# A note on the $\Pi^0_2$–induction rule*

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**Abstract**

It is well–known (due to C. Parsons) that the extension of primitive recursive arithmetic PRA by first–order predicate logic and the rule of $\Pi^0_2$–induction $\Pi^0_2$–IR is $\Pi^0_2$–conservative over PRA. We show that this is no longer true in the presence of function quantifiers and quantifier–free choice for numbers $\mathrm{AC}^{0,0}$–qf. More precisely we show that $\mathcal{T} :=\mathrm{PRA}^2 + \Pi^0_2\text{–IR}+\mathrm{AC}^{0,0}\text{–qf}$ proves the totality of the Ackermann function, where $\mathrm{PRA}^2$ is the extension of PRA by number and function quantifiers and $\Pi^0_2$–IR may contain function parameters. This is true even for $\mathrm{PRA}^2 + \Sigma^0_1\text{–IR}+\Pi^0_2\text{–IR}^- +\mathrm{AC}^{0,0}\text{–qf}$, where $\Pi^0_2\text{–IR}^-$ is the restriction of $\Pi^0_2$–IR without function parameters.

## 1

Let **(PRA)** denote the extension of primitive recursive arithmetic obtained by adding first–order predicate logic.
By the **rule IR** of **induction** we mean

$$\mathbf{IR}\ :\ \frac{A(0)\ ,\ \forall x(A(x)\rightarrow A(x'))}{\forall x\, A(x)},$$

where $x'$ denotes the successor of $x$.
The restriction of **IR** to $\Sigma^0_1$–formulas $\exists v A_0(x,v)$ (resp. to $\Pi^0_2$–formulas $\forall u \exists v A_0(x,u,v)$) is denoted by $\Sigma^0_1$–**IR** (resp. $\Pi^0_2$–**IR**)[1].

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[1] $A_0$ always denotes a quantifier–free formula.

It is well–known that **(PRA)** is closed under $\Pi^0_2$–**IR**. This was proved first by Parsons [7],[8]. Other proofs can be found in [6] and [9].
As a consequence of this fact one has the following rule

$$(1)\quad \begin{cases} \text{If } \mathbf{(PRA)}{+}\Pi^0_2\text{–}\mathbf{IR} \vdash \forall x \exists y\, A_0(x,y), \text{ then there} \\ \text{exists a primitive recursive function } f \text{ such that} \\ \mathbf{(PRA)} \vdash \forall x\, A_0(x, fx), \end{cases}$$

where $A_0(x,y)$ contains only $x, y$ as free variables.

Let $\mathbf{(PRA^2)}$ be the extension of **(PRA)** obtained by the addition of function variables and quantifiers for functions with the usual axioms and rules as well as the schema for explicit definition of functions

$$(ED)\ :\ \exists f \forall \underline{x}(f\underline{x} = t[\,\underline{x}\,]),$$

where $t[\,\underline{x}\,]$ is a term of $\mathbf{(PRA^2)}$ and $\underline{x}$ a tuple of number variables which may occur in $t$ (see e.g. [9]). (For convenience we include variables and quantifiers for $n$-ary number-theoretic functions for each $n$. This theory is a definitional extension of the corresponding theory with unary function variables and quantifiers only, since coding of tuples is possible in **(PRA)**.)
Furthermore the schema **QF–IA** of quantifier–free induction

$$\mathbf{QF\text{–}IA}\ :\ A_0(0) \wedge \forall x(A_0(x) \to A_0(x')) \to \forall x\, A_0(x)$$

is now applicable also to formulas $A_0$ which **contain function variables**. Finally $\mathbf{(PRA^2)}$ has a bounded $\mu$–operator $\mu_b$ with the axioms

$$(\mu_b) : \begin{cases} y \le x \wedge fxy = 0 \to fx(\mu_b fx) = 0, \\ y < \mu_b fx \to fxy \ne 0, \\ \mu_b fx = 0 \vee (fx(\mu_b fx) = 0 \wedge \mu_b fx \le x) \end{cases}$$

(These axioms express that $\mu_b fx = \min y \le x(fxy = 0)$ if such an $y \le x$ exists and $= 0$ otherwise).

