exists g^1 \forall x^0, z^0 \big(fx(gx) =_0 0 \vee fxz \neq_0 0\big).$$

**Lemma 1.12** (*Sh*-interpretation of $\forall f \Sigma^0_1$-$\widehat{\mathsf{CA}}(f)$, [25]**).**

$$\forall f \exists g^1 \forall x^0, z^0\, \big(fx(gx) =_0 0 \vee fxz \neq_0 0\big),$$

*is Sh-interpreted as follows:*

$$\forall f, X^2, Z^2 \ \Big(f\big(Xt_g, t_g(Xt_g)\big) =_0 0 \vee f(Xt_g, Zt_g) \neq_0 0\Big),$$

*where* $t_g :=_1 \Phi_0 X u_{Z,f}^{0(1(0))(0)} 0^0 \mathbf{0}^1$ *and* $u_{Z,f} n^0 v^{1(0)} :=_0 \begin{cases} 1 & \text{if } f(n, Z(v1^0)) \neq_0 0 \\ Z(v1^0) & \text{else} \end{cases}.$

We use this solution to give the *Sh*-interpretation of $\Sigma_1^0$-CA.

**Theorem 1.13** (*Sh*-interpretation of $\Sigma_1^0$-CA)**.** *The schema of arithmetical comprehension over numbers for purely existential formulas (for a given function* $f^{1(0)}$*)*

$$\exists h^1 \forall x^0 (hx =_0 0 \leftrightarrow \exists z\, fxz =_0 0),$$

*is Sh-interpreted as follows (using the clause (S7*) for the conjunction hidden in '↔'):*

$$\begin{aligned} \forall X^{0(10)(1)}, Z^{0(10)(1)} \ [( \ & t_h XZ(X(t_h XZ)(t_z XZ)) =_0 0 \leftarrow \\ & \quad f(X(t_h XZ)(t_z XZ), Z(t_h XZ)(t_z XZ)) =_0 0) \ \wedge \\ (& t_h XZ(X(t_h XZ)(t_z XZ)) =_0 0 \rightarrow \\ & \quad f(X(t_h XZ)(t_z XZ), t_z XZ(X(t_h XZ)(t_z XZ))(Z(t_h XZ)(t_z XZ))) =_0 0 \ )]. \end{aligned}$$

*The witnessing terms are:*

$$t_z := \lambda X^{0(10)(1)}, Z^{0(10)(1)}, a^0, b^0 \ . \ t_g(t_f X)(t_f Z) a,$$
$$t_h := \lambda X^{0(10)(1)}, Z^{0(10)(1)}, n^0 \ . \ \overleftrightarrow{f(n, t_g(t_f X)(t_f Z) n)},$$

*where* $t_f := \lambda X^{0(10)(1)}, g^1 \ . \ X(\lambda n^0 . \overleftrightarrow{f(n, gn)})(\lambda a^0, b^0 . ga)$ *and* $\overleftrightarrow{n^0} :=_0 \begin{cases} 0 & \text{if } \ n =_0 0 \\ 1 & \text{else} \end{cases}.$

*The term* $t_g$ *corresponds to the term defined in proposition 1.12. The only difference is that we give the two type* 2 *arguments of* $t_g$ *explicitly, i.e.,* $t_g$ *stands only for the term* $t_g$ *of type level* 3 *and not for the type* 1 *term* $t_g XZ$ *as above.*

**Proof.** By lemma 1.12 we have that:

$$\forall X^2, Z^2 \ \ ( \ f(X(t_g XZ), t_g XZ(X(t_g XZ))) =_0 0 \vee f(X(t_g XZ), Z(t_g XZ)) \neq_0 0 \ ). \qquad (+)$$

Given any $X_0^{0(10)(1)}$, $Z_0^{0(10)(1)}$ set $X^2 := t_f X_0$ and $Z^2 := t_f Z_0$ to obtain:

$$(*) \ X(t_g XZ) = t_f X_0(t_g XZ) = t_f X_0(t_g(t_f X_0, t_f Z_0)) = X_0(t_h X_0 Z_0)(t_z X_0 Z_0)$$

and - analogously -

$$(**) \ Z(t_g XZ) = Z_0(t_h X_0 Z_0)(t_z X_0 Z_0).$$

- Suppose we have $t_h X_0 Z_0(X_0(t_h X_0 Z_0)(t_z X_0 Z_0)) =_0 0$. It follows by $(*)$ that

  $$t_h X_0 Z_0(X(t_g XZ)) =_0 0$$

  and by definition of $t_h$ that

  $$f(X(t_g XZ), t_g(t_f X_0)(t_f Z_0)(X(t_g XZ))) = 0.$$

  By $(*)$ and definition of $X$ and $Z$ we get

  $$\begin{aligned} & f(X_0(t_h X_0 Z_0)(t_z X_0 Z_0), t_g(t_f X_0)(t_f Z_0)(X_0(t_h X_0 Z_0)(t_z X_0 Z_0))) = \\ & f(X_0(t_h X_0 Z_0)(t_z X_0 Z_0), t_z X_0 Z_0(X_0(t_h X_0 Z_0)(t_z X_0 Z_0))(Z_0(t_h X_0 Z_0)(t_z X_0 Z_0)))) = 0. \end{aligned}$$

- On the other hand, let $f(X_0(t_h X_0 Z_0)(t_z X_0 Z_0), Z_0(t_h X_0 Z_0)(t_z X_0 Z_0)) = 0$. By $(*)$, $(**)$ this yields $f(X(t_g X_0 Z_0), Z(t_g X_0 Y_0)) = 0$, which implies $f(X(t_g XZ), t_g XZ(X(t_g XZ))) = 0$ by (+). Using $(*)$ and the definition of $X$, $Z$, and $t_h$ we obtain

$$f(X_0(t_h X_0 Z_0)(t_z X_0 Z_0), t_g(t_f X_0)(t_f Z_0)(X_0(t_h X_0 Z_0)(t_z X_0 Z_0))) = \\ t_h X_0 Z_0(X_0(t_h X_0 Z_0)(t_z X_0 Z_0))) = 0.$$

□

## 2 Weak König's Lemma

König's Lemma as well as its weakening called Weak König's Lemma (WKL) are well known principles. For general context and definitions we refer e.g. to [33], [28], or [18]. Whereas [28] uses a language with set variables, both [33] and [18] use a formulation with function variables that is more convenient in the context of functional interpretation.

**Definition 2.1** (WKL($\varphi$))**.** For a given $\varphi$, WKL($\varphi$) is the following statement: Every infinite 0/1-tree given by the decision criteria $\varphi$ has an infinite path,

$$\mathsf{WKL}(\varphi) \quad : \quad \mathsf{BinTree}(\varphi) \wedge \forall k \mathsf{Unbounded}(\varphi, k) \to \exists b \Big(\mathsf{BinFunc}(b) \wedge \forall k\ \varphi\big(\bar{b}(k)\big)\Big),$$

where $\mathsf{BinFunc}(b) :\equiv \forall n^0 \big(b(n) =_0 0 \vee b(n) =_0 1\big)$, $\mathsf{BinTree}(\varphi) :\equiv \forall s\big(\ \varphi(s) \to\ s \in \{0,1\}^{<\omega} \wedge \forall t \subseteq s\ \varphi(t)\ \big)$, $\mathsf{Unbounded}(\varphi, k^0) :\equiv\ \ \exists s \in \{0,1\}^k\ \varphi(s)$.[1]
Furthermore, we define the schema $\Pi^0_n$-WKL, as the union of WKL($\varphi$), where $\varphi$ is a $\Pi^0_n$ formula. Also, we write $\Pi^0_n$-WKL($\varphi$) to indicate that we mean the concrete instance WKL($\varphi$) and that $\varphi$ is a $\Pi^0_n$ formula. (Analogously for $\Sigma^0_n$.)

Note that, for every fixed $n \in \mathbb{N}$, we can always reformulate the schema $\Pi^0_n$-WKL as a single $2^{nd}$-order axiom. We will use this fact implicitly. However, in the special case for quantifier-free $\varphi$ we define explicitly:

**Definition 2.2** (WKL $\equiv \forall f$WKL($f$) see also [33])**.** Every infinite binary tree, given by the characteristic function $f$, has an infinite path:

$$\mathsf{WKL}(f) \quad : \quad \mathsf{BinTree}_K(f) \wedge \forall k \exists x\big(\mathrm{lh}(x) =_0 k \wedge f(x) =_0 0\big) \to \\ \exists b \leq_1 \mathbf{1}^1 \Big(\forall k\, f\big(\bar{b}(k)\big) =_0 0\Big),$$

$$\mathsf{BinTree}_K(f^1) :\equiv \forall x, y\big(f(x * y) =_0 0 \to fx =_0 0\big) \ \wedge\ \forall x, n\big(f(x * \langle n \rangle) =_0 0 \to n \leq_0 1\big).$$

We mentioned earlier that the schema $\Pi^0_1$-WKL is equivalent to WKL. More precisely, we have: $\widehat{\mathsf{WE\text{-}HA}}^\omega \upharpoonright\vdash \Pi^0_1\text{-}\mathsf{WKL} \leftrightarrow \mathsf{WKL}$. A proof can be found e.g. in [25] (see also [28]).
Using a construction from [11] (see also Proposition 9.18 in [18]), we can rewrite WKL in a logically somewhat simpler form:

**Definition 2.3.** $\mathsf{WKL}_\Delta \equiv \forall f \mathsf{WKL}_\Delta(f)$, where

$$\mathsf{WKL}_\Delta(f) \quad :\equiv \quad \exists b^1 \forall k^0\ \widehat{\big(\hat{f}\big)}_g (\bar{b}k) =_0 0, \text{ with}$$

[1] We encode finite binary sequences as natural numbers. A natural number $n$ encodes the binary sequence given by all but the first digit of the binary representation of $n+1$ (i.e. $0 = \langle\rangle$, $1 = \langle 0 \rangle$, $2 = \langle 1 \rangle$, $3 = \langle 0, 0 \rangle$, ...). Note that for any $k \in \mathbb{N}$, any $s \in \{0,1\}^k$ is encoded as a natural number $n$ for which the following inequalities hold: $2^k \leq n+1 < 2^{k+1}$.

