Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation*

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Abstract

We consider uniqueness theorems in classical analysis having the form

$$(+)\ \forall u \in U, v_1, v_2 \in V_u \big(G(u, v_1) = 0 = G(u, v_2) \to v_1 = v_2\big),$$

where U, V are complete separable metric spaces, V_u is compact in V and $G : U \times V \to \mathbb{R}$ is a constructive function.
If $(+)$ is proved by arithmetical means from analytical assumptions

$$(++)\ \forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x, y, z) = 0\big)$$

only (where X, Y, Z are complete separable metric spaces, $Y_x \subset Y$ is compact and $F : X \times Y \times Z \to \mathbb{R}$ constructive), then we can extract from the proof of $(++) \to (+)$ an effective modulus of uniqueness, i.e.

$$(+++)\ \forall u \in U, v_1, v_2 \in V_u, k \in \mathbb{N}\big(|G(u, v_1)|, |G(u, v_2)| \le 2^{-\Phi uk} \to d_V(v_1, v_2) \le 2^{-k}\big).$$

Such a modulus Φ can e.g. be used to give a finite algorithm which computes the (uniquely determined) zero of $G(u, \cdot)$ on V_u with prescribed precision if it exists classically.
The extraction of Φ uses a proof–theoretic combination of functional interpretation and pointwise majorization. If the proof of $(++) \to (+)$ uses only simple instances of induction, then Φ is a simple mathematical operation (on a convenient standard representation of X, e.g. on f together with a modulus of uniform continuity for $X = C[0, 1]$).
Various uniqueness theorems in best approximation theory have the form $(+)$ and are proved using only analytical tools of the form $(++)$. We analyse the most common proof of uniqueness for the best Chebycheff approximation of $f \in C[0, 1]$ by polynomials of degree $\le n$ given by de La Vallée Poussin and obtain explicit moduli of uniqueness and uniform constants of strong unicity.
In a subsequent paper two further proofs of this uniqueness will be analysed yielding better estimates (due to the fact that mainly $(++)$–lemmas are used) which allow us to improve results obtained prior by D. Bridges significantly.

1 Introduction and basic notions

Various theorems of classical analysis have the form

$$(1)\ \forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x, y, z) = 0\big),$$

where X, Y, Z are constructively definable complete separable metric spaces (abbreviation: CSM–spaces), (Y_x) is a family of **compact** sets in Y and $F : X \times Y \times Z \to \mathbb{R}$ is an extensional constructive (and therefore continuous) function.

In this paper we consider the following situation:

*The results of this paper form the main part of chapter 8 and the first section of chapter 9 of my dissertation Kohlenbach (90). I am grateful to Prof. H. Luckhardt for many stimulating discussions and helpful suggestions.

Let PA^ω denote the system of Peano arithmetic formulated in the language of all finite types and AC–qf the axiom of choice schema for quantifier–free formulas. Assume that

$$(2)\ PA^\omega + AC\text{–qf} \vdash \forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x,y,z)=0\big) \to \forall u \in U, v \in V_u \exists k \in \mathbb{N} A_1,$$

where $A_1(u,v,k) \in \Sigma^0_1; U, V$ are CSM–spaces and V_u is **compact** in V (X, Y, Z etc. are given in a convenient standard representation, see 3).
Using a proof–theoretic combination of functional interpretation and pointwise majorization, which is developed in Kohlenbach (A)[1], we extract from any given proof of (2) a functional Φ which is primitive recursive in the sense of the Hilbert (26)/Gödel (58) calculus T of functionals of finite type and provides a bound for k which depends neither on the premise nor on v, but only on u:

$$(3)\ HA^\omega \vdash \forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x,y,z)=0\big) \to \forall u \in U, v \in V_u \bigvee_{k=0}^{\Phi u} A_1(u,v,k).$$

Let us assume now for simplicity that $Z = \emptyset$ ($F(x,y,z) =_{\mathbb{R}} 0$ is already a $\forall$–sentence so that this restriction is not very essential).
In a further step we extract a primitive recursive functional Ψ, which reduces the assumption $\forall x \in X \exists y \in Y_x\big(F(x,y)=0\big)$ to its "ε–version" $\forall x \in X, n \in \mathbb{N} \exists y \in Y_x\big(|F(x,y)| \le 2^{-n}\big)$:

$$(4)\ HA^\omega \vdash \forall u \in U\big(\forall x \in X \exists y \in Y_x\big(|F(x,y)| \le 2^{-\Psi u}\big) \to \forall v \in V_u \bigvee_{k=0}^{\Phi u} A_1(u,v,k)\big).$$

In our applications, these ε–versions of (1) are always provable in HA^ω (while the assumptions (1) are usually unprovable in $PA^\omega + AC$–qf). Hence

$$(5)\ HA^\omega \vdash \forall u \in U, v \in V_u \bigvee_{k=0}^{\Phi u} A_1(u,v,k).$$

Thus starting from an ineffective proof of $\forall u \in U, v \in V_u \exists k \in \mathbb{N} A_1(u,v,k)$ we have obtained an effective bound Φ for k (independent of v) together with a constructive verification. The bound Φ is extracted from the classical proof and its verification in (3) may use the assumption (1) while additional work is needed to extract Ψ, which makes possible a totally constructive verification of Φ via (4). This contrasts our treatment with constructive analysis (see e.g. Bishop (67)), which considers **only** constructive proofs.
We remark that our method also works when the assumptions (1) depend on $u \in U$ and thus applies to the more general situation

$$(2)^*\ PA^\omega + AC\text{–qf} \vdash \forall u \in U\big(\forall x \in X \exists y \in Y_{x,u} \forall z \in Z\big(F(u,x,y,z)=0\big) \to \forall v \in V_u \exists k A_1\big)$$

instead of (2).

Examples for (1) are

(α) The intermediate value theorem for $f \in C[0,1]$.

(β) The attainment of the maximum $\forall f \in C[0,1] \exists x_0 \in [0,1]\big(f(x_0) = \sup_{x \in [0,1]} f(x)\big)$.

[1] This is related to the Herbrand–analysis developed in Luckhardt (89) and applied there for number theory. Our method is specially suited for proofs involving higher types which occur in analysis.

(γ) The existence theorem for a best approximation:
$\forall f \in C[0,1] \exists g \in E\big(\|f-g\|_\infty = \text{dist}(f,E)\big)$, where
$E = \{\alpha_1\phi_1 + \ldots + \alpha_n\phi_n : (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n\}$ for $\phi_1,\ldots,\phi_n \in C[0,1]$ linearly independent (Note that $\|f-g\|_\infty = \text{dist}(f,E) \to \|g\|_\infty \leq 2\|f\|_\infty$ since $0 \in E$. Hence $\exists g \in E$ can be restricted to $\exists g \in E_f := \{\tilde{g} \in E : \|\tilde{g}\|_\infty \leq 2\|f\|_\infty\}$, where $E_f$is compact).
More specifically: The existence of an extremal alternant in the case of best approximation by polynomials $p \in P_n$ of degree $\leq n$:

$$(*) \begin{cases} \forall f \in C[0,1] \exists p_b \in P_n, (x_1,\ldots,x_{n+2}) \in [0,1]^{n+2}, j \in \{0,1\} \\ \Big(\|f-p_b\|_\infty = \text{dist}(f,P_n) \wedge \bigwedge\limits_{i=1}^{n+1}(x_{i+1} \geq x_i) \wedge \bigwedge\limits_{i=1}^{n+2}(-1)^{i+j}\big(p_b(x_i)-f(x_i)\big) = \text{dist}(f,P_n)\Big). \end{cases}$$

(δ) The existence theorem of Cauchy–Peano on the solvability of ordinary differential equations.

The mathematical value of the proof–theoretical result sketched above depends on the following questions:

(i) Are there interesting classical theorems which can be brought into the form (2) or (2)*, where

(ii) the bound Φ is of mathematical interest but cannot be obtained directly from the given proof?

(iii) Can the extraction of Φ be carried out from the intuitive proof with reasonable effort (where the verification need not to be constructive and the extraction may follow the mathematical intuition and not step by step the proof–theoretical procedure)?

(iv) Has this bound Φ mathematical properties which allow real mathematical applications (in particular: is it numerically at least as good as known Φ's or even better, instead of being just primitive recursive)?

The results of this paper (and a subsequent one) give positive answers to these questions:
In paragraph 4,5 of this paper we show that a large class of **uniqueness proofs** in classical analysis has (relative to $PA^\omega + AC$–qf) the form (2)*. This holds in particular for best approximation theory. Here the bound Φ yields an a–priori–**modulus of uniqueness**, which e.g. can be used for the computation of best approximations with prescribed precision.
Let $G : U \times V \to \mathbb{R}$ be an extensional constructive function given by a primitive recursive functional in the sense of T and $X, Y, Y_{x,u}, F, U, V, V_u$ as above. We consider uniqueness theorems having the form:

$$(6)\ PA^\omega + AC\text{–qf} \vdash \forall u \in U\Big(\forall x \in X \exists y \in Y_{x,u} F(u,x,y) = 0 \to \underbrace{\forall v_1, v_2 \in V_u\big(G(u,v_1) = 0 = G(u,v_2) \to v_1 =_V v_2\big)}_{i.e. G(u,\cdot) \text{ has at most one zero in } V_u.}\Big)$$

(6) can be easily transformed to

$$(7)\ PA^\omega + AC\text{–qf} \vdash \forall u \in U\Big(\forall x \in X \exists y \in Y_{x,u} F(u,x,y) = 0 \to \forall n \in \mathbb{N}, v_1, v_2 \in V_u \exists k \in \mathbb{N} \big(|G(u,v_1)|, |G(u,v_2)| \leq 2^{-k} \to d_V(v_1,v_2) < 2^{-n}\big)\Big),$$

where d_V denotes the metric on V.
According to (3) we can extract a primitive recursive operation Φ which gives a bound for k (and hence realizes $\exists k$ since the proposition is monotonic in k):

(8) $HA^\omega \vdash \forall u \in U\Big(\forall x \in X \exists y \in Y_{x,u} F(u,x,y) = 0 \to \forall n \in \mathbb{N}, v_1, v_2 \in V_u$

$$\big(|G(u,v_1)|, |G(u,v_2)| \le 2^{-\Phi un} \to d_V(v_1,v_2) < 2^{-n}\big)\Big).$$

We call Φ a modulus of uniqueness for v with $G(u,v) = 0$ (uniform in u). Φ does not depend on v_1, v_2. Using Φ one can construct an algorithm $\Psi \in T$, which computes $v \in V_u$ such that $G(u,v) = 0$ with prescribed precision (if $v \in V_u$ exists):
Assume (+) $\forall u \in U \exists v \in V_u\big(G(u,v) = 0\big)$ and $\forall u \in U, x \in X \exists y \in Y_{x,u}(F = 0)$. Since by assumption G is given by a term in Gödel's T, there exists an algorithm $\chi \in T$ which computes an ε–zero of $G(u,\cdot)$ in V_u (at least by searching through an ε–net of V_u):

(9) $\forall u \in U, k \in \mathbb{N}\big(|G(u,\chi uk)| \le 2^{-k} \wedge \chi uk \in V_u\big)$.

By (8) and (9) $(\Psi un)_{n\in\mathbb{N}}$ with $\Psi un := \chi u(\Phi un)$ is a Cauchy sequence in V_u with modulus of convergence 2^{-n} and $v \in V_u$ as limit such that $G(u,v) = 0$. Hence together with χ, Φ also $\tilde{\Psi} := \lambda u. \lim_{n\to\infty} \Psi un$ is primitive recursive in the sense of Gödel's T and
$\forall u \in U\big(G(u,\tilde{\Psi}u) = 0 \wedge \tilde{\Psi}u \in V_u\big)$.
The argument above uses the existence of $v \in V_u$ such that $G(u,v) = 0$. By a refinement of this reasoning we can verify $\tilde{\Psi}$ in HA^ω under the implicative assumption of the existence of ε–zeroes of $G(u,\cdot)$ (and of ε–zeroes of $F(u,x,\cdot)$) for arbitrary small $\varepsilon > 0$:
(10) $HA^\omega \vdash \forall u \in U\Big(\forall x \in X, m \in \mathbb{N} \exists y \in Y_{x,u}\big(|F(u,x,y)| \le 2^{-m}\big) \wedge$

$$\forall k \in \mathbb{N} \exists v \in V_u\big(|G(u,v)| \le 2^{-k}\big) \to G(u,\tilde{\Psi}u) = 0 \wedge \tilde{\Psi}u \in V_u\Big).$$

In our applications to approximation theory, the ε–premises of (10) can always be proved in HA^ω. Hence

(11) $HA^\omega \vdash \forall u \in U\big(G(u,\tilde{\Psi}u) = 0 \wedge \tilde{\Psi}u \in V_u\big) \wedge \tilde{\Psi} \in C(U,V)$.

All the results above hold analogously for the systems $\widehat{PA}^\omega\!\restriction, \widehat{HA}^\omega\!\restriction$ with restricted induction and elementary recursor constants only instead of PA^ω, HA^ω. Then $\Phi, \tilde{\Psi}$ are even primitive recursive in the sense of Kleene (52) (provable within $\widehat{HA}^\omega\!\restriction$).

Roughly speaking the results (8)–(11) mean: Proof–theoretical analysis of a uniqueness proof (2)* yields a modulus of uniqueness and thus in turn an algorithm $\tilde{\Psi}$ for true higher type existence sentences (+) having the logical form $\forall\exists_{\le 1}\forall$. The verification of $\tilde{\Psi}$, i.e. the proof for $G(u,\tilde{\Psi}u) = 0$, can be made constructive but need not be.
Classically the uniqueness of v implies (using an inconstructive compactness argument) that $(\chi un)_{n\in\mathbb{N}}$ converges to v and that v depends continuously on u. The additional information provided by Φ is an effective rate of convergence of $(\chi un)_{n\in\mathbb{N}}$ and an effective modulus of (pointwise) continuity for $\tilde{\Psi}$.
From recursive analysis it is known that v is computable in u (in the sense of Kleene) if G, U, V etc. are recursively represented (see Kreinovich (79),(82) and Kreisel (82)).
In contrast to such general results our unwinding yields mathematically relevant data $\Phi, \tilde{\Psi}$ which rely on the natural datas of the arithmetical part of the given uniqueness proof, while proofs of analytical lemmas having the form (1) $\forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x,y,z) = 0\big)$ are not used. If these arithmetical parts of the proof use only simple inductions, then Φ will be a **simple** mathematical operation. In fact this happens.

An important special case of (6) is the situation, where $G(u,\cdot)$ assumes its infimum in exactly one point:

(12) $PA^\omega + AC\text{–qf} \vdash \forall u \in U\big(\forall x \in X \exists y \in Y_{x,u} F(u,x,y) = 0 \rightarrow$

$$\forall v_1, v_2 \in V_u \big(G(u,v_1) = \inf_{v \in V_u} G(u,v) = G(u,v_2) \rightarrow v_1 =_V v_2\big)\big).$$

This holds for many uniqueness theorems in best approximation theory. Here we consider the following examples:

(i) Best Chebycheff approximation by polynomials $p \in P_n$ or – more generally – by elements of arbitrary so–called Haar spaces.

(ii) Best approximation of $f \in [0,1]$ by polynomials $p \in P_n$ w.r.t. the norm $\|f\|_1 := \int_0^1 |f(x)|dx$.

(iii) Best approximation in strictly convex spaces, in particular in uniformly convex spaces $(L_p, 1 < p < \infty)$ (especially (L_2)).

(iv) Best approximation of $f \in C[0,1]$ by polynomials having restricted coefficients.

In (i)–(iv) the uniqueness of the best approximation can be proved in $PA^\omega + \forall f \in C[0,1] \exists x_0 \in [0,1]\big(f(x_0) = \sup_{x \in [0,1]} f(x)\big)$. Hence there exist moduli of uniqueness $\Phi \in T$ and algorithms $\tilde{\Psi} \in T$ with prescribed precision for best approximations in these situations. In particular the existence of a best approximation is provable in HA^ω and thus in every framework of constructive analysis (as Bishop (67), intuitionistic analysis or recursive analysis). For (i) and (iii) (for uniformly convex spaces) this was proved first by Bridges (80),(80A) (in the framework of Bishop (67)). (ii),(iii) (for constructive definable strictly convex spaces) and (iv) were established in Kohlenbach (90). Furthermore for every modulus of uniqueness Φ in (i)–(iv), $\Phi + 1$ is a modulus of pointwise continuity for the projection which maps every element of the space considered onto its best approximation (5.4).

In paragraph 6 we carry out the extraction of Φ for (i) with all numerical details from the most common proof (given by de La Vallée Poussin (19)) of the uniqueness of best polynomial approximation w.r.t. the sup norm $\|\cdot\|_\infty$. If the input data f and n are enriched by an estimate $0 < \delta \leq \text{dist}(f, P_n)$ $(\delta \in \mathbb{Q})$ then we obtain an explicit modulus of uniqueness $(\tilde{\Phi} f n \delta) + k$, which is **linear** in $q \hat{=} 2^{-k}$ (6.16,6.17). The existence of such a constant $\tilde{\Phi} f n \delta$, which is called a constant of strong unicity, was first proved ineffectively by Newman/Shapiro (63) (also for arbitrary Haar spaces). Φ and $\tilde{\Phi}$ depend on f only via an estimate $M \geq \|f\|_\infty$ and a modulus of uniform continuity ω_f for f (i.e. $\forall x, y \in [0,1], k \in \mathbb{N}(|x - y| < 2^{-\omega_f(k)} \rightarrow |f(x) - f(y)| < 2^{-k})$). Hence for compact $K \subset C[0,1]$ we obtain (effectively in a common modulus of continuity for all $f \in K$ and an estimate $M_K \geq \|f\|_\infty$) a uniform modulus of uniqueness Φ_K for all $f \in K$.

Furthermore, if there exists a $\delta_K > 0$ with $\delta_K \leq \inf\limits_{f \in K} \text{dist}(f, P_n)$, this yields (eff. in f, n, δ_K) a uniform constant of strong unicity for K (6.19). Henry/Schmidt (76) gave an ineffective proof of the existence of such uniform constants.

The central part of the Φ–extraction is the analysis of the complicated ineffective proof of the so–called alternation theorem, on which the uniqueness proof is based. This analysis yields a new quantitative version of the alternation theorem (see 6.12,6.13).

In a subsequent paper we will analyse two further proofs for the uniqueness of the best Chebycheff

approximation, which are due to Kirchberger (02)/Borel (05) and Young (07)/Rice (64) resp. These proofs make essential use of the alternation theorem from which the uniqueness property is derived quite easily (although in a somewhat more complicated way than in de La Vallée Poussin's proof). However (slight modifications of) these proofs need an application of the alternation theorem only, which has the form $\forall x \in X \exists y \in Y_x \big(F(x,y) = 0\big)$ (namely $(*)$ of (γ) above), while the alternation theorem itself, as it is used in de La Vallée Poussin's proof, has the form
$\forall x \in X \big(\tilde{F}(x) =_{\mathbb{R}} 0 \to \exists y \in Y_x(F(x,y) =_{\mathbb{R}} 0\big)$, which amounts to
$\forall x \in X \exists z \in \mathbb{N} \exists y \in Y_x(A_0(x,z) \to F(x,y) =_{\mathbb{R}} 0)$ with $\mathbb{N}$ non–compact, since $(\tilde{F}(x) =_{\mathbb{R}} 0) \in \Pi^0_1$ (If we assume already the uniqueness theorem, then the alternation theorem and this application are trivially equivalent). Hence by (8) above we are now able to extract moduli of uniqueness from these uniqueness proofs without considering the proof of the alternation theorem or its application. This simplifies the extraction enormously and the results are numerically significantly better. The best modulus is obtained from (a slight simplification) of Young/Rice's proof and generalizes to arbitrary Haar spaces. In particular it allows us to improve estimates already obtained by Bridges (80A),(82),(82A) (who works entirely within the framework of constructive analysis in the sense of Bishop) substantially by an n (=degree approximating polynomials or more general the dimension of the Haar space) in the exponent.
In summary, the proof–theoretical analysis of these three uniqueness proofs shows that it is numerically relevant for the extraction of data to separate those parts from the proof which derive $\forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x,y,z) =_{\mathbb{R}} 0\big)$–lemmas and eventually to modify the given proof so that certain lemmas are only used in this form.

1.1 Basic notions

$E-PA^\omega$ denotes the classical extensional arithmetic in all finite types (i.e. $(E-HA^\omega)^c$ in the notation of Troelstra (73)), where the set $\tilde{T}$ of all finite types is given by the clauses

$$0 \in \tilde{T} \text{ and } \rho, \tau \in \tilde{T} \Rightarrow \rho(\tau) \in \tilde{T}.$$

Each functional of type $\rho(\tau)$ maps objects of type τ to objects of type ρ. Brackets which are uniquely determined are usually omitted, i.e we write e.g. $0(00)$ for $0(0(0))$.
The subset $\tilde{P} \subset \tilde{T}$ of pure types is given by

$$0 \in \tilde{P} \text{ and } \rho \in \tilde{P} \Rightarrow 0(\rho) \in \tilde{P}.$$

Pure types are often denoted by natural numbers: $0(n) := n+1$. The degree $\deg(\rho)$ is defined by $\deg(0) := 0$, $\deg(\rho\tau) := \max(\deg(\rho), \deg(\tau)+1)$.
If the axiom of extensionality for each type is replaced by a quantifier–free rule of extensionality

$$ER\text{–qf}\ \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]}\ ,$$

where A_0 is quantifier–free, then one obtains the system $WE-PA^\omega$. For the corresponding theories with intuitionistic logic only, we write $E-HA^\omega$ and $WE-HA^\omega$. All these systems $\mathcal{T}$ have the same quantifier–free part qf-$\mathcal{T}$ (in the sense of Troelstra (73), 1.6.13), which we call T. T is an extensional version of the Hilbert (26)/Gödel (58) calculus of primitive recursive functionals of finite type.
The systems above all contain recursor constants R_ρ with the defining axioms

$$\begin{cases} R_\rho 0yz =_\rho y \\ R_\rho(Sx^0)yz =_\rho z(R_\rho xyz)x, \end{cases}$$

where y and z are of type ρ and $\rho 0\rho$ and S denotes the successor function.
For $\rho = 0\rho_k \dots \rho_1$, $s =_\rho t$ is used as an abbreviation for $\forall y_1^{\rho_1}, \dots, y_k^{\rho_k}\big(sy_1 \dots y_k =_0 ty_1 \dots y_k\big)$ (or in the quantifier–free calculus T for $sy_1 \dots y_k =_0 ty_1 \dots y_k$) with different variables $y_1, \dots, y_k$ not occurring in s and t.
If the constants R_ρ are replaced by elementary recursor constants $\hat{R}_\rho$ characterized by

$$\begin{cases} \hat{R}_\rho 0 yz\underline{v} =_0 y\underline{v} \\ \hat{R}_\rho(Sx^0)yz\underline{v} =_0 z(\hat{R}xyz\underline{v})\underline{v}, \text{ where } \underline{v} = v_1^{\rho_1}, \dots, v_k^{\rho_k} \text{ such that } y\underline{v} \text{ is of type } 0, \end{cases}$$

and if the schema of full induction is replaced by the axiom of quantifier–free induction

$$\text{(IA)-qf} \ : \ \forall f^1\Big(f0 = 0 \wedge \forall x\big(fx = 0 \to f(Sx) = 0\big) \longrightarrow \forall x(fx = 0)\Big),$$

then one obtains the restricted systems $E-\widehat{PA}^\omega\!\restriction, \dots$ etc. with quantifier–free part $\widehat{PR}$ due to Feferman (77) (The functionals of $\widehat{PR}$ are essentially the primitive recursive functionals in the sense of Kleene (59)).
Within all the systems considered above, it is possible to construct a term $\lambda x^\tau.t^\rho[x] \in T\ (\widehat{PR})$ for each term $t^\rho \in T\ (\widehat{PR})$ such that $\big(\lambda x.t[x]\big)(t'^\tau) =_\rho t[t']$ (see Troelstra (73), 1.6.8,1.8.4). The language of a theory $\mathcal{T}$ is denoted by $\mathcal{L}(\mathcal{T})$.

For functionals x_1^ρ, x_2^ρ we have the following natural inequality relation:

$$\begin{cases} x_1 \le_0 x_2 :\equiv x_1 \le x_2 \text{ (where ` } \le \text{ ' is primitive recursively defined as usual)} \\ x_1 \le_{\rho\tau} x_2 :\equiv \forall y^\tau(x_1 y \le_\rho x_2 y); \quad x_1 \ge_\rho x_2 :\equiv x_2 \le_\rho x_1. \end{cases}$$

"$\forall x \le_\rho y\ A$", "$\exists x \le_\rho y\ A$", "$\bigwedge\limits_{i=0}^{k} A(i)$", "$\bigvee\limits_{i=0}^{k} A(i)$" are abbreviations for "$\forall x(x \le_\rho y \to A)$", "$\exists x(x \le_\rho y \wedge A)$", "$\forall i \le_0 k\ A(i)$", "$\exists i \le_0 k\ A(i)$".
Furthermore "$\forall x; \tilde{x} \le_\rho sxA$" stands for "$\forall x \forall \tilde{x} \le_\rho sxA$", but "$\forall x, \tilde{x} \le_\rho yA$" stands for "$\forall x \le_\rho y \forall \tilde{x} \le_\rho yA$". $A_0, B_0, C_0, \dots$ denote quantifier–free formulas.

The axiom schema of quantifier–free choice is defined by

$$AC\text{–qf} := \bigcup_{\rho,\tau \in \tilde{T}} \{(AC)^{\rho,\tau}\text{–qf})\}, \text{ where}$$

$$(AC)^{\rho,\tau}\text{–qf} \ : \ \forall x^\rho \exists y^\tau A_0(x,y) \to \exists Y^{\tau\rho} \forall x^\rho A_0(x, Yx).$$

By the principle of bounded choice we mean the schema

$$\text{(b–}AC)^{\rho,\tau} \ : \ \forall Z^{\tau\rho}\Big(\forall x^\rho \exists y \le_\tau Zx\, A(x,y,Z) \to \exists Y \le_{\tau\rho} Z \forall x A(x, Yx, Z)\Big).$$

(b–$AC)^{\rho,\tau}$–qf $\Big($(b–$AC)^{\rho,\tau}$–$\forall$, (b–$AC)^{\rho,\tau}$–$\forall^b\Big)$ is (b–$AC)^{\rho,\tau}$ restricted to quantifier–free formulas (formulas having the form $\forall u^\delta A_0$ resp. $\forall u \le_\rho vA_0$) (For a discussion of b–AC see Kohlenbach (B), 3.11,3.12).

Primitive recursive coding: We use the following coding of finite sequences of objects of type 0 (from Troelstra (73)):
$j(x,y) := 2^x(2y+1) \dot{-} 1,\ j_1 z :=minx \le z[\exists y \le z(2^x(2y+1) = Sz)]$,
$j_2 z :=$min$y \le z[\exists x \le z(2^x(2y+1) = Sz)]$.
$\nu_1(x) := x, \nu_{n+1}(x_0,x_1,\ldots,x_n) := j(x_0,\nu_n(x_1,\ldots,x_n))$,

$$j_1^1(x) := x,\ j_i^n(x) := \begin{cases} j_1 \circ (j_2)^{i-1}(x) \text{ if } 1 \le i < n \\ (j_2)^{n-1}(x) \text{ if } 1 < i = n \end{cases} \quad (\text{if } n > 1).$$

It follows that $j_i^n\big(\nu_n(x_1,\ldots,x_n)\big) = x_i\ (1 \le i \le n),\ \nu_n(j_1^n(x),\ldots,j_n^n(x)) = x$.
$<>:= 0,\ < x_0,\ldots,x_n >:= S\big(\nu_2(n,\nu_{n+1}(x_0,\ldots,x_n))\big)$. As an abbreviation we use $\widehat{x} :=< x >$. One can construct primitive recursive functions $*, lth, \Pi$ such that
$< x_0,\ldots,x_n > * < y_0,\ldots,y_m > =< x_0,\ldots,x_n,y_0,\ldots,y_m >,\ lth(< x_0,\ldots,x_n >) = n+1$.

$$\Pi(n,y) = \begin{cases} x_y \text{ if } y \le m, \\ 0^0 \text{ otherwise} \end{cases} \quad \text{for } n =< x_0,\ldots,x_m > .$$

We usually use the notation $(n)_y$ for $\Pi(n,y)$. For functions (i.e. functionals of type 1) a^1, we define $\overline{a}0 :=<>,\ \overline{a}(Sx) := \overline{a}x* < ax >$. Thus for $x \ne 0$ one has $\overline{a}x =< a0,\ldots,a(x \dot{-} 1) >$. $\overline{a}x$ is primitive recursive in a, x
$lth_1\ a^1 := lth\ a0,\ (a^1)_y := \lambda x^0.(ax)_y$.

The weak König's lemma is defined (following Troelstra (74)) to be

$$WKL\ :\ \forall f^1\big(Tf \wedge \forall x^0 \exists n^0(lth\, n = x \wedge fn = 0) \to \exists b \le_1 \lambda k.1 \forall x^0\big(f(\bar{b}x) = 0\big)\big),$$

where

$$Tf :\equiv \forall n,m\big(f(n*m) = 0 \to fn = 0\big) \wedge \forall n,x\big(f(n* < x >) = 0 \to x \le 1\big).$$

Tf asserts that f represents a 0,1–tree (The designation "weak König's lemma (WKL)" is due to H. Friedman).

2 Effective moduli from uniqueness proofs in logical types and spaces

First we recall some results from Kohlenbach (A) on which our investigation in this paper is based:

2.1 Theorem

1) $WE-PA^\omega + AC$–qf$\vdash \forall u^1; v \le_\gamma tu\Big(\forall x^\delta \exists y \le_\rho suvx \forall z^2 A_0(u,v,x,y,z) \to \exists w^2 B_0(u,v,w)\Big)$
$\Rightarrow \exists$ eff. closed terms $\Phi^{21}, \Psi^{21} \in T$ such that
$WE-HA^\omega$+b–$AC^{\delta,\rho}$–$\forall^b \vdash \forall u; v \le tu\Big(\forall x \exists y \le suvx \forall z \le_2 \Psi u\, A_0 \to \exists w \le_2 \Phi u\, B_0\Big)$.