The axiom schema $\mathbf{AC^{0,0}}$–**qf** of quantifier–free choice for functions is given by

$$\mathbf{AC^{0,0}\text{–}qf}\ :\ \forall x \exists y\, A_0(x,y) \to \exists f \forall x\, A_0(x, fx).$$

In **(PRA$^2$)**+$\Sigma^0_1$**-IR**, **(PRA$^2$)**+$\Pi^0_2$**-IR** the rules $\Sigma^0_1$–**IR,** $\Pi^0_2$–**IR** are always understood w.r.t. the language of **(PRA$^2$)**, i.e. instances of $\Sigma^0_1$–**IR,** $\Pi^0_2$–**IR** may contain function variables. If these rules are restricted to instances **without** function variables we write $\Sigma^0_1$–**IR$^-$** , $\Pi^0_2$–**IR$^-$**.

In this note we show that (1) is wrong for **(PRA$^2$)**+$\Pi^0_2$–**IR**+**AC$^{0,0}$–qf** and even for **(PRA$^2$)**+$\Sigma^0_1$–**IR**+$\Pi^0_2$–**IR$^-$**+**AC$^{0,0}$–qf**. In fact both theories prove the totality of the Ackermann function. This shows that the strength of $\Pi^0_2$–**IR** (w.r.t. the growth of provably recursive functions) increases in the presence of the additional analytical principle **AC$^{0,0}$–qf if** $\Pi^0_2$ is understood w.r.t. the extended language with function variables.

Our result provides quite a limit for generalizations of (1) from **(PRA)**+$\Pi^0_2$–**IR** to systems suited for the formalization of fragments of analysis.

This is in sharp contrast to the conservativity of the **axiom of** $\Sigma^0_1$**–induction** $\Sigma^0_1$–**IA**.

The following fact is well–known (see e.g. [2]):

$$(2)\quad \begin{cases} \text{If } \mathbf{(PRA^2)}+\Sigma^0_1\text{–}\mathbf{IA}+\ \mathbf{AC^{0,0}\text{–}qf} \vdash \forall x \exists y\, A_0(x,y), \text{ then there} \\ \text{exists a primitive recursive function } f \text{ such that} \\ \mathbf{(PRA)} \vdash \forall x\, A_0(x, fx), \end{cases}$$

where $A_0(x, y)$ contains only $x, y$ as free variables.

This fact (which also holds for the higher type extension **(PRA$^\omega$)** of **(PRA$^2$)** and full **AC–qf**) can be proved using functional interpretation and subsequent normalization of the resulting term. Here **(PRA$^\omega$)** denotes the theory $\widehat{\text{PA}}^\omega\!\upharpoonright$ from [1]. Indeed one may add also the binary König's lemma **WKL** (For **(PRA$^\omega$)** this was shown first in [4] (see also [3]) with various generalizations. For the special second–order case and **AC$^{0,0}$–qf** instead of full **AC–qf** a proof was given already in [9] ).

We now come to the main result of this note:

**Proposition 1**
**(PRA$^2$)**+$\Sigma^0_1$–**IR**+$\Pi^0_2$–**IR$^-$**+**AC$^{0,0}$–qf** *proves the totality of the Ackermann function.*

***Proof:*** **(PRA**$^2$**)**+$\Sigma^0_1$–**IR** proves (by induction on $x$ with the parameters $f, y_0$):

$$(1)\ \forall x \exists y((y)_0 = y_0 \wedge \forall i < x((y)_{i+1} = f(i, (y)_i)))$$

(Note that the formula $\forall i < x(\ldots)$ can be expressed in a quantifier–free way using $\mu_b$).
**AC**$^{0,0}$**–qf** applied to (1) yields

$$(2)\ \forall f, y_0 \exists g \forall x((gx)_0 = y_0 \wedge \forall i < x((gx)_{i+1} = f(i, (gx)_i))).$$

For $hxi := (gx)_i$ this implies

$$(3)\ hx0 = y_0 \wedge \forall i < x(hxi' = f(i, hxi)).$$

By **QF–IA** applied to $i$ with $x, \tilde{x}, h$ as parameters, one easily proves (for $h$ satisfying (3))

$$(4)\ \forall x, \tilde{x} \forall i \leq \min(x, \tilde{x})(hxi = h\tilde{x}i).$$