$$\widehat{f}\,n := \begin{cases} fn & \text{if } fn \neq 0 \ \vee \ \big(\forall k,l(k*l=n \to fk=0) \wedge \forall i<\mathrm{lh}(n)\ (n_i \leq 1)\big) \\ 1^0 & \text{else,} \end{cases}$$

$$f_g n := \begin{cases} fn & \text{if } f\big(g(\mathrm{lh}(n))\big)=0 \wedge \mathrm{lh}\,\big(g(\mathrm{lh}(n))\big)=\mathrm{lh}(n) \\ 0^0 & \text{else,} \end{cases}$$

$$gk := gfk := \begin{cases} \min n \leq \overline{\mathbf{1}^1}k\ (\mathrm{lh}(n)=k \wedge fn=0) & \text{if such an } n \text{ exists} \\ 0^0 & \text{else.} \end{cases}$$

We have: $\widehat{\text{WE-HA}}^{\omega}\!\upharpoonright \vdash \mathsf{WKL}_\Delta \leftrightarrow \mathsf{WKL}$ (see [11, 18] for the proof).
Howard proves in [10] that one can give the realizing functionals for the Sh-interpretation of WKL using only restricted bar recursion and $\mathcal{T}_0$. This proof is discussed in great detail in [25] and we use it to obtain the Sh-interpretation of $\mathsf{WKL}(f)$:

**Theorem 2.4** (The Sh-interpretation of $\mathsf{WKL}_\Delta$). *The Weak König's lemma for binary trees*

$$\forall f \exists b^1 \forall k^0\ \widehat{\left(\widehat{f}\right)_g}(\bar{b}k) =_0 0$$

*is, provably in* $\mathsf{WE\text{-}HA}^\omega + \mathsf{B'}_{\mathbf{1}}$*, Sh-interpreted as follows:*

$$\forall f, A\ \exists b^1\ \widehat{\left(\widehat{f}\right)_g}\big(\bar{b}(Ab)\big) = 0,$$

*where b is realized by* $b :=_1 B^{\mathsf{WKL}} Af$*:*

$$B^{\mathsf{WKL}}(A,f) := \Big[g\Big(\widehat{\big(\widehat{f}\big)_{gf}}\Big)\Big(K^{\mathsf{WKL}}(A,\varnothing)\Big)\Big],$$

$$K^{\mathsf{WKL}}(A,x) := \begin{cases} 0 & \textit{if } A[x] < \mathrm{lh}(x) \\ 1+\max\big\{K_A(x*0), K_A(x*1)\big\} & \textit{otherwise,} \end{cases}$$

*where g is the same term as we used in the Definition 2.2*[2].

## 2.1 Majorants for Howard's Solution

For the monotone Shoenfield interpretation we need majorants of the terms realizing the Shoenfield interpretation. For the solution of WKL given above these majorants are rather trivial and no longer involve any restricted bar recursors. Note that the monotone interpretation suffices to calibrate the provably total functionals of level $\leq 2$ (see the final section of this paper).
The following proposition is easily verified:

**Proposition 2.5.** *The solution of the Sh-interpretation of* $\mathsf{WKL}(f)$ *is, provably in* $\widehat{\text{WE-HA}}^{\omega}\!\upharpoonright + \mathsf{B'}_{\mathbf{1}}$*, majorized as follows:*

$$K^* :=_{1(0)(2)} \lambda A^2, x^0. A^2\mathbf{1}\ \text{s-maj}_{1(0)(2)}\ K^{\mathsf{WKL}}, \quad B^* :=_{1(2)} \lambda A^2.\mathbf{1}^1\ \text{s-maj}_{1(2)}\ B^{\mathsf{WKL}}.$$

[2] We define $[\cdot]$ analogously for codes of sequences as we did for sequences themselves.

## 3 Interpreting Bolzano-Weierstraß

In this section we will use bar recursion to interpret the Bolzano-Weierstraß theorem.

**Definition 3.1.** *The Bolzano-Weierstraß Principle*
Let $x$ be a sequence in $\mathbb{P} := \prod_{i\in\mathbb{N}}[-k_i, k_i]$ for a known sequence $(k_i)_{i\in\mathbb{N}}$ with $k_i$ in $\mathbb{Q}^+$. Let $d^\omega$ denote the standard product metric (as defined e.g. in [28]):

$$d^\omega(a,b) :=_{\mathbb{R}} \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|a_i - b_i|}{1 + |a_i - b_i|}, \text{ for } a, b \in \mathbb{P}.$$

In the following we tacitly rely on our representation of real numbers by which sequences of real numbers are represented by objects $a^{1(0)}$ and sequences of sequences of real numbers by objects $x^{1(0)(0)}$ (and each such object is a representative of a unique such sequence). We define

$$\mathsf{BW}^\omega_{\mathbb{R}} \quad : \quad \forall x^{1(0)(0)} \in \mathbb{P}^{\mathbb{N}} \underbrace{\exists a^{10} \in \mathbb{P} \ \ \forall k^0 \exists l^0 \geq_0 k \ \ d^\omega(xk, a) \leq_{\mathbb{R}} 2^{-k}}_{\equiv:\mathsf{BW}^\omega_{\mathbb{R}}(x^{1(0)(0)})},$$

where by $x \in \mathbb{P}^{\mathbb{N}}$ we mean a sequence of elements of $\mathbb{P}$ – i.e. a sequence $(x)$ of sequences ($xn \in \mathbb{P}$, $n \in \mathbb{N}$) of real numbers in the corresponding intervals ($(xn)i \in [-k_i, k_i], i \in \mathbb{N}$).[3]

### 3.1 A Simple Proof of BW based on $\Sigma^0_1$-WKL

To demonstrate the main idea of the proof we only treat $\mathsf{BW}^\omega_{\mathbb{R}}$ for sequences of rational numbers in the unit interval $[0,1]$ – denoted by $\mathsf{BW}_{\mathbb{Q}}$ – which obviously is implied by (and in fact equivalent to)

$$\mathsf{BW}'_{\mathbb{Q}} \quad : \quad \forall s^1 \underbrace{\exists a^1 \forall k^0 \exists l^0 \geq_0 k \ |\widetilde{\hat{a}(k+1)} -_{\mathbb{Q}} \widetilde{sl}\ | \leq_{\mathbb{Q}} \langle 2^{-(k+1)} \rangle}_{\equiv:\mathsf{BW}'_{\mathbb{Q}}(s^1)},$$

where $\tilde{n} := \min_{\mathbb{Q}}\{\langle 1\rangle, \max_{\mathbb{Q}}\{\langle 0\rangle, n\}\}$. Consider a tree representation of the unit interval $[0,1]$ which splits the unit interval at level $n$ into $2^n$ intervals of length $2^{-n}$. Note that we can define each node via the path from the root to this interval. This path can be represented by a binary sequence $b$, where the $n$-th element defines which branch to take.
We define a predicate $I(b^0, n^0, m^0)$, which tells us, whether the rational number $r$ encoded by $m = \langle r \rangle$ belongs to an interval defined by such a finite binary sequence $b$ of length $\geq n$, i.e. in an interval of length $2^{-n}$ given by $b$:

$$\begin{aligned} I(b^0, n^0, m^0) :\equiv &\quad n \leq_0 \mathrm{lh}(b) \wedge \left\langle \sum_{i=1}^{n} \frac{b_i}{2^i} \right\rangle \leq_{\mathbb{Q}} m \leq_{\mathbb{Q}} \left\langle \sum_{i=1}^{n} \frac{b_i}{2^i} + \frac{1}{2^n} \right\rangle \\ \Leftrightarrow &\quad n \leq \mathrm{lh}(b) \wedge \sum_{i=1}^{n} \frac{b_i}{2^i} \leq r \leq \sum_{i=1}^{n} \frac{b_i}{2^i} + \frac{1}{2^n}. \end{aligned}$$

We know that for a given finite binary sequence $b$ and an infinite sequence of encodings of rational numbers $s$, there is a function $f_s^{1(0)}$, primitive recursive in $b$ and $s$, such that:

$$f_s(b,k) =_0 0 \ \leftrightarrow \ \Big( k > \mathrm{lh}(b) \wedge I\big(b, \mathrm{lh}(b), \widetilde{sk}\big) \Big).$$

[3] One can easily construct effective transformations which assign to any $x$ of the type $1(0)(0)$ (or $1(0)$) a unique $\tilde{x}$ in $\mathbb{P}^{\mathbb{N}}$ (or $\mathbb{P}$), see theorem 3.6.