Φ and Ψ can be extracted from any given proof of the assumption by functional interpretation combined with pointwise majorization (see Kohlenbach (A),2.2). If type/$z > 2$ then it is still possible to extract the bound Φ (b–$AC^{\delta,\rho}$–$\forall^b$ has then to be replaced by b–$AC^{\delta,\rho}$–$\forall$).

The theorem generalizes to tuples $\underline{u}, \underline{v}, \underline{z}, \underline{w}$ of variables (where $\deg(u_i) \leq 1$, $\deg(w_i, z_i) \leq 2$) and (with a corresponding modification of b–AC–$\forall^b$) also for tuples $\underline{x}, \underline{y}$ instead of the single variables u, v, z, w, x, y. Furthermore we may have a finite conjunction of assumptions having the form $\forall x^\delta \exists y \leq_{rho} suvx \forall z^2 A_0$.

2) An analogous result holds for $\widehat{WE-PA}^{\omega}\restriction, \widehat{PR}$ and $\widehat{WE-HA}^{\omega}\restriction$ instead of $WE-PA^\omega, T$ and $WE-HA^\omega$.

(Note that Φ and Ψ depend on u only!)

Proof: See Kohlenbach (A),2.3,2.4 and 2.13.2.

If all variables have types ≤ 1, 2.1 can be strengthened so that the use of AC in the conclusion can be avoided. Furthermore WKL may be added):

2.2 Theorem

Assume that $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$:

1) $E-PA^\omega + WKL + AC^{\alpha,\beta}\text{–qf} \vdash \forall u^1; v \leq_1 tu \big(\forall x^1 \exists y \leq_1 suvx \forall z^1 A_0 \rightarrow \exists w^1 B_0(u,v,w)\big)$
$\Rightarrow \exists \Phi^2, \Psi^2 \in T$ such that
$WE-HA^\omega \vdash \forall u^1; v \leq_1 tu \big(\forall x \exists y \leq suvx \bigwedge_{i=0}^{\Psi u} A_0(u,v,x,y,\lambda m.(i)_m) \rightarrow \bigvee_{k=0}^{\Phi u} B_0(u,v,\lambda m.(k)_m)\big)$.

This also holds if type$/u, v, x, y, z, w \leq 1$ and generalizes to tuples of variables. Φ, Ψ can be extracted from any proof of the assumption.

2) 1) is also valid for $\widehat{E-PA}^{\omega}\restriction, \widehat{PR}$ and $\widehat{WE-HA}^{\omega}\restriction$.

Proof: The theorem follows from Kohlenbach (A),3.8,4.8.

Furthermore we need two lemmas from Kohlenbach (A) (which are also used in the proof of Kohlenbach (A),3.8 and hence of 2.2):

2.3 Lemma (Kohlenbach (A),3.2)

1) Let $A_0(x^1, \underline{y}) \in \mathcal{L}(WE-PA^\omega)$ be a quantifier–free formula whose free variables are $x^1, \underline{y} = y_1, \ldots, y_n$ and type$/y_i \leq 1$ $(1 \leq i \leq n)$. Then the following holds:

$$WE-HA^\omega \vdash \bigwedge\!\!\!\!\bigvee x^1 A_0(x, \underline{y}) \leftrightarrow \bigwedge\!\!\!\!\bigvee k^0 A_0(\lambda m.(k)_m, \underline{y}).$$

2) An analogous result holds for $\widehat{WE-HA}^{\omega}\restriction$ if $A_0 \in \mathcal{L}(\widehat{WE-PA}^{\omega}\restriction)$.

2.4 Lemma (Kohlenbach (A), 3.5,3.6)

1) Let $A_0(\underline{x},\tilde{x}^1) \in \mathcal{L}(WE-PA^\omega)$ be a quantifier–free formula whose free variables are $\underline{x} = x_1^{\delta_1},\ldots,x_n^{\delta_n}$ and $\tilde{x}^1$, where $\delta_i \le 1$ $(1 \le i \le n)$. Assume that $s^{1\delta_n\ldots\delta_1} \in T$ is closed. Then there are (effectively) quantifier–free formulas $B_0(\underline{x})$ and $C_0(\underline{x})$ (containing only $\underline{x}$ free) such that

   1. $WE-HA^\omega \vdash \forall \tilde{x} \le_1 s\underline{x} A_0(\underline{x},\tilde{x}) \leftrightarrow B_0(\underline{x})$,

   2. $WE-HA^\omega \vdash \exists \tilde{x} \le_1 s\underline{x} A_0(\underline{x},\tilde{x}) \leftrightarrow C_0(\underline{x})$.

   Furthermore one can construct a closed term $\chi \in T$ such that

   $$WE-HA^\omega \vdash \forall \underline{x}\big(\exists \tilde{x} \le s\underline{x} A_0(\underline{x},\tilde{x}) \to A_0(\underline{x},\chi\underline{x}) \wedge \chi\underline{x} \le_1 s\underline{x}\big).$$

2) 1) holds analogously for $\widehat{WE-HA}^\omega\!\restriction$ and $\widehat{PR}$ if $A_0 \in \mathcal{L}(\widehat{WE-PA}^\omega\!\restriction)$.

Using 2.1 we can prove a first result concerning the extraction of moduli of uniqueness:

2.5 Theorem

1) $$WE-PA^\omega + AC\text{–qf} \vdash \forall u^1; v_1,v_2 \le_2 tu\Big(\forall x^\delta \exists y \le_\rho suv_1v_2x\forall z^2 A_0(u,v_1,v_2,x,y,z) \\ \to \big(\forall w^2 B_0(u,v_1,w) \wedge \forall w^2 B_0(u,v_2,w) \to v_1 =_2 v_2\big)\Big)$$

   $\Rightarrow \exists$ effectively closed terms $\Phi^{211}, \Psi^{211} \in T$ such that

   $$WE-HA^\omega\text{+b–}AC^{\delta,\rho}\text{–}\forall^b \vdash \forall u^1; v_1,v_2 \le_2 tu, k^1\Big(\forall x\exists y \le suv_1v_2x\forall z \le_2 \Psi uk\ A_0 \\ \to \big(\forall w \le_2 \Phi uk\big(B_0(u,v_1,w) \wedge B_0(u,v_2,w)\big) \to v_1k =_0 v_2k\big)\Big).$$

2) An analogous result holds for $\widehat{WE-PA}^\omega\!\restriction, \widehat{PR}, \widehat{WE-HA}^\omega$.

Φ and Ψ can be extracted by functional interpretation and pointwise majorization from any given proof of the assumption. The types of z,v_1,v_2,w (u) may be also ≤ 2 (≤ 1).

Proof:

1) $$WE-PA^\omega + AC\text{–qf} \vdash \forall u^1; v_1,v_2 \le_2 tu\Big(\forall x\exists y \le suv_1v_2x\forall z^2 A_0 \to \\ \big(\forall w^2 B_0(u,v_1,w) \wedge \forall w^2 B_0(u,v_2,w) \to \forall k^1(v_1k =_0 v_2k)\big)\Big) \Rightarrow$$

   $$WE-PA^\omega + AC\text{–qf}\vdash \forall u; v_1,v_2 \le tu; k\Big(\forall x\exists y \le suv_1v_2x\forall z^2 A_0 \to \\ \exists w^2\big(B_0(u,v_1,w) \wedge B_0(u,v_2,w) \to v_1k = v_2k\big)\Big).$$

   By 2.1.1 one can extract $\Phi,\Psi \in T$ such that

   $$WE-HA^\omega\text{+b–}AC^{\delta,\rho}\text{–}\forall^b \vdash \forall u; v_1,v_2 \le tu; k\Big(\forall x\exists y \le suv_1v_2x\forall z \le_2 \Psi uk\ A_0 \to \\ \exists w \le_2 \Phi uk\big(B_0(u,v_1,w) \wedge B_0(u,v_2,w) \to v_1k = v_2k\big)\Big).$$

   This implies the theorem.

2) can be proved analogously using 2.1.2.

2.6 Remark

If z is of arbitrary type then the bound Φ can still be extracted (but b–$AC^{\delta,\rho}$–$\forall^b$ has to be replaced by
b–$AC^{\delta,\rho}$–$\forall$).

Using 2.2,2.3 and $\forall k^0(\overline{v_1^1}k = \overline{v_2^1}k) \leftrightarrow v_1 =_1 v_2$, the proof of 2.5 implies the following

2.7 Corollary

Assume that $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$:

1) $E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf $\vdash \forall u^1; v_1, v_2 \leq_1 tu\Big(\forall x^1 \exists y \leq_1 suv_1v_2x \forall z^0, \tilde{z}^1 A_0 \to$
$$\big(\forall w^0, \tilde{w}^1 B_0(u, v_1, w, \tilde{w}) \wedge \forall w^0, \tilde{w}^1 B_0(u, v_2, w, \tilde{w}) \to v_1 =_1 v_2\big)\Big)$$
$\Rightarrow \exists$ closed terms $\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi} \in T$ such that
$$WE-HA^\omega \vdash \forall u; v_1, v_2 \leq tu; k\Big(\forall x \exists y \leq suv_1v_2x \bigwedge_{\substack{z \leq_0 \Psi uk \\ j \leq_0 \tilde{\Psi} uk}} A_0(u, v_1, v_2, x, y, z, \lambda m.(j)_m)$$
$$\to \Big(\bigwedge_{\substack{w \leq_0 \Phi uk \\ i \leq_0 \tilde{\Phi} uk}} \big(B_0(u, v_1, w, \lambda m.(i)_m) \wedge B_0(u, v_2, w, \lambda m.(i)_m)\big) \to \overline{v_1}k =_0 \overline{v_2}k\Big)\Big).$$

2) The same result holds for $\widehat{E-PA}^\omega\!\upharpoonright, \widehat{PR}$ and $\widehat{WE-HA}^\omega\!\upharpoonright$.

2.7 contains as a special case:

2.8 Corollary

1) $E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf $\vdash \forall x^1 \exists y \leq_1 sx \forall z^0 A_0(x, y, z) \to$
$$\forall u^1; v_1, v_2 \leq_1 tu\big(\forall w^0 B_0(u, v_1, w) \wedge \forall w^0 B_0(u, v_2, w) \to v_1 =_1 v_2\big)$$
$\Rightarrow \exists$ closed terms $\Phi, \Psi \in T$ such that
$$WE-HA^\omega \vdash \forall u^1, k^0\big(\forall x \exists y \leq sx \bigwedge_{z=0}^{\Psi uk} A_0(x, y, z) \to$$
$$\forall v_1, v_2 \leq tu\big(\bigwedge_{w=0}^{\Phi uk} (B_0(u, v_2, w) \wedge B_0(u, v_2, w)) \to \overline{v}_1 k =_0 \overline{v}_2\big)\big).$$

2) The same result holds for $\widehat{E-PA}^\omega\!\upharpoonright, \widehat{PR}, \widehat{WE-HA}^\omega\!\upharpoonright$.

2.9 Remark

1) The extraction of Φ, Ψ in 2.5,2.8 and – if $\tilde{z}^1, \tilde{w}^1$ are omitted – also in 2.7 uses not the continuity of primitive recursive functionals (of type 2) but only majorizability of primitive recursive functionals of higher type and functional interpretation.

2) For our applications the 2.8 (instead of 2.7) is sufficient. However it is sometimes possible to reduce the complexity of (+) $\forall x^1 \exists y \leq_1 sx \forall z^0 A_0(x, y, z)$ if we allow that this assumption may depend on u, v_1, v_2 (as in 2.7) which simplifies the proof-theoretic analysis (e.g. for each u, v_1, v_2 only a special instance of "$\forall x$" may be needed).

3) For some proof it is much easier to extract a somewhat weaker modulus $\widehat{\Phi}$ such that $\forall v, v' \leq tu\big(\bigwedge\limits_{w=0}^{\widehat{\Phi}uk}(B_0(u,v',w) \wedge \forall w B_0(u,v,w) \to \overline{v}'k =_0 \overline{v}k\big)$. If there exists a $v_u \leq tu$ such that $\forall w B_0(u,v_u,w)$ then $\widehat{\Phi}u$ is already a modulus in the sense of 2.8 since $\overline{v}_1 k = \overline{v}k \wedge \overline{v}_2 k = \overline{v}k$ implies $\overline{v}_1 k = \overline{v}_2 k$. This strategy will be useful if the given proof uses the assumption $\forall w B_0(u,v_1,w)$ and the assumption $\forall w B_0(u,v_2,w)$ in a different way in order to conclude that $v_1 = v_2$. This is the case in the uniqueness proofs given by Kirchberger (02)/Borel(05) and Young(07/Rice(64) for best Chebycheff approximation (mentioned in the introduction).

We call the functional Φ from 2.5 and 2.7 (and in the case of 2.7 also $\tilde{\Phi}$) a **modulus of uniqueness** uniform in u for v such that $\forall w B_0(u,v,w)$. Φuk is a bound for the instances $w_1,\ldots,w_n$ of "$\forall w^0$" which are needed in order to conclude that $\overline{v}_1 k =_0 \overline{v}_2 k$ (i.e. v_1, v_2 as elements of the Baire space have distance $\leq \frac{1}{k}$) from $\bigwedge\limits_{i=0}^{n} B_0(u,v_1,w_i) \wedge B_0(u,v_2,w_i)$.

Now let's assume that v_u is an element (and hence by uniqueness **the** element) $\leq_1 tu$ such that $\forall w^0 B_0(u,v_u,w)$. In order to compute $\overline{v}_u k$, it is sufficient to construct a $v_{u,k}$ with $\bigwedge\limits_{w=0}^{\Phi uk} B_0(u,v_{u,k},w)$. Since $\bigwedge\limits_{w=0}^{\Phi uk} B_0(u,v_u,w)$, it follows that $\overline{v}_u k =_0 \overline{v}_{u,k} k$. Note that it is crucial here that Φ does not depend on v_u.

If $\forall u^1 \exists v \leq_1 tu \forall w^0 B_0(u,v,w)$ (and hence a fortiori $\forall u^1, k^0 \exists v \leq_1 tu \bigwedge\limits_{w=0}^{k} B_0(u,v,w)$) true, then using 2.4 one can construct a functional $\chi \in T$ such that $\forall u, k\big(\bigwedge\limits_{w=0}^{k} B_0(u,\chi uk, w) \wedge \chi uk \leq tu\big)$. Put together we have $\overline{\chi u(\Phi uk)}k =_0 \overline{v}_u k$ and hence $v_u = \lambda k.\big(\chi u(\Phi u(k+1))\big)(k)$, i.e. using the modulus Φ one can compute v_u primitive recursively (in the sense of T resp. $\widehat{PR}$) in u.

The reasoning above supposes the existence of v_u. A more careful argument yields that the assumption of 2.7 together with $\forall u^1, k^0 \exists v \leq_1 tu \bigwedge\limits_{w=0}^{k} B_0(u,v,w)$ already implies the verification of $\lambda k.\big(\chi u(\Phi u(k+1))\big)(k)$ and hence the existence of v_u (Thus the existence of the solution v_u follows constructively from the existence of "ε–solutions" for arbitrary $\varepsilon > 0$):

2.10 Theorem

1) $WE-PA^\omega + AC$–qf $\vdash \forall u^1; v_1, v_2 \leq_1 tu\Big(\forall x^\delta \exists y \leq_\rho suv_1v_2x \forall z^2 A_0(u,v_1,v_2,x,y,z) \to$

$$\big(\forall w^0, \tilde{w}^1 B_0(u,v_1,w,\tilde{w}) \wedge \forall w^0, \tilde{w}^1 B_0(u,v_2,w,\tilde{w}) \to v_1 =_1 v_2\big)\Big)$$

$\Rightarrow \exists$ a closed term $\Phi^{11} \in T$ such that

$WE-HA^\omega$+b–$AC^{\delta,\rho}$–$\forall^b \vdash \forall u\Big(\forall v_1, v_2 \leq_1 tu; x^\delta, z^2 \exists y \leq suv_1v_2x \forall \tilde{z} \leq_2 zA_0 \wedge$

$$\forall n^0 \exists v \leq_1 tu \bigwedge_{w,i=0}^{n} B_0(u,v,w,\lambda m.(i)_m) \to \forall w, \tilde{w} B_0(u,\Phi u, w, \tilde{w}) \wedge \Phi u \leq_1 tu\Big).$$

2) Analogously for $\widehat{WE-PA^\omega}\!\restriction, \widehat{PR}$ and $\widehat{WE-HA^\omega}\!\restriction$.

Given a proof of the assumption, Φ can be constructed using functional interpretation and majorization.

Proof:

1) By 2.3 it is sufficient to consider $\forall w^0 B_0(u,v,w)$ instead of $\forall w^0, \tilde{w}^1 B_0(u,v,w,\tilde{w})$. The assumption and 2.5 yield effectively the existence of functionals $\Phi_0, \Psi \in T$ such that

$$(1)\ WE-HA^\omega\text{+b–}AC^{\delta,\rho}\text{–}\forall^b \vdash \forall u^1; v_1, v_2 \le_1 tu; k^0 \Big(\forall x \exists y \le suv_1v_2x \forall z \le_2 \Psi uk\ A_0 \rightarrow \big(\bigwedge_{w=0}^{\Phi uk}(B_0(u,v_1,w)\wedge B_0(u,v_2,w)) \rightarrow v_1k =_0 v_2k\big)\Big).$$

By 2.4 one can construct a $\chi \in T$ such that

$$(2)\ WE-HA^\omega \vdash \forall u^1, n^0\big(\exists v \le_1 tu \bigwedge_{w=0}^{n} B_0(u,v,w) \rightarrow \bigwedge_{w=0}^{n} B_0(u,\chi un,w) \wedge \chi un \le tu\big).$$

Define $\Phi := \lambda u^1, k^0.[\chi u(\Phi_0 uk)](k)$. Now assume $\bigwedge_{w=0}^{\Phi_0 uk} B_0(u,v,w)$ for a $v \le_1 tu$. Then (2) implies $\bigwedge_{w=0}^{\Phi_0 uk} B_0(u,\chi u(\Phi_0 uk),w)$. Together with (1) this yields

$$(3)\ WE-HA^\omega\text{+b–}AC^{\delta,\rho}\text{–}\forall^b \vdash \forall u,k\Big(\forall v_1, v_2 \le tu; x\exists y \le suv_1v_2x\forall z \le_2 \Psi uk\ A_0 \rightarrow \forall v \le tu\big(\bigwedge_{w=0}^{\Phi_0 uk} B_0(u,v,w) \rightarrow vk = (\Phi u)k\big)\Big).$$

It remains to show that

$$WE-HA^\omega\text{+b–}AC^{\delta,\rho}\text{–}\forall^b \vdash \forall u\big(\forall k; v_1,v_2 \le tu; x\exists y \le suv_1v_2x\forall z \le_2 \Psi uk\ A_0 \wedge \forall n \exists v \le tu \bigwedge_{w=0}^{n} B_0(u,v,w) \rightarrow \forall w B_0(u,\Phi u,w)\wedge \Phi u \le tu\big).$$

By Troelstra (73),1.6.14 there exists a closed term $p \in T$ such that

$$WE-HA^\omega \vdash \forall u^1, v^1, w^0\big(puvw =_0 0 \leftrightarrow B_0(u,v,w)\big).$$

Furthermore one can construct a modulus of continuity $\tilde{p} \in T$ (with respect to the variable v) for p (Troelstra (73),2.7.8), i.e.

$$(4)\ WE-HA^\omega \vdash \forall u,v,v',w\big(\overline{v}(\tilde{p}uvw) =_0 \overline{v'}(\tilde{p}uvw) \rightarrow puvw =_0 puv'w\big).$$

Let u^1 and w_0^0 be arbitrary but fixed and assume that

$\forall k; v_1, v_2 \le tu; x\exists y \le suv_1v_2x\forall z \le_2 \Psi uk\ A_0(u,v_1,v_2,x,y,z)$ and

$\forall n \exists v \le tu \bigwedge_{w=0}^{n} B_0(u,v,w)$:

Define $n_0 := \tilde{p}u(\Phi u)w_0,\ n_1 :=\max\{\Phi_0 un | n < n_0\},\ n_2 :=\max(n_1, w_0)$. (2) implies

$$(5)\ \bigwedge_{w=0}^{n_2} B_0(u,\chi un_2,w) \wedge \chi un_2 \le tu$$

and hence (since $w_0 \le n_2$) (6) $B_0(u,\chi un_2,w_0)$. Since $n_2 \ge \Phi_0 un$ for $n < n_0$ it follows from (5) that

$$(7)\ \forall n < n_0 \bigwedge_{w=0}^{\Phi_0 un} B_0(u,\chi un_2,w).$$

(3) and (7) yield

$$\forall n < n_0\big((\chi u n_2)n =_0 (\Phi u)n\big), \text{ hence } (\overline{\chi u n_2})n_0 = (\overline{\Phi u})n_0.$$

(6),(4) and the n_0–definition imply $B_0(u, \Phi u, w_0)$. Furthermore from (2) and the assumption we can conclude that $\Phi u \leq_1 tu$.
Since the reasoning above can be carried out in $WE-HA^\omega$, the theorem follows.

2) is proved analogously. In Kohlenbach (90) it is shown that one can construct a modulus of pointwise continuity $\tilde{p} \in \widehat{PR}$ if $p \in \widehat{PR}$.

2.11 Remark to the proof of 2.10

1) Instead of using the modulus $\tilde{p}$ of pointwise continuity for p we could have used a modulus $\hat{p}$ of uniform continuity on $\{v^1 | v \leq_1 tu\}$. Such a modulus can be extracted from proofs of the extensionality of $p \in T$ w.r.t. $=_1$ (and the argument v) using majorization (see Kohlenbach (B)) whereas the construction of a pointwise modulus involves a quite complicated formalization of standard reduction sequences of terms (see Troelstra (73),2.7.8).

2) In applications to concrete mathematical uniqueness theorems one will extract χ from a given proof of $\forall n \exists v \leq tu \bigwedge\limits_{w=0}^{n} B_0(u,v,w)$ (which can usually be given in $WE-PA^\omega$ in our applications) and so obtain a better result than the one obtained from 2.4 (which uses bounded search).

2.12 Corollary

1)

$$WE-PA^\omega + AC\text{–qf} \vdash \forall u^1\big(\forall x^\delta \exists y \leq_\rho sux \forall z^2 A_0(u,x,y,z) \rightarrow \exists! v \leq_1 tu \forall w^0, \tilde{w}^1 B_0(u,v,w,\tilde{w})\big)$$

$\Rightarrow \exists$ a closed term $\Phi^{11} \in T$ such that
$WE-HA^\omega$+b–$AC^{\delta,\rho}$–$\forall^b \vdash \forall u\big(\forall x, z^2 \exists y \leq sux \forall \tilde{z} \leq_2 z A_0 \rightarrow \forall w, \tilde{w} B_0(u, \Phi u, w, \tilde{w}) \wedge$
$\Phi u \leq_1 tu\big)$.

2) 1) also holds for $\widehat{WE-PA^\omega}\restriction, \widehat{PR}$ and $\widehat{WE-HA^\omega}\restriction$ instead of $WE-PA^\omega, T$ and $WE-HA^\omega$.

Proof: 1) In particular the assumption implies

$$WE-PA^\omega + AC\text{–qf} \vdash \forall u\big(\forall x \exists y \leq sux \forall z A_0 \rightarrow \exists v \leq tu \forall w, \tilde{w} B_0\big)$$

and therefore a fortiori

$$(*)\ WE-PA^\omega + AC\text{–qf} \vdash \forall u\Big(\forall x \exists y \leq sux \forall z A_0 \rightarrow \forall n \exists v \leq tu \bigwedge_{w,i=0}^{n} B_0(u,v,w,\lambda m.(i)_m)\Big).$$

Let $t_{B_0} \in T$ be such that $t_{B_0}(u,v,w,\tilde{w}) =_0 0 \leftrightarrow B_0(u,v,w,\tilde{w})$. Since t_{B_0} is (provable in $WE-HA^\omega$) extensional in v w.r.t. $=_1$ (see Troelstra (73),2.7.3), $(*)$ implies

$$WE-PA^\omega + AC\text{–qf} \vdash \forall u\Big(\forall x \exists y \leq sux \forall z A_0 \rightarrow \forall n \exists v^1 \bigwedge_{w,i=0}^{n} B_0(u, \min_1(v,tu), w, \lambda m.(i)_m)\Big).$$

By 2.1 it follows that

$$WE-HA^\omega\text{+b–}AC^{\delta,\rho}\text{–}\forall^b \vdash \forall u\Big(\forall x,z\exists y\le sux\forall\tilde z\le_2 zA_0 \to \forall n\exists v^1 \bigwedge_{w,i=0}^{n} B_0(u,\min\nolimits_1(v,tu),w,\lambda m.(i)_m)\Big).$$

2.12 now follows from 2.10.
2) is proved analogously.

In our mathematical applications, the types of x,y,z,w are always ≤ 1. In this case the use of b–AC in the conclusion of 2.10 can be avoided and $WE-PA^\omega$ be replaced by $E-PA^\omega$ (as in 2.7). Furthermore, WKL may be added:

2.13 Theorem

Assume $(\alpha=0\wedge\beta\le 1)$ or $(\alpha=1\wedge\beta=0)$:

1) $E-PA^\omega+WKL+AC^{\alpha,\beta}\text{–qf} \vdash \forall u^1; v_1,v_2\le_1 tu\Big(\forall x^1\exists y\le_1 suv_1v_2x\forall z^0,\tilde z^1 A_0 \to (\forall w^0,\tilde w^1 B_0(u,v_1,w,\tilde w)\wedge\forall w,\tilde w B_0(u,v_2,w,\tilde w)\to v_1=_1 v_2)\Big)$

   $\Rightarrow \exists$ a closed term $\Phi^{11}\in T$ such that

   $$WE-HA^\omega \vdash \forall u\Big(\forall v_1,v_2\le tu; x,k\exists y\le suv_1v_2x \bigwedge_{z,j=0}^{k} A_0(u,v_1,v_2,x,y,z,\lambda m.(j)_m) \wedge\forall n\exists v\le_1 tu \bigwedge_{w,i=0}^{n} B_0(u,v,w,\lambda m.(i)_m)\to\forall w,\tilde w B_0(u,\Phi u,w,\tilde w)\wedge\Phi u\le_1 tu\Big).$$

2) An analogous result holds for $\widehat{E-PA}^\omega\!\restriction, \widehat{PR}$ and $\widehat{WE-HA}^\omega\!\restriction$.

Proof: The assertion follows as in the proof of 2.10 but with the use of 2.7 instead of 2.5.

In the following, we show that the uniqueness and boundedness of v as well as the assumption that $\forall w^0 B_0\in\Pi^0_1$ are necessary for the validity of 2.12:

1) **Necessity of uniqueness:**
   $PA\vdash\forall u\exists v\le 1\forall wB_0(u,v,w) \not\Rightarrow \big(\exists$ a recursive $f:\forall u,wB_0(u,fu,w)$ is true$\big)$
   ($B_0\in\mathcal{L}(PA)$ quantifier–free).
   Proof: From recursion theory (see e.g. Rogers (67)) it is well known that there are recursive enumerable sets $A=\{x|\exists y(\alpha(x,y)=0)\}$ and $B=\{x|\exists y(\beta(x,y)=0)\}$ (α,β primitive recursive) such that
   a) $PA\vdash\forall u\Big(\forall w\big(\alpha(u,v)\ne 0\big)\vee\forall w\big(\beta(u,w)\ne 0\big)\Big)$ (i.e. $A\cap B=\emptyset$) and
   b) there exists no recursive function $f:\mathbb{N}\to\mathbb{N}$ such that

   $$\forall u\Big([fu=0\to\forall w\big(\alpha(u,w)\ne 0\big)]\wedge[fu\ne 0\to\forall w\big(\beta(u,w)\ne 0\big)]\Big)$$

   (i.e. A,B are recursive inseparable).
   a) $\Rightarrow PA\vdash\forall u\exists v\le 1\forall w,\tilde w B_0(u,v,w,\tilde w)$, where

$B_0(u,v,w,\tilde{w}) :\equiv [v=0 \to \alpha(u,w)\neq 0] \wedge [v\neq 0 \to \beta(u,\tilde{w})\neq 0]$. Now let $f\in \mathbb{N}^{\mathbb{N}}$ be such that $\forall u,w,\tilde{w} B_0(u,fu,w,\tilde{w})$,i.e.

$$[fu=0 \to \forall w\big(\alpha(u,w)\neq 0\big)] \wedge [fu\neq 0 \to \forall w\big(\beta(u,w)\neq 0\big)] \text{ for all } u.$$

By b), f is not recursive.

2) **Necessity of the boundedness of v:**
$PA\vdash \forall u\exists! v\forall w B_0(u,v,w) \not\Rightarrow (\exists f \text{rec.}:\forall u,wB_0(u,fu,w) \text{ is true})$.
Proof: Define

$$B(u,v) :\equiv [v\neq 0\wedge Tuu(v\dot{-}1)]\vee[v=0\wedge\neg\exists wTuuw],$$

where T denotes Kleene's T–predicate.

$$B(u,v)\leftrightarrow \forall wB_0(u,v,w), \text{ where } B_0(u,v,w) :\equiv [v\neq 0\wedge Tuu(v\dot{-}1)]\vee[v=0\wedge\neg Tuuw].$$

It is clear that $PA\vdash\forall u\exists!v\forall wB_0(u,v,w)$. Let$f\in\mathbb{N}^{\mathbb{N}}$ be such that $\forall u,wB_0(u,fu,w)$. Then $fu\neq 0\leftrightarrow \exists wTuuw$. The recursive undecidability of the halting problem implies that f is not recursive.

3) **Necessity of the assumption that $\forall wB_0$ does not contain a positive $\exists$–quantifier:**
$PA\vdash\forall u\exists!v\leq 1\forall w\exists zB_0(u,v,w,z)\not\Rightarrow(\exists f \text{ rec.: } \forall u,w\exists zB_0(u,fu,w,z)$.

