EFFECTIVE ASYMPTOTIC REGULARITY FOR ONE-PARAMETER NONEXPANSIVE SEMIGROUPS

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Abstract. We give explicit bounds on the computation of approximate common fixed points of one-parameter strongly continuous semigroups of nonexpansive mappings on a subset C of a general Banach space. Moreover, we provide the first explicit and highly uniform rate of convergence for an iterative procedure to compute such points for convex C. Our results are obtained by a logical analysis of the proof (proof mining) of a theorem by T. Suzuki.

1. Introduction and Preliminaries

In this paper we give a quantitative analysis of a theorem due to Suzuki [18] which states that in order to compute a common fixed point of a one-parameter strongly continuous semigroup of nonexpansive mappings it is sufficient to compute a fixed point of a single nonexpansive mapping which is derived from this semigroup. As a corollary to this we get an explicit and highly uniform rate of asymptotic regularity for the semigroup. Such semigroups play a central role in the study of abstract Cauchy problems (see e.g. [3, 4, 16, 2] for the classical theory).
Suzuki's proof which we analyze here is not effective and is based on a number-theoretic density result. This makes our extraction of explicit bounds highly non-trivial and so our paper is also a significant new contribution to the so-called 'Proof Mining' program (going back to pioneering ideas of Georg Kreisel in the 50's) which uses tools from logic (applied proof theory) to extract new quantitative constructive information by logical analysis of prima facie noneffective proofs. The information is 'hidden' behind an implicit use of quantifiers in the proof, and its extraction is guaranteed by certain logical metatheorems if the statement proved is of a certain logical form (for instance here a $\forall\exists$ statement) and proved within a suitable deductive framework ([10, 6, 9]). The resulting quantitative form of the given theorem then comes again with an ordinary proof in analysis (as in this paper) which makes no reference to any tools from logic. Within the past 15 years, proof mining has been applied by the first author and his collaborators to various fields of mathematics, including approximation theory, ergodic theory, fixed point theory, nonlinear analysis, and (recently) PDE theory (see e.g. [8, 9, 11, 13]).

In this section we recall some basic definitions, introduce certain preliminary concepts and state our main result.

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By $\mathbb{N}$ we denote the set of natural numbers $\{1, 2, ...\}$, by $\mathbb{Z}$ the set of integers and by $\mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+$ the sets of non-negative integers, rationals and reals respectively.

Definition 1. *Given a Banach space E and a subset $C \subseteq E$, a mapping $T : C \to E$ is nonexpansive if*

$$\forall x, y \in C \ (\|Tx - Ty\| \leq \|x - y\|) .$$

Definition 2. *A family $\{T(t) : t \geq 0\}$ of self-mappings $T(t) : C \to C$ for a subset C of a Banach space E is called a one-parameter strongly continuous semigroup of nonexpansive mappings (or nonexpansive semigroup for short) if the following conditions hold:*

(1) *for all $t \geq 0$, $T(t)$ is a nonexpansive mapping on C,*
(2) *$T(s) \circ T(t) = T(s+t)$,*
(3) *for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is continuous.*

Our main result will be a quantitative version of the following theorem by Suzuki in [18] :

Theorem 1. *(Theorem 1 in [18]) Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on a subset $C \subseteq E$ for some Banach space E. Let $F(T(t))$ denote the set of fixed points of $T(t)$. Let $\alpha, \beta \in \mathbb{R}^+$ satisfying $\alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. Then for all $\lambda \in (0, 1)$ we have:*

$$\bigcap_{t \geq 0} F(T(t)) = F(\lambda T(\alpha) + (1 - \lambda)T(\beta)),$$

where

$$\lambda T(\alpha) + (1 - \lambda)T(\beta)$$

is a mapping from C into E defined by

$$(\lambda T(\alpha) + (1 - \lambda)T(\beta))x = \lambda T(\alpha)x + (1 - \lambda)T(\beta)x$$

for $x \in C$.