Now, by $\Sigma^0_1$-$\mathsf{CA}(f_s)$ we obtain a function $g_s$, s.t. $\forall b^0 \Big( g_s b =_0 0 \leftrightarrow \exists k^0 \big(f_s(b,k) =_0 0\big) \Big)$.
In other words we have for all $b^0$:

$$g_s(b) =_0 0 \ \leftrightarrow \exists k^0 >_0 \mathrm{lh}(b) \ \ I\big(b, \mathrm{lh}(b), \widetilde{sk}\big). \tag{+}$$

To show $\mathsf{BinTree}(g_s)$, consider any finite binary sequence $b$:

$$\begin{aligned} g_s(b) =_0 0 \ \wedge \ x \subseteq b &\rightarrow \exists k^0 >_0 \mathrm{lh}(b) \ \ I\big(b, \mathrm{lh}(b), \widetilde{sk}\big) \ \wedge \ \mathrm{lh}(x) \leq_0 \mathrm{lh}(b) \ \wedge \ x \subseteq b \\ &\rightarrow \exists k^0 >_0 \mathrm{lh}(x) \ \ I\big(b, \mathrm{lh}(x), \widetilde{sk}\big) \ \wedge \ x \subseteq b \\ &\rightarrow \exists k^0 >_0 \mathrm{lh}(x) \ \ I\big(x, \mathrm{lh}(x), \widetilde{sk}\big) \rightarrow g_s(x) =_0 0. \end{aligned}$$

To show

$$\forall k \exists x \big( \mathrm{lh}(x) =_0 k \wedge g_s(x) =_0 0\big) \tag{++}$$

just consider any given natural number $k$. By the definition of our tree, it splits the $[0,1]$ interval at any level, in particular on level $k$, completely. Therefore, we have: $I(b, \mathrm{lh}(b), \widetilde{sk})$ and $\mathrm{lh}(b) =_0 k$ for a suitable $b$. As we started with arbitrary $k$, this implies (++).
Now, we can apply $\mathsf{WKL}(g_s)$ to get:

$$\exists b^1 \big(\mathsf{BinFunc}(b) \wedge \forall k \ g_s(\bar{b}k) =_0 0\big). \tag{$*$}$$

Note that in $(*)$ (and from now on) $b^1$ is a binary function and $g_s$ takes the encoding of the initial segment, $\langle b(0), \ldots, b(k-1)\rangle$, of this infinite sequence as its type 0 argument. Using (+) we can conclude that $(*)$ is equivalent to: $\exists b^1 \leq \mathbf{1} \ \forall n \ \exists k > n \ \ I(\bar{b}n, n, \widetilde{\ sk\ })$. This means that $\mathsf{BW}'_{\mathbb{Q}}(s)$ is satisfied by $\hat{a}$ where $a$ is defined as:
$a(n^0) :=_{\mathbb{Q}} \left\langle \sum_{i=1}^{n+1} \frac{b(i-1)}{2^i} \ + \frac{1}{2^{n+2}} \right\rangle$ provided that $\widehat{a} =_1 a$.
It, therefore, remains to show that $a$ represents a real number in the sense of Definition 1.3. W.l.o.g, at this point, we use $r, =, |\cdot|, \ldots$ directly instead of the proper syntactic form $\langle r\rangle, =_{\mathbb{Q}}, |\cdot|_{\mathbb{Q}}, \ldots$ to achieve better readability. To prove $a =_1 \hat{a} \in \mathbb{R}$, take any natural number $n$. We have:

$$\begin{aligned} |a(n) - a(n+1)| &= \left| \sum_{i=1}^{n+2} \frac{b(i-1)}{2^i} \ + \frac{1}{2^{n+3}} - \left( \sum_{i=1}^{n+1} \frac{b(i-1)}{2^i} \ + \frac{1}{2^{n+2}} \right) \right| \\ &= \left| \frac{b(n+1)}{2^{n+2}} + \frac{1}{2^{n+3}} - \frac{1}{2^{n+2}} \right| = 2^{-(n+3)} < 2^{-(n+1)}, \end{aligned}$$

which concludes the proof.

The only relevant difference for the general case (i.e. for sequences in $\mathbb{P}$) is the definition of $f_s$. If we wanted each node $i$ at level $n$ to define a subspace $\mathbb{P}^n_i \subseteq \mathbb{P}$ such that $\exists a^n_i \in \mathbb{P}^n_i \forall b \in \mathbb{P}^n_i d^{\omega}(a,b) \leq 2^{-n}$, then the number of children couldn't be bounded by a constant.
It turns out that it is simpler to define a representation of $\mathbb{P}$ by a binary tree, where any infinite path defines a single element of $\mathbb{P}$ and provide a function which returns the sufficient level to satisfy the condition above.
We define such a tree as follows. We start by splitting the first dimension into two halves, i.e. the two children represent the spaces $\mathbb{P}^1_0 = [-k_0, 0] \times \prod_{j=1}^{\infty}[-k_j, k_j]$ and $\mathbb{P}^1_1 = [0, k_0] \times \prod_{j=1}^{\infty}[-k_j, k_j]$. Next two levels arise by first splitting the new intervals in the first dimension and then splitting the second dimension into two halves. At level $\frac{l(l+1)}{2}$ we create the next $l+1$ levels by splitting the new intervals for the first $l$ dimensions and by splitting the original interval for the $(l+1)^{\text{th}}$ dimension. We define formally:

**Definition 3.2.** Let $w$ be the primitive recursive function representing the number of times we split dimension $d$ up to level $n$:

$$w(n^0, d^0) := \max_{\mathbb{N}}\{l : l > 0 \wedge 1 + \frac{(d+l)(d+l-1)}{2} + d \leq n \ \vee \ l = 0\}.$$

For an encoding of a finite binary sequence $b = \langle b_0, b_1, \ldots, b_{n-1}\rangle$ we define

- $D(b^0, d^0) := \langle b_{d(d+1)/2+d}, b_{(d+1)(d+2)/2+d}, \ldots, b_{(d+w(n,d)-1)(d+w(n,d))/2+d}\rangle$
(the code for the splittings of dimension $d$ corresponding to the node defined by $b$
– for $w(n,d) = 0$ we define $D$ to be the empty sequence
– using the Cantor pairing function we could also write $D(b^0, d^0) := \langle b_{\langle i,d\rangle} : i \in \{0, \ldots, w(n,d)-1\}\rangle$),
- $\mathbb{P}^b_d := \begin{cases} [-k_d + \sum_{i=0}^{w(n,d)} \frac{D(b,d)(i)}{2^i} k_d, k_d - \sum_{i=0}^{w(n,d)} \frac{1-D(b,d)(i)}{2^i} k_d] & \text{if } d \leq w(n,0), \\ [-k_d, k_d] & \text{else.} \end{cases}$
(the partition of the dimension $d$ relevant at the node defined by $b$),
- $\mathbb{P}^b := \prod_{i=0}^{\infty} \mathbb{P}^b_i$
(the subspace corresponding to the node defined by $b$).

For a sequence $s \subseteq \mathbb{P}$ we define $f^{\mathbf{1}}_s$ as follows:

$$f_s(b,k) := \begin{cases} 0 & \text{if} \quad k > \mathrm{lh}(b) \ \wedge \ \forall i < \mathrm{lh}(b)\ b(i) \leq 1 \ \wedge \\ & \qquad \bigwedge_{i=0}^{w(\mathrm{lh}(b),0)} (\widetilde{sk})ik \in_{\mathbb{Q}} \mathbb{P}^b_i, \\ 1 & \text{else.} \end{cases}$$

From now on our notation refers to this definition.

**Lemma 3.3.** *Define the functions*

$$\mathrm{lv}^1_d(n) :=_{\mathbb{N}} 1 + \frac{(\lceil \log_2(k_d(n+2))\rceil + n + 1)(\lceil \log_2(k_d(n+2))\rceil + n)}{2} + d,$$
$$\mathrm{lv}(n) :=_{\mathbb{N}} \max_{\mathbb{N}}\{\mathrm{lv}_d(n) : d \leq n+1\}.$$

*Then the following holds for all finite binary sequences $b$ and $n \in \mathbb{N}$:*

$$\mathrm{lh}(b) \geq \mathrm{lv}(n) \to \forall x, y \in \mathbb{P}^b \ \left(d^\omega(x,y) \leq 2^{-n}\right).$$

*For the specific $x$, s.t. $x_d$ is the center of $\mathbb{P}^b_d$ for all dimensions $d$ we have even:*

$$A^{\mathsf{BW}}([b]) :=_{1(0)} \lambda d^0, n^0. \left\langle -k_d + \sum_{i=0}^{w(\mathrm{lv}(n),d)-1} \frac{D(\overline{[b]}(\mathrm{lv}(n)), d)(i)}{2^i} k_d + \frac{1}{2^{w(\mathrm{lv}(n),d)}} k_d \right\rangle,$$

$$\mathrm{lh}(b) \geq \mathrm{lv}(n) \to \forall y \in \mathbb{P}^b \ \left(d^\omega(A^{\mathsf{BW}}([b]), y) \leq 2^{-n-1}\right).$$

*Moreover, we have*

$$f_s(b,k) = 0 \wedge \mathrm{lh}(b) \geq \mathrm{lv}(n) \quad \to \quad d^\omega((\widetilde{sk}), A^{\mathsf{BW}}([b])) < 2^{-n}.$$

**Proof.** W.l.o.g let $l := \mathrm{lh}(b) = \mathrm{lv}(n)$. By definition we have $w(l,d) \geq \lceil \log_2(k_d(n+2))\rceil + n + 1 - d$. This means $\lceil \log_2(k_d(n+2))\rceil - w(l,d) \leq d - n - 1$.
So $(n+2)k_d 2^{-w(l,d)} \leq 2^{-n-1+d}$ and $2k_d - \sum_{i=0}^{w(l,d)} 2^{-i} k_d \leq \frac{2^{-n-1+d}}{n+2}$.
By definition of $\mathbb{P}^b_d$ we obtain for all $d \leq n+1 (\leq w(l,0))$: $|\mathbb{P}^b_d| \leq \frac{2^{-n-1+d}}{n+2}$,[4]

[4] By $|[a,b]|$, $a, b \in \mathbb{Q}$ we mean the length of the rational interval $[a,b]$.

which implies $\sum_{d=0}^{n+1} 2^{-d}|\mathbb{P}^b_d| \le 2^{-n-1}$ and $\sum_{d=0}^{\infty} 2^{-d}\frac{|\mathbb{P}^b_d|}{1+|\mathbb{P}^b_d|} \le 2^{-n}$.

To show $d^\omega((\widetilde{sk}), A^{\mathsf{BW}}([b])) < 2^{-n}$ suppose $\bigwedge_{i=0}^{w(\mathrm{lh}(b),0)} (\widetilde{sk})ik \in \mathbb{P}^b_i$. This implies there is an $y \in \mathbb{P}^b$ s.t. $\forall d\ |y_d - (\widetilde{sk})_d| \le 2^{-k}$. Therefore $d^\omega((\widetilde{sk}), A^{\mathsf{BW}}([b])) \le 2^{-n-1} + 2^{-k}$ and since $k > \mathrm{lh}(b) \ge \mathrm{lv}(n) \ge n+2$ also $d^\omega((\widetilde{sk}), A^{\mathsf{BW}}([b])) < 2^{-n}$.