Proof: We consider again the example from 1)
$PA\vdash\forall u\exists v\leq 1\forall w,\tilde{w}B_0'(u,v,w,\tilde{w})$
where $B_0'(u,v,w,\tilde{w}) :\equiv [v=0\to\alpha(u,w)\neq 0]\wedge[v\neq 0\to\beta(u,\tilde{w})\neq 0]$.
Hence
$PA\vdash\forall u\exists!v\leq 1\Big(\forall w,\tilde{w}B_0'(u,v,w,\tilde{w})\wedge(v\neq 0\to\neg\forall w,\tilde{w}B_0'(u,0,w,\tilde{w}))\Big)$ and therefore
$PA\vdash\forall u\exists!v\leq 1\forall w,\tilde{w}\exists z,\tilde{z}B_0(u,v,w,\tilde{w},z,\tilde{z})$ where
$B_0(u,v,w,\tilde{w},z,\tilde{z}) :\equiv B_0'(u,v,w,\tilde{w})\wedge(v\neq 0\to\neg B_0'(u,0,z,\tilde{z}))$.
Let $f\in\mathbb{N}^{\mathbb{N}}$ be such that $\forall u,w,\tilde{w}\exists z,\tilde{z}B_0(u,fu,w,\tilde{w},z,\tilde{z})$.
Then $\forall u\big([fu=0\to\forall w(\alpha(u,w)\neq 0)]\wedge[fu\neq 0\to\forall w(\beta(u,w)\neq 0)]\big)$.
Therefore (as in 1) f is not recursive.
Remark: The proof above shows that even for $B\in\Delta_2^0$ (instead of $B\in\Pi_1^0$) it may happen that
$PA\vdash\forall u\exists!v\leq 1B(u,v)\not\Rightarrow(\exists f \text{ rec.: } \forall uB(u,fu) \text{ is true})$,
since $B(u,v) :\equiv\forall w,\tilde{w}B_0'(u,v,w,\tilde{w})\wedge(v\neq 0\to\neg\forall w,\tilde{w}B_0'(u,0,w,\tilde{w}))\in\Delta_2^0$.

3 Representation of complete separable metric spaces and compact metric spaces in the finite type system $WE-PA^\omega$

In order to generalize the logical results of 2 to complete separable metric spaces (instead of the special Baire space $\mathbb{N}^{\mathbb{N}}$) and compact metric spaces (instead of the Cantor space $2^{\mathbb{N}}$ or $\{v\in\mathbb{N}^{\mathbb{N}}|v\leq_1 s\}$) we have to show how such spaces can be represented in the system $\widehat{WE-PA}^\omega\!\restriction$ (and hence in $WE-PA^\omega$). In doing so we make free use of ideas and constructions from Brown/Simpson (86), Troelstra (77), Troelstra/van Dalen (88) and Beeson (85) (see also Kreitz/Weihrauch(85)).

We develop this representation in some detail since, for the application of our proof–theoretical extraction of effective data, the logical form of the representation is crucial. In particular we have to construct a representation for which the quantifiers $\bigwedge\!\!\!\bigvee x \in X$ and $\bigwedge\!\!\!\bigvee y \in K$ (X complete separable, K compact) reduce to the "logical" quantifiers $\bigwedge\!\!\!\bigvee x^1$ and $\bigwedge\!\!\!\bigvee x \leq_1 s$ respectively without introducing new quantifiers. The use of finite types allows some simplifications not possible within second order languages with set-variables (as used e.g. in Brown/Simpson (86)).
Since H. Friedman, S. Simpson, D. Brown, R.L. Smith, Shioji, Tanaka and others have shown that substantial parts of classical analysis (and algebra) can be carried out in (second order fragments of) $WE-PA^{\omega}+WKL+AC^{0,0}$–qf and even in $\widehat{WE-PA}^{\omega}\!\restriction + WKL + AC^{0,0}$–qf (see e.g. Simpson (85) and Shioji/Tanaka (90)), this reduction makes our extraction applicable to quite a lot of non–trivial proofs in classical analysis.

Real numbers in $\widehat{WE-PA}^{\omega}\!\restriction$

We introduce real numbers as Cauchy sequences of rational numbers with fixed Cauchy modulus 2^{-n}. To this end we first have to define the ordered field $(\mathbb{Q},+,\cdot,0,1,<)$ of rational numbers within $\widehat{WE-PA}^{\omega}\!\restriction$:
Rational numbers are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers (i.e. type-0 objects): $j(n,m)$ represents the rational number $\frac{\frac{n}{2}}{m+1}$ if n is even, and the negative rational number $-\frac{\frac{n+1}{2}}{m+1}$ otherwise. Since we use a surjective pairing function j, each number can be conceived as code of a uniquely determined rational number. We define an equality relation $=_{\mathbb{Q}}$ on the representatives of the rational numbers, i.e. on $\mathbb{N}$, to be

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \frac{\frac{j_1 n_1}{2}}{j_2 n_1 + 1} = \frac{\frac{j_1 n_2}{2}}{j_2 n_2 + 1} \text{ if } j_1 n_1 \text{ and } j_2 n_2 \text{ both are even}$$

and analogous in the remaining cases, where $\frac{a}{b} = \frac{c}{d}$ is defined to hold if $ad =_0 cb$ when $bd > 0$.
In order to express the statement that n represents the rational r, we write $n =_{\mathbb{Q}} \langle r \rangle$ or simply $n = \langle r \rangle$. Of course $\langle \cdot \rangle$ is not a function of r since r possesses infinitely many representatives. Rational numbers are, strictly, speaking equivalence classes on $\mathbb{N}$ w.r.t. $=_{\mathbb{Q}}$. By using only their representatives and $=_{\mathbb{Q}}$ one can avoid formally introducing the set $\mathbb{Q}$ of all these equivalence classes. On $\mathbb{N}$ one can easily define primitive recursive operations $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and predicates $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ (in the sense of Kleene) such that e.g. $\langle r_1 \rangle +_{\mathbb{Q}} \langle r_2 \rangle =_{\mathbb{Q}} \langle r_3 \rangle$ iff $r_1 + r_2 = r_3$ for the rational numbers r_1, r_2, r_3 which are represented by $\langle r_1 \rangle, \langle r_2 \rangle, \langle r_3 \rangle$ (analogous for $\cdot_{\mathbb{Q}}, <_{\mathbb{Q}}, \leq_{\mathbb{Q}}$). $\mathbb{N}$ can be naturally embedded into our representation of $\mathbb{Q}$ via $n \mapsto \langle n \rangle := j(2n,0); 0_{\mathbb{Q}} := \langle 0 \rangle, 1_{\mathbb{Q}} := \langle 1 \rangle$. It can easily be verified (within $\widehat{WE-PA}^{\omega}\!\restriction$) that $(\mathbb{N}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, 0_{\mathbb{Q}}, 1_{\mathbb{Q}}, <_{\mathbb{Q}})$ is an ordered field (which represents $(\mathbb{Q},+,\cdot,0,1,<)$ in $\widehat{WE-PA}^{\omega}\!\restriction$).
Each function $f : \mathbb{N} \to \mathbb{N}$ (i.e. each functional of type 1) can be conceived of as an infinite sequence of codes of rationals and therefore as representative of an infinite sequence of rationals.
Real numbers are represented by functions f such that

$$(*)\ \forall n \big(|fn -_{\mathbb{Q}} f(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-n-1} \rangle \big), \text{ hence}$$

$$\forall n \forall k > m \geq n \big(|fm -_{\mathbb{Q}} fk|_{\mathbb{Q}} \leq_{\mathbb{Q}} \Sigma_{i=m}^{k-1} |fi -_{\mathbb{Q}} f(i+1)|_{\mathbb{Q}} \leq_{\mathbb{Q}} \Sigma_{i=n}^{\infty} |fi -_{\mathbb{Q}} f(i+1)|_{\mathbb{Q}} < \langle 2^{-n} \rangle \big).$$

Each f which satisfies $(*)$ therefore represents a Cauchy sequence of rationals with Cauchy modulus 2^{-n}. In order to guarantee that each function f codes a real number, we introduce the following construction which is primitive recursive (in the sense of Kleene (52) and hence in the sense of $\widehat{PR}$ and T) construction:

$$(**)\ \widehat{f}n := \begin{cases} fn \text{ if } \forall k < n\big(|fk -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-k-1}\rangle\big), \\ fk \text{ for the least } k < n \text{ such that } |fk -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} \geq_{\mathbb{Q}} \langle 2^{-k-1}\rangle \text{ otherwise.} \end{cases}$$

$\widehat{f}$ always satisfies $(*)$. If $(*)$ is already valid for f then $\forall n(fn =_0 \widehat{f}n)$. Thus each function f codes a uniquely determined real number, namely the real number which is given by the Cauchy sequence coded by $\widehat{f}$. In the other direction, if f represents a Cauchy sequence of rationals with modulus 2^{-n}, then $gn := f(n+1)$ satisfies $(*)$ and therefore represents the real number, given by f, in our sense. This shows that nothing is lost by our restriction of sequences satisfying $(*)$. The construction enables us to reduce quantifiers ranging over $\mathbb{R}$ to $\forall f^1$ resp. $\exists f^1$. This also holds for the operations on $\mathbb{R}$ below.
On the representatives (in the sense above) of real numbers (i.e. on the functionals of type 1) f_1, f_2 we define an equivalence relation $=_{\mathbb{R}}$ by

$$f_1 =_{\mathbb{R}} f_2 :\equiv \forall n\big(|\widehat{f}_1(n+1) -_{\mathbb{Q}} \widehat{f}_2(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-n}\rangle\big).$$

$f_1 =_{\mathbb{R}} f_2$ holds iff f_1 and f_2 represent the same real number (w.r.t. the usual identity relation on the reals).
In contrast to $=_{\mathbb{Q}}$, the relation $=_{\mathbb{R}}$ is not decidable, i.e. not a Π^0_0–predicate, but it is a Π^0_1–predicate.

$$f_1 <_{\mathbb{R}} f_2 :\equiv \exists n\big(\widehat{f}_2(n+1) -_{\mathbb{Q}} \widehat{f}_1(n+1) \geq_{\mathbb{Q}} \langle 2^{-n}\rangle\big) \in \Sigma^0_1,$$

$$f_1 \leq_{\mathbb{R}} f_2 :\equiv \neg(f_2 <_{\mathbb{R}} f_1) \in \Pi^0_1.$$

It is not difficult to define primitive recursive functionals $+_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}$ etc. (in the sense of Kleene) on our codes of real numbers, which represent the elementary operations $+, -, \cdot$ etc. on $\mathbb{R}$: For example, define $f_1 +_{\mathbb{R}} f_2$ by

$$(f_1 +_{\mathbb{R}} f_2)(k) := \widehat{f}_1(k+1) +_{\mathbb{Q}} \widehat{f}_2(k+1).$$

Then $f_1 +_{\mathbb{R}} f_2 =_{\mathbb{R}} f_3$ holds iff $x_1 + x_2 = x_3$ for the real numbers x_1, x_2, x_3 which are represented by f_1, f_2, f_3. $+_{\mathbb{R}}$ is a functional of type $1(1)(1)$. $-_{\mathbb{R}}$ is defined analogously.
If $n = \langle r\rangle$ codes the rational number r, then $\lambda k.n$ represents r as a real number. $0_{\mathbb{R}} := \lambda k.0_{\mathbb{Q}}$, $1_{\mathbb{R}} := \lambda k.1_{\mathbb{Q}}$.
$\mathbb{R}$ denotes the set of all equivalence classes on the set of functions f w.r.t. $=_{\mathbb{R}}$. As in the case of $\mathbb{Q}$, we use $\mathbb{R}$ only informally and deal exclusively with the representatives and the operations defined on them. $(\mathbb{N}^{\mathbb{N}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, <_{\mathbb{R}})$ is an Archimedean ordered field (provable in $\widehat{WE-PA}^{\omega}\!\upharpoonright$), which represents $(\mathbb{R}, +, \cdot, 0, 1, <)$ in $\widehat{WE-PA}^{\omega}\!\upharpoonright$.
One easily verifies the following fact:

3.1 Lemma

$\forall k\big(|f -_{\mathbb{R}} \lambda n.\widehat{f}(k)|_{\mathbb{R}} <_{\mathbb{R}} \langle 2^{-k}\rangle\big)$.

Each functional Φ^{10} can be conceived of as an infinite sequence of codes of real numbers and therefore as a representative of a sequence of real numbers. We have the following Cauchy completeness:

3.2 Lemma

$$\widehat{WE-PA^\omega}\upharpoonright \vdash \forall\Phi^{10}\big(\forall n; m,k\geq n(|\Phi m -_{\mathbb{R}} \Phi k|_{\mathbb{R}} \leq_{\mathbb{R}} \langle 2^{-n}\rangle) \to \exists f^1 \forall n(|\Phi n -_{\mathbb{R}} f|_{\mathbb{R}} \leq_{\mathbb{R}} \langle 2^{-n}\rangle)\big).$$

f can be defined primitive recursively in Φ : $fk := \Phi(\widehat{k+3})(k+3)$.

Complete separable metric and normed spaces in $\widehat{WE-PA^\omega}\upharpoonright$

Complete separable metric spaces, which we call: CSM–spaces, may be conceived of as completions $(\widehat{X},\widehat{d})$ of countable metric spaces (X,d). Such countable spaces are given as follows: Assume that the elements of X are coded by natural numbers and that f_X is a (prim. rec.) enumeration of the set $\langle X\rangle$ of all these codes. (X,d) can now be represented by a pseudo metric d_X (more precisely: a functional of type 100, which represents a pseudo metric) on $\mathbb{N}$ such that $d_X(n,m) =_{\mathbb{R}} \langle d(x,y)\rangle$, where $f_X n$ and $f_X m$ are codes of $x,y\in X$ and $\langle d(x,y)\rangle$ is a representative of the real number $d(x,y)$. Note that d_X is in general not a metric on $\mathbb{N}$: $d_X(n,m) =_{\mathbb{R}} 0_{\mathbb{R}} \not\Rightarrow n =_0 m$.
In the following we assume that $d_X(n,m) =_1 d_X\widehat{(n,m)}$ (for otherwise we simply could define $d'_X(n,m) := d_X\widehat{(n,m)})$. The completion $(\widehat{X},\widehat{d})$ of (X,d) is now represented as the completion of $(\mathbb{N},d_X)$: An element of this completion is given by a function h satisfying

$$(*)\ \forall n\big(d_X(hn,h(n+1)) <_{\mathbb{R}} \langle 7\cdot 2^{-n-1}\rangle\big).$$

$(*)$ implies that $(hn)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{N},d_X)$ with modulus 2^{-n+3}. As for the representation of $\mathbb{R}$ above, we want to have that **each** function h represents a (uniquely determined) element of the completion. If we would try to define $\widehat{h}$ in the same way as we defined $\widehat{f}$, then this operation would not be primitive recursive in h since $<_{\mathbb{R}}$ is (in contrast to $<_{\mathbb{Q}}$) not decidable. In order to avoid this difficulty we first modify $(*)$ to

$$(**)\ \forall n\big([d_X(hn,h(n+1))](n+1) <_{\mathbb{Q}} \langle 6\cdot 2^{-n-1}\rangle\big).$$

Now $\widehat{h}$ can be defined primitive recursively in h by

$$\widehat{h}(n) := \begin{cases} h(n) \text{ if } \forall k<n\big([d_X(hk,h(k+1))](k+1) <_{\mathbb{Q}} \langle 6\cdot 2^{-k-1}\rangle\big), \\ h(k) \text{ for the least } k<n:\ [d_X(hk,h(k+1))](k+1) \geq_{\mathbb{Q}} \langle 6\cdot 2^{-k-1}\rangle, \text{ otherwise.}\end{cases}$$

(In the following we always refer to this definition of).$\widehat{h}$ fulfils $(**)$ and therefore $(*)$ for all h. If $(**)$ is already valid for h, then $\forall n(hn =_0 \widehat{h}n)$. Hence each h may be thought of as being a representative of a (uniquely determined) element of the completion of $(\mathbb{N},d_X)$, namely of that element, which is represented by $\widehat{h}$. In the other direction, each representative of a Cauchy sequence in $(\mathbb{N},d_X)$ with Cauchy modul 2^{-n} fulfils $(**)$ and is therefore also a representative in our sense (of the element given by this Cauchy sequence).
The construction $\widehat{h}$ enables us to extend the pseudo metric d_X to a pseudo metric $\widehat{d}_X$ on $\mathbb{N}^{\mathbb{N}}$:

$$\widehat{d}_X^{1(1)(1)}(h_1,h_2)(n) :=_0 [d_X(\widehat{h}_1(n+5),\widehat{h}_2(n+5))](n+5).$$

By the summmand $+5$, the "right" rate of convergence of $\widehat{d}_X(h_1,h_2)$ is ensured:

$$\forall k\big(|\widehat{d}_X(h_1,h_2)(k) -_{\mathbb{Q}} \widehat{d}_X(h_1,h_2)(k+1)| <_{\mathbb{Q}} \langle 2^{-k-1}\rangle\big).$$

Furthermore we have

3.3 Lemma

$\forall k\big(\widehat{d}_X(f,\lambda m.\widehat{f}(k)) <_{\mathbb{R}} \langle 2^{-k+3}\rangle\big)$.

$(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$ is the completion of $(\mathbb{N},d_X)$:

3.4 Lemma

$$\widehat{WE-PA}^{\omega}\upharpoonright\ \vdash \forall\Phi^{10}\Big(\forall n;m,k\geq n\big(\widehat{d}_X(\Phi m,\Phi k)\leq_{\mathbb{R}}\langle 2^{-n}\rangle\big)$$
$$\to \exists f^1\forall n\big(\widehat{d}_X(\Phi n,f)\leq_{\mathbb{R}}\langle 2^{-n}\rangle\big)\Big).$$

f can be defined primitive recursively in Φ : $fk:=\Phi(\widehat{k+5})(k+5)$.
$(\mathbb{N},d_X)$ is canonically embedded into $(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$ by
$i:\mathbb{N}\to,\ i(n):=\lambda k^0.n$.
$d_X(n,m)=_{\mathbb{R}}\widehat{d}_X(i(n),i(m))$ for all m,n.

$(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$ is called **standard representation** of $(\widehat{X},\widehat{d})$. If one defines an equality relation $=_{\widehat{X}}$ on the set $\mathbb{N}^{\mathbb{N}}$ by

$$h_1=_{\widehat{X}}h_2:\equiv\big(\widehat{d}_X(h_1,h_2)=_{\mathbb{R}}0_{\mathbb{R}}\big),$$

then $\widehat{d}_X$ defines a metric on the equivalence classes with respect to $=_{\widehat{X}}$. This metric space is isometric to $(\widehat{X},\widehat{d})$, which justifies the expression "standard representation of $(\widehat{X},\widehat{d})$".

3.5 Examples

1) $[0,1]^n$ endowed with the Euclidean metric
$\widehat{d}_E\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big):=\big(\Sigma_{i=1}^n|x_i-y_i|^2\big)^{\frac{1}{2}}$ is the completion of the space $[0,1]^n\cap\mathbb{Q}^n$ with the metric
$d_E\big((r_1,\ldots,r_n),(\tilde{r}_1,\ldots,\tilde{r}_n)\big):=\big(\Sigma_{i=1}^n|r_i-\tilde{r}_i|^2\big)^{\frac{1}{2}}$ on $[0,1]^n\cap\mathbb{Q}^n$.

2) The space $C[0,1]$ of all continuous functions $f:[0,1]\to\mathbb{R}$ together with the sup metric $\widehat{d}_\infty$ is the completion of (A,d_∞) where A is the set of all finite tuples of rational numbers and
$d_\infty\big((r_0,\ldots,r_m),(\tilde{r}_0,\ldots,\tilde{r}_n)\big):=\sup\limits_{x\in[0,1]}|(r_mx^m+\ldots+r_1x+r_0)-(\tilde{r}_nx^n+\ldots+\tilde{r}_1x+\tilde{r}_0)|$.

3) Let L_p denote (as usual) the space of functions $f:[0,1]\to\mathbb{R}$ such that $|f|^p$ is integrable on $[0,1]$ $(1\leq p<\infty)$:
On the set A of 2), we define the metric
$d_p\big((r_0,\ldots,r_m),(\tilde{r}_0,\ldots,\tilde{r}_n)\big):=\big(\int_0^1|(r_mx^m+\ldots+r_0)-(\tilde{r}_nx^n+\ldots+\tilde{r}_0)|^pdx\big)^{\frac{1}{p}}$.
L_p is given in $\widehat{WE-PA}^{\omega}\upharpoonright$ as the standard representation of the completion $(\widehat{A},\widehat{d}_p)$ of (A,d_p).

3.6 Definition

A CSM–space $(\widehat{X},\widehat{d})$ is called T–definable ($\widehat{PR}$–definable), if it possesses a standard representation $(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$ with $\widehat{d}_X^{1(1)(1)}\in T$ $(\widehat{PR})$ (i.e. $d_X^{100}\in T$ $(\widehat{PR})$) and $\widehat{d}_X$ represents **provable** in $WE-PA^\omega$ (resp. $\widehat{WE-PA}^{\omega}\upharpoonright$) a pseudo–metric on $\mathbb{N}^{\mathbb{N}}$.

3.7 Remark

The metric spaces 3.5.1–3.5.3 are all $\widehat{PR}$–definable. Furthermore the definitions of these spaces in 3.5 are classically equivalent to the usual mathematical definitions. However, this equivalence is in general not provable in $\widehat{WE-PA}^{\omega}\upharpoonright(+AC^{0,0}\text{–qf})$ or $WE-PA^{\omega}(+AC^{0,0}\text{–qf})$. If e.g. $C[0,1]$ is as the set of all pointwise continuous functions on $[0,1]$, then one needs WKL to show that these functions are uniformly continuous. Then one applies the Weierstraß approximation theorem (which holds constructively for uniform continuous functions) in order to construct the representation 3.5.2. Our representation 3.5.2 of $C[0,1]$ is constructively equivalent to the definition of $C[0,1]$as the set of all uniformly continuous functions $f:[0,1]\to \mathbb{R}$, where f is given together with a modulus $\omega_f:\mathbb{N}\to\mathbb{N}$ of uniform continuity, i.e.

$$\forall n\in\mathbb{N}, x,y\in[0,1]\big(|x-y|<2^{-\omega_f(n)}\to|fx-fy|<2^{-n}\big).$$

This is the definition of $C[0,1]$ which we use when we carry out the extraction of moduli of uniqueness from concrete mathematical proofs. The enriching of data by such a modulus ω_f is more convenient in practice than the presentation of f as a Cauchy sequence of polynomials having rational coefficients, since such a sequence is in general quite complicated to construct, whereas a modulus ω_f can often easily be written down. Thus, standard representations of CSM–spaces are only used for proving the general metatheorems (as 4.1–4.6,5.1 below). For the unwinding of concrete proofs, the most useful definition of these objects is used.

3.8 Definition

Let $(\widehat{X},\widehat{d})$ be a CSM–space. Then for each $x_0\in\widehat{X},\ r\in\mathbb{R}^+$, the open ball with radius r and center x_0 is defined by $B(x_0,r):=\Big\{x\in\widehat{X}:\widehat{d}(x,x_0)<r\Big\}$.

3.9 Lemma

Let $(\widehat{X},\widehat{d})$ be a CSM–space.

1) Assume that h_{x_0},h_y represent $x_0,y\in\widehat{X}$ in $(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$. Then the following implication holds

$$\overline{h}_{x_0}(m+1)=\overline{h}_y(m+1)\Rightarrow y\in B(x_0,2^{-m+4}).$$

2) To each $x_0\in\widehat{X},\ m\in\mathbb{N}$ there exists an $n\in\mathbb{N}$ such that for all $y\in B(x_0,2^{-m})$ there exists a representative h_y in $(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$ with $\overline{h}_y m=_0 n$.

Proof:

1) $\widehat{d}_X\big(h_{x_0},i(\widehat{h}_{x_0}(m))\big)<_{\mathbb{R}}\langle 2^{-m+3}\rangle$ and $\widehat{d}_X\big(h_y,i\underbrace{(\widehat{h}_y(m))}_{\widehat{h}_{x_0(m)}}\big)<_{\mathbb{R}}\langle 2^{-m+3}\rangle$ (3.3,3.4) implies

$\widehat{d}_X(h_{x_0},h_y)<_{\mathbb{R}}\langle 2^{-m+3}+2^{-m+3}\rangle=\langle 2^{-m+4}\rangle$.

2) Let h_{x_0},h_y be representatives of x_0,y in $(\mathbb{N}^{\mathbb{N}},\widehat{d}_X)$ and assume without loss of generality that $m\geq 1$. Then

$$n:=<\widehat{h}_{x_0}(3),\ldots,\widehat{h}_{x_0}(m+2)>\ \text{ and }\ \tilde{h}_y(k):=\begin{cases}\widehat{h}_{x_0}(k+3)\text{ if }k<m,\\ \widehat{h}_y(k+3)\text{ if }k\geq m,\end{cases}$$

fulfil the lemma:
$\overline{\tilde{h}}_y m =_0 n$. Furthermore $\tilde{h}_y$ represents y, i.e. $\widehat{d}_X(\tilde{h}_y, h_y)) =_{\mathbb{R}} 0_{\mathbb{R}}$: It is sufficient to show that $\widehat{\tilde{h}}_y =_1 \tilde{h}_y$. i.e.
$(*)\ \forall k\Big(d_X\big(\tilde{h}_y(k), \tilde{h}_y(k+1)\big)(k+1) <_{\mathbb{Q}} \langle 6\cdot 2^{-k-1}\rangle\Big)$, which can be verified using 3.3,3.4 (the only problematic case is when $k = m-1$).

The proof of 3.9 can be carried out in $\widehat{WE-PA}^{\omega}\restriction$, i.e.

3.10 Corollary

For each CSM–space $(\widehat{X}, \widehat{d})$ given in standard representation $(\mathbb{N}^{\mathbb{N}}, \widehat{d}_X)$, $\widehat{WE-PA}^{\omega}\restriction$ proves

1) $\forall h_0^1, h^1, m^0\big(\overline{h}_0(m+1) =_0 \overline{h}(m+1) \to \widehat{d}_X(h_0, h) <_{\mathbb{R}} \langle 2^{-m+4}\rangle\big)$ and

2) $\forall h_0^1, m^0 \exists n^0 \forall h^1\big(\widehat{d}_X(h_0, h) <_{\mathbb{R}} \langle 2^{-m}\rangle \to \exists \tilde{h}\big(\widehat{d}_X(h, \tilde{h}) =_{\mathbb{R}} 0_{\mathbb{R}} \wedge \overline{\tilde{h}}m =_0 n\big)\big)$.

3.11 Definition

Let $\widehat{X}$ and $\widehat{Y}$ be CSM–spaces[2] A closed term $\Phi^{1(1)} \in T\ (\widehat{PR})$ represents provable in $WE-PA^{\omega}$ $(\widehat{WE-PA}^{\omega}\restriction)$ a function $\widehat{X} \to \widehat{Y}$ if
$WE-PA^{\omega} \vdash \forall f_1^1, f_2^1\big(f_1 =_{\widehat{X}} f_2 \to \Phi f_1 =_{\widehat{y}} \Phi f_2\big)$.
$(\widehat{WE-PA}^{\omega}\restriction)$.

3.12 Remark

1) 3.11 is justified by the following fact: If Φ represents a function $\widehat{X} \to \widehat{Y}$ in the sense of 3.11, then Φ induces (provable in $WE-PA^{\omega}$ or in $\widehat{WE-PA}^{\omega}\restriction$) a function on the equivalence classes on $\mathbb{N}^{\mathbb{N}}$ w.r.t. $=_{\widehat{X}}$ and $=_{\widehat{Y}}$. Modulo an isometry between $\widehat{X}, \widehat{Y}$ and their standard representations this function in turn induces a function $\widehat{X} \to \widehat{Y}$.

2) Using negative translation it follows that
$WE-PA^{\omega} \vdash \forall f_1^1, f_2^1\big(f_1 =_{\widehat{X}} f_2 \to \Phi f_1 =_{\widehat{Y}} \Phi f_2\big)$ implies
$WE-HA^{\omega} \vdash \forall f_1^1, f_2^1\big(f_1 =_{\widehat{X}} f_2 \to \Phi f_1 =_{\widehat{Y}} \Phi f_2\big)$. This also holds for $\widehat{WE-PA}^{\omega}\restriction$ and $\widehat{WE-HA}^{\omega}\restriction$.

3.13 Definition

A function $F : \widehat{X} \to \widehat{Y}$ is called T–definable ($\widehat{PR}$-def.) if a closed term $\Phi_F^{1(1)} \in T\ (\widehat{PR})$ exists such that

1) Φ_F represents provable in $WE-PA^{\omega}$ $(\widehat{WE-PA}^{\omega}\restriction)$ a function $\widehat{X} \to \widehat{Y}$ and

[2]For notational simplicity we often write $\widehat{X}$ instead of $(\widehat{X}, \widehat{d})$.

2) the function $\widehat{X} \to \widehat{Y}$ induced by Φ_F equals F, i.e.
$\forall x \in \widehat{X}, y \in \widehat{Y}, \langle x\rangle^1, \langle y\rangle^1 \big(\langle x\rangle$ represents $x \wedge \langle y\rangle$ represents $y \to$
$$(\Phi_F(\langle x\rangle) =_{\widehat{Y}} \langle y\rangle \leftrightarrow F(x) = y)\big).$$

3.14 Remarks and conventions

The function F in 3.13 may be given in set–theoretical terms which are not expressible within $WE-PA^\omega$. In particular, 2) may be unprovable in $WE-PA^\omega$. In the following, if we say that a certain sentence involving $\widehat{X}, \widehat{Y}, F$ holds provable in $WE-PA^\omega$, we always mean that the corresponding statement expressed in terms of the standard representations of $\widehat{X}, \widehat{Y}$ and Φ_F is provable in $WE-PA^\omega$. We always consider $\widehat{X}, \widehat{Y}$ as given with fixed standard representations $(\mathbb{N}^\mathbb{N}, \widehat{d}_X)$, $(\mathbb{N}^\mathbb{N}, \widehat{d}_Y)$ and F as represented by a fixed functional Φ_F. Assume e.g. that $\widehat{X}$ and $\widehat{Y}$ are T–definable CSM–spaces and that $F : \widehat{X} \times \widehat{Y} \to \mathbb{R}$ is a T–definable function. Then the sentence $\forall x \in \widehat{X} \exists y \in \widehat{Y} \big(F(x,y) = 0\big)$ is represented in $WE-PA^\omega$ by
$(*)$ $\forall x^1 \exists y^1 \big(\Phi_F(x,y) =_\mathbb{R} 0_\mathbb{R}\big)$ with $\Phi_F \in T$. $(*)$ has the logical form
$(**)$ $\forall x^1 \exists y^1 \forall k^0 A_0(x,y,k)$ where $A_0 \in \mathcal{L}(WE-PA^\omega)$ is quantifier–free.
If $\mathcal{T}$ is a theory in the language of $WE-PA^\omega$, then
$\mathcal{T} \vdash \forall x \in \widehat{X} \exists y \in \widehat{Y} \big(F(x,y) = 0\big)$ stands for
$\mathcal{T} \vdash \forall x^1 \exists y^1 \forall k^0 A_0(x,y,k)$.

