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1 Why were you initially drawn to the foundations of mathematics and/or the philosophy of mathematics?

At the age of 13 or so some initial interest in philosophy and Aristotelian logic was prompted by my classes in Ancient Greek language which was a main emphasis of study at my high school.

My real interest in the foundations of mathematics, however, started at the age of 17 during my last year at high school. Our mathematics teacher had the idea to have each of us to write an extended essay on some period in the history of mathematics. He designed a list of 20 topics starting from ancient mathematics to the beginning 20th century. The very day the topics could be chosen I was ill and could not attend school. When I finally was back in school I had to learn that only topic no. 20 on ‘Cantor, Dedekind, Hilbert’ was left, apparently because everybody had figured out that a topic touching on comparatively recent mathematics would be more difficult to deal with than, say, Babylonian mathematics. After I had overcome some initial shock I went to the university library in Frankfurt to get hold of the collected works of G. Cantor, D. Hilbert as well as R. Dedekind’s ‘Was sind und was sollen die Zahlen’ and some popular treatments of the ‘foundational crisis’ at the early 20’s century. Immediately, I got excited about the topic. After having finished the essay I was determined to study philosophy and mathematics with the aim to become a logician. During my first 1-2 years at the University of Frankfurt I focused on philosophy but I soon realized that in order to become able to prove new results in logic (rather than just discussing the early history of mathematical logic) I would need a solid background in mathematics. My teacher in Mathematical Logic at Frankfurt was Professor H. Luckhardt who pointed me towards the writings of A.S. Troelstra (most notably [19]). Via Luckhardt I also got under the influence of G. Kreisel’s views on logic and, in particular, proof theory (see e.g. [15, 16]). Much of my subsequent work has been stimulated by Kreisel’s ideas on the unwinding of proofs in mathematics and the necessity to pay for the latter some ‘entrance fee’ by learning enough core mathematics. So I was glad to get during my PhD studies an assistant position in the group ‘Real Analysis and Potential Theory’ with Professor J. Bliedtner and after that in ‘Analytic Number Theory and Functional Analysis’ with Professor W. Schwarz.

2 What examples from your work (or the work of others) illustrate the use of mathematics for philosophy?

Well, as an ancient use of mathematics in philosophy one may point to the refutation of the Pythagorean philosophy by the proof of the irrationality of the length of the diagonal in the unit square.

From my work I can only address possible uses of mathematical logic for the philosophy of mathematics: while in my area originally pre-formal concepts from the philosophy of mathematics have been instrumental to guide certain mathematical developments (see the next question), applications in the opposite direction in my view mainly consist in correcting errors in some philosophically inclined foundational debates. This e.g. concerns the topic of constructive foundations of mathematics. A common view is that a (fully) constructive proof per se is superior (contains more information) than a prima facie classical argument. Experience from the unwinding of classical arguments, however, shows that usually only small parts of a proof contribute to the computational content of a proof while the bulk of the proof may well be proven ineffectively. Sometimes, using a more ineffective argument helps to push further parts of a proof into lemmas which are not needed to be analyzed at all. Replacing ineffective analytic or geometric steps in a proof by constructive induction arguments can result in bounds that are much less good than those extractable from the original classical proofs (see e.g. [7]). Often even ineffective parts in a proof which **do** matter for the computational content can be systematically transformed in such a way that computationally relevant information can be read off. E.g. this can be done by suitable forms of proof interpretations, notably of Gödel's functional ('Dialectica') interpretation ([14]). In this sense already the original proof can be seen as **implicitly** containing additional computational content even in cases where the latter seems absent at first.

As a justification for a global approach to constructivity (based on intuitionistic logic) often useful properties of the resulting constructive formal systems are pointed to: e.g. usually systems of intuitionistic analysis satisfy the so-called fan rule which, for instance, allows one to infer uniform convergence from pointwise convergence on compact spaces. However, it can be shown that this property also holds for such systems if one adds the binary ('weak') König's lemma WKL or even König's lemma KL (and - since we can include the axiom of choice schema - also the uniform versions of these principles) as well as the Markov principle (in the absence of the Markov principle even full comprehension in all types for all negated formulas may be added, [8, 9, 12]). This is remarkable since it was the very intuitionistic rejection of WKL and related principles which led to the formulation of the fan principle in intuitionistic mathematics. Hence if the motivation for finding a constructive proof is to be able to use the fan rule then there is no reason not to allow e.g. WKL in such a proof despite of the fact that this principle is not effective as long as the proof is constructive relative to WKL. If the matrix in the instance of the fan rule at hand is only purely existential than even full classical logic can be allowed and one still has the closure under the corresponding instance of the rule ([5, 7]). In fact, in my work I have shown that e.g. proofs of uniqueness theorems in best approximation theory that are based on WKL carry just as much computational content (namely a so-called modulus of uniqueness or 'rate of strong unicity') as fully constructive proofs (even of strong positive reformulations of) uniqueness (see [5, 6, 13]).

In connection with the common misconception that a constructive proof always is better than a classical argument it has been overlooked that occasionally it pays off to view a constructive proof as a classical one which first needs a constructivization via negative translation: consider an implication $A \to B$ where A is a $\forall\exists\forall$-('Π^0_3')-sentence while B a $\forall\exists$-('Π^0_2)'-sentence and suppose that $A \to B$ is provable in intuitionistic ('Heyting') arithmetic HA. The straightforward constructive interpretation of (a constructive proof of) such an implication (as spelled out e.g. by Kreisel's modified realizability interpretation) is that it provides a procedure F that transforms any witnessing ('Skolem') function f for the premise A into a witnessing functions g for the conclusion. This interpretation, however, is very weak in cases where A does not have an effective witnessing function f. If one views that proof of $A \to B$ instead as a classical proof and applies negative translation to it, then the situation improves: both in the premise A as well as in the conclusion B the existential quantifiers get weakened by a double negation in front. However, this weakening of the premise is much more severe than that of the conclusion. From the latter one easily recovers B using the aforementioned Markov principle. Using now instead of modified realizability a technique supporting the Markov principle, namely Gödel's functional interpretation, one obtains a witness function for the conclusion in a witness of the so-called no-counterexample interpretation ([15]) of the premise A which is a much weaker (usually subrecursive of low complexity) input than the (in general noncomputable) Skolem function seemingly required by the original constructive reading of the implication.