□

Furthermore, we need to show the following property of our tree representation of $\mathbb{P}$.

**Lemma 3.4.** *At any level $n \in \mathbb{N}$, the union of all spaces corresponding to the paths of length $n$ is the whole space $\mathbb{P}$:*

$$\bigcup_{b \in \{b^0 : \mathrm{lh}(b) = n \wedge \bigwedge_{i=0}^{n} b(i) \le_0 1\}} \mathbb{P}^b = \mathbb{P}.$$

**Proof.** Let $I_d$ denote the set of indices within a given, arbitrary long binary sequences $b$ used by $D$ to generate the subsequence $D(b,d)$ ( $I_d = \{(d+i)(d+i+1)/2 + d = \langle i, d\rangle : i \in \mathbb{N}\}$). Since the Cantor pairing function is bijective, it follows that for $d_1 \ne d_2$ the intersection $I_{d_1} \cap I_{d_2}$ is empty and we can choose the binary sequences for each dimension independently.
Therefore it suffices to show the following (we scale by $2k_d$ and shift by $\frac{1}{2}$):

$$\forall n \forall x \in [0,1] \exists b\ \big(\mathrm{lh}(b) = n \wedge x \in [\sum_{i=1}^{n} \frac{b(i-1)}{2^i}, 1 - \sum_{i=1}^{n} \frac{1-b(i-1)}{2^i}]\big).$$

This holds for any $n$ and $x$ when we choose $b$ as the following binary sequence:

$$b(i) := \begin{cases} 1 & \text{if } x \ge \sum_{j=1}^{i} \frac{b(j-1)}{2^j} + \frac{1}{2^{i+1}}, \\ 0 & \text{else.} \end{cases}$$

□

## 3.2 Functional Interpretation of $\mathsf{BW}^\omega_{\mathbb{R}}$

From now on we consider $s$ to be an infinite sequence of points in $\mathbb{P}$, and $f_s$ to be the characteristic function of the corresponding tree (as defined in 3.2 above). From section 2, we know that using an appropriate formula $\varphi^{\mathsf{WKL}}$ we can write $\mathsf{WKL}$ as

$$\forall (h^1) \mathsf{WKL}_\Delta(h) \equiv \forall (h^1) \exists b^1 \forall k\ \varphi^{\mathsf{WKL}}(h,b,k) \equiv \forall (h^1) \exists b^1 \forall k\ \left(\widehat{h}\right)_{g(h)}(\bar{b}k) =_0 0,$$

where $\varphi^{\mathsf{WKL}}$ is quantifier-free.
We introduce following notations for $\Sigma^0_1$-$\mathsf{CA}$:

$$\begin{aligned} &\exists g^1 \forall x^0 \varphi^{\mathsf{CA}(f)}_{\Sigma^0_1}(x, gx) \\ &\qquad \equiv \exists g^1 \forall x^0 \big(gx =_0 0 \leftrightarrow \exists z^0\ f(x,z) =_0 0\big) \\ \Leftrightarrow\ &\exists g^1 \forall x^0 \forall z_2^0 \exists z_1^0 \varphi^{\mathsf{CA}(f)}(x, gx, z_1, z_2) \\ &\qquad \equiv\ \exists g^1 \forall x^0 \forall z_2^0 \exists z_1^0 \big((gx =_0 0 \to f(x,z_1) =_0 0) \wedge (gx =_0 0 \leftarrow f(x,z_2) =_0 0)\big), \end{aligned}$$

where $\varphi^{\mathsf{CA}(f)}$ is a quantifier-free formula.
The essential step in the proof above is the following implication:

$$(\Sigma^0_1\text{-}\mathsf{CA}(f) \wedge \mathsf{WKL}) \quad \to \quad \Big(\exists g, b \forall x, k\ \big(\ \varphi^{\mathsf{CA}(f)}_{\Sigma^0_1}(x, gx) \wedge \varphi^{\mathsf{WKL}}(g,b,k)\ \big)\Big), \qquad (+)$$

since its conclusion is essentially the same as $\mathsf{WKL}(\psi_f)$, where $\psi_f(k^0) \leftrightarrow \exists n^0 f(k,n) =_0 0$ and $k$ is the variable which is bound by the last for-all quantifier in WKL. Moreover, for $f_s$ as defined above it actually directly implies BW, whereas considered as a schema for arbitrary $f^{\mathbf{1}}$ it corresponds to $\Sigma^0_1$-WKL.
The Sh-interpretation of (+) using this representation and applying QF-AC is as follows:

$$\begin{aligned}
&\exists G, Z_1, B, H', X', Z_2', K' \;\; \forall B', Z_1', G', X, Z_2, K \\
&\quad \Big( \big( \varphi^{\mathsf{CA}(f)} \big( X'(G'X'Z_2')(Z_1'X'Z_2'), G'X'Z_2'(X'(G'X'Z_2')(Z_1'X'Z_2')), Z_1'X'Z_2'(X'(G'X'Z_2')(Z_1'X'Z_2'))(Z_2'(G'X'Z_2')(Z_1'X'Z_2')), \\
&\qquad\qquad Z_2'(G'X'Z_2')(Z_1'X'Z_2') \big) \wedge \varphi^{\mathsf{WKL}}\big(H', B'H'K', K'(B'H'K')\big) \big) \to \\
&\qquad \big( \varphi^{\mathsf{CA}(f)}\big(XGZ_1B, G(XGZ_1B), Z_1(XGZ_1B)(Z_2GZ_1B), Z_2GZ_1B\big) \wedge \varphi^{\mathsf{WKL}}(G, B, KGZ_1B) \big) \Big)
\end{aligned}$$

where, again, each exists-variable (i.e. $G$, $Z_1$, $B$, $H'$, $X'$, $Z_2'$, and $K'$) may depend on any for-all-variable (i.e. $B'$, $Z_1'$, $G'$, $X$, $Z_2$, and $K$). E.g. by $G$ we mean in fact $(GB'Z_1'G'XZ_2K)$. This interpretation yields the following functional equations:

$$X'(G'X'Z_2')(Z_1'X'Z_2') = XGZ_1B, \qquad H' = G, \quad (1{,}5)$$
$$G'X'Z_2'(X'(G'X'Z_2')(Z_1'X'Z_2')) = G(XGZ_1B), \qquad B'H'K' = B, \quad (2{,}6)$$
$$Z_1'X'Z_2'(X'(G'X'Z_2')(Z_1'X'Z_2'))(Z_2'(G'X'Z_2')(Z_1'X'Z_2')) = Z_1(XGZ_1B)(Z_2GZ_1B), \qquad K'(B'H'K') = KGZ_1B, \quad (3{,}7)$$
$$Z_2'(G'X'Z_2')(Z_1'X'Z_2') = Z_2GZ_1B. \quad (4)$$

We use a very similar approach to the one used by Gerhardy in [6] to solve such equations for finite DNS. First, we conclude from (5) and (6) that $B = B'GK'$ and from (1) and (2) that $G = G'X'Z_2'$. Using (6), we can set $K'$ to $\lambda b.KGZ_1b$ according to (7). This is not that trivial for $X'$ and $Z_2'$. However, as pointed out by Gerhardy in [6], in the presence of the $\lambda g$ and $\lambda z_1$, which as we know will stand for the input of $G$ and $Z_1$, the objects $X'$ and $Z_2'$ become well definable terms:

$$t_{X'} := \lambda g, z_1.Xgz_1(B'g(\lambda b.Kgz_1b)), \quad t_{Z_2'} := \lambda g, z_1.Z_2gz_1(B'g(\lambda b.Kgz_1b)).$$

This makes the rest of our terms we need well defined. This is easy to see since for each term all dependencies are only on the terms defined above:

$$\begin{aligned}
t_{Z_1} &:= Z_1' t_{X'} t_{Z_2'}, & t_{H'} &:= t_G, \\
t_G &:= G' t_{X'} t_{Z_2'}, & t_{K'} &:= \lambda b.K t_G t_{Z_1} b, \\
t_B &:= B' t_G t_{K'}.
\end{aligned}$$

We have found the realizing terms for the Shoenfield interpretation $(+)^{Sh}$ of $(+)$ for any $G'$, $Z_1'$ and $B'$. To finally obtain the Shoenfield interpretation of ($\Sigma^0_1$-$\mathsf{WKL}(\varphi)$) we just need to define these three functionals in such a way that the assumptions $\varphi^{\mathsf{WKL}}$ and $\varphi^{\mathsf{CA}(f)}$ are always true.
For $\varphi^{\mathsf{CA}(f)}$, as we know from the functional interpretation of $\Sigma^0_1$-CA (see section 1.5), we get:

$$G'^{\mathbf{3}} = t_h, \quad Z_1'^{\mathbf{3}} = t_z,$$

where $t_h$ and $t_z$ are defined as in the Sh-interpretation of $\Sigma^0_1$-CA (see corollary 1.13).
For $B'$, from the interpretation of WKL, we know the following equality holds:

$$\underbrace{\overline{B'H'K'}\big(K'(B'H'K')\big)}_{\text{as above}} = \underbrace{\overline{B}(AB)}_{\text{as in section 2}}.$$

We use the same notation as we used to define $B$ in section 2 and define:

$$B' := \lambda h.\lambda A.\big[F_h\big(K_A(\varnothing)\big)\big],$$

where $F_h$ and $K_A$ are defined as in the Sh-interpretation of WKL (see Theorem 2.4). The terms defined above, using these definitions for $G'$ and $B'$, then satisfy the Shoenfield interpretation of the conclusion of $(+)$ :

$$\forall X, Z_2, K\ (\ \varphi^{\mathsf{CA}(f)}(Xt_Gt_{Z_1}t_B, t_G(Xt_Gt_{Z_1}t_B), t_{Z_1}(Xt_Gt_{Z_1}t_B)(Z_2t_Gt_{Z_1}t_B), Z_2t_Gt_{Z_1}t_B) \wedge \varphi^{\mathsf{WKL}}(t_G, t_B, Kt_Gt_{Z_1}t_B)\ ).$$