3.15 Proposition

If $F : \widehat{X} \to \widehat{Y}$ is a T–definable ($\widehat{PR}$–definable) function, then F possesses provable in $WE-HA^\omega$ ($\widehat{WE-HA}^\omega\!\restriction$) a modulus $\omega_F^{001} \in T$ ($\widehat{PR}$) of pointwise continuity, i.e.
$WE-HA^\omega \vdash \forall f_0^1, f^1, n^0 \big(\widehat{d}_X(f_0, f) <_\mathbb{R} \langle 2^{-\omega_F f_0 n}\rangle \to \widehat{d}_Y(\Phi_F f_0, \Phi_F f) <_\mathbb{R} \langle 2^{-n}\rangle\big)$ (Analogous for $\widehat{WE-HA}^\omega\!\restriction$).
Note that ω_f is in general not extensional in f_0 with respect to $=_{\widehat{X}}$.

Proof: The proposition follows from 3.10 and the fact that each functional of type 1(1) in T ($\widehat{PR}$) possesses a modulus of pointwise continuity in T ($\widehat{PR}$) (provable in $WE-HA^\omega$ resp. $\widehat{WE-HA}^\omega\!\restriction$); see Troelstra (73) (3.7.8) and Kohlenbach (90) (3.22.1).

Complete separable **normed** spaces are represented in $\widehat{WE-HA}^\omega\!\restriction$ as special metric spaces with the metric $d(x,y) := \|x-y\|$. In addition to the metric also the vector space operations have to be represented in $\widehat{PR}$ or T. The spaces in 3.5 are also $\widehat{PR}$–definable as normed spaces (over $\mathbb{R}$).
Furthermore one can introduce bounded linear functionals and continuous linear functionals between CSM-spaces. The equivalence of these concepts is provable in $\widehat{WE-PA}^\omega\!\restriction + AC^{0,0}$–qf (see Brown/Simpson (86)).

Substantial parts of classical analysis and functional analysis can be formalized in $RCA_0 + WKL$ and therefore in $\widehat{WE-PA}^\omega\!\restriction + WKL + AC^{0,0}$–qf (This also holds for algebra: See e.g. Friedman/Simpson/Smith (83) and Hatzikiriakou/Simpson (89)).
In particular the following theorems are provable in this system:

1) Each $f \in C[0,1]$ attains a maximum value in [0,1] (Simpson (84)).

2) The Cauchy–Peano theorem on the existence of local solutions of systems of ordinary differential equations (Simpson (84)).

3) The Hahn–Banach theorem for separable Banach spaces (Brown/Simpson (86)).

4) Alaoglu's theorem for separable Banach spaces (Brown (87)).

5) The fixpoint theorems of Brouwer, Tychonoff–Schauder and Markov–Kakutani (Shioji/Tanaka (90)).

6) The intermediate value theorem for $f \in C[0,1]$ is even provable in $\widehat{WE-PA}^{\omega}$+AC^{0,0}–qf (see Simpson (85)).

1)–6) are provable also for the classical definitions of $C[0,1]$ as the set of pointwise continuous functions. For our definition of $C[0,1]$ one can easily prove in $\widehat{WE-HA}^{\omega}\upharpoonright$ that for every $f \in C[0,1]$ there exist $\sup\limits_{x\in[0,1]} f(x)$ and $\inf\limits_{x\in[0,1]} f(x)$.

Compact metric and normed spaces in $\widehat{WE-PA}^{\omega}\upharpoonright$

Compact spaces are usually defined via the Heine–Borel covering property. However this property is unprovable in $WE-PA^{\omega}+AC$–qf even for the most common spaces as e.g. $[0,1] \subset \mathbb{R}$ (see e.g. Friedman (75)). Therefore we use a different definition of compactness due to Brouwer, which is common in constructive mathematics and classically (with the so–called fan principle even intuitionistically) equivalent to the usual Heine–Borel compactness (see e.g. Brown (90) and Troelstra/van Dalen (88)):

3.16 Definition

A CSM–space $(\widehat{X},\widehat{d})$ is called **compact** if it is totally bounded, i.e. if there exist functions $x_{(\cdot,\cdot)} : \mathbb{N}\times\mathbb{N} \to \widehat{X},\ \alpha : \mathbb{N} \to \mathbb{N}$, which yield a finite sequence $x_{k,0},\ldots,x_{k,\alpha(k)}$ in $\widehat{X}$ for each $k \in \mathbb{N}$ such that
$\forall x \in \widehat{X}, k\exists i \leq \alpha(k)\big(\widehat{d}(x,x_{k,i}) < 2^{-k}\big)$.

3.17 Remark

Each compact metric space is separable: The countable set
$X := \bigcup_{k\in\mathbb{N}} \{x_{k,0},\ldots,x_{k,\alpha(k)}\}$ is dense in $\widehat{X}$.

Since by 3.17 compact metric spaces are special CSM–spaces, we could use the standard representation for CSM–spaces described above. However, we need a special form of this representation, which has the property that all representatives are bounded by some function M, i.e. are elements of $\{x \in \mathbb{N}^{\mathbb{N}} | \forall n(xn \leq M(n))\}$, and every element of this set represents a uniquely determined point in the compact space:
Let $(\widehat{X},\widehat{d})$ be a compact metric space with a 2^{-k}–net $x_{k,0},\ldots,x_{k,\alpha(k)}$ in $\widehat{X}$ for each $k \in \mathbb{N}$. We code $x_{k,i}$ by $j(k,i)$ (If n is not of the form $j(k,i)$ with $i \leq \alpha(k)$, then n codes the element $x_{0,\alpha(0)}$). Now we

consider the standard representation $(\mathbb{N}^{\mathbb{N}}, \widehat{d}_X)$ of $(\widehat{X}, \widehat{d})$ w.r.t. $X := \bigcup_{k\in\mathbb{N}} \{x_{k,0}, \ldots, x_{k,\alpha(k)}\}$ using this coding. For each $f \in \mathbb{N}^{\mathbb{N}}$ there exists (effectively) a function $g \leq_1 M$ such that f and g represent the same element in $\widehat{X}$, i.e. $\widehat{d}_X(f,g) =_{\mathbb{R}} 0_{\mathbb{R}}$, where $M(k) :=\max\{j(k+2,0), \ldots, j(k+2, \alpha(k+2))\}$: Define $\tilde{f}(k) := \widehat{f}(k+1)$, then $\forall k\big(d_X(\tilde{f}(k), \tilde{f}(k+1)) <_{\mathbb{R}} \langle 7 \cdot 2^{-k-2}\rangle\big)$. For every $k \in \mathbb{N}$ exists (effectively in α and d_X) a $i_k \leq M(k)$ such that $d_X(i_k, \tilde{f}(k)) <_{\mathbb{R}} \langle 2^{-k-1}\rangle$: Define i_k as the least $i \leq M_\alpha(k)$ such that

$(*)$ $d_X(i, \tilde{f}k)(k+3) <_{\mathbb{Q}} \langle 2^{-k-2}\rangle + \langle 2^{-k-3}\rangle$

(such an $i \leq M(k)$ always exists since, by M–definition, there exists an $i \leq M(k)$ with $d_X(i, \tilde{f}k) <_{\mathbb{R}} \langle 2^{-k-2}\rangle$).

The construction of i_k implies

$d_X(i, \tilde{f}k) <_{\mathbb{R}} \langle 2^{-k-2}\rangle + \langle 2^{-k-3}\rangle + \langle 2^{-k-3}\rangle = \langle 2^{-k-1}\rangle$.

Hence

$$\begin{aligned} d_X(i_k, i_{k+1}) &\leq_{\mathbb{R}} d_X(i_k, \tilde{f}k) + d_X(\tilde{f}k, \tilde{f}(k+1)) + d_X(\tilde{f}(k+1), i_{k+1}) \\ &<_{\mathbb{R}} \langle 2^{-k-1}\rangle + \langle 7 \cdot 2^{-k-2}\rangle + \langle 2^{-k-2}\rangle = \langle 5 \cdot 2^{-k-1}\rangle \end{aligned}$$

and therefore $d_X(i_k, i_{k+1})(k+1) <_{\mathbb{Q}} \langle 5 \cdot 2^{-k-1}\rangle + \langle 2^{-k-1}\rangle = \langle 6 \cdot 2^{-k-1}\rangle$,

i.e. $\widehat{g} =_1 g$ for $gk := i_k$. It follows that g represents the same element of $\widehat{X}$ as f and $\forall k \in \mathbb{N}(gk \leq M(k))$.

$\{f \in \mathbb{N}^{\mathbb{N}} | f \leq_1 M\}$ together with the restriction of $\widehat{d}_X$ on $\{\ldots\}$, which we also denote by $\widehat{d}_X$, represents the compact metric space $(\widehat{X}, \widehat{d})$.

3.18 Definition

$\Big(\{f \in \mathbb{N}^{\mathbb{N}} : f \leq_1 M\}, \widehat{d}_X\Big)$ is called standard representation of the compact metric space $(\widehat{X}, \widehat{d})$.

A compact metric space $(\widehat{X}, \widehat{d})$ is called T–definable ($\widehat{PR}$–definable), if it possesses a standard representation $\Big(\{f \in \mathbb{N}^{\mathbb{N}} : f \leq_1 M\}, \widehat{d}_X\Big)$ where $M^1, \widehat{d}_X^{1(1)(1)} \in T$ ($\widehat{PR}$), i.e. $M^1, d_X^{100} \in T$ ($\widehat{PR}$), and if d_X represents **provable** in $WE-PA^\omega$ (resp. $\widehat{WE-PA^\omega}\restriction$) a pseudo–metric on $\langle X\rangle$ such that

$WE-PA^\omega \vdash \forall i \in \langle X\rangle, k\exists j \leq M(k)\big(d_X(i,j) < 2^{-k-2}\big)$.

3.19 Examples

1) $[0,1]^n (\subset \mathbb{R}^n)$ is a $\widehat{PR}$–definable compact metric space (w.r.t. the Euclidean metric d_E).

2) Let K, C be positive rational numbers such that $K > C > 0$. Then
   $M := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq K \wedge |x_i - x_j| \geq C\ (1 \leq i < j \leq n)\}$
   (endowed with the Euclidean metric) is a $\widehat{PR}$–definable compact metric space.

3.20 Remarks and conventions

3.14 holds analogously if $\widehat{Y}$ is a compact metric space:

Let $\widehat{X}$ be a CSM–space, K a compact metric space and $F : \widehat{X} \times K \to \mathbb{R}$ a T–definable ($\widehat{PR}$–definable) function on the T ($\widehat{PR}$)–definable spaces $\widehat{X}, \widehat{K}$. Then the sentence $\forall x \in \widehat{X} \exists y \in K\big(F(x,y) = 0\big)$ is represented in $WE-PA^\omega$ ($\widehat{WE-PA^\omega}\restriction$) by

(*) $\forall x^1 \exists y \leq_1 M\big(\Phi_F(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}\big)$ with $M, \Phi_F \in T\ (\widehat{PR})$. (*) has the logical form

(**) $\forall x^1 \exists \leq_1 M \forall k^0 A_0(x,y,k)$ where $A_0 \in \mathcal{L}(WE-PA^\omega)$ $(\in \mathcal{L}(\widehat{WE-PA}^\omega\upharpoonright)$) is quantifier–free. If $\mathcal{T}$ is a theory in the language of $WE-PA^\omega$ resp. $\widehat{WE-PA}^\omega\upharpoonright$ then

$\mathcal{T} \vdash \forall x \in \widehat{X} \exists y \in K\big(F(x,y)=0\big)$ stands for $\mathcal{T} \vdash \forall x^1 \exists y \leq_1 M \forall k^0 A_0(x,y,k)$.

3.21 Proposition

If $F : K \to \widehat{X}$ is a T–definable function on the T–definable compact metric space K into a T–definable CSM–space $\widehat{X}$, then F possesses provable in $WE-PA^\omega$ a modulus $\omega_F^1 \in T$ of uniform continuity, i.e.

$WE-HA^\omega \vdash \forall f_1, f_2 \leq_1 M, n^0\big(\widehat{d}_K(f_1,f_2) <_{\mathbb{R}} \langle 2^{-\omega_F(n)}\rangle \to \widehat{d}_X(\Phi_F f_1, \Phi_F f_2) <_{\mathbb{R}} \langle 2^{-n}\rangle\big)$.

An analogous result holds for $\widehat{WE-HA}^\omega\upharpoonright, \widehat{PR}$.

Proof: Similar to the proof of 3.15 using the fact that each functional of type 1(1) in T $(\widehat{PR})$ possesses a modulus of uniform continuity on $\{f^1 : f \leq_1 M\}$ in T $(\widehat{PR})$ (see Kohlenbach (B) for a simple construction of such a modulus).

3.22 Definition

Let $(\widehat{X},\widehat{d})$,$(\widehat{Y},\widehat{\rho})$ be T–definable ($\widehat{PR}$–definable) CSM–spaces with standard representations $(\mathbb{N}^{\mathbb{N}}, \widehat{d}_X)$ and $(\mathbb{N}^{\mathbb{N}}, \widehat{\rho}_Y)$. An $\widehat{X}$–family $(Y_x)_{x\in\widehat{X}}$ of compact subsets $Y_x \subset \widehat{Y}$ is called T–definable ($\widehat{PR}$–definable) if functionals $s^{1(1)}, \Delta^{1(1)(1)} \in T$ $(\widehat{PR})$ exist which transform representatives of elements of Y_x uniformly in x into representatives in $\widehat{Y}$:

1) If $\langle x\rangle$ represents $x \in \widehat{X}$ then for each $f \leq_1 s(\langle x\rangle)$ the function $\Delta f(\langle x\rangle)$ represents an element of $Y_x \subset \widehat{Y}$ in $(\mathbb{N}^{\mathbb{N}}, \widehat{\rho}_Y)$.

2) For every $x \in \widehat{X}, y \in \widehat{Y}$ there exists an $f \leq_1 s(\langle x\rangle)$ such that $\Delta f(\langle x\rangle)$ represents y in $(\mathbb{N}^{\mathbb{N}}, \widehat{\rho}_Y)$ if $y \in Y_x$ (Here $\langle x\rangle$ denotes a representative of x).

3) The subset $A_x \subset \mathbb{N}^{\mathbb{N}}$ given by $A_x := \{\Delta f x^1 | f \leq_1 sx\}$ (more precisely the set of $=_{\widehat{Y}}$–equivalence classes generated by the elements of A_x) is provable in $WE-PA^\omega$ $(\widehat{WE-PA}^\omega\upharpoonright)$ compact in $(\mathbb{N}^{\mathbb{N}}, \widehat{d}_Y)$ for all x^1.

4) A_x is (provable in $WE-PA^\omega$ resp. $\widehat{WE-PA}^\omega\upharpoonright$) extensional w.r.t. $=_{\widehat{X}}$, i.e.

$x_1 =_{\widehat{X}} x_2 \wedge y \in A_{x_1} \to \exists \tilde{y} \in A_{x_2}(\Delta f y =_{\widehat{Y}} \Delta f \tilde{y})$.

3.23 Example

Let$F : C[0,1] \to \mathbb{R}$ be a T–definable ($\widehat{PR}$–definable) function and $\|\cdot\|_E$ denote the Euclidean norm on $\mathbb{R}^n$. Then

$(Y_f)_{f\in C[0,1]}$ with $Y_f := \{(a_1,\ldots,a_n) \in \mathbb{R}^n : \|(a_1,\ldots,a_n)\|_E \leq F(f)\} \subset \mathbb{R}^n$ is a T–definable ($\widehat{PR}$–definable) $C[0,1]$–family of compact subsets of $\mathbb{R}^n$.

3.24 Remarks

3.14 and 3.20 hold analogously if $(Y_x)_{x\in\widehat{X}}$ is a T–definable $\widehat{X}$–family of compact sets in $\widehat{Y}$: Let $\widehat{X},\widehat{Y}$ be CSM–spaces. If $(Y_x)_{x\in\widehat{X}}$ is a T–definable $\widehat{X}$–family of compact sets in $\widehat{Y}$ and $F:\widehat{X}\times\widehat{Y}\to\mathbb{R}$ is a T-definable function on the T–def. CSM–spaces $\widehat{X},\widehat{Y}$, then the sentence $\forall x\in\widehat{X}\exists y\in Y_x F(x,y)=0$ is represented in $WE-PA^\omega$ by
$(*)$ $\forall x^1\exists y\le_1 sx\big(\tilde{\Phi}_F(x,y)=_{\mathbb{R}}0\big)$ with $s,\tilde{\Phi}_F\in T$, where $\tilde{\Phi}_F(x,y):=_1\Phi_F(x,\Delta(y,sx))$, with Φ_F,Δ from the definition of F, Y_x resp.
$(*)$ has the logical form $\forall x^1\exists y\le_1 sx\forall k^0 A_0(x,y,k)$ where $A_0\in\mathcal{L}(WE-PA^\omega)$ is quantifier–free. An analogous remark holds for $\widehat{PR}$ and $\widehat{WE-PA}^\omega\!\upharpoonright$ instead of T and $WE-PA^\omega$.

3.25 Proposition

1) If $F:\widehat{X}\times\widehat{Y}\to\mathbb{R}$ is as in 3.24, then $F(x,\cdot)$ is (uniform in x) for each $x\in X$ uniformly continuous on Y_x and possesses provable in $WE-HA^\omega$ a modulus $\omega_F(x)$ of uniform continuity on Y_x, i.e. $\omega_F^{1(1)}\in T$ such that
$WE-HA^\omega\vdash\forall x\in\widehat{X},y_1,y_2\in Y_x,n\in\mathbb{N}\big(\widehat{\rho}(y_1,y_2)<2^{-\omega_F(x,n)}\to\widehat{d}(Fy_1,Fy_2)<2^{-n}\big)$, where ω_F is in general not extensional in x w.r.t. $=_{\widehat{X}}$.

2) The T–definability of F implies the T–definability of $\tilde{F}(x):=\inf_{y\in Y_x}F(x,y)$.

3) $WE-HA^\omega\vdash\forall x\in\widehat{X},k\in\mathbb{N}\exists y_0\in Y_x\big(|F(x,y_0)-\inf_{y\in Y_x}F(x,y)|<2^{-k}\big)$.

An analogous result holds for $\widehat{PR},\widehat{WE-HA}^\omega\!\upharpoonright$.

Proof: 1) By Kohlenbach (B,2.15,2.5.1) one can construct a functional $\Delta^*\in T$ (for Δ from the definition of (Y_x)) such that
$(+)$ $WE-HA^\omega\vdash\forall x^1;y\le_1 sx\big(\Delta^*(sx)x\ge_1\Delta yx\big)$. Define $A_x:=\big\{y^1:y\le_1\lambda k.\big(\Delta^*(sx)x\big)(k+3)\big\}$. From the proof of 3.9.2 and $(+)$ it follows that

$$WE-HA^\omega\vdash\forall y_1,y_2\le_1 sx\big(\widehat{d}_Y(\Delta y_1x,\Delta y_2x)<2^{-m}\to \exists f_{y_1,x},f_{y_2,x}\in A_x(\bigwedge_{i=1,2}\Delta y_ix=f_{y_i,x}\wedge\overline{f_{y_1,x}}m=\overline{f_{y_2,x}}m)\big).$$

The proposition now follows from the fact that each functional $\Phi^{1(1)}\in T$ possesses a modulus $\tilde{\Phi}x$ ($\tilde{\Phi}\in T$) of uniform continuity on A_x together with 3.10.1, 2) and 3) follow easily from 1).

4 Effective moduli from uniqueness proofs in classical analysis

In this paragraph we generalize the results 2.10 and 2.13 from the logical spaces $\mathbb{N}^{\mathbb{N}},2^{\mathbb{N}}$ to arbitrary (constructively representable) CSM–spaces X and X–families $(Y_x)_{x\in X}$ of compact sets in a CSM–space Y. Throughout this paragraph we make the following general assumptions:
X,Y,U,V,W are always T–definable resp. $\widehat{PR}$–definable CSM-spaces. $(Y_{x,u})_{x\in X,u\in U}$, $(V_u)_{u\in U}$ are T– resp. $\widehat{PR}$–definable X–families of compact sets in Y,V. Furthermore $F:U\times V\times X\times Y\times Z\to\mathbb{R}$ and $G:U\times V\times W\to\mathbb{R}$ are T–definable ($\widehat{PR}$–definable) functions (Together with X_1,X_2 also

$X_1 \times X_2$ is a T–definable ($\widehat{PR}$-def.) CSM space w.r.t. the product metric). $(z_n), (w_n)$ are T–definable ($\widehat{PR}$–definable) sequences in Z, W, which are provable within $WE-PA^\omega$ ($\widehat{WE-PA^\omega}\!\restriction$) dense in Z, W (such sequences exist by the very definition of T–definable ($\widehat{PR}$–definable) CSM–spaces.

4.1 Theorem

Assume $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$.

1) Let $A_1(u^1, v^1, k^0, m^0) \in \mathcal{L}(WE-PA^\omega)$ be a formula having the form $\exists \underline{l}^{0/1} A_0$ where A_0 is quantifier–free, which contains only u, v, k, m as free variables. Furthermore, assume that A_1 is "mathematically" extensional w.r.t. $=_U, =_V$ in u, v (provable in $E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf).[3] Then the following rule holds:

$$E-PA^\omega + WKL + AC^{\alpha,\beta}\text{–qf} \vdash \forall u \in U, v \in V_u \Big(\forall x \in X \exists y \in Y_{x,u} \forall z \in Z \big(F(u,v,x,y,z) = 0\big)$$
$$\to \forall k \in \mathbb{N} \exists m \in \mathbb{N} A_1(u,v,k,m)\Big)$$

$\Rightarrow \exists$ closed terms $\Phi, \Psi, \tilde{\Psi} \in T$ such that

$$WE-HA^\omega \vdash \forall u \in U, v \in V_u, k \in \mathbb{N} \Big(\forall x \in X \exists y \in Y_{x,u} \bigwedge_{i=0}^{\tilde{\Psi}uk} \big(|F(u,v,x,y,z_i)| \leq 2^{-\Psi uk}\big)$$
$$\to \bigvee_{m=0}^{\Phi uk} A_1(u,v,k,m)\Big).$$

Φ, Ψ and $\tilde{\Psi}$ can be extracted from the assumption by functional interpretation combined with pointwise majorization.

2) A result analogous to 1) holds for $\widehat{PR}, \widehat{E-PA^\omega}\!\restriction, \widehat{WE-HA^\omega}\!\restriction$ instead of $T, E-PA^\omega, WE-HA^\omega$.

Proof: 1) By the continuity of F (3.15) the assumption implies

$$E-PA^\omega + WKL + AC^{\alpha,\beta}\text{–qf} \vdash \forall u \in U, v \in V_u \Big(\forall x \in X \exists y \in Y_{x,u} \forall i, n \big(|F(u,v,x,y,z_i)| \leq 2^{-n}\big)$$
$$\to \forall k \in \mathbb{N} \exists m \in \mathbb{N} A_1(u,v,k,m)\Big).$$

This sentence has the logical form (3.24)

$$E-PA^\omega + WKL + AC^{\alpha,\beta}\text{–qf} \vdash \forall u^1; v \leq_1 tu \big(\forall x^1 \exists y \leq_1 sux \forall i^0, n^0 B_1(u,v,x,y,i,n)$$
$$\to \forall k^0 \exists m^0 A_1(u, \Delta vu, k, m)\big),$$

where $s, t \in T$ are closed terms and $B_1 \in \Pi^0_1$, $A_1 \in \Sigma^0_1$. 2.2.1 yields the extractability of closed terms $\Phi, \Psi, \tilde{\Psi} \in T$ such that

$$WE-HA^\omega \vdash \forall u; v \leq tu; k \Big(\forall x \exists y \leq sux \bigwedge_{\substack{i \leq \tilde{\Psi}uk \\ n \leq \Psi uk}} B_1(u,v,x,y,i,n) \to \bigvee_{m=0}^{\Phi uk} A_1(u, \Delta uv, k, m)\Big),$$

i.e. the assertion of the theorem. 2) is proved analogously using 2.2.2.

4.2 Remark

1) The assumption "$E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf$\vdash \forall u \in U, v \in V_u(\ldots)$" in 4.1 (and in the following) means (according to 3.14) that the **representation** of this proposition by using

[3] Hence A_1 represents a property of the elements of U, V, which we denote also by A_1.

the standard representation of $U, X, Y_{x,u}, Z, V_u$ and the representation of F in T is provable in $E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf.

The functionals Φ, Ψ and $\tilde{\Psi}$ operate on the representatives of elements of U and are in general not extensional with respect to $u_1 =_U u_2$ $\Big(\equiv d_U(u_1,u_2)=0\Big)$. In concrete mathematical applications these functionals will be extensional in natural enrichments of the data e.g. in $f \in C[0,1]$ together with a modulus of uniform continuity on $[0,1]$.

2) The assumption that A_1 is provable in $E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf extensional with respect to $=_U, =_V$ in u, v in 4.1 implies that this extensionality is also provable in $WE-HA^\omega$: $F :\equiv \forall u_1^1, u_2^1, v_1^1, v_2^1, k, m\big(u_1 =_U u_2 \wedge v_1 =_V v_2 \wedge A_1(u_1,v_1,k,m) \to A_1(u_2,v_2,k,m)\big)$ is (classically) equivalent to a $\forall \underline{u}^1 \exists \underline{k}^0 F_0$–sentence, where the implication $\forall \underline{u}^1 \exists \underline{k}^0 F_0 \to F$ holds intuitionistically. 2.2 includes as a special case that
$E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf$\vdash \forall \underline{u}^1 \exists \underline{k}^0 F_0$ implies $WE-HA^\omega \vdash \forall \underline{u}^1 \exists \underline{k}^0 F_0$ and therefore $WE-HA^\omega \vdash F$.

Next we apply 4.1 to uniqueness theorems:

4.3 Theorem

For α, β as in 4.1, the following rule holds:
1)$E-PA^\omega + WKL + AC^{\alpha,\beta}$–qf$\vdash$

$$\forall u \in U, v_1, v_2 \in V_u\Big(\forall x \in X \exists y \in Y_{x,u} \forall z \in Z\big(F(u,v_1,v_2,x,y,z)=0\big) \to \big(\forall w \in W(G(u,v_1,w)=0=G(u,v_2,w)) \to v_1=v_2\big)\Big)$$

$\Rightarrow \exists$ closed terms $\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi} \in T$ such that

$$WE-HA^\omega \vdash \forall u \in U, v_1, v_2 \in V_u\Big(\forall x \in X \exists y \in Y_{x,u} \forall z \in Z \bigwedge_{i=0}^{\tilde{\Psi}uk}\big(|F(\ldots,z_i)| \le 2^{-\Psi uk}\big) \to \Big(\bigwedge_{j=0}^{\tilde{\Phi}uk}\big(|G(u,v_1,w_j)|, |G(u,v_2,w_j)| \le 2^{-\Phi uk}\big) \to d_V(v_1,v_2) \le 2^{-k}\Big)\Big).$$

$\Phi, \tilde{\Phi}, \Psi$ and $\tilde{\Psi}$ can be extracted as in 4.1.
2) The theorem also holds for $\widehat{E-PA}^\omega\!\upharpoonright, \widehat{PR}, \widehat{WE-HA}^\omega\!\upharpoonright$.

Proof: 1) The assumption implies

$$E-PA^\omega + WKL + AC^{\alpha,\beta} - \text{qf} \vdash \forall u \in U, v_1, v_2 \in V_u\Big(\forall x \in X \exists y \in Y_{x,u} \forall i, m\big(|F(\ldots,z_i)| \le 2^{-m}\big) \to \big(\forall j, n(|G(u,v_1,w_j)|, |G(u,v_2,w_j)| \le 2^{-n}) \to \forall k \in \mathbb{N}(d_V(v_1,v_2) < 2^{-k})\Big).$$

Therefore

$$E-PA^\omega + WKL + AC^{\alpha,\beta} - \text{qf} \vdash \forall u \in U, v_1, v_2 \in V_u\Big(\forall x \in X \exists y \in Y_{x,u} \forall i, m(|F| \le 2^{-m}) \to \forall k \exists j, n\big(|G(u,v_1,w_j)|, |G(u,v_2,w_j)| \le 2^{-n} \to d_V(v_1,v_2) < 2^{-k}\big)\Big).$$

The Formula $(*)$ $\big(|G(u,v_1,w_j)|, |G(u,v_2,w_j)| \le 2^{-n} \to d_V(v_1,v_2) < 2^{-k}\big)$ has the logical form $B(u,v_1,j,n) \wedge B(u,v_2,j,n) \to C(v_1,v_2,k)$ where $B \in \Pi_1^0, C \in \Sigma_1^0$. $(*)$ is logically equivalent to a prenex normal form $D(u,v_1,v_2,j,n,k) \in \Sigma^0_1$ (The implication $D \to (B(v_1)\wedge B(v_2)\to C)$ even holds intuitionistically). 4.1 yields closed terms $\Phi,\tilde{\Phi},\Psi,\tilde{\Psi}\in T$ such that

$$WE-HA^\omega \vdash \forall u\in U, v_1,v_2\in V_u, k\in\mathbb{N}\Big(\forall x\in X\exists y\in Y_{x,u}\bigwedge_{i=0}^{\tilde{\Psi}uk}\big(|F(\ldots,z_i)|\le 2^{-\Psi uk}\big) \to \bigvee_{\substack{j\le\tilde{\Phi}uk\\ n\le\Phi uk}} D(u,v_1,v_2,j,n,k)\Big);$$

hence the theorem follows.
3) follows analogously.