The inclusion

$$\bigcap_{t \geq 0} F(T(t)) \subseteq F(\lambda T(\alpha) + (1 - \lambda)T(\beta))$$

is trivial.

We will extract a bound from (the proof of) the nontrivial inclusion

$$\bigcap_{t \geq 0} F(T(t)) \supseteq F(\lambda T(\alpha) + (1 - \lambda)T(\beta))$$

in the following sense: notice that the above inclusion states

$$\forall q \in C \ ((\lambda T(\alpha) + (1 - \lambda)T(\beta))q = q \to \forall t \geq 0 \ T(t)q = q)$$

which can be written as

$$\forall q \in C \ \forall m \in \mathbb{N} \ \forall t \geq 0 \ \exists k \in \mathbb{N}$$
$$(\|(\lambda T(\alpha) + (1 - \lambda)T(\beta))q - q\| \leq 2^{-k} \to \|T(t)q - q\| < 2^{-m}).$$

The above statement is of the form $\forall\exists$. Therefore, as guaranteed by general logical metatheorems due to the first author (see Chapter 17 in [9] and also [10, 6]), it is possible to extract a computable bound $\Psi > 0$ depending on bounds on the input data so that

$$\forall b \in \mathbb{N} \ \forall q \in C_b \ \forall M \in \mathbb{N} \ \forall t \in [0, M] \ \forall m \in \mathbb{N}$$

$$(\|(\lambda T(\alpha)+(1-\lambda)T(\beta))q-q\| \leq \Psi(M,m,b,...) \to \|T(t)q-q\| < 2^{-m}),$$

where, given $b \in \mathbb{N}$, $C_b := \{q \in C : \|q\| \leq b\}$. For this, however, we need (in order to apply the logical metatheorems) to strengthen the concept of 'strongly continuous semigroup of nonexpansive mappings' by imposing an equicontinuity condition (witnessed by a modulus ω):

Definition 3. *We say that a nonexpansive semigroup $\{T(t) : t \geq 0\}$ on a subset C of a Banach space E is uniformly equicontinuous if the mapping $t \mapsto T(t)q$ is uniformly continuous on each compact interval $[0,K]$ for all $K \in \mathbb{N}$ and given a $b \in \mathbb{N}$ it has a common modulus of continuity for all $q \in C_b$. Namely if there exists a function $\omega : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ so that*

$$\forall b \in \mathbb{N}\ \forall q \in C_b\ \forall m \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall t,t' \in [0,K]$$

$$(|t-t'| < 2^{-\omega_{K,b}(m)} \to \|T(t)q-T(t')q\| < 2^{-m}).$$

We call ω a modulus of uniform equicontinuity for the nonexpansive semigroup $\{T(t) : t \geq 0\}$.

In the following we will assume uniform equicontinuity as defined above for the nonexpansive semigroup $\{T(t) : t \geq 0\}$.
In fact, in the literature one may find easily examples where this is fulfilled. For instance, in [15] the following nonexpansive semigroup is studied (referring to [14] where it is attributed to G.F. Webb):

Let $E = C = C_{[0,1]}$ and for $f \in C_{[0,1]}$ and $x \in [0,1]$ define:

$$[T(t)f](x) := \begin{cases} t+f(x) \text{ if } f(x) \geq 0, \\ t+\frac{1}{2}f(x) \text{ if } f(x) < 0 \text{ and } t+\frac{1}{2}f(x) \geq 0, \\ 2t+f(x) \text{ if } t+\frac{1}{2}f(x) < 0. \end{cases}$$

It is easy to see that the above semigroup is nonexpansive and that $\omega(m) := m+1$ can be taken as a common modulus of uniform continuity for $t \mapsto T(t)q$ on $[0,\infty)$ (uniformly on the whole space E).
We also note that in the case of the original definition of a strongly continuous semigroup, our bound will still apply replacing $\omega_{K,b}$ by any modulus $\omega_{K,q}$ of uniform continuity (on $[0,K]$) of $t \mapsto T(t)q$ (see the corollary at the end of Section 3). It is only the independence of the bound from q which requires the extra equicontinuity assumption.