Using that $\forall X^{0(10)(1)}, Z^{0(10)(1)}, a^0, b^0\ t_zXZab =_0 t_zXZa0 = t_Za0$ we conclude:

**Lemma 3.5.**
*The principle (which essentially represents[5] $\Sigma^0_1$-$\mathsf{WKL}(\varphi)$):*

$$\exists g^1, b^1 \forall x^0, k^0\ \big(\varphi^{\mathsf{CA}(f)}_{\Sigma^0_1}(x, gx) \wedge \varphi^{\mathsf{WKL}}(g, b, k)\big),$$

*is Sh-interpreted by ($\tau = 01(10)1$ [6]):*

$$\forall X^\tau, Z^\tau, K^\tau\ \Big(\ \varphi^{\mathsf{CA}(f)}(Xt_Gt_Zt_B, t_G(Xt_Gt_Zt_B), t_Z(Xt_Gt_Zt_B)0, Zt_Gt_Zt_B) \wedge \varphi^{\mathsf{WKL}}(t_G, t_B, Kt_Gt_Zt_B)\ \Big),$$

*where*

$$t_B :=_1 B^{\mathsf{WKL}}(\lambda b.Kt_Gt_Zb, t_G), \qquad t'_X :=_{0(10)1} \lambda g^1, z^{10}.Xgz(B^{\mathsf{WKL}}(\lambda b.Kgzb, g)),$$
$$t_Z :=_{10} t_zt'_Xt'_Z, \qquad t'_Z :=_{0(10)1} \lambda g^1, z^{10}.Zgz(B^{\mathsf{WKL}}(\lambda b.Kgzb, g)),$$
$$t_G :=_1 t_ht'_Xt'_Z,$$

*The remaining terms are defined as in previous sections.*[7]

**Theorem 3.6.** *The Bolzano-Weierstraß principle $\mathsf{BW}^\omega_{\mathbb{R}}$ for an infinite sequence $s$ of elements in $\mathbb{P}$ (let $\rho = (10)0$ and $\sigma = 0(10)1$, $\widetilde{x^{1(0)}} := \lambda i^0.\min_{\mathbb{R}}(-k_i, \max_{\mathbb{R}}(k_i, xi))$, recall $\mathbb{P} = \prod_{i\in\mathbb{N}}[-k_i, k_i]$):*

$$\forall s^\rho \exists a^{1(0)} \forall m^0 \exists l^0 >_0 m\ \ d^\omega(\widetilde{sl}, \tilde{a}) <_{\mathbb{R}} 2^{-m}.$$

*is Sh-interpreted[8] by:*

$$\forall s^\rho, M^\sigma \exists L^1, a^{1(0)} \underbrace{\big(\ L(MLa) >_0 MLa\ \wedge\ d^\omega(\widetilde{s(L(MLa))}, \tilde{a}) <_{\mathbb{R}} 2^{-MLa}\ \big)}_{\mathsf{BW}_{Sh}(a, MLa, L(MLa), s) :=}$$

*where L and a are realized by the terms $t^{\mathbf{3}}_L$ and $t^{\mathbf{3}}_A$ (we use the notation from 3.2 and 3.3):*

$$t_L(s^\rho, M^\sigma) :=_1\ \lambda n^0.t_Z(\overline{t_B}(\mathrm{lv}(n)))0,$$

$$t_A(s^\rho, M^\sigma) :=_{1(0)} A^{\mathsf{BW}}(t_B) =\ \lambda d^0, n^0.\Big\langle -k_d + \sum_{i=0}^{w(\mathrm{lv}(n),d)-1} \frac{D(\overline{t_B}(\mathrm{lv}(n)), d)(i)}{2^i} k_d + \frac{1}{2^{w(\mathrm{lv}(n),d)}} k_d \Big\rangle,$$

*Here $t_B$ is defined as above, i.e. (see Definition 2.3 on page 11 for $B^{\mathsf{WKL}}$):*

$$t_B =_1 B^{\mathsf{WKL}}(M't_Gt_Z, t_G),$$

[5] It trivially implies $\Sigma^0_1$-$\mathsf{WKL}(\varphi)$ (provably in $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^\omega\!\restriction$). Let us note, however, that the actual computation of the witnesses for $\Sigma^0_1$-$\mathsf{WKL}(\varphi)$ still involves some highly non-trivial technical work.

[6] Or, in a more illustrative notation: $\tau = (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$.

[7] $B^{\mathsf{WKL}}$ in Theorem 2.4, and $t_z, t_h$ in Theorem 1.13.

[8] The quantifier in $<_{\mathbb{R}}$ is irrelevant, see the remark after the theorem.

*where as before (with* $f := f_s$*,* $z^- :\equiv_1 \lambda n.z^{10}n0$*,* $g^+ :\equiv_{10} \lambda a, b\, .g^1 a$ *and* $g\restriction_{f_s} :\equiv \lambda n.\overleftrightarrow{f_s(n, gn)}$*):*

$$t_Z =_{10} \left(\Phi_0\big(\lambda g^1.X(g\restriction_{f_s})g^+(B^{\mathsf{WKL}}(M'(g\restriction_{f_s})g^+, g\restriction_{f_s}))\big)u^{0(1(0))(0)}_{\left(\lambda g^1.Z(g\restriction_{f_s})g^+(B^{\mathsf{WKL}}(M'(g\restriction_{f_s})g^+, g\restriction_{f_s}))\right), f_s} 0^0\mathbf{0}^1\right)^+$$

$t_G =_1 ((t_Z)^-)\restriction_{f_s}$.

*Here* $t_B$ *and* $t_Z$ *are shortcuts for* $t_B XZM'$ *and* $t_Z XZM'$ *with fixed X, Z:*

$$X :=_\tau \lambda g^1, z^{10}, b^1\, .\, B(M'gzb, g, z^-, b), \quad Z :=_\tau \lambda g^1, z^{10}, b^1\, .\, N(M'gzb, g, z^-, b), \quad ,$$

*where* $(M')^\tau$ *(recall* $\tau = 01(10)1$ *) is defined for any given* $M^\sigma$ *similarly as X and Z as follows:*

$$M'(g^1, z^{10}, b^1) :=_0 \mathrm{lv}\left(M\underbrace{\left(\lambda n^0.z^-(\bar{b}(\mathrm{lv}(n)))\right)}_{\sim t_L}\underbrace{\left(A^{\mathsf{BW}}(b)\right)}_{\sim t_A}\right). \tag{1}$$

*The terms B,* $X_n$[9] *and N are primitive recursive, though not trivial, case distinctions:*

$$B(m^0, g^1, z^1, b^1) :=_0 \begin{cases} \min_0\left\{x^0 \;\middle|\; \begin{array}{l}\mathrm{lh}(x) \leq_0 m \;\wedge\; x \in \{0,1\}^{\mathrm{lh}(x)} \;\wedge \\ \neg\varphi^{\mathsf{CA}}(x, gx, zx, Nmgzb)\end{array}\right\} & \textit{if it exists,} \\ X_n(m, g, b) & \textit{else,} \end{cases}$$

$$X_n(m^0, g^1, b^1) :=_0 \begin{cases} \bar{b}m & \textit{if } g(\bar{b}m) =_0 0, \\ \min_0\{x^0 \,|\, \mathrm{lh}(x) = m \wedge f_s(x, m+1) =_0 0\} & \textit{else,} \end{cases}$$

$$N(m^0, g^1, z^1, b^1) :=_0 \begin{cases} m+1 & \textit{if } f_s(X_n(m, g, b), m+1) =_0 0, \\ z(X_n mgb) & \textit{else.} \end{cases}$$

**Proof.** Unwinding $\varphi^{\mathsf{CA}}$ and $\varphi^{\mathsf{WKL}}$ we get by lemma 3.5:

$$\begin{aligned} &\forall K^\tau, X^\tau, Z^\tau\ \big((t_G XZK(X(t_G XZK)(t_Z XZK)(t_B XZK))\big) =_0 0 \to \\ &\qquad f_s\big((X(t_G XZK)(t_Z XZK)(t_B XZK)), t_Z XZK(X(t_G XZK)(t_Z XZK)(t_B XZK))0\big) =_0 0\big) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\forall K^\tau, X^\tau, Z^\tau\ \big((t_G XZK(X(t_G XZK)(t_Z XZK)(t_B XZK))\big) =_0 0 \leftarrow \\ &\qquad f_s\big(X(t_G XZK)(t_Z XZK)(t_B XZK), Z(t_G XZK)(t_Z XZK)(t_B XZK)\big) =_0 0\big) \end{aligned} \tag{3}$$

and

$$\forall K^\tau, X^\tau, Z^\tau\ \overbrace{\big(\widehat{t_G XZK}\big)_{g(t_G XZK)}}\ \big(\overline{t_B XZK}(K(t_G XZK)(t_Z XZK)(t_B XZK))\big) =_0 0. \tag{4}$$

Fix an arbitrary $M^\sigma$.
We set $X$, $Z$ and $K :=_\tau M'$ – see (1) – as in the theorem. We will use the following abbreviations:

$$\begin{aligned} x_0 &:\equiv_0 X(t_G XZK)(t_Z XZK)(t_B XZK), & \gamma &:\equiv_1 t_G XZK, \\ z_0 &:\equiv_0 Z(t_G XZK)(t_Z XZK)(t_B XZK), & z &:\equiv_1 \lambda n^0.t_Z XZKn0, \\ k_0 &:\equiv_0 K(t_G XZK)(t_Z XZK)(t_B XZK), & b &:\equiv_1 t_B XZK. \end{aligned}$$

Note that by (2) the equality $\gamma(x_0) =_0 0$ implies $f_s(x_0, zx_0) =_0 0$ and thereby $\mathsf{BinFunc}([x_0])$. We will not be able to show $\mathsf{BinTree}(\gamma)$ but fortunately we need only to show:

$$\forall x^0\ \Big(\big(x \subseteq x_0 \;\wedge\; \gamma(x_0) = 0\big) \to \gamma(x) =_0 0\Big) \tag{5}$$

[9] The existence of the required $x$ follows from Lemma 3.4.

and

$$\mathrm{lh}(x_0) =_0 k_0 \ \wedge\ \gamma(x_0) =_0 0. \tag{6}$$

Note that if $x_0$ was not equal to $X_n(k_0, \gamma, b)$ then

$$\neg\varphi^{\mathsf{CA}}(x_0, \gamma x_0, z x_0, \underbrace{N(k_0, \gamma, z, b)}_{=z_0})$$

would hold, which is a contradiction to (2) or a contradiction to (3). So we can assume that $x_0 = X_n(k_0, \gamma, b)$. Similarly, we have that

$$(\gamma(q) =_0 0 \to f_s(q, z(q)) =_0 0) \ \wedge\ (\gamma(q) =_0 0 \leftarrow f_s(q, z_0) =_0 0). \tag{7}$$

Suppose namely (7) would not hold for some $q'$ with $\mathrm{lh}(q') \leq k_0$, then $x_0$ is equal to such a $q'$ by the definition of $X$ and we get a contradiction to (2) $\wedge$ (3) again.