The next theorem shows, how the moduli $\Phi,\tilde{\Phi}$ of 4.3 can be used to compute $v\in V_u$ with $\forall w\in W\big(G(u,v,w)=0\big)$ (uniformly in u):

4.4 Theorem

Let α,β be as above.
1) $E-PA^\omega+WKL+AC^{\alpha,\beta}$–qf $\vdash \forall u\in U, v_1,v_2\in V_u\big(\forall x\in X\exists y\in Y_{x,u}\forall z\in Z(F=0) \to \big(\forall w\in W(G(u,v_1,w)=0=G(u,v_2,w))\to v_1=v_2\big)\big)$

$\Rightarrow \exists$ a closed term $\Phi\in T$ such that

$$WE-HA^\omega\vdash \forall u\in U\Big(\forall v_1,v_2\in V_u, x\in X, l,m\exists y\in Y_{x,u}\bigwedge_{i=0}^{l}(|F(\ldots,z_i)|\le 2^{-m}) \wedge\forall h,n\exists v\in V_u\bigwedge_{j=0}^{h}(|G(u,v,w_j)|\le 2^{-n})\to\forall w\in W(G(u,\Phi u,w)=0)\wedge \Phi u\in V_u\Big)$$

2) An analogous result is valid for $\widehat{E-PA}^\omega\!\restriction, \widehat{PR}$ and $\widehat{WE-HA}^\omega\!\restriction$.

4.5 Corollary

Let α,β be as above.

1) $E-PA^\omega+WKL+AC^{\alpha,\beta}$–qf $\vdash \forall x\in X\exists y\in Y_x\forall z\in Z\big(F(x,y,z)=0\big)\to \forall u\in U,v_1,v_2\in V_u\Big(\forall w\in W\big(G(u,v_1,w)=0=G(u,v_2,w)\big)\to v_1=v_2\Big)$

   $\Rightarrow\exists$ a closed term $\Phi\in T$ such that

   $$WE-HA^\omega\vdash\forall x\in X,l,m\exists y\in Y_x\bigwedge_{i=0}^{l}\big(|F(x,y,z_i)|\le 2^{-m}\big)\wedge \forall u\in U,h,n\exists v\in V_u\bigwedge_{j=0}^{h}\big(|G(u,v,w_j)|\le 2^{-n}\big)\to\forall u\in U,w\in W\big(G(u,\Phi u,w)=0\big)\wedge\Phi\in C(U,V)$$

   ($\Phi\in C(U,V)$ means that Φ represents a continuous function $U\to V$).

2) Analogous for $\widehat{E-PA}^\omega\!\restriction,\widehat{PR},\widehat{WE-HA}^\omega\!\restriction$.

Proof of 4.4: 1) By 4.3 the assumption of 4.4 implies the extractability of closed terms $\Phi_0,\tilde{\Phi}_0\in T$ such that

(1) $WE-HA^\omega\vdash\forall u\in U,v_1,v_2\in V_u\Big(\forall x\in X\exists y\in Y_{x,u}\forall z\in Z\bigwedge_{i=0}^{l}\big(|F(\ldots,z_i)|\le 2^{-m}\big)$

$$\rightarrow \Big(\bigwedge_{j=0}^{\tilde{\Phi}_0 uk} \big(|G(u,v_1,w_j)|, |G(u,v_2,w_j)| \le 2^{-\Phi_0 uk}\big) \rightarrow d_V(v_1,v_2) < 2^{-k-1}\big)\Big).$$

Let $u \in U$ and assume

(2) $\forall h, n \exists v \in V_u \bigwedge_{j=0}^{h} \big(|G(u,v,w_j)| \le 2^{-n-2}$. (2) has the form

$\forall h^0, n^0 \exists v \le_1 tu \bigwedge_{j=0}^{h} \big(|\Phi_G(u, \Delta vu, w_j)| \le_{\mathbb{R}} \langle 2^{-n-2}\rangle\big)$, where $t, \Phi_G, \Delta \in T$. This implies

$$\forall h^0, n^0 \exists v \le_1 tu \underbrace{\bigwedge_{j=0}^{h} \Big(|\Phi_G(u, \widehat{\Delta v} u, w_j)|(n+2) \le_{\mathbb{Q}} \langle 2^{-n-1}\rangle\Big)}_{A_0(u,v,h,n):\equiv}.$$

A_0 is quantifier–free and by 2.4 it follows (effectively) the existence of a closed term $\chi \in T$ such that $WE-HA^\omega \vdash \forall u^1 \big(\forall h^0, n^0 \exists v \le_1 tuA_0(u,v,h,n) \rightarrow \forall h, nA_0(u, \chi uhn, h, n) \wedge \chi uhn \le_1 tu\big)$.

Define $\tilde{\chi} uhn := \Delta(\chi uhn, u)$. This yields

$$(3)\ WE-HA^\omega \vdash \forall u \in U\Big(\forall h, n\exists v \in V_u \bigwedge_{j=0}^{h} \big(|G(u,v,w_j)| \le 2^{-n}\big) \rightarrow$$

$$\forall h, n\big(\bigwedge_{j=0}^{h} \big(|G(u, \tilde{\chi} uhn, w_j)| \le 2^{-n}\big) \wedge \tilde{\chi} uhn \in V_u\big)\Big).$$

Define $\Phi := \lambda u, k.\widehat{\Phi_1 uk}(k+5) \in T$, where $\Phi_1 := \lambda u, k.[\tilde{\chi} u(\tilde{\Phi}_0 u(k+5))(\Phi_0 u(k+5))]$.

Φ fulfils the claim of the theorem: Assume

$\forall v_1, v_2 \in V_u, x \in X, l, m \exists y \in Y_{x,u} \bigwedge_{i=0}^{l} \big(|F(\ldots, z_i)| \le 2^{-m}\big)$ and

$\forall h, n \exists v \in V_u \bigwedge_{j=0}^{h} \big(|G(u,v,w_j)| \le 2^{-n}\big)$.

Then (1) and (3) yield

$$(4)\ \forall k \in \mathbb{N}, v_1, v_2 \in V_u \Big(\bigwedge_{j=0}^{\tilde{\Phi}_0 uk} \big(|G(u,v_1,w_j)|, |G(u,v_2,w_j)| \le 2^{-\Phi_0 uk}\big)$$

$$\rightarrow d_V(v_1,v_2) < 2^{-k-1}\Big)$$

and

(5) $\forall h, n \bigwedge_{j=0}^{h} \big(|G(u, \tilde{\chi} uhn, w_j)| \le 2^{-n}\big)$.

(4) and (5) imply that $\big(\tilde{\chi} u(\tilde{\Phi}_0 uk)(\Phi_0 uk)\big)_{k \in \mathbb{N}}$ is a Cauchy sequence in V_u with modulus 2^{-k}:

For $m, n \ge k$ define $\tilde{c} :=$max$(\tilde{\Phi}_0 um, \tilde{\Phi}_0 un)$, $c :=$max$(\Phi_0 um, \Phi_0 un)$.

(4) and (5) yield $d_V\big(\tilde{\chi} u(\tilde{\Phi}_0 um)(\Phi_0 um), \tilde{\chi} u\tilde{c}c\big) < 2^{-m-1}$ and

$d_V\big(\tilde{\chi} u(\tilde{\Phi}_0 un)(\Phi_0 un), \tilde{\chi} u\tilde{c}c\big) < 2^{-n-1}$, and hence

$d_V\big(\tilde{\chi} u(\tilde{\Phi}_0 um)(\Phi_0 um), \tilde{\chi} u(\tilde{\Phi}_0 un)(\Phi_0 un)\big) < 2^{-m-1} + 2^{-n-1} \le 2^{-k-1} + 2^{-k-1} = 2^{-k}$.

By 3.4 and the definition of Φ, it follows that

(6) $\Phi u = \lim_{k \to \infty} \tilde{\chi} u(\tilde{\Phi}_0 uk)(\Phi_0 uk)$ with modulus of convergence 2^{-k}.

We have to show that $\forall w \in W\big(G(u, \Phi u, w) = 0\big)$ (Provable in $WE-HA^\omega$): Let $k, m \in \mathbb{N}$ be arbitrary but fixed. $G(u, \cdot, w_k)$ is continuous in Φu with a modulus of continuity $\omega_{u,k}$ (for short: ω) (3.15).

$a :=\max\big(k, \tilde{\Phi}_0 u(\omega(m)+1)\big), b :=\max\big(m, \Phi_0 u(\omega(m)+1)\big)$. (5) implies

(7) $\bigwedge\limits_{j=0}^{a} \big(|G(u, \tilde{\chi}uab, w_j)| \leq 2^{-b}\big)$ and therefore in particular

(8) $|G(u, \tilde{\chi}uab, w_k)| \leq 2^{-m}$. (7) yields (together with (4),(5))

(9) $d_V\big(\tilde{\chi}uab, \tilde{\chi}u\big(\tilde{\Phi}_0 u(\omega(m)+1)\big)\big(\Phi_0 u(\omega(m)+1)\big)\big) < 2^{-\omega(m)-1}$. By (6) we have

(10) $d_V\big(\Phi u, \tilde{\chi}u\big(\tilde{\Phi}_0 u(\omega(m)+1)\big)\big(\Phi_0 u(\omega(m)+1)\big)\big) \leq 2^{-\omega(m)-1}$. (9) and (10) imply

(11) $d_V(\Phi u, \tilde{\chi}uab) < 2^{-\omega(m)}$.

By (8) and (11) and the definition of ω it follows that $|G(u, \Phi u, w_k)| \leq 2^{-m} + 2^{-m}$. Since $k, m \in \mathbb{N}$ were arbitrary, we have $\forall k\big(G(u, \Phi u, w_k) = 0\big)$ and therefore $\forall w \in W\big(G(u, \Phi u, w) = 0\big)$ by the continuity of $G(u, \Phi u, \cdot)$ in w (3.15). Since $\tilde{\chi}uk \in V_u$, (6) implies $\Phi u \in V_u$ for all $u \in U$ (V_u is closed since V_u is compact in the CSM–space V).
2) is proved analogously.

Proof of 4.5: 1) By the assumptions of 4.5 and by theorem 4.4 we can construct a functional $\Phi \in T$ such that $\forall u \in U, w \in W\big(G(u, \Phi u, w) = 0\big)$. It remains to show that $\Phi \in C(U, V)$ (provable in $WE-HA^\omega$): Because of 3.15 it suffices to show that Φ is extensional w.r.t. $=_U, =_V$, i.e.
$WE-HA^\omega \vdash \forall u_1^1, u_2^1(u_1 =_U u_2 \to \Phi u_1 =_V \Phi u_2)$: Assume $u_1 =_U u_2$. We have
$(*)$ $\forall w \in W\big(G(u_1, \Phi u_1, w) = 0\big) \wedge \forall w \in W\big(G(u_2, \Phi u_2, w) = 0\big)$. By the extensionality of G (3.13), $(*)$ implies $\forall w \in W\big(G(u_2, \Phi u_1, w) = 0\big)$. The uniqueness of $v \in V_{u_2}$ such that $\forall w\big(G(u_2, v, w) = 0\big)$ and the extensionality of V_u in u (3.22) yield, together with $(*)$, $\Phi u_1 =_V \Phi u_2$ (Since 4.4 applied to the assumptions of 4.5 implies that the uniqueness of v is provable in $WE-HA^\omega$ and since the extensionality of G and V_u is by assumption provable in $WE-PA^\omega$, we have shown the extensionality of Φ in $WE-PA^\omega$ and therefore, by negative translation, also in $WE-HA^\omega$).
2) is proved analogously.

4.6 Corollary

Let α, β be as before.

1) $$E-PA^\omega + WKL + AC^{\alpha,\beta}\text{–qf} \vdash \forall u \in U\Big(\forall x \in X \exists y \in Y_{x,u} \forall z \in Z\big(F(u,x,y,z) = 0\big) \to \exists! v \in V_u \forall w \in W\big(G(u,v,w) = 0\big)\Big)$$

$\Rightarrow \exists$ a closed term $\Phi \in T$ such that

$$WE-HA^\omega \vdash \forall u \in U\big(\forall x \in X, l, m \in \mathbb{N} \exists y \in Y_{x,u} \bigwedge_{i=0}^{l} \big(|F(u,x,y,z_i)| \leq 2^{-m}\big) \to \forall w \in W\big(G(u, \Phi u, w) = 0\big) \wedge \Phi u \in V_u\big).$$

2) An analogous result holds for $\widehat{E-PA}^{\omega}\!\restriction, \widehat{PR}, \widehat{WE-HA}^{\omega}\!\restriction$.

Φ can be extracted from any given proof of the assumption by functional interpretation and majorization.

Proof: The assertion follows immediately from the proof of 4.4 (see the proof of 2.12 above).

5 General applications to approximation theory

In the examples from best approximation theory discussed below we are concerned with the computation of a uniquely determined extremal point of a function $G(u,\cdot): V \to \mathbb{R}$ on a compact set $V_u \subset V$ (uniformly in $u \in U$). This situation can be reduced to the one in 4.3–4.6:
By 3.25.2 $\tilde{G}(u,v) := G(u,v) - \inf\limits_{\tilde{v}\in V_u} G(u,\tilde{v})$ is T–definable if G is, and for $v \in V_u$ we have $\tilde{G}(u,v) = 0 \leftrightarrow G(u,v) = \inf\limits_{\tilde{v}\in V_u} G(u,\tilde{v})$. Hence 4.3–4.6 can be applied to
"$G(u,v) = \inf\limits_{\tilde{v}\in V_u} G(u,\tilde{v})$" instead of "$\forall w \in W\big(G(u,v,w)=0\big)$". In particular, since (by 3.25.2)
$WE-HA^\omega \vdash \forall u \in U, m \in \mathbb{N} \exists v \in V_u\big(|\tilde{G}(u,v)| \le 2^{-m}\big)$, 4.4 applied to $\tilde{G}$ yields

5.1 Theorem

For $\alpha = 0 \wedge \beta \le 1$ or $\alpha = 1 \wedge \beta = 0$ the following rule holds:
1) $E-PA^\omega + WKL + AC^{\alpha,\beta}\text{–qf} \vdash \forall u \in U, v_1, v_2 \in V_u\Big(\forall x \in X \exists y \in Y_{x,u} \forall z \in Z\big(F(\ldots,z)=0\big)$

$$\to \big(G(u,v_1) = \inf_{v\in V_u} G(u,v) = G(u,v_2) \to v_1 = v_2\big)\Big)$$

$\Rightarrow \exists$ a closed term $\Phi \in T$ such that

$$WE-HA^\omega \vdash \forall u \in U\big(\forall v_1, v_2 \in V_u, x \in X, l, m \exists y \in Y_{x,u} \bigwedge_{i=0}^{l} \big(|F(\ldots,z_i)| \le 2^{-m}\big)$$

$$\to G(u,\Phi u) = \inf_{v\in V_u} G(u,v) \wedge \Phi u \in V_u\Big).$$

2) An analogous result holds for $\widehat{E-PA}^\omega\restriction, \widehat{PR}$ and $\widehat{WE-HA}^\omega\restriction$ instead of $E-PA^\omega, T$ and $WE-HA^\omega$.

The theory of best approximation offers an interesting class of examples for applications of 4.3 and 5.1 since

1) non–trivial and non–constructive proofs of the uniqueness of the best approximation are central,
2) the uniqueness theorems have the logical form which is supposed in 4.3 and 5.1,
3) various uniqueness proofs in approximation theory make essential use of lemmas having the logical form $(*)$ $\forall x \in X \exists y \in Y_x \forall z \in Z\big(F(x,y,z)=0\big)$ and are formalizable in $E-PA^\omega + AC^{0,0}$–qf+lemmas $(*)$ and
4) numerical data as moduli of uniqueness (which are investigated in approximation theory under the heading "strong unicity") as well as algorithms for the computation of best approximations are of practical interest.

One of the theoretical fundamentals of best approximation theory is the following classical existence theorem:

5.2 Theorem

Let (X,d) be a metric space and K a compact non–void subset of X. Then each $x \in X$ possesses a best approximation in K, i.e.
$\forall x \in X \exists y_b \in K\big(d(x,y_b) = \text{dist}(x,K)\big)$, where $\text{dist}(x,K) := \inf\limits_{y\in K} d(x,y)$.

Proof: $d(x,\cdot) : X \to \mathbb{R}^+$ is continuous (w.r.t. the topology induced by d and the Euclidean topology on $\mathbb{R}$). Hence $d(x,\cdot)$ assumes its infimum on K.

For the linear case the following theorem holds (see e.g. Achieser (47))

5.3 Theorem

Let $(X, \|\cdot\|)$ be a real normed space and $E \subset X$ a finite dimensional subspace of X. Then $\forall x \in X \exists y_b \in E\big(\text{dist}(x,E) = \|x - y_b\|\big)$.

Proof: Assume $\|x-y\| \le \|x-0\| (= \|x\|)$, where 0 is the zero vector in E ($x \in X, y \in E$). Then $\|y\| \le 2\|x\|$ since $\|y\| - \|x\| \le \|x-y\|$. Hence $\text{dist}(x,E) = \text{dist}(x,K_x)$, where $K_x := \{y \in E : \|y\| \le 2\|x\|\}$ is compact in E (since E is finite dimensional). Therefore 5.3 follows from 5.2.

For the usual examples of spaces X and E in approximation theory as e.g. $(X, \|\cdot\|) := (C[0,1], \|\cdot\|_\infty)$ and $E :=$ {algebraic polynomials over $\mathbb{R}$ of degree $\le n$} $(=: P_n)$, $\tilde{K}_x := \{y \in E : \|y\| \le 2\|x\| + 1\}$ is a $\widehat{PR}$–definable family of compact sets in X and thus $\text{dist}(x,E) = \text{dist}(x,\tilde{K}_x)$ (see the proof of 5.3) is a $\widehat{PR}$–definable function in x. Hence if the uniqueness of the best approximation of every $x \in X$ in E can be proved within $E-PA^\omega + WKL + AC^{0,0}$+assumptions having the form $\forall u \in U \exists v \in V_u \forall z \in Z\big(F(u,v,z) = 0\big)$, then by 4.3 we can extract moduli of uniqueness $\Phi \in T$, i.e.

$$(1)\ \forall x \in X, y_1, y_2 \in \tilde{K}_x, k \in \mathbb{N}\big(\|y_1 - x\|, \|y_2 - x\| - \text{dist}(x,E) \le 2^{-\Phi xk} \to \|y_1 - y_2\| \le 2^{-k}\big).$$

Φ can easily extended to $q \in \mathbb{Q}^*_+$ instead of $k \in \mathbb{N}$:

$$(2)\ \forall x \in X, y_1, y_2 \in \tilde{K}_x, q \in \mathbb{Q}^*_+\big(\|y_1 - x\|, \|y_2 - x\| - \text{dist}(x,E) \le \tilde{\Phi} xq \to \|y_1 - y_2\| \le q\big),$$

where $\tilde{\Phi} xq := 2^{-\Phi xk}$ for the least $k \in \mathbb{N}$ such that $q \ge 2^{-k}$.
Furthermore $\tilde{\Phi}$ can be generalized to a modulus of uniqueness on E instead of $\tilde{K}_x$ (although E is not compact in X!):
Define $\widehat{\Phi} xq := \min(1, \tilde{\Phi} xq)$:
Claim: $\forall x \in X, y_1, y_2 \in E, q \in \mathbb{Q}^*_+\big(\|y_1 - x\|, \|y_2 - x\| - \text{dist}(x,E) \le \widehat{\Phi} xq \to \|y_1 - y_2\| \le q\big)$.

Proof: For $y_1, y_2 \in \tilde{K}_x$ the claim follows from (2). Assume $y_1 \notin \tilde{K}_x$. Then $\|y_1\| > 2\|x\| + 1$. If $\|y_1 - x\| \le \text{dist}(x,E) + \widehat{\Phi} xq \le \|x\| + 1$, then $\|y_1\| \le 2\|x\| + 1$ which is a contradiction.

As was shown above, moduli of uniqueness allow the construction of finite algorithms for the best approximation with prescribed precision (5.1).
The importance of the notion "modulus of uniqueness" in the context of approximation theory is further indicated by the following proposition (which is provable in $\widehat{WE-HA^\omega}\!\upharpoonright$):

5.4 Proposition

Let $(X, \|\cdot\|)$ be a complete separable normed space, $E \subset X$ a finite dimensional subspace. Assume that every $x \in X$ possesses a uniquely determined best approximation in E and that the operation Φ is a modulus of uniqueness, i.e. (in particular)

$\forall x \in X, y, y_b \in E, q \in \mathbb{Q}_+^* \big(\|x-y\| - \text{dist}(x,E) \le \Phi x q \wedge \|x - y_b\| = \text{dist}(x,E) \to \|y - y_b\| \le q \big)$.
Then the following holds
1) $\frac{1}{2} \cdot \Phi$ is a modulus of pointwise continuity for the projection $\mathcal{P} : X \to E$ which maps $x \in X$ to its best approximation $y_b \in E$, i.e.
$\forall x, x_0 \in X, q \in \mathbb{Q}_+^* \big(\|x - x_0\| \le \frac{1}{2}\Phi x_0 q \to \|\mathcal{P}x - \mathcal{P}x_0\| \le q \big)$.

If Φ is linear in q,i.e. $\Phi x q = q \cdot \gamma(x)$, then
2) $\gamma(x)$ is a "constant of strong unicity", i.e.
$\forall x \in X, y \in E\big(\|x-y\| \ge \|x-y_b\| + \gamma(x) \cdot \|y - y_b\| \big)$, where y_b is the best approximation of x in E,
3) $\lambda(x) := \frac{2}{\gamma(x)}$ is a pointwise Lipschitz constant for $\mathcal{P}$, i.e.
$\forall x, x_0 \in X \big(\|\mathcal{P}x - \mathcal{P}x_0\| \le \lambda(x_0) \cdot \|x - x_0\| \big)$.

Proof: 1) One easily verifies that $\text{dist}(\cdot, E)$ is Lipschitz continuous in x with Lipschitz constant 1, i.e.
$(*)$ $\|x - x_0\| \le \varepsilon \to |\text{dist}(x,E) - \text{dist}(x_0,E)| \le \varepsilon$. Now assume $\|x - x_0\| \le \frac{1}{2}\Phi x_0 q$. Then

$$(1)\ \|x_0 - \mathcal{P}x\| \le \|x_0 - x\| + \|x - \mathcal{P}x\| = \text{dist}(x,E) + \|x_0 - x\| \overset{(*)}{\le} \text{dist}(x_0,E) + \tfrac{1}{2}\Phi x_0 q + \|x - x_0\| \le \text{dist}(x_0,E) + \Phi x_0 q$$

and (2) $\|x_0 - \mathcal{P}x_0\| \le \text{dist}(x_0,E)$. Since Φ is a modulus of uniqueness it follows that $\|\mathcal{P}x - \mathcal{P}x_0\| \le q$.
2) The assumption that Φ is a modulus of uniqueness implies
$\|y - y_b\| \ge q \to \|x - y\| \ge \underbrace{\text{dist}(x,E)}_{=\|x-y_b\|} + \gamma(x) \cdot q \cdot \frac{n}{n+1}$ for all $n \in \mathbb{N}$. $q := \|y - y_b\|$ yields
$\forall n \in \mathbb{N}\big(\|x-y\| \ge \|x - y_b\| + \gamma(x) \cdot \|y - y_b\| \cdot \frac{n}{n+1} \big)$. Hence $\|x - y\| \ge \|x - y_b\| + \gamma(x) \cdot \|y - y_b\|$.
3) follows immediately from 1).

5.5 Remark

1) In the case of Chebycheff approximation the existence of a constant of strong unicity $\gamma(x) > 0$ which fulfils 5.4.2 was proved non–constructively in Newman/Shapiro (63) (The name "constant of strong unicity" is due to Cheney (66)). In the literature the name constant of strong unicity often refers to the least $\gamma(x) > 0$ which satisfies 5.4.2.

2) The fact that $\lambda(x) = \frac{2}{\gamma(x)}$ is a Lipschitz constant for the projection $\mathcal{P}$ if $\gamma(x)$ is a constant of strong unicity, was first established in Cheney (66).

In many cases, the uniqueness of the best approximation can be proved classically:

5.6 Examples

1) For every $f \in C[0,1]$ there exists a uniquely determined algebraic polynomial $p_{b,f} \in P_n :=$ {algebraic polynomials having degree $\le n$ with coefficients in $\mathbb{R}$} such that
$\|f - p_{b,f}\|_\infty = \text{dist}(f, P_n) = \text{dist}(f, Lin_{\mathbb{R}}(1, x, \ldots, x^n))$, where $\|\cdot\|_\infty$ denotes the sup norm and $\text{dist}(f, P_n) := \inf\limits_{p \in P_n} \|f - p\|_\infty$.
More general: An n–tuple $(\phi_1, \ldots, \phi_n)$ of functions $\in C[0,1]$ is called "Chebycheff system" if it satisfies the so–called "Haar condition", i.e. if each generalized polynomial
$\alpha_1\phi_1 + \ldots + \alpha_n\phi_n$ $(\alpha_1, \ldots, \alpha_n \in \mathbb{R})$, which is not the zero polynomial on $[0,1]$, has at most

$n-1$ distinct zeroes. It is well known that:
$[\forall f \in C[0,1]$ (f has exactly one best approximation in $Lin_{\mathbb{R}}(\phi_1,\ldots,\phi_n)) \Leftrightarrow$
$(\phi_1,\ldots,\phi_n)$ is a Chebycheff system] (Haar (18)).

2) For $f \in C[0,1]$ the norm $\|\cdot\|_1$ is defined as $\|f\|_1 := \int_0^1 |f(x)|dx$.
Theorem (D. Jackson (21)): $\forall f \in C[0,1]\exists! p_{b,f} \in P_n\big(\|f-p_{b,f}\|_1 = \mathrm{dist}_1(f,P_n)\big)$, where $\mathrm{dist}_1(f,P_n) := \inf\limits_{p\in P_n} \|f-p\|_1$.
This theorem also generalizes to arbitrary Chebycheff systems.

3) A normed space $(X,\|\cdot\|)$ is called strictly convex if
$\forall x,y \in X\big(\|x\| = \|y\| = \|\frac{1}{2}(x+y)\| = 1 \to x = y\big)$. The following theorem (due to M. Krein) holds:
If X is strict convex and $E \subset X$ finite dimensional then
$\forall x \in X \exists! b_x \in E\big(\|x-b_x\| = \mathrm{dist}(x,E)\big)$. Examples: Hilbert spaces, L_p $(1<p<\infty)$. Note that neither $\big(C[0,1],\|\cdot\|_\infty\big)$ nor $\big(C[0,1],\|\cdot\|_1\big)$ or $\big(L_1,\|\cdot\|_1\big)$ are strict convex.

Whereas in 1)–3) the set of approximations is a linear space, it is only a convex subset of a linear space in the next example:

4) Chebycheff approximation by polynomials having bounded coefficients:
Let $k_i \in \mathbb{N}$, $0 \le i \le p$ be such that $0 < k_0 < k_1 < \ldots < k_p \le n$ $(p \le n)$ and $a_1,\ldots,a_p,b_1,\ldots,b_p \in \overline{\mathbb{R}} := \mathbb{R}\cup\{-\infty,+\infty\}$ such that
(i) $a_i \neq +\infty$, (ii) $b_i \neq -\infty$, (iii) $a_i \le b_i$ for $i = 0,1,\ldots,p$.
$K_n := \big\{\Sigma_{i=0}^n c_i x^i | a_i \le c_{k_i} \le b_i; i = 0,1,\ldots,p\big\}$. K_n is a convex subset of P_n.
Theorem (J.A. Roulier/G.D. Taylor (71)):
$\forall f \in C[0,1]\exists! p_{b,f} \in K_n\big(\|f-p_{b,f}\|_\infty = \mathrm{dist}(f,K_n)\big)$.
4.3 and 5.1 can be applied to this situation since K_n can be replaced by a compact subset: Let $p_0 \in K_n$. Then
$\forall f \in C[0,1], p \in K_n : \|f-p\|_\infty = \mathrm{dist}(f,K_n) \to$
$\|f-p\|_\infty \le \|f-p_0\|_\infty \le \|f\|_\infty + \|p_0\|_\infty \to$
$\|p\|_\infty \le 2\|f\|_\infty + \|p_0\|_\infty$. Hence $\mathrm{dist}(f,K_n) = \mathrm{dist}(f,\tilde{K}_n)$ where $\tilde{K}_n := \{p \in K_n : \|p\|_\infty \le 2\|f\|_\infty + \|p_0\|_\infty\}$.

In the following we discuss the examples 1) and 3) in more detail. For 2) and 4) we refer to Kohlenbach (90) where it is shown that classical proofs of the uniqueness in these cases can be formalized in $WE-PA^\omega + \forall f \in C[0,1]\exists x_0 \in [0,1]\big(F(x_0) = \sup\limits_{x\in[0,1]} f(x)\big)$ (For 2) the uniqueness proof from Cheney (66) is used).

1) Linear Chebycheff approximation

In this paper we restrict ourselves to the case of Chebycheff approximation by algebraic polynomials $p \in P_n$. $(P_n,\|\cdot\|_\infty)$ can be identified with $(\mathbb{R}^{n+1},\|\cdot\|_\infty)$ via coefficients, where $\|(c_0,\ldots,c_n)\|_\infty := \sup\limits_{x\in[0,1]} |c_n x^n + \ldots c_1 x + c_0|$. Since $\|\cdot\|_{\max}$ and $\|\cdot\|_\infty$ are equivalent norms on $\mathbb{R}^{n+1}$, a rate of convergence w.r.t. $\|\cdot\|_{\max}$ yields a rate of convergence w.r.t. $\|\cdot\|_\infty$ (by multiplication with a suitable constant) and vice versa. 5.3 implies the existence of a polynomial $p_{b,f} \in P_n$ such that $\|f-p_{b,f}\|_\infty = \mathrm{dist}(f,P_n)$.

This was first proved in Kirchberger (02), where also the uniqueness of $p_{b,f}$ is shown (based on prior work by Chebycheff (59)). The most common uniqueness proof in the literature appeared first in de La Vallée Poussin (19). A more explicit presentation of this proof is given in Natanson (49) on which the following outline is based. In 6 we will give an explicit proof–theoretical analysis of this proof in the sense above and extract a modulus of uniqueness with all numerical details.