Define $\tilde{h}x := hx'x$. Then (3) and (4) yield

$$(5)\ \tilde{h}0 = y_0 \wedge \forall x(\tilde{h}x' = f(x, \tilde{h}x)).$$

Let

$$A :\equiv \exists y_0 A_0(0, y_0) \wedge \forall x(\exists y_1 A_0(x, y_1) \rightarrow \exists y_2 A_0(x', y_2)) \rightarrow \forall x \exists y\, A_0(x, y)$$

be an arbitrary instance of $\Sigma^0_1$–**IA**.
**(PRA**$^2$)+**AC**$^{0,0}$**–qf** proves that $A$ is equivalent to

$$\exists y_0 A_0(0, y_0) \wedge \exists f \forall x, y_1(A_0(x, y_1) \rightarrow A_0(x', fxy_1)) \rightarrow \forall x \exists y\, A_0(x, y).$$

Assume

$$\exists y_0 A_0(0, y_0) \wedge \exists f \forall x, y_1(A_0(x, y_1) \rightarrow A_0(x', fxy_1)).$$

To $y_0, f$ choose $\tilde{h}$ such that (5) is satisfied.
Using **QF–IA** one easily shows that

$$\forall x\ A_0(x, \tilde{h}x)$$

and therefore

$$\forall x \exists y\ A_0(x, y).$$

Thus **(PRA**$^2$**)**+$\Sigma^0_1$–**IR**+**AC**$^{0,0}$**–qf** proves every instance of $\Sigma^0_1$–**IA**. However it is well–known that already the first–order fragment of **(PRA**$^2$)+$\Sigma^0_1$–**IA**+$\Pi^0_2$–**IR** and so a fortiori **(PRA**$^2$) + $\Sigma^0_1$–**IA**+$\Pi^0_2$–**IR**$^-$ proves the totality of the Ackermann function (see [7] ).

**Corollary 2**
**(PRA$^2$)**+$\Pi^0_2$–**IR**+**AC$^{0,0}$–qf** *proves the totality of the Ackermann function and therefore is not* $\Pi^0_2$*–conservative over* **(PRA$^2$)**.

Cor.2 taken together with the rule (2) from the introduction shows that in some sense **(PRA)**+$\Pi^0_2$–**IR** is closer to systems proving the totality of the Ackermann function than **(PRA)**+$\Sigma^0_1$–**IA** is. This is not apparent from Parson's first–order result (used in the proof of our proposition above) that **(PRA)**+$\Pi^0_2$–**IR**+$\Sigma^0_1$–**IA** proves the totality of the Ackermann function (whereas both **(PRA)**+$\Pi^0_2$–**IR** and **(PRA)**+$\Sigma^0_1$–**IA** do not), since this result is completely symmetrical w.r.t. $\Pi^0_2$–**IR** and $\Sigma^0_1$–**IA**.

As the proof of the proposition shows, the reason for this phenomenon is that in the presence of function variables, **AC$^{0,0}$–qf** allows to derive $\Sigma^0_1$–**IA** from $\Pi^0_2$–**IR** ( and even from $\Sigma^0_1$–**IR**) but $\Sigma^0_1$–**IA** plus **AC$^{0,0}$–qf** does not lead to a system which is closed under $\Pi^0_2$–**IR** (not even under $\Pi^0_2$–**IR**$^-$).

The derivability of $\Sigma^0_1$–**IA** in **(PRA$^2$)**+$\Sigma^0_1$–**IR**+**AC$^{0,0}$–qf** shown in the proof of proposition 1 rests on three facts

(i) **(PRA$^2$)**+$\Sigma^0_1$–**IR** suffices to introduce the $\Pi^0_2$–form of the iteration of a function $f$

(ii) From (i) one derives – using **AC$^{0,0}$–qf** – the existence of a function $\tilde{h}$ which is the iteration of $f$

(iii) Using such an iteration function one can prove the $\Pi^0_2$–Herbrand normal form of $\Sigma^0_1$–**IA** which – again by **AC$^{0,0}$–qf** – implies $\Sigma^0_1$–**IA**.