- To prove (5) suppose:

$$x \subseteq x_0 \quad \wedge \tag{8a}$$
$$\gamma(x_0) = 0 \tag{8b}$$

  holds for some $x$. Together with (8b) we obtain from (7):

$$f_s\big(x_0, z(x_0)\big) =_0 0,$$

  since $\mathrm{lh}(x_0) \leq k_0$ (by the definition of $X$). We follow the definition of $Z$. We see that either $z_0$ directly equals $z(X_n(k_0, \gamma, b))$ or we have $f_s(X_n(k_0, \gamma, b), k_0 + 1) = 0$ and it equals $k_0 + 1$. This means that in both cases we obtain (using that $x_0 = X_n(k_0, \gamma, b)$)

$$f_s(x_0, z_0) =_0 0.$$

  By the definition of $f_s$, see also section 3.1, and (8a) this implies

$$f_s(x, z_0) =_0 0.$$

  From (8a) we get $\mathrm{lh}(x) \leq k_0$ and by (7) we obtain

$$\gamma(x) = 0,$$

  which concludes the proof of (5).

- Recall that $x_0 = X_n(k_0, \gamma, b)$. This proves the first part of (6):

$$\mathrm{lh}(x_0) =_0 k_0.$$

  Now we follow the definition of $X_n$. Either $\gamma(\overline{b}(k_0)) = 0$ and therefore $x_0 = \overline{b}(k_0)$ and we obtain (6) immediately, or we have that:

$$f_s\big(X_n(k_0, \gamma, b), k_0 + 1\big) = 0.$$

  In that case, we can infer that

$$z_0 = N(k_0, \gamma, z, b) = k_0 + 1,$$

  and we get

$$f_s(x_0, z_0) = 0.$$

  Finally, applying (3) concludes the proof of (6).

This concludes the proofs of (5) and (6).
To show $\gamma\big(\overline{b}(k_0)\big) = 0$ assume towards contradiction that

$$\gamma\big(\overline{b}(k_0)\big) \neq 0.$$

If so, then by definition 2.3 we have also that

$$\widehat{\gamma}\,\big(\overline{b}(k_0)\big) \neq 0.$$

By (6) we know that $\gamma(x_0) = 0$. Using (5) we know that also $\widehat{\gamma}\,(x_0) = 0$ and since $\mathrm{lh}(x_0) = k_0 = \mathrm{lh}(\overline{b}(k_0))$ we have that

$$\widehat{\gamma}_{g\gamma}\big(\overline{b}(k_0)\big) \neq 0.$$

By definition 2.3 this is a contradiction to (4) and we obtain (recall that we started with an arbitrary $M$):

$$\forall M^{\sigma}\ \gamma\big(\overline{b}(k_0)\big) =_0 0.$$

This implies that $x_0 = \overline{b}(k_0)$ (by the definition of $X_n$) and therefore it follows by (2) that:

$$\forall M^{\tau}\ \ f_s\big(\ \overline{b}(k_0), z(\overline{b}k_0)\ \big) =_0 0.$$

Using the terms $t_L$ and $t_A$ in the short notation (i.e. $t_L$ instead of $t_L sM$, $t_B$ instead of $t_B XZM'$ and similarly for $t_A$ and $t_Z$) this becomes

$$\forall M^{\sigma}\ \ f_s\big(\ \overline{t_B}(\mathrm{lv}(Mt_Lt_A)),\ t_Z(\overline{t_B}(\mathrm{lv}(Mt_Lt_A)))0\ \big) =_0 0.$$

This implies by Lemma 3.3 (note that $t_L(Mt_Lt_A) \equiv t_Z(\overline{t_B}(\mathrm{lv}(Mt_Lt_A)))0$):

$$\forall M^{\sigma}\ \Big(\ t_L(Mt_Lt_A) > Mt_Lt_A\ \wedge\ d^{\omega}(\widetilde{t_A}, s(\widetilde{t_L(Mt_Lt_A)})) <_{\mathbb{R}} 2^{-Mt_Lt_A}\Big).$$

Finally observe that for all $n^0$ we have $(t_A(s,M))n =_{\mathbb{R}} (\widetilde{t_A(s,M)})n$ and that

$$\begin{aligned}
t_Z &=_{10} t_z t'_X t'_Z = \big(t_g(t_f t'_X)(t_f t'_Z)\big)^{+} \\
&= \lambda a, b\ .\ \underbrace{t_g(\lambda g^1.X(g\restriction_{f_s})g^{+}(B^{\mathsf{WKL}}(M'(g\restriction_{f_s})g^{+}, g\restriction_{f_s})))(\lambda g^1.Z(g\restriction_{f_s})g^{+}(B^{\mathsf{WKL}}(M'(g\restriction_{f_s})g^{+}, g\restriction_{f_s})))a}_{=\Phi_0\left(\lambda g^1.X(g\restriction_{f_s})g^{+}\left(B^{\mathsf{WKL}}(M'(g\restriction_{f_s})g^{+}, g\restriction_{f_s})\right)\right)u^{0(1(0))(0)}_{\left(\lambda g^1.Z(g\restriction_{f_s})g^{+}\left(B^{\mathsf{WKL}}(M'(g\restriction_{f_s})g^{+}, g\restriction_{f_s})\right)\right), f_s}{}^{0^0\mathbf{0}^1 a}}, \\
t_G &=_1 t_h t'_X t'_Z = \big(t_g(t_f t'_X)(t_f t'_Z)\big)\restriction_{f_s}.
\end{aligned}$$

□

**Remark.** *From the proof we actually see how to realize the hidden quantifier in* $<_{\mathbb{R}}$. *Namely before using Lemma 3.3 we can simply apply the definition of* $f_s$ *and obtain the following equivalent universal formula (for* $\rho = (10)0$ *and* $\sigma = 0(10)1$ *and writing* $t_L$ *instead of* $t_L sM$ *and* $t_A$ *instead of* $t_A sM$*):*

$$\forall s^{\rho}, M^{\sigma}\big(t_{L(Mt_Lt_A)} >_0 Mt_Lt_A \wedge \bigwedge_{d=0}^{w(\mathrm{lv}(Mt_Lt_A))} \big(s(\widetilde{t_{L(Mt_Lt_A)}})\big)d\big(t_{L(Mt_Lt_A)}\big) \in_{\mathbb{Q}} \mathbb{P}^{\overline{t_B}(\mathrm{lv}(Mt_Lt_A))}.$$

**Remark.** *One can also extend Theorem 3.6 to cover the case where the sequence* $(k_d)$ *is a sequence in* $\mathbb{R}_+$ *rather than* $\mathbb{Q}_+$ *provided that one adapts the definition of* $f_s$ *in such a way that also the boundaries of the interval* $[-k_d, k_d]$ *are replaced by suitable rational approximations etc. Since the application of our analysis of the Bolzano-Weierstraß theorem given in [20] (referred to in the introduction) only uses the monotone functional interpretation given below (Theorem 3.10) as will be usually the case, we restrict ourselves in this paper to treat the case with real* $k_d$ *only in that context where things are particularly simple (and no approximation of the type mentioned above is needed).*

Usually, the Bolzano-Weierstraß theorem is formulated to state the existence of a converging subsequence rather than the existence of a cluster point. The next theorem gives the solution for the Shoenfield interpretation of this formulation:

**Theorem 3.7.** *The version of the Bolzano-Weierstraß theorem stating the existence of a convergent subsequence is Sh-interpreted as follows (where* $t_L, t_A$ *are as in theorem 3.6):*

$$\forall s, M \exists f, p (f(Mfp+1) > f(Mfp) \wedge d^{\omega}(\tilde{p}, s(\widetilde{f(Mfp+1)})) < 2^{-Mfp})$$

*Let* $M'$ *be obtained from* $M$ *by*

$$M'la := \begin{cases} l^{M(\lambda n.l^n 0)a}0 & \textit{if } \big(\forall n < M(\lambda n.l^n 0)a\big)\ \big(l^{n+1}0 > l^n 0\big), \\ l^{\min_0\{n:\ l^{n+1}0 \le l^n 0\}}0 & \textit{else.} \end{cases}$$

*Define the functionals*

$$t_F sM := \lambda n.(t_L sM')^n 0 \quad \textit{and} \quad t_P sM := t_A sM',$$

*then the following holds:*

$$\begin{aligned} \forall s, M \quad ( & \ t_F sM(M(t_F sM)(t_P sM)+1) > t_F sM(M(t_F sM)(t_P sM)) \wedge \\ & d^{\omega}(s(t_F sM(\widetilde{M(t_F sM)}(t_P sM)+1)), \widetilde{t_P sM}) < 2^{-M(t_F sM)(t_P sM)} \ ). \end{aligned}$$