Let (A) denote the sentence

$$(A)\ \forall f \in C[0,1], a,b \in [0,1], a < b \exists x_0 \in [a,b](f(x_0) = \sup_{x\in[a,b]} f(x))$$

and take $f \in C[0,1], n \in \mathbb{N}$ as fixed but arbitrary in the following. $E_{n,f} :=\mathrm{dist}(f, P_n)$. We show that the uniqueness proof from de La Vallée Poussin/Natanson can be formalized in $WE-PA^\omega + (A)$. Since A is logically equivalent to a $\forall x \in X \exists y \in K_x A_1$–sentence with $A_1 \in \Pi^0_1$, we can apply 4.3 and 5.1.

In the following we write for simplicity $\|\cdot\|$ instead of $\|\cdot\|_\infty$.

Since the uniqueness of $p_{b,f}$ is trivial for $E_{n,f} = 0 (p_{b,f} = f!)$, we may assume that $E_{n,f} > 0$. Let $p_b \in P_n$ be a best approximation of f, i.e. $\|f - p_b\| = E_{n,f}$.

$|p_b(x) - f(x)| \in C[0,1]$ assumes its maximum on $[0,1]$ (A), i.e. $\exists x_0 \in [0,1]\big(|p_b(x_0) - f(x_0)| = E_{n,f}\big)$. A point with this property is called an extremal point (abbreviation: (e)–point) of $p_b - f$. An (e)–point x_0 is a (+)–point if $p_b(x_0) - f(x_0) = E_{n,f}$ and a (–)–point if $p_b(x_0) - f(x_0) = -E_{n,f}$.

5.7 Lemma (A)

There exist always both (+)–points and (–)–points.

Proof: Assume without loss of generality that no (–)–point exists, i.e.
$\forall x \in [0,1]\big(p_b(x) - f(x) > -E_{n,f}\big)$. (A) yields that $\inf_{x\in[0,1]}\big(p_b(x) - f(x)\big) > -E_{n,f}$. Define $h := \frac{1}{2}\big(\inf_{x\in[0,1]}\big(p_b(x) - f(x)\big) + E_{n,f}\big)$. Then $h > 0$ and $\inf_{x\in[0,1]}\big(p_b(x) - f(x)\big) = -E_{n,f} + 2h$. Hence $-E_{n,f} + 2h \le p_b(x) - f(x) \le E_{n,f}$ and therefore $-E_{n,f} + h \le p_b(x) - h - f(x) \le E_{n,f} - h$ for all $x \in [0,1]$, which implies that $p_b - h \in P_n$ is a better approximation of f than p_b, contradiction.

5.8 Remark

The proof of 5.7 can be formalized in $WE-PA^\omega + (A)$ (and hence e.g. in $WE-PA^\omega + WKL + AC^{0,0}$–qf). (A) is used essentially: Even for $n = 0$, 5.7 implies
$\forall f \in C[0,1] \exists x_0 \in [0,1]\big(f(x_0) = \sup_{x\in[0,1]} f(x)\big)$ and hence (A) (relative to $WE-HA^\omega$): Let m resp. M denote the infimum resp. supremum of f on $[0,1]$. Then $E_{0,f} =\mathrm{dist}(f, P_0) = \frac{1}{2}(M - n)$ and $\frac{1}{2}(M + n)$ is the best approximation of f in P_0. 5.7 yields the existence of an $x_0 \in [0,1]$ such that $f(x_0) - \frac{1}{2}(M + n) = \frac{1}{2}(M - n)$, i.e. $f(x_0) = M$ (This argument is taken from Bridges (82A)).

5.9 Theorem (P.L. Chebycheff,Kirchberger)

In $[0,1]$ there exists a sequence of $n+2$ points $x_1 < x_2 < \ldots < x_{n+2}$ which are alternating (+)–points and (–)–points, i.e.
$\bigwedge_{i=1}^{n+2}\big(p_b(x_i) - f(x_i) = (-1)^{i+j} E_{n,f}\big)$ for $j = 0$ or $j = 1$. Such a system of points is called alternant of

$p_b - f$.

Proof (Natanson (49)): Since $f \in C[0,1]$ is uniformly continuous on $[0,1]$ we can construct points $\xi_0 = 0 < \xi_1 < \xi_2 < \ldots < \xi_s = 1$ such that
$\sup_{x\in I_i}\big(p_b(x)-f(x)\big) - \inf_{x\in I_i}\big(p_b(x)-f(x)\big) < \frac{1}{2}E_{n,f}$, where $I_i = [\xi_i, \xi_{i+1}]$ for $i = 0,\ldots,s-1$. I_i is called (e)–interval if it contains an (e)–point of $p_b - f$. If I_i is an (e)–interval then $p_b(x) - f(x) \neq 0$ for all $x \in I_i$. An (e)–interval I_i is called (+)–interval if $\forall x \in I_i(p_b(x) - f(x) > 0)$ and (−)–interval if $\forall x \in I_i(p_b(x) - f(x) < 0)$. We number the (e)–intervals consecutively from left to right $I_{j_1},\ldots,I_{j_N}$ and assume without loss of generality that I_{j_1} is a (+)–interval. Therefore we get the following schema

$$(*)\begin{cases} I_{j_1},\ldots,I_{j_{k_1}} & (+)\text{-intervals}\\ I_{j_{k_1+1}},\ldots,I_{j_{k_2}} & (-)\text{-intervals}\\ \vdots & \\ I_{j_{k_{m-1}+1}},\ldots,I_{j_{k_m}} & (-1)^{-m-1}\text{-intervals } (k_m = N).\end{cases}$$

5.7 implies $m \geq 2$. We show that $m \geq n+2$.
Assume: $(**)$ $m < n+2$. $p_b - f$ has different signs in $I_{j_{k_1}}$ and $I_{j_{k_1+1}}$. Hence the right boundary of $I_{j_{k_1}}$ does not coincide with the left boundary of $I_{j_{k_1+1}}$. Therefore there exists a point $z_1 \in [0,1]$ which is strictly greater than all points of $I_{j_{k_1}}$ and strictly less than all points of $I_{j_{k_1+1}}$ (we write: $I_{j_{k_1}} < z_1 < I_{j_{k_1+1}}$). Analogously, one shows the existence of $z_2, z_3, \ldots, z_{m-1} \in [0,1]$ such that $I_{j_{k_i}} < z_i < I_{j_{k_i+1}}$ for $i = 2,\ldots,m-1$.
Define $\rho(x) := (z_1 - x)(z_2 - x)\cdot\ldots\cdot(z_{m-1} - x)$. $(**)$ implies that $\rho(x) \in P_n$. Since no interval $I_{j_1},\ldots,I_{j_N}$ contains one of the zeroes $z_1,\ldots,z_{m-1}$ of $\rho(x)$, $\rho(x)$ has constant sign on each I_{j_i} $(1 \leq i \leq N)$. This sign is the same as the sign of $p_b(x) - f(x)$ on I_{j_i}.
Now consider an interval I_i of the partition $I_1,\ldots,I_s$ of $[0,1]$ which is not an (e)–interval, i.e. $\forall x \in I_i(|p_b(x) - f(x)| < E_{n,f})$. (A) yields that $\sup_{x\in I_i}|p_b(x) - f(x)| < E_{n,f}$ and therefore
$E^* := \max\big\{\sup_{x\in I_i}|p_b(x) - f(x)| : I_i \text{ is not an (e)–interval}\big\} < E_{n,f}$. Define $R := \sup_{x\in[0,1]}|\rho(x)|$. For sufficiently small $\lambda > 0$ one has $\lambda R < E_{n,f} - E^*$ and $\lambda R < \frac{1}{2}E_{n,f}$. $Q(x) := p_b(x) - \lambda\rho(x) \in P_n$.
We show: $\forall x \in [0,1]\big(|Q(x) - f(x)| < E_{n,f}\big)$ (Using (A) this implies $\|Q - f\| < E_{n,f}$ which contradicts the definition of $E_{n,f}$):
Case 1: I_i is not an (e)–interval:

$$|Q(x) - f(x)| \leq |p_b(x) - f(x)| + \lambda|\rho(x)| \leq E^* + \lambda R < E_{n,f} \text{ for all } x \in I_i.$$

Case 2: I_i is an (e)–interval: Let $x \in I_i$. $p_b(x) - f(x)$ and $\lambda\cdot\rho(x)$ have the same sign and $|p_b(x) - f(x)| > \lambda\cdot|\rho(x)|$, since $|p_b(x) - f(x)| \geq \frac{1}{2}E_{n,f}$ and $\lambda\cdot|\rho(x)| < \frac{1}{2}E_{n,f}$. It follows that
$|Q(x) - f(x)| = |p_b(x) - f(x) - \lambda\rho(x)| = |p_b(x) - f(x)| - \lambda\cdot|\rho(x)|$. Hence
$|Q(x) - f(x)| \leq E_{n,f} - \lambda\cdot|\rho(x)| < E_{n,f}$, since $\rho(x) \neq 0$ on (e)–intervals.

5.10 Corollary

There exists at most one (and therefore by 5.3 exactly one) polynomial of best approximation for $f \in C[0,1]$ in P_n.

Proof: Assume that $p_1, p_2 \in P_n$ are best approximations of f, i.e. $\|p_1 - f\| = \|p_2 - f\| = E_{n,f}$.

Then $p(x) := \frac{p_1(x)+p_2(x)}{2} \in P_n$ is also a best approximation of f. By 5.9 there exists an alternant $x_1 < x_2 < \ldots < x_{n+2}$ for $p(x) - f(x)$ in $[0,1]$. Assume that x_k is a (+)–point of $p(x) - f(x)$. Then $\frac{p_1(x_k)-f(x_k)}{2} + \frac{p_2(x_k)-f(x_k)}{2} = E_{n,f}$. Since $p_2(x_k) - f(x_k) \le E_{n,f}$ it follows that $\frac{p_1(x_k)-f(x_k)}{2} + \frac{E_{n,f}}{2} \ge E_{n,f}$. Hence $p_1(x_k) - f(x_k) \ge E_{n,f}$ and therefore $p_1(x_k) - f(x_k) = E_{n,f}$ (since p_1 is a best approximation). Analogously one shows that $p_2(x_k) - f(x_k) = E_{n,f}$. Thus $p_1(x_k) = p_2(x_k)$ for the (+)–points of the alternant $x_1, \ldots, x_{n+2}$. An analogous argument establishes this for the (–)–points of $x_1, \ldots, x_{n+2}$. Hence $p_1, p_2 \in P_n$ coincide on $n+2$ distinct points, which implies $p_1 = p_2$.

5.11 Remark

The argument used in the proof of 5.10 to derive the uniqueness of the best approximation from the existence of $n+2$ distinct points x_i such that $|p_b(x_i) - f(x_i)| = E_{n,f}$ appeared first in Tonelli (08).

Under the assumption of 5.9, 5.10 is provable in $\widehat{WE-HA}^\omega$. Therefore it remains to show that 5.9 can be proved within $WE-PA^\omega + (A)$: The proof of 5.9 uses essentially (A) and classical logic. Furthermore for the definition of E^* and the schema $(*)$ we need the following schema of "pre"–comprehension:

$$(+)\ \forall k^0, x_0^1, \ldots, x_k^1 \exists i_0^0, \ldots, i_k^0 \bigwedge_{j=0}^{k} \big(i_j = 0 \leftrightarrow A(x_j)\big),$$

which can be expressed in $\mathcal{L}(WE-PA^\omega)$ via primitive recursive coding of tuples as follows

$$(++)\ \forall k^0, x^1 \Big(lth_1 x =_0 k \to \exists i^0 \big(lth\ i = k \wedge \forall j^0 \big(j < k \to [(i)_j =_0 0 \leftrightarrow A((x)_j)]\big)\big)\Big),$$

where $A \in \mathcal{L}(WE-PA^\omega)$. $(++)$ can easily be proved in $WE-PA^\omega$ using classical logic and induction on k. By 5.7 and $(++)$ it follows that for every $n \in \mathbb{N}, f \in C[0,1]$ the schema $(*)$ (in the proof of 5.9) and E^* exist, which leads using (A) to a contradiction to the assumption $(**)$. Hence we have shown

$WE-PA^\omega + (A) \vdash \forall f \in C[0,1], n \in \mathbb{N}, (c_0, \ldots, c_n), (\tilde{c}_0, \ldots, \tilde{c}_n) \in \mathbb{R}^{n+1}$

$$\big(\|f - (c_n x^n + \ldots + c_0)\| = \text{dist}(f, P_n) = \|f - (\tilde{c}_n x^n + \ldots + \tilde{c}_0)\| \to \bigwedge_{i=0}^{n} c_i = \tilde{c}_i\big).$$

As was seen above, P_n can be replaced by $\tilde{K}_{f,n} := \{p \in P_n : \|p\| \le 2\|f\| + 1\}$, and a modulus of uniqueness for $p_1, p_2 \in \tilde{K}_{f,n}$ can be easily extended to a modulus on P_n. $p \in \tilde{K}_{f,n}$ implies that the coefficients $c_0, \ldots, c_n$ of p are bounded $|c_i| \le \chi(f,n)$, where χ is a primitive recursive function on f and n (see below). Applying 4.3 to the result above yields a modulus of uniqueness Ψ on $A_{f,n} := \Big\{(c_0, \ldots, c_n) : \bigwedge_{i=0}^{n} |c_i| \le \chi(f,n)\Big\} \subset \mathbb{R}^{n+1}$, which extends to a modulus on $\mathbb{R}^{n+1}$ (provable in $WE-HA^\omega + (A^\varepsilon)$, where

$(A^\varepsilon):\ \forall f \in C[0,1], a, b \in [0,1], a < b, k \in \mathbb{N} \exists x_0 \in [a,b] \big(|f(x_0) - \sup_{x \in [a,b]} f(x)| < 2^{-k}\big)$.

Since this "ε–version" of (A) is provable in $WE-HA^\omega$ it follows that

5.12 Theorem

$WE-HA^\omega \vdash \forall f \in C[0,1], n \in \mathbb{N}, k \in \mathbb{N}, (c_0, \ldots, c_n), (\tilde{c}_0, \ldots, \tilde{c}_n) \in \mathbb{R}^{n+1}$

$$\big(\|f - (c_n x^n + \ldots + c_1 x + c_0)\|, \|f - (\tilde{c}_n x^n + \ldots + \tilde{c}_1 x + \tilde{c}_0)\| \le \text{dist}(f, P_n) + 2^{-\Psi f n k}$$

$$\rightarrow \bigwedge_{i=0}^{n} |c_i - \tilde{c}_i| \leq 2^{-k}\big),$$

where the functional $\Psi^{0001} \in T$ is an operation on the standard representation of $C[0,1]$ and n,k, i.e. primitive recursive in f, n, k where f is given together with a modulus of uniform continuity on $[0,1]$.

Analogously, the application of 5.1,5.3 yields an algorithm Φ for computation of the coefficients of the best approximation with prescribed precision:

5.13 Theorem

$WE - HA^{\omega} \vdash \forall f \in C[0,1], n \in \mathbb{N}\big(\Phi fn \in A_{f,n} \subset \mathbb{R}^{n+1} \wedge \|f - \widehat{\Phi fn}\| = \mathrm{dist}(f, P_n)\big)$,

where $(c_0, \widehat{\ldots}, c_n)$ denotes the polynomial $c_n x^n + \ldots + c_0$ and $\Phi \in T$. By the uniqueness of the best approximation Φ is extensional in f and thus represents a function $C[0,1] \to \mathbb{R}^{n+1}$, which is pointwise continuous with modulus of continuity $\Psi + 1$ (5.4.1,5.11).

An explicit formula for the operation χ can be given using Markov's inequality $\|p'\| \leq 2n^2 \cdot \|p\|$ for all $p \in P_n$, where p' denotes the derivative of p (see e.g. Meinardus (64),p. 65):

$p = \Sigma_{i=0}^{n} c_i x^i \wedge \|p\| \leq M \in \mathbb{N} \to$

$\|p^{(i)}\| \leq 2^i \cdot n^2(n-1)^2 \cdot \ldots \cdot (n-i+1)^2 M \stackrel{(p^{(i)}(0)=i!c_i)}{\longrightarrow}$

$|c_i| \leq \frac{2^i \cdot n^2(n-1)^2 \cdot \ldots \cdot (n-i+1)^2}{i!} M \leq \frac{2^i \cdot (n^2)^i}{i!} M \leq 2Mn^{2i}.$

Hence we can define $\chi(f,n) := 2(2\|f\|+1)n^{2n}$.

In the next paragraph we extract explicitely a modulus Ψ of uniqueness from the above uniqueness proof. The result is not only $\in T$ (and even $\in \widehat{PR}$) but a very elementary operation in f, n and k and for $\mathrm{dist}(f, P_n) > 0$ linear in k, i.e. a constant of strong unicity (in the sense of 5.4.2).

6 Explicit extraction of moduli of alternation and uniqueness, constants of strong unicity, Lipschitz constants and moduli of continuity from the uniqueness proof for best Chebycheff approximation from de La Vallée Poussin (19) and Natanson (49)

In this paragraph we carry out with all numerical details the extraction of an effective modulus of uniqueness from the proof of the uniqueness of the best approximation of $f \in C[0,1]$ in P_n, which is formulated in 5. More precisely we construct an operation $\Psi : C[0,1] \times \mathbb{N} \times \mathbb{Q}_+^* \to \mathbb{Q}_+^*$ such that for all $f \in C[0,1], n \in \mathbb{N}$

$\forall q \in \mathbb{Q}_+^*, p_1, p_2 \in P_n\big(\|f - p_1\|, \|f - p_2\| \leq \mathrm{dist}(f, P_n) + \Psi fnq \to \|p_1 - p_2\| \leq q\big)$.

Ψ can easily be modified to yield a modulus of uniqueness for the coefficients of p_1, p_2 in the norm $\|\cdot\|_{\max}$: By the Markov inequality it follows that $\|p\| \leq q \to |c_i| \leq 2n^{2n} \cdot q$ for $i = 0, \ldots, n$; $p \in P_n, p(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_0$.

Under the assumption $\mathrm{dist}(f, P_n) \geq r > 0$ ($r \in \mathbb{Q}_+^*$) we obtain a modulus of uniqueness $\tilde{\Psi}$ which is linear in q, i.e. $\tilde{\Psi} fnrq = C(f,n,r) \cdot q$ and hence (by 5.4.2) a constant of strong unicity. By 5.4.1,5.4.3 $\frac{1}{2}\Psi$ resp. $\frac{2}{C(f,n,r)}$ is a modulus of pointwise continuity resp. a Lipschitz constant for the Chebycheff projection.

The key part of the extraction of these moduli is the analysis of the proof of the alternation theorem 5.9 which yields a new quantitative version of the ε–weakening of this theorem: We construct an effective operation χ which is linear in $q \in \mathbb{Q}_+^*$ such that

$$\forall f \in C[0,1], p \in P_n, q, \varepsilon \in \mathbb{Q}_+^* \big(\varepsilon, q < \mathrm{dist}(f, P_n) \wedge \|f-p\| \leq \mathrm{dist}(f, P_n) + (\chi f n k q)\cdot \varepsilon \to$$
$$\exists \text{ an } \varepsilon\text{–alternant of length } k \text{ for } p-f\big)$$

($k = 2, \ldots, n+2$; see theorem 6.13 below).

The moduli Ψ, χ resp. $\tilde{\Psi}$ are simple constructions in the data f (where f is always endowed with a modulus of uniform continuity on $[0,1]$) and n, q resp. f, n, q plus $r \in \mathbb{Q}_+^*$ such that $r \leq \mathrm{dist}(f, P_n)$. In particular $\Psi, \chi, \tilde{\Psi}$ do not depend on the best approximation or the alternation points. Thus, these moduli are a–priori estimates.

6.1 Notation

$\tilde{K}_{f,n} := \{p \in P_n : \|p\| \leq \frac{5}{2}\|f\|\}$, $E_{n,f} := \mathrm{dist}(f, P_n)$.

Let $f \in C[0,1], n \in \mathbb{N}, p \in P_n$ and $1 \leq k \leq n+2$. $(x_1, \ldots, x_k) \in [0,1]^k$ is called alternant of length k for $p - f$ if

$x_1 < x_2 < \ldots < x_k \wedge \bigwedge\limits_{i=1}^{k} \big((-1)^{i+j}(p(x_i) - f(x_i)) = E_{n,f}\big)$ for $j = 0$ or $j = 1$. $\omega_f : \mathbb{Q}_+^* \to \mathbb{Q}_+^*$ is called modulus of uniform continuity of $f \in C[0,1]$ on $[0,1]$ if

$\forall q \in \mathbb{Q}_+^*, x, y \in [0,1]\big(|x-y| < \omega_f(q) \to |fx - fy| < q\big)$.

Logical analysis if the uniqueness proof from de La Vallee Poussin (19)/Natanson (49)
The uniqueness proof from 5 can be split into the following part:

1) $\forall p_1, p_2 \in \tilde{K}_{f,n}\big(\|f - p_1\| = E_{n,f} = \|f - p_2\| \to \|f - \frac{p_1+p_2}{2}\| = E_{n,f}\big)$,

2) $\forall p \in \tilde{K}_{n,f}\big(\|f-p\| = E_{n,f} \to \exists (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}$
$\big((x_1, \ldots, x_{n+1})$ is an alternant for $p - f$ of length $n+1)\big)$,

3) $\forall p_1, p_2 \in \tilde{K}_{f,n}, (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}\big(\|p_1 - f\| = E_{n,f} = \|p_2 - f\|$
$\wedge (x_1, \ldots, x_{n+1})$ is an alternant for $\frac{p_1+p_2}{2} - f \to \bigwedge\limits_{i=1}^{n+1} p_1(x_i) = p_2(x_i)\big)$,

4) $\forall p_1, p_2 \in \tilde{K}_{f,n}, (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}\big(\bigwedge\limits_{i=1}^{n} (x_{i+1} > x_i) \wedge \bigwedge\limits_{i=1}^{n+1} (p_1(x_i) = p_2(x_i))$
$\to p_1 = p_2\big)$,

5) $\forall p \in P_n\big(\|f-p\| = E_{n,f} \to p \in \tilde{K}_{f,n}\big)$.

Claim: 1)–4) $\to \forall p_1, p_2 \in \tilde{K}_{f,n}\big(\|f - p_1\| = E_{n,f} = \|f-p_2\| \to p_1 = p_2\big)$
$\overset{5)}{\to} \forall p_1, p_2 \in P_n\big(\|f-p_1\| = E_{n,f} = \|f - p_2\| \to p_1 = p_2\big)$.

Proof: 1) $\to \|f - \frac{p_1+p_2}{2}\| = E_{n,f} \overset{2)}{\to} \exists (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1} : (x_1, \ldots, x_{n+1})$ is an alternant for $\frac{p_1+p_2}{2} - f \overset{3)}{\to} \exists (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}\big(\bigwedge\limits_{i=1}^{n} (x_{i+1} > x_i) \wedge \bigwedge\limits_{i=1}^{n+1} p_1(x_i) = p_2(x_i)\big) \overset{4)}{\to} p_1 = p_2$.

6.2 Remark

1) The alternation theorem 5.9 yields the existence of an alternant of length $n+2$ in 2). However, in the proof of 5.10 only $n+1$ points are necessary. This can be utilized in the extraction of Ψ and yields a numerical improvement.

2) The restriction to $\tilde{K}_{n,f}$ instead of P_n and the use of 5) are necessary only for the extraction of Ψ (in order to apply 4.3) but not for the proof of the uniqueness.

The main step in the proof of the uniqueness is the proof of the alternation theorem 2). If $E_{n,f} > 0$, then

$\bigwedge\limits_{i=1}^{n} (x_{i+1} \geq x_i) \wedge \exists j \in \{0,1\} \bigwedge\limits_{i=1}^{n+1} \big((-1)^{i+j}(p(x_i) - f(x_i)) = E_{n,f}\big)$ implies that $(x_1, \ldots, x_{n+1})$ is an alternant of length $n+1$ for $p - f$, i.e.

$\bigwedge\limits_{i=1}^{n} (x_{i+1} > x_i) \wedge \exists j \in \{0,1\} \bigwedge\limits_{i=1}^{n+1} \big((-1)^{i+j}(p(x_i) - f(x_i)) = E_{n,f}\big)$.

Since the alternation theorem and the uniqueness of the best approximation are trivial for $E_{n,f} = 0$, 2) can be replaced by

$$2.1)\ \forall p \in \tilde{K}_{f,n}\Big(\|p - f\| = E_{n,f} \to \exists (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\}$$
$$\big(\bigwedge\limits_{i=1}^{n} (x_{i+1} \geq x_i) \wedge \bigwedge\limits_{i=1}^{n+1} \big((-1)^{i+j}(p(x_i) - f(x_i)) = E_{n,f}\big)\big)\Big) \text{ and}$$

$$2.2)\ \bigwedge\limits_{i=1}^{n} (x_{i+1} \geq x_i) \wedge \bigwedge\limits_{i=1}^{n+1} \big(|(-1)^{i+j}(p(x_i) - f(x_i)) - E_{n,f}| < E_{n,f}\big) \to \bigwedge\limits_{i=1}^{n} (x_{i+1} > x_i).$$

2.1) has (modulo the standard representation of the spaces involved and the coding of finite tuples of variables) the form

$(*)$ $\forall x^1 \big(\forall k^0 A_0(k,x) \to \exists y \leq sx \forall z^0 B_0\big)$, where A_0, B_0 are quantifier–free. Because of the premise "$\forall k^0 A_0(k,x)$" $(*)$ is not an admissible assumption in 2.12 (in contrast to (A)), i.e. it is not enough to consider the proof of "$(*) \to$ uniqueness of best approximation". We also have to consider the proof of $(A) \to (*)$ in $WE-PA^\omega$. In Kohlenbach (A, Thm. 4.17) it is shown that it is sufficient to analyse the proof of $(A) \to \varepsilon$–weakening of $(*)$, where the ε–weakening of $(*)$ is

$\forall x^1, z^0 \big(\forall k^0 A_0(k,x) \to \exists y \leq sx \bigwedge\limits_{i=0}^{z} B_0\big)$. From the proof of this weakening one extracts a functional χ such that

$\forall x, z \big(\bigwedge\limits_{k=0}^{\chi xz} A_0(k,x) \to \exists y \leq sx \bigwedge\limits_{i=0}^{z} B_0\big)$. Applying the functional obtained from the proof of "$(*) \to$ uniqueness' to χ yields the modulus of uniqueness. In our situation, this strategy means that we have to analyse the proof of the following ε-weakening of the alternation theorem (under the assumption (A)):

$$\widehat{2}.(i) \forall p \in \tilde{K}_{f,n}\Big(\|f - p\| = E_{n,f} \to \forall q \in \mathbb{Q}_+^* \exists (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\}$$
$$\big(\bigwedge\limits_{i=1}^{n+1} \big(|(-1)^{i+j}(p(x_i) - f(x_i)) - E_{n,f} < q\big) \wedge \bigwedge\limits_{i=1}^{n} (x_{i+1} > x_i)\big)\Big).$$

By simple manipulations (as e.g. writing $\forall q \in \mathbb{Q}_+^* \big(\|f - p\| \leq E_{n,f} + q\big)$ instead of $\|f - p\| = E_{n,f}$) and a suitable (partial) prenexing, 1),$\widehat{2}$),3) and 4) are equivalent to

$1^*)$ $\forall p_1, p_2 \in \tilde{K}_{f,n}, q \in \mathbb{Q}_+^* \exists r \in \mathbb{Q}_+^* \big(\|f - p_1\|, \|f - p_2\| \leq E_{n,f} + r \to \|f - \frac{p_1+p_2}{2}\| < E_{n,f} + q\big)$.