In the context of predicative foundations of mathematics ([20]) special emphasis has been given to the fact that most function spaces used in scientifically applicable functional analysis are separable and so to a large extent can be treated based on arithmetical comprehension ([2]). Similarly, in the program of so-called reverse mathematics (initiated H. Friedman and S. Simpson,[18]) one works - following Hilbert and Bernays ([4]) - in systems (fragments of 2nd order arithmetic) whose very language is so restricted that only separable spaces can be represented (Hilbert and Bernays actually do allow 3rd order parameters). Similar restrictions (to separable spaces) are put forward in constructive mathematics (e.g. Bishop's constructive analysis [1]) as an instance of the general rule of avoiding 'pseudo-generality' and also play an important role in intuitionistic analysis via so-called standard representation of complete separable ('Polish') metric spaces. Recent work in the logical analysis of proofs, however, has shown that even if one is interested primarily in separable spaces it can be crucial to observe that a given proof does not use separability: general logical metatheorems ([10]) guarantee the extractability of uniform bounds that are independent not only from parameters in compact spaces but from parameters in metrically bounded spaces (as well as of self-mappings of such spaces) as long as only general facts about the class of the spaces in question are used which have a strong uniformity built-in. This applies to metric, hyperbolic, CAT(0), normed, uniformly convex and inner product spaces among many others. However, the uniform version of separability translates into total boundedness of metrically bounded subspaces and hence - modulo completeness - compactness. So in order to be able to diagnose uniformity in the absence of compactness it is critical to observe that separability is not used. For much refined recent metatheorem which not even require the boundedness of the whole underlying space but only bounds on local distances see [3]. A survey on the numerous applications of this approach in metric fixed point theory is given in [11].

So while it **is** of foundational interest that large parts of mathematics can be formalized in weak predicative systems due to the separability of most spaces of interest it would, nevertheless, be philosophically misguided to restrict ones attention in the philosophy of mathematics to those parts only.

3 What is the proper role of philosophy of mathematics in relation to logic, foundations of mathematics, the traditional core areas of mathematics, and science?

One main direction of influence from the philosophy of mathematics to the foundations of logic and mathematical logic has been the result of bringing concepts first developed in philosophy (not only of mathematics) into a formal context and deriving clear cut mathematical consequences from this transition. A particularly striking example is the Liar paradox which, when phrased in different formal contexts, has played an enormous role in the development of mathematical logic:

1) When stated in a formalized language, say of Peano arithmetic PA, it implies the undefinability of a truth predicate for PA in the language of PA (A. Tarski).

2) When applied to the proof predicate for PA it yields Gödel's first incompleteness theorem.

3) When truth is replaced by 'terminates' it yields the undecidability of the halting problem (A. Turing, A. Church).

Another example of a deep mathematical use of a concept from the philosophy of the mathematics is the notion of predicative definability and its (impredicative) use by Gödel to construct the constructible hierarchy and thereby the (relative) consistency of the continuum hypothesis.

Leibniz' idea of possible worlds led to the development of Kripke semantics for modal and intuitionistic logic and so, in a sense, to the notion of forcing (discovered originally by P. Cohen). The latter also has some similarities with the concept of the development of mathematical knowledge in stages as formulated in Brouwer's theory of the 'creative subject'.

Different formalizations of the informal (so-called Brouwer-Heyting-Kolmogorov) semantics of intuitionistic logic have led to numerous important technical devices such as realizability and functional interpretations and play (in the form of the 'proofs-as-programs' paradigm) an increasing role in computer science.

4 What do you consider the most neglected topics and/or contributions in late 20th century philosophy of mathematics?

I am not familiar enough with the recent and current literature on the philosophy of mathematics to comment on this question.

5 What are the most important open problems in the philosophy of mathematics and what are the prospects for progress?

In my view one of the most important problem in the philosophy of mathematics as well as the foundations of mathematics is to explain why in actually existing core mathematics apparently only a tiny part (both in terms of proof theoretic strength as well as w.r.t. the logical complexity involved) of the vast resources provided by formal systems such as Zermelo Fraenkel set theory is exploited. This concerns the facts that (i) only weak set existence axioms are used and (ii) only formulas of low quantifier complexity show up in ordinary mathematics: despite of the incompleteness of PA usually small fragments of PA suffice to proof statements that can be expressed in the language of PA. Reverse mathematics has shown that large parts of mathematics can be carried out in systems conservative over primitive recursive arithmetics PRA ([17]) and even larger parts in systems conservative over PA ([2]). Although proofs in logic in general have underlying Herbrand disjunctions whose length is nonelementary in the basic proof data (R. Statman, V. Orevkov), in practice such disjunctions usually are rather simple and short. The question now is whether these empirical observations reflect a general mathematical fact, namely that proof theoretic strength and logical complexity have no real relevance for mathematics, or whether it indicates that there are relevant parts of (future) mathematics which simply have been overlooked just because they require a better understanding of strong set theoretic concepts and statements of high logical complexity. Some exciting work of H. Friedman points to the latter though to give a definitive answer seems to be premature.

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