**Proof.** Consider any given $s$ and $M$ and define $M'$ as in the theorem being proved. We denote $t_L sM'$, $t_A sM'$ and $M(\lambda n.(t_L sM')^n 0)(t_A sM')$ by $l_0$, $a_0$ and $m_0$. Unwinding the terms $t_P$ and $t_F$ in the statement of the theorem leads to:

$$\forall s, M \quad (\ l_0^{m_0+1}0 > l_0^{m_0}0 \ \wedge d^{\omega}(\tilde{a}_0, s(\widetilde{l_0^{m_0+1}0})) \le 2^{-m_0}\ ).$$

1. If $(\forall n < m_0)\ (l_0^{n+1}0 > l_0^n 0)$ then the claim follows directly from theorem 3.6 applied to $s$ and $M'$ (i.e. $M'l_0a_0$ becomes $l_0^{m_0}0$ ) and its proof:

$$l_0(l_0^{m_0}0) > l_0^{m_0}0 \ \wedge d^{\omega}(s(\widetilde{l_0(l_0^{m_0}0)}), \tilde{a}_0) <_{\mathbb{R}} 2^{-l_0^{m_0}0} \qquad (3.6 \text{ for } M')$$

   where the proof allows us to omit the final step from:

$$l_0(l_0^{m_0}0) >_0 l_0^{m_0}0 \ \wedge \bigwedge_{d=0}^{l_0^{m_0}0} |(a_0)(l_0^{m_0}0+1)d -_{\mathbb{Q}} s(\widetilde{l_0(l_0^{m_0}0)})d(l_0^{m_0}0+1)| \quad <_{\mathbb{Q}} \quad 2^{-(l_0^{m_0}0+2)}.$$

   To see that $l_0^{m_0} \ge m_0$ recall that $\forall n < m_0\ l_0^{n+1}0 > l_0^n 0$.

2. Otherwise we have $l_0^{i+1}0 \le l_0^i 0$ for some $i < m_0$ and $M'l_0a_0 = l_0^i 0$. However this is a contradiction to theorem 3.6 which implies

$$l_0^{i+1}0 = l_0(l_0^i 0) = l_0(M'l_0a_0) > M'l_0a_0 = l_0^i 0.$$

□

**Remark.** *Analogously to the previous remark, the proof of Theorem 3.6 gives us a more strict Sh-interpretation.*

Applying the majorization results from section 2.1 we obtain a (much easier) solution to the **monotone** Shoenfield interpretation of $\mathsf{BW}^{\omega}_{\mathbb{R}}$ (which is all the information on $\mathsf{BW}^{\omega}_{\mathbb{R}}$ one needs to extract uniform bounds from proofs that use $\mathsf{BW}^{\omega}_{\mathbb{R}}$).
Instead of having to majorize explicitly the complicated construction of $t_A$ we can rely on the fact that elements in compact intervals $[-K_d, K_d]$ with $K_d \in \mathbb{N}$ have a representations $x \leq_1 N_{K_d}$, where $N_m(k)$ is a fixed (non-decreasing) primitive recursive function in $m, k$ which can be taken as (see [18], p.93)

$$N_m(k) := j(m2^{k+3} + 1, 2^{k+2} - 1) \text{ for the Cantor pairing function } j.$$

From the proof of Theorem 3.6 it follows that instead of $(A^{\mathsf{BW}}(b))(d)$ we can take any other representative of the same real number, in particular the representative $\leq N_{K_d}$ from the construction in [18](p.93) for $K_d \geq k_d$. As a result, we can replace $t_A(s, M)(d)$ by another representative that is – for all $s, M$ – majorized by $N_{K_d}$. In fact, we may even allow $k_d$ to be a real number where then in the definition of $f_s$ the clause '$(\widetilde{sk})ik \in_{\mathbb{Q}} \mathbb{P}^b_i$' has to be replaced by '$(\widetilde{sk})i \in_{\mathbb{R}} \mathbb{P}^b_i$'. The resulting function $f_s$ then no longer is computable but still trivially majorizable by the constant-1 function.

**Remark.** *The shape of $N_m$ above is due to the particular Cauchy representation used in [18]. If one uses the so-called signed-digit representation, it can be improved to the constant-$(2m+3)$ function, see prop. 14 in [3].*
Before we state the monotone functional interpretation of $\mathsf{BW}^{\omega}_{\mathbb{R}}$ we first need, however, a simple lemma:

**Lemma 3.8.** *Define (for $\mathbb{N} \ni K_d \geq k_d$ for all $d \in \mathbb{N}$)*

$$\mathrm{lv}^*(n) := (\max_{d \leq n+1} \{K_d, n\} + 2)^4.$$

*Then* $\mathrm{lv}^*$ s-maj lv.

**Proof.** observe that for $d \leq n+1$ we have that $\mathrm{lv}_d(n) \leq n + 2 + (\lceil k_d(n+2) \rceil + n)^2 \leq (n+2) + (n+2)^2(\lceil k_d + 1 \rceil)^2 \leq (n+2)^2(\lceil k_d + 2 \rceil)^2$. Moreover $\mathrm{lv}^*$ is non-decreasing. □ To give the monotone version of Theorem 3.7, we need the following definition:

**Definition 3.9.** A majorant for $\Phi_0$ is given by (see also [18] or [2]):

$$\Phi_0^* y u^{0(1(0))(0)} n x :=_1 \max\left( (\Phi_0 y^{\mathsf{m}} u_x^{0(1(0))(0)})^{\mathsf{M}} n x, x^{\mathsf{M}} \right),$$

where

$$\begin{aligned} w^{\mathsf{M}}(n) &:=_\rho \max\{w(i) : i \leq_0 n\} \quad (\text{for } w^{\rho 0}, \rho \in \boldsymbol{T}), \\ y^{\mathsf{m}}(x) &:=_0 y(x^{\mathsf{M}}), \\ u_x(n, v) &:=_1 \max\{x^{\mathsf{M}}, v(unv)\}. \end{aligned}$$

**Theorem 3.10.** *Let $(k_d)$ be a sequence of non-negative reals and $(K_d)$ be a sequence of natural numbers with $K_d \geq k_d$ for all $d \in \mathbb{N}$. The realizing terms of the Sh-interpretation of the Bolzano-Weierstraß principle $\mathsf{BW}^{\omega}_{\mathbb{R}}$ for an infinite sequence $s$ of elements in $\mathbb{P}$ (defined by $(k_d)$, see also Theorem 3.6 above) can be majorized by terms $t_L^*$ and $t_A^*$, i.e. $t_L^*$ and $t_A^*$ satisfy the* *monotone Sh-interpretation of* $\mathsf{BW}^\omega_\mathbb{R}$*. The term* $t^*_L$ *depends only on* $M$ *(but not on* $s$*) and* $t^*_A$ *even is independent from both* $s$ *and* $M$*, indeed:*

$$t^*_L(M) :=_1 \lambda n^0.\mathbf{\Phi}^*_0(X^*M)u^{*0(1(0))(0)}_{(Z^*M)}0^0\mathbf{0}^1\big(\bar{\mathbf{1}}(\mathrm{lv}^*(n))\big),$$
$$t^*_A :=_{1(0)} \lambda d^0, n^0.N_{K_d},$$

*where*

$$N_m(k) := j(m2^{k+3}+1, 2^{k+2}-1)$$

*and* $X^*M$ *and* $Z^*M$ *are defined primitive recursively in* $M$*:*

$$X^*M := \lambda g^1\,.\,\bar{\mathbf{1}}(M'g), \qquad Z^*M := \lambda g^1\,.\,\max_0\big(\,M'g,\, g(\bar{\mathbf{1}}(M'g))\,\big).$$

*The term* $M'$ *is a similar primitive recursive modification of* $M$ *as before:*

$$M'g := \mathrm{lv}^*\Big(M\,\Big(\lambda n^0.g\big(\bar{\mathbf{1}}(\mathrm{lv}^*(n))\big)\Big)\,(t^*_A)\Big).$$

*The majorized versions of* $u$ *and* lv*, are given by:*

$$u^*_Z n^0 v^{1(0)} := \max_0\big(1, Z(v1)\big), \quad \mathrm{lv}^*(n) := (\max_{d\le n+1}{}_\mathbb{N}\{K_d, n\}+2)^4.$$

Though it is not entirely obvious how we obtain this theorem, the actual steps are purely elementary.

Similarly, we can obtain the majorized version of Theorem 3.7:

**Theorem 3.11.** *The terms* $t^*_F$ *and* $t^*_P$ *satisfy the monotone Sh-interpretation of* $\mathsf{BW}^\omega_\mathbb{R}$ *stating the existence of a converging subsequence (see also Theorem 3.7 above):*

$$t^*_F M := \lambda n.(t^*_L M'')^n 0, \quad t^*_P := t^*_A,$$

*with* $t^*_L$ *and* $t^*_A$ *defined as in the Theorem 3.10 and the term and where the term* $M''$ *is a similar primitive recursive modification of* $M$ *as in Theorem 3.7:*

$$M''la := l^{M(\lambda n.l^n(0))a}(0).$$

**Proof.** For $M^*$ s-maj $M$, we have that $(M^*)''$ s-maj $M', M''$. Hence $t^*_L((M^*)'')(n) \geq t_L sM'(n)$ and the function $t^*_L((M^*)'')$ is non-decreasing. Therefore $t^*_F M^*$ is non-decreasing as well and majorizes $t_F sM$. □

## 3.3 Analysis of the Complexity of the Realizers

Theorem 3.10 implies that we can use only a single application of $\mathsf{B}_{0,1}$ on primitive recursive functionals and primitive recursion to obtain the realizing terms for $\mathsf{BW}^\omega_\mathbb{R}$. Now, we can investigate how the principle $\mathsf{BW}^\omega_\mathbb{R}$ does affect the complexity of the realizers of a given theorem proved using this principle. Depending on the way the $\mathsf{BW}^\omega_\mathbb{R}$-principle is used in such a proof, we get the results stated in theorems 3.12 and 3.15 respectively.