The presence of function variables in $\Sigma^0_1$–**IR** is used only to define the iteration $\tilde{h}$ of $f$, i.e. $\forall x\,(\tilde{h}0 = y_0 \wedge \tilde{h}x' = f(x, \tilde{h}x))$, **uniformly in the function parameter** $f$.

Hence if we add an iteration functional $\Phi_{it}$ to **(PRA$^2$)** together with the axioms

$$(\Phi_{it})\ :\ \begin{cases} \Phi_{it}fy0 = y \\ \Phi_{it}fyx' = f(x, \Phi_{it}fyx), \end{cases}$$

we obtain

**Proposition 3** $(\mathbf{PRA}^2)+(\Phi_{it})+\Pi^0_2$–$\mathbf{IR}^-$+$\mathbf{AC}^{0,0}$–$\mathbf{qf}$ *proves the totality of the Ackermann function and therefore is not* $\Pi^0_2$*–conservative over* $(\mathbf{PRA}^2)$.

As we have seen in this note the addition of $\mathbf{AC}^{0,0}$–$\mathbf{qf}$ to $\Pi^0_2$–$\mathbf{IR}$ in the second–order context of $(\mathbf{PRA}^2)$ destroys the $\Pi^0_2$–conservativity over $\mathbf{PRA}$.

This also happens already for $\Pi^0_2$–$\mathbf{IR}^-$ if the comprehension schema

$$\Pi^0_1\text{–}\mathbf{CA}^- \ : \ \exists f \forall x (fx = 0 \leftrightarrow A(x)),$$

where $A \in \Pi^0_1$ contains no function parameters, is added:

By applying $\mathbf{QF}$–$\mathbf{IA}$ to the comprehension functions, $(\mathbf{PRA}^2)+\Pi^0_1$–$\mathbf{CA}^-$ proves every function parameter free instance of $\Sigma^0_1$–$\mathbf{IA}$ (i.e. $\Sigma^0_1$–$\mathbf{IA}^-$) which together with $\Pi^0_2$–$\mathbf{IR}^-$ yields (using Parson's first–order result mentioned above) the totality of the Ackermann function. Hence

**Proposition 4** $(\mathbf{PRA}^2)+\Pi^0_1$–$\mathbf{CA}^-+\Pi^0_2$–$\mathbf{IR}^-$ *proves the totality of the Ackermann function and therefore is not* $\Pi^0_2$*–conservative over* $\mathbf{PRA}$.

This proposition refutes a theorem stated in [6] as well as cor. 5.9 (and its generalizations thm.5.8, 5.13 and 5.14) stated in [9] where this conservativity is claimed for a certain theory $\mathbf{BT} \supset (\mathbf{PRA}^2)$ even when $\mathbf{WKL}$ is added.

**Remark:** The proof of prop.3 essentially uses the fact that function parameters are allowed to occur in the schema $\mathbf{QF}$–$\mathbf{IA}$ of $(\mathbf{PRA}^2)$ (The proof does not use $\mu_b$). However to forbid the occurrence of function parameters in this schema does not help to repair the claims in [6],[9] mentioned above: If one has a coding functional $\Phi f x := \overline{f}x$ together with the axiom

$$\forall f, x, y (y < x \to (\overline{f}x)_y = fy),$$

then still the totality of the Ackermann function can be shown to be provable. But such a coding (used explicitly in [6],[9]) is necessary in order to deal with $\mathbf{WKL}$ (as formulated in [6], [9]).

On the other hand one can prove by methods different from those used in [6] and [9] that the provably recursive functions of $(\mathbf{PRA}^2)+\Pi^0_1$–$\mathbf{CA}^-$ are primitive recursive. Indeed one can show much more general results for extensions of $(\mathbf{PRA}^2)$ to finite types (which however must not contain $\Phi_{it}$) with $\Delta^0_2$–$\mathbf{IA}^-$, $\mathbf{WKL}$, full $\mathbf{AC}$–$\mathbf{qf}$ and many other analytical principles added and for sentences involving higher types (instead of $\Pi^0_2$–sentences). These results and a discussion of the reasons for the failure of the methods used in [6] and [9] (which – at least as they stand there – can not be used to yield our positive results) are developed in chapters 11,12 of [5] and will be published in a paper under preparation.

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