$2^*)(i)\ \forall p \in \tilde{K}_{n,f}, q \in \mathbb{Q}^*_+ \exists r \in \mathbb{Q}^*_+ \Big(\|f-p\| \le E_{n,f} + r \to \exists (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\}$

$$\big(\bigwedge_{i=1}^{n+1} \big(|(-1)^{i+j}(p(x_i)-f(x_i)) - E_{n,f}| < q\big) \wedge \bigwedge_{i=1}^{n} (x_{i+1} > x_i)\big)\Big).$$

$(ii)\ \forall p \in \tilde{K}_{f,n}, (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, q \in \mathbb{Q}^*_+, j \in \{0,1\}\, \exists l \in \mathbb{Q}^*_+ \Big(\bigwedge_{i=1}^{n} (x_{i+1} \ge x_i) \wedge$

$$\bigwedge_{i=1}^{n+1} \big(|(-1)^{i+j}(p(x_i)-f(x_i)) - E_{n,f}| + q \le E_{n,f}\big) \to \bigwedge_{i=1}^{n} (x_{i+1} - x_i > l)\Big).$$

$3^*)\ \forall p_1, p_2 \in \tilde{K}_{f,n}, (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, q \in \mathbb{Q}^*_+, j \in \{0,1\}\, \exists r, l \in \mathbb{Q}^*_+ \Big(\|f-p_1\|, \|f-p_2\| \le E_{n,f} + r \wedge$

$$\bigwedge_{i=1}^{n+1} \Big(|(-1)^{i+j}\big(\tfrac{p_1(x_i)+p_2(x_i)}{2} - f(x_i)\big) - E_{n,f}| \le l\Big) \to \bigwedge_{i=1}^{n+1} \big(|p_1(x_i) - p_2(x_i)| < q\big)\Big).$$

$4^*)\ \forall p_1, p_2 \in \tilde{K}_{f,n}, (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, r, q \in \mathbb{Q}^*_+ \exists l \in \mathbb{Q}^*_+$

$$\big(\bigwedge_{i=1}^{n} (x_{i+1} - x_i) \ge r\big) \wedge \bigwedge_{i=1}^{n+1} (|p_1(x_i) - p_2(x_i)| \le l) \to \|p_1 - p_2\| < q\big).$$

Since the matrix of the sentences $1^*)$–$4^*)$ is equivalent to a $\exists^{0/1}A_0$–formula A and A is monotonic in the $\exists$-data "$\exists r \in \mathbb{Q}^*_+$" and "$\exists l \in \mathbb{Q}^*_+$", we can apply 4.1 in order to extract functionals $\Phi_1, \Phi_2, \tilde{\Phi}_2, \Phi_3, \tilde{\Phi}_3, \Phi_4 \in T$ which realize "$\exists r$" and "$\exists l$" and depend only on f, n, q (resp. f, n, r, q in 4^*))

For notational simplicity, we omit the arguments f and n of the functionals in the following:

$1^{**})\ \forall p_1, p_2 \in \tilde{K}_{f,n}, q \in \mathbb{Q}^*_+ \big(\|f-p_1\|, \|f-p_2\| \le E_{n,f} + \Phi_1 q \to \|f - \frac{p_1+p_2}{2}\| < E_{n,f} + q\big).$

$2^{**}(i)\ \forall p \in \tilde{K}_{f,n}, q \in \mathbb{Q}^*_+ \Big(\|f-p\| \le E_{n,f} + \Phi_2 q \to \exists (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\}$

$$\big(\bigwedge_{i=1}^{n+1} \big(|(-1)^{i+j}(p(x_i)-f(x_i)) - E_{n,f}| < q\big) \wedge \bigwedge_{i=1}^{n} (x_{i+1} > x_i)\big)\Big),$$

(ii) $\forall p \in \tilde{K}_{f,n}, (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\}, q \in \mathbb{Q}^*_+ \Big(\bigwedge_{i=1}^{n} (x_{i+1} \ge x_i) \wedge$

$$\bigwedge_{i=1}^{n+1} \big(|(-1)^{i+j}(p(x_i)-f(x_i)) - E_{n,f}| + q \le E_{n,f}\big) \to \bigwedge_{i=1}^{n} (x_{i+1} - x_i \ge \tilde{\Phi}_2 q)\Big).$$

$3^{**})\ \forall p_1, p_2 \in \tilde{K}_{f,n}, (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\}, q \in \mathbb{Q}^*_+$

$$\Big(\|f-p_1\|, \|f-p_2\| \le E_{n,f} + \Phi_3 q \wedge \bigwedge_{i=1}^{n+1} \Big(|(-1)^{i+j}\big(\tfrac{p_1(x_i)+p_2(x_i)}{2} - f(x_i)\big) - E_{n,f}| \le \tilde{\Phi}_3 q\Big)$$

$$\to \bigwedge_{i=1}^{n+1} \big(|p_1(x_i) - p_2(x_i)| < q\big)\Big).$$

$4^{**})\ \forall p_1, p_2 \in \tilde{K}_{f,n}, (x_1,\ldots,x_{n+1}) \in [0,1]^{n+1}, r, q, \in \mathbb{Q}^*_+ \big(\bigwedge_{i=1}^{n} (x_{i+1} - x_i \ge r) \wedge$

$\bigwedge_{i=1}^{n+1} (|p_1(x_i) - p_2(x_i)| \le \Phi_4 r q) \to \|p_1 - p_2\| < q\big)$. ($\Phi_i q, \Phi_4 r q \in \mathbb{Q}^*_+$ for all $q, r \in \mathbb{Q}^*_+$; i=1,2,3).

For $f \in C[0,1], n \in \mathbb{N}, q, l \in \mathbb{Q}^*_+$ define

$$\Phi l q := \min\Big\{\Phi_1[\Phi_2 \min\Big(\tfrac{l}{4}, \tilde{\Phi}_3\big(\Phi_4(\tilde{\Phi}_2(\tfrac{3l}{4}))q\big)\Big)], \Phi_3\big(\Phi_4(\tilde{\Phi}_2(\tfrac{3l}{4}))q\big)\Big\} \in \mathbb{Q}^*_+.$$

6.3 Proposition

$\forall p_1, p_2 \in \tilde{K}_{f,n}, l, q \in \mathbb{Q}\big(l \le E_{n,f} \wedge \|f - p_1\|, \|f - p_2\| \le E_{n,f} + \Phi l q \to \|p_1 - p_2\| \le q\big).$

Proof: Assumption $\overset{1^{**})}{\to} \|f - \frac{p_1+p_2}{2}\| \le E_{n,f} + \Phi_2 c$, where $c := \min\Big(\frac{l}{4}, \tilde{\Phi}_3\big(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q\big)\Big)$

$\overset{2^{**})}{\to} \exists (x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}, j \in \{0,1\} \bigwedge\limits_{i=1}^{n+1} \Big(|(-1)^{i+j}\big(\frac{p_1(x_i)+p_2(x_i)}{2} - f(x_i)\big) - E_{n,f}| < c\Big)$

and $(*)$ $\bigwedge\limits_{i=1}^{n+1} \big(x_{i+1} - x_i > \tilde{\Phi}_2(\frac{3l}{4})\big)$ (since $|(-1)^{i+j}\big(\frac{p_1(x_i)+p_2(x_i)}{2} - f(x_i)\big) - E_{n,f}| < \frac{l}{4} \le \frac{E_{n,f}}{4}$ implies $|\ldots| + \frac{3l}{4} \le E_{n,f}$)

$\overset{3^{**})}{\to} \bigwedge\limits_{i=1}^{n+1} \Big(|p_1(x_i) - p_2(x_i)| < \Phi_4\big(\tilde{\Phi}_2(\frac{3l}{4})\big)q\Big)$ (since by Φ–definition)

$\|f - p_{1/2}\| \le E_{n,f} + \Phi_3\big(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q\big)$)

$\overset{4^{**}),(*)}{\longrightarrow} \|p_1 - p_2\| < q.$

Φ can be easily modified to yield a modulus Φ^* which no longer depend on l and thus avoids the assumption "$l \le E_{n,f}$":
$\Phi^* q := \min\big(\frac{q}{4}, \Phi(\frac{q}{4})q\big).$

6.4 Proposition

$\forall p_1, p_2 \in \tilde{K}_{f,n}, q \in \mathbb{Q}_+^*\big(\|f - p_1\|, \|f - p_2\| \le E_{n,f} + \Phi^* q \to \|p_1 - p_2\| \le q\big).$

Proof: Case 1: $E_{n,f} \ge \frac{q}{4}$. Then the assertion follows immediately from 6.3.
Case 2: $E_{n,f} < \frac{q}{4}$: $\|f - p_1\|, \|f - p_2\| \le E_{n,f} + \Phi^* q < \frac{q}{4} + \frac{q}{4} = \frac{q}{2} \to$
$\|p_1 - p_2\| \le \|p_1 - f\| + \|f - p_2\| < \frac{q}{2} + \frac{q}{2} = q.$

The extraction of $\Phi_1, \Phi_3, \tilde{\Phi}_3$ as well as the functionals themselves are very simple. Φ_4 is obtained from Lagrange's interpolation formula. The extraction of $\tilde{\Phi}_2$ uses essentially majorization. The main part of our extraction however is the construction of Φ_2. While the other steps in the uniqueness proof are constructive, the proof of the alternation theorem (from which Φ_2 is gained) is essentially inconstructive both by using classical logic as well as the ineffective theorem (A). Nevertheless, we succeed in extracting Φ_2 from the given ineffective proof without global constructivation but by following those data which are computationally relevant for the construction of Φ_2.

6.5 Definition

For $n \in \mathbb{N}, f \in C[0,1], p \in P_n$ and $\varepsilon \in \mathbb{Q}_+^*$ we define:

1) $x \in [0,1]$ is an ε–(e)–point ("ε–extremal–point") of $p - f$ if
$E_{n,f} - \varepsilon \le |p(x) - f(x)| \le E_{n,f} + \varepsilon.$

2) $x \in [0,1]$ is an ε–(+)–point (ε–(–)–point) of $p - f$ if
$E_{n,f} - \varepsilon \le p(x) - f(x) \le E_{n,f} + \varepsilon$ $(-E_{n,f} - \varepsilon \le p(x) - f(x) \le -E_{n,f} + \varepsilon)$.

3) $(x_1, \ldots, x_k) \in [0,1]^k$ $(1 \le k \le n+2)$ is an ε–alternant of $p - f$ (having length k) if $x_1 < \ldots < x_k$

and $\bigwedge_{i=1}^{k} |(-1)^{i+j}(p(x_i) - f(x_i)) - E_{n,f}| \leq \varepsilon$ for $j = 0$ or $j = 1$.

4) $p \in P_n$ is an ε–best approximation of f if $\|p - f\| \leq E_{n,f} + \varepsilon$.

6.6 Remark

1) Formally 6.5.3 is defined for every $\varepsilon \geq 0$. However, it does make sense only for $0 \leq \varepsilon < E_{n,f}$ since only in this case the sign of $p(x_i) - f(x_i)$ alternates for $i = 1, \ldots, k$.

2) For $\varepsilon = 0$ 6.5 coincides with the usual definitions of (e)–point,(+),(–)–point, alternant and best approximation.

3) A definition which is similar to 6.5 can be found in Bridges (80A).

Extraction of $\Phi_1, \Phi_2, \tilde{\Phi}_2, \Phi_3, \tilde{\Phi}_3, \Phi_4$

Ad Φ_1: The extraction is trivial: Define $\Phi_1 q := q$.

$\|f - p_{1/2}\| \leq E_{n,f} + q \to \left\{ \begin{array}{c} -E_{n,f} - q \leq f(x) - p_1(x) \leq E_{n,f} + q \\ -E_{n,f} - q \leq f(x) - p_2(x) \leq E_{n,f} + q \end{array} \right\}$

$\to -2E_{n,f} - 2q \leq 2f(x) - \big(p_1(x) + p_2(x)\big) \leq 2E_{n,f} + 2q$

$\to -E_{n,f} - q \leq f(x) - \frac{p_1(x)+p_2(x)}{2} \leq E_{n,f} + q$ for all $x \in [0,1]$.

Hence $\|f - \frac{p_1+p_2}{2}\| \leq E_{n,f} + q$.

Ad $\Phi_3, \tilde{\Phi}_3$: Again the extraction is obvious: $\Phi_3 q := \frac{q}{4}, \tilde{\Phi}_3 q := \frac{q}{4}$.

Assume $\|f - p_1\|, \|f - p_2\| \leq E_{n,f} + fracq4$. Let $(x_1, \ldots, x_{n+1}) \in [0,1]^{n+1}$ be a $\frac{q}{4}$–alternant of $\frac{p_1+p_2}{2} - f$ and let x_i be a $\frac{q}{4}$–(+)–point:

$E_{n,f} - \frac{q}{4} \leq \frac{p_1(x_i)+p_2(x_i)}{2} - f(x_i) = \frac{p_1(x_i)-f(x_i)}{2} + \frac{p_2(x_i)-f(x_i)}{2}$

$\overset{p_2(x_i)-f(x_i) \leq E_{n,f}+\frac{q}{4}}{\longrightarrow} E_{n,f} - \frac{q}{4} \leq \frac{p_1(x_i)-f(x_i)}{2} + \frac{E_{n,f}+\frac{q}{4}}{2}$

$\to 2E_{n,f} - 2\frac{q}{4} \leq p_1(x_i) - f(x_i) + E_{n,f} + \frac{q}{4}$

$\to$ (1) $E_{n,f} - \frac{q}{2} - \frac{q}{4} \leq p_1(x_i) - f(x_i) \leq E_{n,f} + \frac{q}{4}$.

Analogously, one shows that

(2) $E_{n,f} - \frac{q}{2} - \frac{q}{4} \leq p_2(x_i) - f(x_i) \leq E_{n,f} + \frac{q}{4}$.

(1) and (2) imply $|p_1(x_i) - p_2(x_i)| \leq q$. If x_i is an $\frac{q}{4}$–(–)–point, then the reasoning is analogous.

Ad Φ_4: By Lagrange we have for all $p \in P_n$

$(*)$ $p(x) = \sum_{i=1}^{n+1} l_i(x) p(x_i)$, where $l_i(x) := \frac{\prod\limits_{j=1, j\neq i}^{n+1} (x-x_j)}{\prod\limits_{j=1, j\neq i}^{n+1} (x_i-x_j)}$ $(1 \leq i \leq n+1)$.

Since $x_{i+1} - x_i \geq r$ (by the assumption) it follows that

$(**)$ $|l_i(x)| \leq \frac{1}{\prod\limits_{j=1, j\neq i}^{n+1} r\cdot|i-j|} \leq \frac{1}{r^n (i-1)!(n-i+1)!}$ for all $x \in [0,1]$.

For $p_1 - p_2$ such that $\bigwedge_{i=1}^{n+1} \big(|p_1(x_i) - p_2(x_i)|\big) \leq q$, $(*)$ implies

$p_1(x)-p_2(x)=\sum\limits_{i=1}^{n+1} l_i(x)\big(p_1(x_i)-p_2(x_i)\big)$ and therefore

$|p_1(x)-p_2(x)| \le \sum\limits_{i=1}^{n+1} |l_i(x)|\cdot|p_1(x_i)-p_2(x_i)| \le q\cdot\sum\limits_{i=1}^{n+1}|l_i(x)|$

$(***)\ \stackrel{(**)}{\le}\ q\cdot\sum\limits_{i=1}^{n+1}\frac{1}{r^n(i-1)!(n-i+1)!}$: One easily verifies that $(i-1)!(n-i+1)! \ge \lfloor\frac{n}{2}\rfloor!\lceil\frac{n}{2}\rceil!\ (1\le i\le n+1)$.

Hence $|p_1(x)-p_2(x)| \le q\cdot\frac{n+1}{\lfloor\frac{n}{2}\rfloor!\lceil\frac{n}{2}\rceil!r^n}$. Define $\Phi_4 rq := \frac{\lfloor\frac{n}{2}\rfloor!\lceil\frac{n}{2}\rceil!r^n}{n+1}\cdot q$. Φ_4 does not depend on f! This is due to the fact that $(***)$ holds for all $p_1,p_2\in P_n$ and not only for $p_1,p_2\in\tilde{K}_{f,n}\subset P_n$. Put together the reasoning above yields

$\bigwedge\limits_{i=1}^{n}(x_{i+1}-x_i\ge r)\wedge\bigwedge\limits_{i=1}^{n+1}\big(|p_1(x_i)-p_2(x_i)|\le\Phi_4 rq\big)\to\|p_1-p_2\|\le q$ for all $p_1,p_2\in P_n,(x_1,\ldots,x_{n+1})\in[0,1]^{n+1},r,q\in\mathbb{Q}_+^*$.

Ad $\tilde{\Phi}_2$: Assume $\bigwedge\limits_{i=1}^{k-1}(x_{i+1}\ge x_i)\wedge\bigwedge\limits_{i=1}^{k}\big(|(-1)^{i+j}(p(x_i)-f(x_i))-E_{n,f}|+q\le E_{n,f}\big)$, where $j=0$ or $j=1$. One easily shows that $\bigwedge\limits_{i=1}^{k-1}\big(|(p-f)(x_{i+1})-(p-f)(x_i)|\ge 2q\big)$. Let $\omega_{p,f}$ be a modulus of uniform continuity for $p-f$ on $[0,1]$. Then $\bigwedge\limits_{i=1}^{k-1}(x_{i+1}-x_i\ge\omega_{p,f}(2q))$. Define $\widehat{\Phi}_2 qp := \omega_{p,f}(2q)$. In the following we use majorization to obtain a $\tilde{\Phi}_2$ from $\widehat{\Phi}_2$ such that
$\forall p\in\tilde{K}_{f,n},q\in\mathbb{Q}_+^*\big(0<\tilde{\Phi}_2 q\le\widehat{\Phi}_2 qp\big)$. $\widehat{\Phi}_2$ depends on p via $\omega_{p,f}$. Hence we obtain $\tilde{\Phi}_2$ by replacing $\omega_{p,f}$ in $\widehat{\Phi}_2$ by a uniform modulus of continuity for all $p-f$, where $p\in\tilde{K}_{f,n}$: Assume $p(x)=c_nx^n+\ldots c_1x+c_0\in P_n, x,y\in[0,1]$. By the Markov inequality $p\in\tilde{K}_{f,n}$ implies $\|p'\|\le 2n^2\big(\frac{5}{2}\|f\|\big)=5n^2\|f\|$, where p' is the derivative of p. The mean value theorem yields that p is Lipschitz continuous with the Lipschitz constant $5n^2\|f\|$. Thus $\omega_n(q):=\frac{q}{5n^2\|f\|}$ is a uniform modulus of continuity for all $p\in\tilde{K}_{f,n}$ on $[0,1]$ (if $n\ge1$; for $n=0$ define $\omega_n(q):=1$). Let ω_f be a modulus of uniform continuity for $f\in C[0,1]$. Then $\tilde{\omega}_{f,n}(q):=\min\big(\omega_n(\frac{q}{2}),\omega_f(\frac{q}{2})\big)$ is a uniform modulus of uniform continuity for $\Big\{p-f|p\in\tilde{K}_{f,n}\Big\}$ on $[0,1]$.
Define $\tilde{\Phi}_2 q:=\tilde{\omega}_{f,n}(2q)$. $\tilde{\Phi}_2$ fulfils 2^{**}). [4]

Ad Φ_2: For the extraction of Φ_2 we need the following majorization:

6.7 Lemma

Let $z_1,\ldots,z_n\in[0,1]$ $(n\ge1)$ be real numbers with
$\bigwedge\limits_{i=1}^{n-1}(z_{i+1}-z_i\ge2\alpha)$ for a fixed $\alpha\in\mathbb{Q}_+^*,\alpha\le1$ and let K be defined by $K:=\Big\{x\in[0,1]:\bigwedge\limits_{i=1}^{n}(|z_i-x|\ge\frac{\alpha}{2})\Big\}$.

[4] More precisely: Not $\tilde{\Phi}_2$ but its variant where $\|f\|$ is replaced by $\chi(f)$ such that $\|f\|\le\chi(f)\in\mathbb{Q}_+^*$, since $\tilde{\Phi}_2\notin\mathbb{Q}$ in general. I ts clear that such a bound $\chi(f)$ for $\|f\|$ can be easily computed (since f is supposed to be endowed with a modulus of uniform continuity ω_f). Note that $\chi(f)$ may depend on ω_f.

Then for $p(x) := (z_1 - x)(z_2 - x) \cdot \ldots \cdot (z_n - x) \in P_n$ the following holds:

$$\inf_{x\in K} |p(x)| \geq \prod_{i=1}^{n} |2i - \frac{1}{2} - n| \cdot \alpha^n = \prod_{i=1}^{\lceil \frac{n}{2} \rceil} (2i - \frac{3}{2}) \cdot \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2i - \frac{1}{2}) \cdot \alpha^n.$$

(Note that the right side does not depend on z_i).

Proof: For $n = 1$ the lemma is trivial. Thus assume $n > 1$. Define $\tilde{z}_i := 2(i-1)\alpha$ $(1 \leq i \leq n)$ $\tilde{p}(x) := (\tilde{z}_1 - x)(\tilde{z}_2 - x) \cdot \ldots \cdot (\tilde{z}_n - x)$, $\tilde{K} := \left\{ x \in [0,1] : \bigwedge_{i=1}^{n} (|\tilde{z}_i - x| \geq \frac{\alpha}{2}) \right\}$.

i) Claim: $\inf_{x\in\tilde{K}} |\tilde{p}(x)| \leq \inf_{x\in K} |p(x)|$. The claim is proved by showing

$\forall x \in K \exists \tilde{x} \in \tilde{K}\big(|\tilde{p}(\tilde{x})| \leq |p(x)|\big)$. This is not hard to verify if one treats the cases $0 \leq x \leq z_1 - \frac{\alpha}{2}$, $z_n + \frac{\alpha}{2} \leq x \leq 1$ and $\exists i(1 \leq i \leq n-1 \wedge z_i + \frac{\alpha}{2} \leq x \leq z_{i+1} - \frac{\alpha}{2})$ separately.

ii) Claim: $\inf_{x\in\tilde{K}} |\tilde{p}(x)| = \prod_{i=1}^{n} |2i - \frac{1}{2} - n| \cdot \alpha^n = \prod_{i=1}^{\lceil \frac{n}{2} \rceil} (2i - \frac{3}{2}) \cdot \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2i - \frac{1}{2}) \cdot \alpha^n$.

Case 1: n even. We show that $\inf_{x\in\tilde{K}} |\tilde{p}(x)| = |\tilde{p}(\tilde{z}_{\frac{n}{2}} + \frac{\alpha}{2})| = |\tilde{p}(\tilde{z}_{frac{n}{2}+1} - \frac{\alpha}{2}|$. Let $x_0 \in \tilde{K}$ and assume $x_0 \leq \tilde{z}_{\frac{n}{2}} - \frac{\alpha}{2}$. Then $|\tilde{p}(x_0')| < |\tilde{p}(x_0)|$ for $x_0' := x_0 + 2\alpha \in \tilde{K}$ since $\bigwedge_{i=1}^{n-1} (\tilde{z}_i - x_0| = |\tilde{z}_{i+1} - x_0'|)$ and $|\tilde{z}_n - x_0| > |\tilde{z}_1 - x_0'|$. Analogously for $x_0 \geq \tilde{z}_{\frac{n}{2}+1} + \frac{\alpha}{2}$ and $x_0' := x_0 - 2\alpha$.

One easily verifies for $i = 0, \ldots, \frac{n}{2} - 1$:

$|(\tilde{z}_{\frac{n}{2}-i} - x)(\tilde{z}_{\frac{n}{2}+1+i} - x)|$ assumes its minimum on $I := [\tilde{z}_{\frac{n}{2}} + \frac{\alpha}{2}, \tilde{z}_{\frac{n}{2}+1} - \frac{\alpha}{2}]$ at the point $x = \tilde{z}_{\frac{n}{2}} + \frac{\alpha}{2}$. Hence $|\tilde{p}(x)|$ assumes its minimum on I at this point. Together with the reasoning above we can conclude $|\tilde{p}(\tilde{z}_{\frac{n}{2}} + \frac{\alpha}{2})| = \inf_{x\in\tilde{K}} |\tilde{p}(x)|$.

Since $\tilde{z}_{\frac{n}{2}} + \frac{\alpha}{2} = (n - \frac{3}{2})\alpha \wedge |\tilde{z}_i - (n - \frac{3}{2})\alpha| = \alpha|2i - \frac{1}{2} - n|$ it follows that $|\tilde{p}(\tilde{z}_{\frac{n}{2}} + \frac{\alpha}{2})| = \prod_{i=1}^{n} |2i - \frac{1}{2} - n| \cdot \alpha^n$.

Furthermore, $\prod_{i=\frac{n}{2}+1}^{n} |2i - \frac{1}{2} - n| = \prod_{i=1}^{\frac{n}{2}} (2i - \frac{1}{2})$ and $\prod_{i=1}^{\frac{n}{2}} |2i - \frac{1}{2} - n| = \prod_{i=1}^{\frac{n}{2}} (2i - \frac{3}{2})$.

Case 2: n odd. We show $|\tilde{p}(\tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2})| = |\tilde{p}(\tilde{z}_{\lceil \frac{n}{2} \rceil} + \frac{\alpha}{2})| = \inf_{x\in\tilde{K}} |\tilde{p}(x)|$ (This implies (as in case 1) the claim since $\tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2} = (n - \frac{3}{2})\alpha$).

a) $0 \leq x < \tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2} \to |\tilde{p}(x)| > |\tilde{p}(\tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2})|$:

$\tilde{p}(x) = \tilde{q}(x) \cdot (\tilde{z}_n - x)$, where $\tilde{q}(x) = (\tilde{z}_1 - x) \cdot \ldots \cdot (\tilde{z}_{n-1} - x)$. $0 \leq x < \tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2}$ implies (by case 1, since $n-1$ is even)

$|\tilde{q}(x)| \overset{!}{\geq} |\tilde{q}\big(\tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2}\big)| \wedge |\tilde{z}_n - x| > |\tilde{z}_n - \big(\tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2}\big)|$ and therefore $|\tilde{p}(x)| > |\tilde{p}\big(\tilde{z}_{\lceil \frac{n}{2} \rceil} - \frac{\alpha}{2}\big)|$.

(! holds since $\lceil \frac{n}{2} \rceil = \frac{n-1}{2} + 1$ and $|\tilde{q}\big(\tilde{z}_{\frac{n-1}{2}+1} - \frac{\alpha}{2}\big)| = \inf_{x\in\tilde{K}} |\tilde{q}(x)|$).

b) $1 \geq x > \tilde{z}_{\lceil \frac{n}{2} \rceil} + \frac{\alpha}{2} \to |\tilde{p}(x)| > |\tilde{p}\big(\tilde{z}_{\lceil \frac{n}{2} \rceil} + \frac{\alpha}{2}\big)|$ follows analogously (consider $\tilde{p}(x) = (\tilde{z}_1 - x) \cdot \tilde{\tilde{q}}$).

6.8 Remark

1) The proof of 6.7 shows that the estimate is optimal (as an estimate which depends on α and n only), since the polynomial $\tilde{p}(x)$ is admissible.

2) The estimate $\inf\limits_{x\in K}|p(x)| \geq \left(\frac{\alpha}{2}\right)^n$ is trivial.

The following lemma can be easily verified:

6.9 Lemma

Assume $0 \leq \alpha \leq \frac{1}{n}$. Then for all $j, k \leq n$:

$$k < j \rightarrow \prod_{i=1}^{\lceil \frac{j}{2} \rceil}(2i-\tfrac{3}{2}) \cdot \prod_{i=1}^{\lfloor \frac{j}{2} \rfloor}(2i-\tfrac{1}{2}) \cdot \alpha^j < \prod_{i=1}^{\lceil \frac{k}{2} \rceil}(2i-\tfrac{3}{2}) \cdot \prod_{i=1}^{\lfloor \frac{k}{2} \rfloor}(2i-\tfrac{1}{2}) \cdot \alpha^k \leq 1.$$

6.10 Lemma

Let $q \in \mathbb{Q}_+^*$ with $E_{n,f} \geq q$. Then $\tilde{\omega}_{f,n}(\frac{q}{2}) \leq \frac{1}{2(n+1)}$ (Here and in he following $\tilde{\omega}_{f,n}$ denotes the uniform modulus of continuity for all $p - f$ with $p \in \tilde{K}_{f,n}$ from the construction of $\tilde{\Phi}_2$).

Proof: We divide $[0,1]$ into subintervals of length $\tilde{\omega}_{f,n}(\frac{q}{2})$ (the last one may have shorter length). The amplitude of $p_b - f$ on each of these subintervals is at most $\frac{E_{n,f}}{2}$ (where $p_b \in \tilde{K}_{f,n}$ is the best approximation of f in P_n). By the alternation theorem the number of these subintervals is at least $2(n+1)$.

6.11 Lemma

For $f \in C[0,1], n \in \mathbb{N}, p \in P_n, 0 \leq \varepsilon < E_{n,f}$ assume $\|p - f\| \leq E_{n,f} + \varepsilon$. Then there exist both ε–(+)–points and ε–(-)–points of $p - f$.

Proof: The proof is very similar to the proof of 5.7.

6.12 Theorem

Assume that $0 \leq \varepsilon < \frac{E_{n,f}}{4}$, $0 < q \leq E_{n,f}$ $(\varepsilon, q \in \mathbb{Q})$ and $2 \leq k \leq n+2$. If $p_\varepsilon \in P_n$ is a $\prod_{i=1}^{\lfloor \frac{k-2}{2} \rfloor}(2i-\frac{1}{2}) \cdot \prod_{i=1}^{\lceil \frac{k-2}{2} \rceil}(2i-\frac{3}{2}) \cdot \min\left(\frac{1}{n}, \omega_{f,p_\varepsilon}(\frac{q}{2})\right)^{k-2} \cdot \varepsilon$–best approximation of $f \in C[0,1]$, where ω_{f,p_ε} is a modulus of uniform continuity of $p_\varepsilon - f$, then there exists an ε–alternant having length k for $p_\varepsilon - f$.