**Theorem 3.12.** Program extraction for proofs based on an instance of $\mathsf{BW}^\omega_\mathbb{R}$.
*For* $n \geq 1$*, given a proof using* $\mathsf{BW}^\omega_\mathbb{R}$ *on a known sequence* $sx$ *of elements in* $\mathbb{P}$ *specified by a closed term* $s$ *of* $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$*, i.e.* $s \in (\mathbb{P}^\mathbb{N})^\mathbb{N}$ *defines a sequence of such sequences (where the bounds* $(k_d)$ *used in forming* $\mathbb{P}$ *are also given by a term* $rx$ *that may depend on* $x$*), and a quantifier-free formula* $\varphi_{qf}(x,y)$ *containing only* $x,y$ *as free variables we have: from a proof*

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction\; + \mathsf{QF\text{-}AC} \,+\, \Sigma^0_n\text{-}\mathsf{IA} \;\vdash\; \forall x^0\big(\mathsf{BW}^\omega_\mathbb{R}(sx) \to \exists y^0 \varphi_{qf}(x,y)\big),$$

*we can extract a function* $f \in \mathcal{T}_n$ *by Sh-interpretation s.t.*

$$\mathcal{S}^\omega \;\models\; \forall x^0 \varphi_{qf}(x, f(x)).$$

**Remark.**

1. *In particular, for $n = 1$ we obtain a $<\omega^{\omega^{\omega}}$- recursive realizer. Moreover, for the special case $n = 0$, the 2nd author showed in [13] that when some weaker systems than* $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\restriction$ + QF-AC *are used, namely* $\mathsf{G}_\infty\mathsf{A}^\omega$ + QF-AC, *one obtains even $<\omega^{\omega}$-recursive realizers. Note that this case is not covered by Theorem 3.12 above.*

2. *Instead of* $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\restriction$ *we may also have the system resulting from this by adding the recursors $R_\rho$ with $deg(\rho) \leq n - 1$.*

**Corollary 3.13.** *Theorem 3.12 also holds with* $\widehat{\mathsf{E\text{-}PA}}^{\omega}\restriction$ *instead of* $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\restriction$ *once we restrict* QF-AC *to the types $(\rho, \tau) := (1, 0)$ and $(\rho, \tau) := (0, 1)$.*

**Proof.** Apply elimination of extensionality to theorem 3.12 (see [22] or [18]). □

**Proof of Theorem 3.12.** By soundness of $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\restriction$ (see e.g. [18] and [30]) we know that one can construct closed terms $t_M$ and $t_Y$ in $T_{n-1}$ realizing the Sh-interpretation of the analyzed proof:

$$\forall x, L, a\, (\mathsf{BW}_{Sh}(a, t_M xLa, L(t_M xLa), sx) \to \varphi_{Sh}(x, t_Y xLa)).$$

Now, using the solution terms $t_A, t_L$ for the Shoenfield interpretation of $\mathsf{BW}^\omega_{\mathbb{R}}$ from theorem 3.6 we define

$$a := t_A(sx, t_M x) \text{ and } L := t_L(sx, t_M x)$$

and obtain that

$$\mathsf{BW}_{Sh}(a, t_M xLa, L(t_M xLa), sx).$$

Hence for

$$tx := t_Y xLa = t_Y x(t_L(sx, t_M x))(t_A(sx, t_M x))$$

we have that $\varphi_{qf}(x, tx)$.
Now let $t^*_M, t^*_Y$ be majorants for $t_M, t_Y$ in $T_{n-1}$. Then –using the majorants $t^*_L, t^*_A$ of $t_L, t_A$ from theorem 3.10 – we define

$$t^* x := t^*_Y x(t^*_L(t^*_M x))(t^*_A(t^*_M x)).$$

Since $x$ s-maj$_0$ $x$ we get that $t^* x \geq tx$ for all $x$. It, therefore, suffices to show that the function denoted by $t^*$ can be defined in $T_n$ : Using Parsons' result from [23] (p. 361) we know that $t^*_M x$ has computational sizes $< \omega_n(\omega)$. This fact allows us to apply Howard's proposition, see [10] (p. 23), from which it follows that $t^*_{L}(t^*_M x)y^0$ has computational size $< \omega_{n+1}(\omega)$ and so (using Parson's result again) can be defined by a term in $T_n$ which finishes the proof as $t^*_A$ even is in $T_0$.

□

The Theorem 3.12 is optimal in the following sense:

**Proposition 3.14.** *Any function $h$ given by a closed term in $\mathcal{T}_n$ can be proven to be total in* $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\restriction$ + QF-AC + $\Sigma^0_n$-IA *using a concrete instance of* $\mathsf{BW}(s)$ *for a suitable closed term $s^1$ in $\mathcal{T}_0$.*

**Proof.**
In [17] (Proposition 5.5), the second author gave a construction of a functional $F^2$ such that (relative to $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\restriction$) $\forall f^{1(0)}(\mathsf{PCM}(F(f)) \to \Pi^0_1\text{-}\widehat{\mathsf{CA}}(f))$, where $\mathsf{PCM}(f^{1(0)})$ is the principle of monotone convergence defined as follows:

$$\forall n\big(0 \leq_{\mathbb{R}} f(n+1) \leq_{\mathbb{R}} f(n)\big) \to \exists g \forall k \forall m_1, m_2 \geq gk \left(|fm_1 -_{\mathbb{R}} fm_2| \leq \frac{1}{k+1}\right).$$

Moreover, in the presence of QF-AC we also have $\forall f^{1(0)}(\mathsf{BW}_{\mathbb{R}}(G(f)) \rightarrow \mathsf{PCM}(f))$ for a suitable functional $G$. Hence there are functionals $F_\exists$, and $F_\forall$ such that (relative to $\widehat{\text{WE-PA}}^{\omega}\!\restriction\ + \text{QF-AC}$) for any function $f^{1(0)}$ the instance $\mathsf{BW}(F_\exists f)$, resp. $\mathsf{BW}(F_\forall f)$, of BW implies the instance $\Sigma^0_1$-$\mathsf{CA}(f)$ of $\Sigma^0_1$-CA, resp. $\Pi^0_1$-$\mathsf{CA}(f)$ of $\Pi^0_1$-CA.
By Parsons's results in [24] we know that $\widehat{\text{WE-PA}}^{\omega}\!\restriction\ + \text{QF-AC}$ proves the $\Pi^0_2$ sentence stating the totality of $h$ using additionally just finitely many, say $m$, instances of $\Sigma^0_{n+1}$-IA with number parameters only.
Assume that $n$ is odd. Each of these $m$ instances can be reformulated as an instance of $\Sigma^0_n$-IA with number and function parameters plus the instances $\forall \underline{a}^0 \Pi^0_1$-$\mathsf{CA}(t'_i(\underline{a}))$ of $\Pi^0_1$-comprehension, where the term $t'_i \in \mathcal{T}_0$ ($i \in \{1 \ldots m\}$) corresponds to the quantifier-free matrix of the induction formula in the $i^{th}$ instance of $\Sigma^0_{n+1}$-IA (i.e. $t'_i$ essentially is the characteristic term of that matrix). Furthermore, for suitable closed terms $t_i \in \mathcal{T}_0$ each formula $\forall \underline{a}^0 \Pi^0_1$-$\mathsf{CA}(t'_i(\underline{a}))$ is equivalent to an instance $\Pi^0_1$-$\mathsf{CA}(t_i)$ of $\Pi^0_1$-CA. Moreover, $\bigwedge_{i=0}^{m} \Pi^0_1$-$\mathsf{CA}(t_i)$ is equivalent to $\Pi^0_1$-$\mathsf{CA}(t)$ for a suitable $t \in \mathcal{T}_0$(see [14] for this).
Analogously, if $n$ is even we obtain the equivalence to $\Sigma^0_1$-$\mathsf{CA}(t)$.
Finally, $\Pi^0_1$-$\mathsf{CA}(t)$ and $\Sigma^0_1$-$\mathsf{CA}(t)$ are derivable from the instances $\mathsf{BW}(F_\forall t)$ resp. $\mathsf{BW}(F_\exists t)$ of BW.

□

Theorem 3.12 should be contrasted to the case of proofs based on the full (2nd order closure – also w.r.t. the bounding sequence $(k_d)$ from $\mathbb{P}$ – of the) Bolzano-Weierstraß principle BW where the following (equally optimal) result follows from the literature:

**Theorem 3.15.** *Given a proof using* $\mathsf{BW}^{\omega}_{\mathbb{R}}$ *on any given sequence in* $\mathbb{P}$*: Let* $\varphi_{qf}(x, y)$ *be a quantifier-free formula containing only the free variables* $x, y$*. Then from a proof*

$$\widehat{\text{WE-PA}}^{\omega}\!\restriction\ + \text{QF-AC} + \Sigma^0_\infty\text{-IA} + \mathsf{BW}^{\omega}_{\mathbb{R}} \ \vdash\ \forall x^1 \exists y^0 \varphi_{qf}(x, y)$$

*we can extract by Sh-interpretation a closed term* $t^1 \in \mathcal{T}$ *s.t.*

$$\mathcal{S}^{\omega} \ \models\ \forall x^0 \varphi_{qf}(x, t(x)).$$

**Proof.** Over $\widehat{\text{WE-PA}}^{\omega}\!\restriction\ + \text{QF-AC}$ the schema of arithmetical comprehension $\mathsf{CA}^0_{ar}$ clearly implies both $\mathsf{BW}^{\omega}_{\mathbb{R}}$ as well as $\Sigma^0_\infty$-IA. The system $\widehat{\text{WE-PA}}^{\omega}\!\restriction\ + \text{QF-AC} + \mathsf{CA}^0_{ar}$, however, has a *Sh*-interpretation by terms in $\mathcal{T}_0 + \mathsf{B}_{0,1}$ (see e.g. [18], Theorem 11.14). Hence we get a term $t \in \mathcal{T}_0 + \mathsf{B}_{0,1}$ satisfying $\forall x^1 \varphi_{qf}(x, t(x))$. Finally, using Corollary 4.4.1 from [16], we can conclude that $t$ can be rewritten as a functional in $\mathcal{T}$.

□

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