Proof: We may assume that $\omega_{f,p_\varepsilon}(\frac{q}{2}) \leq \frac{1}{n}$ (For otherwise we define $\omega'_{f,p_\varepsilon}(q) := \min\left(\frac{1}{n}, \omega_{f,p_\varepsilon}(q)\right)$). $C_{k-2} = \prod_{i=1}^{\lfloor \frac{k-2}{2} \rfloor}(2i-\frac{1}{2}) \cdot \prod_{i=1}^{\lceil \frac{k-2}{2} \rceil}(2i-\frac{3}{2}) \cdot \omega_{f,p_\varepsilon}(\frac{q}{2})^{k-2}$. We divide $[0,1]$ into subintervals $I_1, \ldots, I_s$ of length $\omega_{f,p_\varepsilon}(\frac{q}{2})$ (I_s may have shorter length). The amplitude of $p_\varepsilon - f$ on $I_i = [\xi_i, \xi_{i+1}]$ is $\leq \frac{q}{2}$. I_i is called ε–(e)–interval if it contains an ε–(e)–point of $p_\varepsilon - f$. Since $\varepsilon < \frac{E_{n,f}}{2}$, $(p_\varepsilon - f)(x)$ is distinct from 0 on every ε–(e)–interval and therefore has constant sign. An ε–(e)–interval I_i is an ε–(+)–interval (ε–(-)–interval) if $p_\varepsilon(x) - f(x) > 0$ ($p_\varepsilon(x) - f(x) < 0$) for all $x \in I_i$ (Thus an ε–(+)–interval contains an ε–(+)–point but no ε–(-)–point). Enumerating all ε–(e)–intervals from left to right $(I_{j_1}, \ldots, I_{j_N})$

we obtain the following schema (assuming without loss of generality that I_{j_1} is an ε–(+)–interval):

$$(*)\begin{cases} I_{j_1},\dots,I_{j_{k_1}} & \varepsilon\text{–(+)–intervals} \\ I_{j_{k_1+1}},\dots,I_{j_{k_2}} & \varepsilon\text{–(-)–intervals} \\ \vdots & \\ I_{j_{k_{m-1}+1}},\dots,I_{j_{k_m}} & \varepsilon-(-1)^{m-1}\text{–intervals}\ \ (k_m=N). \end{cases}$$

6.9 implies $\|f-p_\varepsilon\| \le E_{n,f}\cdot C_{k-2}\cdot\varepsilon \le E_{n,f}+\varepsilon$. By 6.11 the schema $(*)$ consists of at least two groups (with at least one ε–(+)–interval resp. an ε–(-)–interval). Thus $m\ge 2$. We show $m\ge k$: Assume on the contrary $m<k$ $(**)$. $p_\varepsilon - f$ has different sign on $I_{j_{k_1}}$ and $I_{j_{k_1+1}}$. Hence the right endpoint of $I_{j_{k_1}}$ is distinct from the left endpoint of $I_{j_{k_1+1}}$. Thus there exists at least one interval $I_{i_1}=[\xi_{i_1},\xi_{i_1+1}]$ of $I_1,\dots,I_s$ which lies between $I_{j_{k_1}}$ and $I_{j_{k_1+1}}$ and is not an ε–(e)–interval. Define $z_1 := \frac{\xi_{i_1}+\xi_{i_1+1}}{2}$. Analogously, we define $z_2,\dots,z_{m-1}\in[0,1]$ by $z_l := \frac{\xi_{i_l}+\xi_{i_l+1}}{2}$, where $I_{i_l}=[\xi_{i_l},\xi_{i_l+1}]$ is an interval lying between $I_{j_{k_l}}$ and $I_{j_{k_l+1}}$ $(1\le l\le m-1)$.
$\rho(x) := (z_1-x)(z_2-x)\cdot\ldots\cdot(z_{m-1}-x)$. By our assumption $(**)$ it follows that $\rho(x)\in P_n$ (since $m-1\le k-2\le n$). $z_1,\dots,z_{m-1}$ are the only zeroes of $\rho(x)$. Since none of the ε–(e)-intervals contains one of these zeroes, $\rho(x)$ has constant sign on each I_{j_i}. This sign equals the sign of $p_\varepsilon(x)-f(x)$ on I_{j_i}.
$(***)$ $R := \sup\limits_{x\in[0,1]}|\rho(x)|\le 1$ (since $z_1,\dots,z_{m-1}\in[0,1]$). By the definition of z_l we have $z_{l+1}-z_l\ge 2\omega_{f,p_\varepsilon}(\frac{q}{2})$. Furthermore, $|z_l-x|\ge\frac{\omega_{f,p_\varepsilon}(\frac{q}{2})}{2})$ $(1\le l\le m-1)$ for every x which lies in an ε–(e)–interval. Hence 6.7 implies

$$(****)\begin{cases} \text{If x is element of an } \varepsilon\text{–(e)–interval, then} \\ |\rho(x)|\ge C_{m-1}\overset{6.9}{\ge} C_{k-2} \text{ and therefore } |\varepsilon\rho(x)|=\varepsilon|\rho(x)|\ge\varepsilon\cdot C_{k-2} \end{cases}.$$

Let I_i be an interval of $I_1,\dots,i_s$ which is not an ε–(e)–interval. Then $E^i := \sup\limits_{x\in I_i}|p_\varepsilon(x)-f(x)| < E_{n,f}-\varepsilon$. Let E^* be the maximum of these E^i (for all non–ε–(e)–intervals I_i): $E^*<E_{n,f}-\varepsilon$. Choose $\lambda>0$ so small that
$\lambda R < E_{n,f}-E^*-\varepsilon$ and $\lambda R\le\frac{E_{n,f}}{4}-\varepsilon R$ $(\frac{E_{n,f}}{4}-\varepsilon R>0$, since $R\le 1$ and $\varepsilon<\frac{E_{n,f}}{4})$. Define $Q(x) := p_\varepsilon(x)-(\lambda+\varepsilon)\rho(x)\in P_n$. We have to show that
$\forall x\in[0,1](|Q(x)-f(x)|<E_{n,f})$ $(\Rightarrow \|Q-f\|<E_{n,f}$ contradicting the definition of $E_{n,f})$:
Case 1: I_i is not an ε–(e)–interval,

$$x\in I_i\to|Q(x)-f(x)|\le|p_\varepsilon(x)-f(x)|+(\varepsilon+\lambda)|\rho(x)|\le E^*+\varepsilon|\rho(x)|+\lambda|\rho(x)|$$
$$\overset{(***)}{<} E^*+\varepsilon|\rho(x)|+E_{n,f}-E^*-\varepsilon\le\varepsilon\cdot R-\varepsilon+E_{n,f}\overset{(***)}{\le} E_{n,f}.$$

Case 2: I_i is an ε–(e)–interval, $x\in I_i$:
$p_\varepsilon(x)-f(x)$ and $(\varepsilon+\lambda)\rho(x)$ have the same sign and $|p_\varepsilon(x)-f(x)|>(\varepsilon+\lambda)|\rho(x)|$ since $|p_\varepsilon(x)-f(x)|\ge E_{n,f}-\varepsilon-\frac{E_{n,f}}{2}>\frac{E_{n,f}}{4}$ and $(\varepsilon+\lambda)|\rho(x)|\le\varepsilon R+\lambda R\le\frac{E_{n,f}}{4}$. Hence

$$|Q(x)-f(x)|=|p_\varepsilon(x)-f(x)-(\varepsilon+\lambda)\rho(x)|=|p_\varepsilon(x)-f(x)|-(\varepsilon+\lambda)|\rho(x)|$$
$$\le E_{n,f}+\varepsilon\cdot C_{k-2}-\underbrace{\varepsilon|\rho(x)|}_{\overset{(****)}{\ge}\varepsilon\cdot C_{k-2}}-\underbrace{\lambda|\rho(x)|}_{>0}<E_{n,f}.$$

6.13 Corollary ("ε–alternation theorem")

Suppose that $0 < q \le E_{n,f}$, $0 \le \varepsilon < E_{n,f}$ $(q, \varepsilon \in \mathbb{Q})$ and $2 \le k \le n+2$. Let χ be defined by

$$\chi fnqk := \begin{cases} 1 & \text{if } k = 2 \\ \frac{1}{4}\prod_{i=1}^{\lfloor \frac{k-2}{2} \rfloor}(2i - \frac{1}{2}) \cdot \prod_{i=1}^{\lceil \frac{k-2}{2} \rceil}(2i - \frac{3}{2}) \cdot \tilde{\omega}_{f,n}(\frac{q}{2})^{k-2} & \text{if } k > 2. \end{cases}$$

If $p \in P_n$ is a $(\chi fnqk) \cdot \varepsilon$–best approximation of $f \in C[0,1]$, then there exists an ε–alternant of length k for $p - f$ in $[0,1]$ (Note that $\chi fnqk$ is independent of p!). If $\varepsilon < \frac{E_{n,f}}{4}$ then the factor $\frac{1}{4}$ can be omitted.

Proof: If $k = 2$, then the corollary follows from 6.11. Thus we may assume $k > 2$:
Case 1: $p \in \tilde{K}_{f,n}$. By construction $\tilde{\omega}_{f,n}$ is a modulus of uniform continuity for $p - f$ on $[0,1]$. Since $\frac{\varepsilon}{4} < \frac{E_{n,f}}{4}$ the corollary follows from 6.12 (By 6.10 we have $\tilde{\omega}_{f,n}(\frac{q}{2}) \le \frac{1}{4}$).
Case 2: $p \notin \tilde{K}_{f,n}$. In this case we have $\|p\| > \frac{5}{2}\|f\|$ and therefore
$(+)$ $\|f - p\| > \frac{3}{2}\|f\| \ge \frac{3}{2}E_{n,f}$. p cannot be a $(\chi fnqk) \cdot \varepsilon$–best approximation of f (for every $\varepsilon < E_{n,f}$: $\|p - f\| \le E_{n,f} + \frac{1}{4}(\ldots) \cdot \varepsilon \le E_{n,f} + \frac{\varepsilon}{4}$ implies (by $(+)$) $\frac{\varepsilon}{4} > \frac{E_{n,f}}{2}$.

6.14 Remark

1) 6.13 immediately implies that also $\tilde{\chi} fn\varepsilon k := (\chi fn\varepsilon k) \cdot \varepsilon$ is an alternation modulus, i.e. if $p \in P_n$ is a $\tilde{\chi} fn\varepsilon k$–best approximation of f then there exists an ε–alternant of length k for $p - f$ in $[0,1]$. $\tilde{\chi}$ does no longer depend on q but is not linear in ε.

2) For $\varepsilon = 0$ the proof of 6.12 transforms into the classical proof of the usual alternation theorem 5.9.

By 6.14.1 $\tilde{\chi}$ fulfils the demands for Φ_2. Nevertheless we use χ instead of $\tilde{\chi}$ for the construction of the modulus of uniqueness Φ although it depends in addition to f, n, ε also on an estimate $0 < q \le E_{n,f}$. The reason for this is that χ is linear in ε and so the whole modulus of uniqueness will be linear in ε which is an important property in view of 5.4. Furthermore the construction of Φ^* from Φ in 6.4 yields a modulus which no longer depends on q (but is not linear). Thus we define

$$\Phi_2 rq := \frac{1}{4}\prod_{i=1}^{\lfloor \frac{n \dot{-} 1}{2} \rfloor}(2i - \tfrac{1}{2}) \cdot \prod_{i=1}^{\lceil \frac{n \dot{-} 1}{2} \rceil}(2i - \tfrac{3}{2}) \cdot \tilde{\omega}_{f,n}(\tfrac{r}{2})^{n \dot{-} 1} \cdot q = \tfrac{1}{4}\tilde{C}_{n \dot{-} 1} \cdot \tilde{\omega}_{f,n}(\tfrac{r}{2})^{n \dot{-} 1} \cdot q$$

for $r \in \mathbb{Q}^*_+, q \in \mathbb{Q}_+$.
By 6.13 we have for all $f \in C[0,1], n \in \mathbb{N}, p \in P_n$ and $q \in \mathbb{Q}, r \in \mathbb{Q}^*_+$ such that $q < E_{n,f}, r \le E_{n,f}$:
$\|f - p\| \le E_{n,f} + \Phi_2 rq \to \exists q$–alternant of length $n+1$ for $p - f$.

6.15 Lemma

Let $\omega_{f,n}$ be a modulus of uniform continuity on $[0,1]$ for $p_b - f$, where p_b is the best approximation of f in P_n. Then for $f \in C[0,1], n \in \mathbb{N}, 0 < q \le E_{n,f}$:
$k(q) := \frac{\lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !}{n+1} \cdot \omega_{f,n}(\frac{3q}{2})^n \le 1.$

Proof: The lemma follows easily from the fact that $\omega_{f,n}(\frac{3q}{2}) \le \frac{1}{n+1}$, which is proved similar to 6.10.

Now we are ready to define our modulus of uniqueness (using 6.3 and the fact that $\Phi_1 q = q$):

$$\Phi lrq := \min\left\{ \Phi_2 r\Big(\min\big(\frac{l}{4}, \tilde{\Phi}_3(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q)\big)\Big), \Phi_3\big(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q\big) \right\}.$$

By 6.3 Φ is a modulus of uniqueness for $p_1, p_2 \in \tilde{K}_{f,n}$ if $0 < r, l \le E_{n,f}$. By 6.9 and 6.10 we have
$\Phi_2 r\Big(\min\big(\frac{l}{4}, \tilde{\Phi}_3(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q)\big)\Big) \le \min\big(\frac{l}{4}, \tilde{\Phi}_3(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q)\big)$
$\le \tilde{\Phi}_3(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q) = \Phi_3(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q)$ since $\tilde{\Phi}_3 q = \frac{q}{4} = \Phi_3 q$.
Hence $\Phi lrq = \Phi_2 r\Big(\min\big(\frac{l}{4}, \tilde{\Phi}_3(\Phi_4(\tilde{\Phi}_2(\frac{3l}{4}))q)\big)\Big)$
$= \frac{1}{4} \cdot \tilde{C}_{n \dot{-} 1} \cdot \tilde{\omega}_{f,n}(\frac{r}{2})^{n \dot{-} 1} \cdot \min\big(\frac{l}{4}, \tilde{k}(l) \cdot q\big)$, where

$$\tilde{k}(l) := \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{n+1} \cdot \tilde{\omega}_{f,n}(\frac{3l}{2})^n.$$

Furthermore $\min\big(\frac{l}{4}, \frac{1}{4}\tilde{k}(l) \cdot q\big)$ can be replaced by $\min\big(\frac{E_{n,f}}{4}, \frac{1}{4}\tilde{k}(l) \cdot q\big)$ since the minimum with $\frac{l}{4}$ is only used in 6.3 to derive
$|(-1)^{i+j}\big(\frac{p_1(x_i)+p_2(x_i)}{2} - f(x_i)\big) - E_{n,f}| + \frac{3l}{4} \le E_{n,f}$ from $| - -" - - | < \frac{l}{4} \le \frac{E_{n,f}}{4}$. But the former follows also from $| - -" - - | < \frac{E_{n,f}}{4}$ and $l \le E_{n,f}$.
Replacing $\frac{1}{4}\tilde{k}(l)$ by $\frac{1}{10}\tilde{k}(l)$ makes Φlrq only smaller and therefore a fortiori yields a modulus of uniqueness. Finally we identify r and l. Thus Φ is redefined as follows
$\tilde{\Phi} lq := \frac{1}{4}\tilde{C}_{n \dot{-} 1} \cdot \tilde{\omega}_{f,n}(\frac{l}{2})^{n \dot{-} 1} \cdot \min\big(\frac{E_{n,f}}{4}, \frac{1}{10}\tilde{k}(l) \cdot q\big)$.
Now suppose that $\frac{1}{10}q < \frac{E_{n,f}}{4}$. Then (by 6.15) $\frac{1}{10}\tilde{k}(l) \cdot q < \frac{E_{n,f}}{4}$ and
$\tilde{\Phi} lq = \frac{1}{4}\tilde{C}_{n \dot{-} 1} \cdot \tilde{\omega}_{f,n}(\frac{l}{2})^{n \dot{-} 1} \cdot \frac{1}{10}\tilde{k}(l) \cdot q$. In this case we can omit the factor $\frac{1}{4}$ by 6.13. In the following we show that the resulting functional is in fact a modulus of uniqueness for arbitrary $q > 0$ and $p_1, p_2 \in P_n$:

6.16 Theorem

Proof–theoretic analysis of the uniqueness proof from de La Vallée Poussin (19) yields:
For $f \in C[0,1], n \in \mathbb{N}, l \in \mathbb{Q}_+^*$ define

$\Psi fnl := \frac{1}{10(n+1)} \prod_{i=1}^{\lfloor \frac{n \dot{-} 1}{2} \rfloor} (2i - \frac{1}{2}) \cdot \prod_{i=1}^{\lceil \frac{n \dot{-} 1}{2} \rceil} (2i - \frac{3}{2}) \cdot \lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil ! \tilde{\omega}_{f,n}(\frac{l}{2})^{n \dot{-} 1} \tilde{\omega}_{f,n}(\frac{3l}{2})^n$. Then
$\forall l \in \mathbb{Q}_+^*, p_1, p_2 \in P_n, q \in \mathbb{Q}_+ \big(l \le E_{n,f} \wedge \|f - p_1\|, \|f - p_2\| \le E_{n,f} + (\Psi fnl) \cdot q \to \|p_1 - p_2\| \le q\big)$
for all $f \in C[0,1], n \in \mathbb{N}$.

Proof: **Case 1**: $\frac{1}{10}q < \frac{E_{n,f}}{4}$: If $p_1, p_2 \in \tilde{K}_{f,n}$, then the theorem follows from the reasoning above. Thus suppose w.l.g. that $p_1 \notin \tilde{K}_{f,n}$. Then $\|p_1\| \ge \frac{5}{2}\|f\|$ and therefore $\|p_1 - f\| \ge \frac{3}{2}\|f\| \ge \frac{3}{2}E_{n,f}$. Hence $\|p_1 - f\| \le E_{n,f} + (\Psi fnl) \cdot q$ implies $(\Psi fnl) \cdot q > \frac{E_{n,f}}{2}$, which contradicts that fact that $(\Psi fnl) \cdot q \le \frac{1}{10}q < \frac{E_{n,f}}{4}$. Thus p_1 cannot be a $(\Psi fnl) \cdot q$–best approximation of f.
Case 2: $\frac{1}{10}q \ge \frac{E_{n,f}}{4}$, i.e. $E_{n,f} \le \frac{2}{5}q$.
$\|f - p_1\|, \|f - p_2\| \le E_{n,f} + (\Psi fnl) \cdot q \le E_{n,f} + \frac{1}{10}q$
$\to \|p_1 - p_2\| \le \|p_1 - f\| + \|f - p_2\| \le 2E_{n,f} + \frac{1}{5}q \le \frac{4}{5}q + \frac{1}{5}q = q$.

6.17 Corollary

Let Ψfnl be defined as in 6.16 and $l \in \mathbb{Q}$ such that $0 < l \leq E_{n,f}$.

1) Ψfnl is a constant of strong unicity, i.e.
$\forall f \in C[0,1], n \in \mathbb{N}, p \in P_n\big(\|f-p\| \geq \|f-p_b\| + (\Psi fnl)\cdot\|p-p_b\|\big)$, where p_b is the best approximation of f in P_n.

2) $\lambda_l(f,n) := \frac{2}{\Psi fnl}$ is a pointwise Lipschitz constant for the Chebycheff projection $\mathcal{P}$:
$\forall f, f_0 \in C[0,1], n \in \mathbb{N}\big(\|\mathcal{P}(f,n)-\mathcal{P}(f_0,n)\| \leq \lambda_l(f_0,n)\cdot\|f-f_0\|\big)$.

3) $\tilde{\Psi} fnq := \min\big(\frac{q}{4}, \Psi fn(\frac{q}{4}\cdot q)\big)$ is a modulus of uniqueness: $\forall p_1,p_2 \in P_n, q \in \mathbb{Q}^*_+\big(\|f-p_1\|,\|f-p_2\| \leq E_{n,f} + \tilde{\Psi} fnq \to \|p_1-p_2\| \leq q\big)$.

4) $\frac{1}{2}\tilde{\Psi} fnq$ ($\tilde{\Psi}$ as in 3)) is a modulus of pointwise continuity of $\mathcal{P}$.

Note that $\tilde{\Psi}$ is a constructive operation in f, n, q and does not depend on any estimate $0 < l \leq E_{n,f}$.

Proof: 1) and 2) follow immediately from 6.16 and 5.4.2,5.4.3.
3) follows from 6.16. and the proof of 6.4.
4) follows from 3) and 5.4.1.

6.18 Remark

1) The extraction of the functionals $\Psi_1, \Phi_2, \tilde{\Phi}_2, \Phi_3, \tilde{\Phi}_3, \Phi_4$ and the proofs of 6.16 and 6.17 can be carried out in $WE-PA^\omega + (A)$. Hence, by 4.1, these results are even provable in $WE-HA^\omega$ plus the ε-weakening of (A) and thus in $WE-HA^\omega$ alone (since the ε-weakening of (A) is provable in $WE-HA^\omega$.

2) The modulus $\tilde{\omega}_{f,n}$ in $\Psi, \tilde{\Psi}$ may also be replaced for $p_1, p_2 \in P_n$ by a modulus of uniform continuity ω_{f,p_1,p_2} of $\frac{p_1+p_2}{2} - f$, if
$\tilde{C}_{f,k-2}\cdot\omega_{f,p_1,p_2}(\frac{E_{n,f}}{2})^{k-2} \leq \tilde{C}_{f,j-2}\cdot\omega_{f,p_1,p_2}(\frac{E_{n,f}}{2})^{j-2}$ for $2 \leq j \leq k \leq n+2$, thus in particular if $\omega_{f,p_1,p_2}(\frac{E_{n,f}}{2}) \leq \frac{1}{n}$. The moduli which result under this replacement may be numerically much better than $\Psi, \tilde{\Psi}$ for concrete p_1, p_2 but are no longer a–priori–estimates.

Let A be a subset of $C[0,1]$ and assume that $\omega_{A,n}$ is a uniform modulus of continuity for all $p-f$ where $f \in A$ and $p \in \tilde{K}_{f,n}$. Assume furthermore that $0 < l_A \leq \inf\limits_{f\in A} E_{n,f}$. Then the moduli $\Psi_A(\tilde{\Psi}_A)$ obtained from Ψ ($\tilde{\Psi}$) by replacing $\tilde{\omega}_{f,n}$ by $\omega_{A,n}$ and l by l_A are common moduli of uniqueness for all $f \in A$.
In particular if $K \subset C[0,1]$ is compact (w.r.t. $\|\cdot\|_\infty$). Then there exists (classically) a common modulus of continuity ω_K and a constant $M_K \in \mathbb{Q}^*_+$ such that $\|f\| \leq M_K$ for all $f \in K$.
Define $\omega_{K,n}(q) := \min\big(\omega_{n,M_K}(\frac{q}{2}, \omega_K(\frac{q}{2})\big)$, where $\omega_{n,M_K} := \frac{q}{5n^2M_K}$. As in the case of $\tilde{\omega}_{f,n}$ it follows that $\omega_{K,n}$ is a common modulus of continuity for all $p-f$ with $f \in K$ and $p \in \tilde{K}_{f,n}$. Put together we have shown the following theorem:

6.19 Theorem

Let $K \subset C[0,1]$ be compact (w.r.t. $\|\cdot\|_\infty$) with a common modulus of continuity ω_K for all $f \in K$ and $M_K \in \mathbb{Q}_+^*$ such that $\forall f \in K(\|f\| \le M_K)$ and define $\omega_{K,n}$ as above. Let Ψ_K denote the result obtained from Ψ by replacing $\tilde{\omega}_{f,n}$ by $\omega_{K,n}$ and l by $l_K \in \mathbb{Q}_+^*$ such that $0 < l_K \le \inf\limits_{f\in K} E_{n,f}$. Then

1) $\forall p_1, p_2 \in P_n, q \in \mathbb{Q}_+ \big(\|f - p_1\|, \|f - p_2\| \le E_{n,f} + (\Psi_K n l_k) \cdot q \to \|p_1 - p_2\| \le q\big)$.

2) $\Psi_K n l_K$ (resp. $\lambda_K := \frac{2}{\Psi_K}$) is a common constant of strong unicity (Lipschitz constant) for all $f \in K$.

3) $\tilde{\Psi}_K nq := \min\big(\frac{q}{4}, \Psi_K n(\frac{q}{4}) \cdot q\big)$ (resp. $\frac{1}{2}\tilde{\Psi}_K nq$) is a common modulus of uniqueness (pointwise continuity for the Chebycheff projection) for all $f \in K$.

Note that $\tilde{\Psi}_K$ is also defined if $\inf\limits_{f\in K} E_{n,f} = 0$!.

Correction and remark to Kohlenbach(A): Replace the condition '$(\alpha = 0 \wedge \beta$ arbitrary) or $(\alpha = 1 \wedge \beta = 0)$' on $AC^{\alpha,\beta} - qf$ in 2.7, 3.8, 4.9.2 and 4.17 by '$(\alpha = 0 \wedge \beta \le 1)$ or $(\alpha = 1 \wedge \beta = 0)$'. This condition, which is needed only for the elimination of extensionality (for $WE - PA^\omega$ instead of $E - PA^\omega$ full $AC - qf$ is allowed), is not a real restriction since 3.8, 4.9.2 and 4.17 only deal with purely analytical sentences.

References

Achieser, N.I.
Vorlesungen über Approximationstheorie. Akademie–Verlag, Berlin (1967) (Translation from russian edition 1947).

Beeson, M.J.
Foundations of constructive mathematics. Springer–Verlag, Berlin Heidelberg New–York Tokyo (1985).

Borel, E.
Lecons sur les Fonctions de Variables Réelles. Gauthier–Villars, Paris (1905).

Bridges, D.S.
On the foundation of best approximation theory. J. Approximation Theory **28**, pp. 273–292 (1980).

Bridges, D.S.
A constructive development of Chebyshev approximation theory.
J. Approximation Theory **30**, pp. 99–120 (1980 A).

Bridges, D.S.
Lipschitz constants and moduli of continuity for the Chebyshev projection.
Proc. Amer. Math. Soc. **85**, pp. 557–561 (1982).

Bridges, D.S.
Recent progess in constructive approximation theory. In: Troelstra, A.S./van Dalen, D. (eds.) The L.E.J. Brouwer Centenary Symposium. North–Holland, Amsterdam (1982 A).

Bishop, E.
Foudations of constructive analysis. McGraw–Hill, New–York (1967).

Bishop, E./Bridges D.S.
Constructive analysis. Springer–Verlag, Berlin (1985).

Brown, D.K.
Functional analysis in weak subsystems of second order arithmetic. Ph.D. Thesis, Pennsylvania State University, University Park (1987).

Brown, D.K.
Notions of closed subsets of a complete separable metric space in weak subsystems of second order arithmetic. In: Sieg, W. (ed.), Logic and computation, Contemporary Mathematics vol. 106, AMS, pp. 39–50 (1990).

Brown, D.K./Simpson, S.G.
Which set existence axioms are needed to prove the separable Hahn–Banach theorem? Ann. of Pure and Appl. Logic **31**, pp. 123–144 (1986).

Chebycheff, P.L.
Sur les questions de minima qui se rattachent a la reprèsentation approximative des fonctions. Oeuvres I, pp. 273–378 (1859).

Cheney, E.W.
Introduction to approximation theory. McGraw–Hill, New–York (1966).

Feferman. S. Theories of finite type. In: Barwise, J. (ed.), Handbook of Mathematical Logic, North–Holland, Amsterdam, pp. 913–972 (1977).

Friedman, H.
Some systems of second order arithmetic and their use. Proc. of the 1974 International Congress of Mathematicians, Vancouver 1974, Vol. 1, Canadian Mathematical Congress, pp. 235–242 (1975).

Friedman, H.
Systems of second order arithmetic with restricted induction (abstract). J. Symbolic Logic **41**, pp. 558–559 (1976).

Friedman, H./Simpson, S.G./Smith, R.L.
Countable algebra and set existence axioms. Ann. of Pure and Applied Logic **25**, pp. 141–181 (1983).

Gödel, K.
Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280–287 (1958).

Haar, A.
Die Minkowskische Geometrie und die Annäherung an stetige Funktionen. Math. Ann. **78**, pp. 294–311 (1918).

Hatzikiriakou, K./Simpson, S.G.
Countable valued fields in weak subsystems of second order arithmetic. Ann. Pure and Applied Logic **41**, pp. 27–32 (1989).

Henry, M.S./Schmidt, D.
Continuity theorems for the product approximation operator. In: Law, A.G./Sahney, B.N. (eds.), Theory of Approximation with Applications, Academic Press, New–York (1976).

Hilbert, D.
Über das Unendliche. Math. Ann. **95**, pp. 161–190.

Jackson, D.
Note on a class of polynomials of approximation. Trans. Amer. Math. Soc. **22**, pp. 320–326 (1921).

Kirchberger, P.
Über Tschebychefsche Annäherungsmethoden. Dissertation, Göttingen (1902).

Kleene, S.C.
Introduction to metamathematics. North–Holland, Amsterdam (1952).

Kleene, S.C.
Recursive functionals and quantifiers of finite types I. Trans. Amer. Math. Soc. **91**, pp. 1-52 (1959).

Ko, K.–I.
On the computational complexity of best Chebyshev approximations. Journal of Complexity **2**, pp. 95–120 (1986).

Kohlenbach, U.
Theorie der majorisierbaren und stetigen Funktionale und ihre Anwendung bei der Extraktion von Schranken aus inkonstruktiven Beweisen: Effektive Eindeutigkeitsmodule bei besten Approximationen aus ineffektiven Eindeutigkeitsbeweisen. Dissertation, Frankfurt/Main, pp. xxii+278 (1990).

Kohlenbach, U.
Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. To appear in: J. Symbolic Logic (A).

Kohlenbach, U.
Pointwise hereditary majorization and some applications. To appear in: Arch. Math. Logic (B).

Kreinovich, V.Ja.
Categories of space–time models (russian). Candidate Thesis, Novosobirsk, Gos. Univ., Novosobirsk (1979).

Kreinovich, V.Ja.
Review of "Bridges, D.S.: Constructive functional analysis". Zentralblatt fr Mathematik **401**, 03027.

Kreisel, G.
Review of "Pour–El, M.B./Richards, I.: A computable ordinary differential equation which possesses no computable solution". J. Symbolic Logic **47**, pp. 900–902 (1982).

Luckhardt, H.
Extensional Gödel functional interpretation. A consistency proof of classical analysis. Springer Lecture Notes in Mathematics **306** (1973).

Luckhardt, H.
Herbrand–Analysen zweier Beweise des Satzes von Roth: Polynomiale Anzahlschranken. J. Symbolic Logic **54**, pp. 234–263.

Markoff, A.A.
Sur une question posee par Mendeleieff. Izvestia Akademii Nauk SSSR **62**, pp. 1–24.

Meinardus, G.
Approximation von Funktionen und ihre numerische Behandlung. Springer Tracts in Natural Philosophy **4**, Berlin (1964).

Natanson, I.P.
Konstruktive Funktionenthoie. Akademie–Verlag, Berlin (Tranlation of the original russian edition 1947).

Newman, D.J./Shapiro, H.S.
Some theorems on Cebysev approximation. Duke Math. J. **30**, pp. 673–682 (1963).

Rice, J.R.
The approximation of functions, vol.1. Addison–Wesley, Reading, Mass. (1964).

Roulier, J.A./Taylor, G.D.
Uniform approximation by polynomials having bounded coefficients. Abh. Math. Sem. Univ. Hamburg **36**, pp. 126–135 (1971).

Schönhage, A.
Approximationstheorie. Walter de Gruyter and Co., Berlin (1971).

Shioji, N./Tanaka, K.
Fixed point theory in weak second–order arithmetic. Ann. Pure and Appl. Logic **47**, pp. 167–188 (1990).

Simpson, S.G.
Which set existence axioms are needed to prove the Cauchy–Peano theorem for ordinary differential equations? J. Symbolic Logic **49**, pp. 783–802 (1984).

Simpson, S.G.
Reverse mathematics. In: Nerode, A./Share, R. (eds.), Proceedings of Symposia in Pure Mathematics **42**, AMS, Providence, R.I., pp. 461-471 (1985).

Tonelli, L.
I polinomi d'approssimazione di Tchebychev. Annali di Matematica Pura ed Applicata **15**, pp. 47–119 (1908).

Troelstra, A.S.
(Ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).

Troelstra, A.S.
Note on the fan theorem. J. Symbolic Logic **39**, pp. 584–596 (1974).

Troelstra, A.S./van Dalen, D.
Constructivism in mathematics: An introduction. Vol. I,II. North–Holland, Amsterdam (1988).

Weihrauch, K./Kreitz, C.

Representations of the real numbers and of the open subsets of the set of real numbers. Ann. Pure and Appl. Logic **35**, pp. 247–260.

Young, J.W.

General theory of approximation by functions involving a given number of arbitrary parameters. Trans. Amer. Math. Soc. **8**, pp. 331–344.