We will achieve the above bound extraction by proof mining on the proof of Theorem 1 in [18].

As Suzuki's theorem makes an irrationality assumption on γ we will need a quantitative version of this assumption in our quantitative analysis of his proof:

Definition 4. *Let $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. A function $f_\gamma : \mathbb{N} \to \mathbb{N}$ s.t.*

$$\forall p \in \mathbb{N}, p' \in \mathbb{Z}^+\ (|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)}) \quad (I)$$

is called an effective irrationality measure for γ.

Remark 1. *Since $\gamma > 0$, (I) can easily be seen to imply the claim also for $p' \in \mathbb{Z}$.*

Example: We may choose in Suzuki's theorem $\alpha := \sqrt{2}, \beta := 2$ so that $\gamma := \alpha/\beta \in (0,1) \setminus \mathbb{Q}$. It is easy to see that in this case we can take $f_\gamma(p) := 4p^2$.

We are now ready to state the main result of this paper:

Theorem 2. *(Quantitative version of Theorem 1 in* [18]*). Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on $C \subseteq E$ for some Banach space E. Let $\alpha, \beta \in \mathbb{R}^+$ with $0 < \alpha < \beta$. Let $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with an effective irrationality measure f_γ. Let*

$$S : C \to E,\ S := \lambda T(\alpha) + (1-\lambda)T(\beta)$$

with $\lambda \in (0,1)$. Moreover, assume that $\{T(t) : t \geq 0\}$ is uniformly equicontinuous with a modulus of uniform equicontinuity ω. Let $\Lambda \in \mathbb{N}$ be such that $1/\Lambda \leq \lambda, 1-\lambda$. Then

$$\forall b \in \mathbb{N}\ \forall M \in \mathbb{N} \forall q \in C_b\ \forall m \in \mathbb{N}$$

$$(\|Sq - q\| \leq \Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) \to \forall t \in [0, M]\, (\|T(t)q - q\| < 2^{-m})),$$

where

$$\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) = \frac{2^{-m}}{8(\sum_{i=1}^{\phi(k,f_\gamma)-1} \Lambda^i + 1)(1 + MN)}$$

and $N \in \mathbb{N}$ so that $\beta \geq 1/N$, $\mathbb{N} \ni D \geq \beta$, $k := D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1} \in \mathbb{N}$ and

$$\phi(k, f) := \max\{2f(i-j) + 6\ :\ 0 \leq j < i \leq k+1\} \in \mathbb{N}.$$

Together with a well-known quantitative asymptotic regularity result for the Krasnoselskii iteration $\{x_n\}$ of S (in the case where C is convex and $\{x_n\}$ is bounded, see [1]) we derive from this in the final section an explicit asymptotic regularity result for the computation of approximate common fixed points for each given initial part $[0, M]$ of the nonexpansive semigroup (see Corollary 1).

2. Quantitative lemmas

We will start by giving quantitative versions of several preliminary lemmas in [18]. We will then make use of our quantitative versions of the lemmas to extract our quantitative version of Theorem 1 in [18], i.e. our Theorem 2. In the following, as in [18], for $x \in \mathbb{R}$ we denote with $[x] \in \mathbb{Z}$ the largest integer not exceeding $x \in \mathbb{R}$. Notice that for all $t \in \mathbb{R}, z \in \mathbb{Z}$ we have

$$[t + z] = [t] + z.\ (II)$$

Throughout this section, α, β will always be real numbers with $0 < \alpha < \beta$ and $\gamma := \alpha/\beta$.

Definition 5. *Let $\theta \in [0,1]$.*

(1) *Define a sequence $\{\tilde{A}_n\}$ of subsets of $[0,1]$ by $\tilde{A}_1 = \{\theta\}$ and*

$$\tilde{A}_{n+1} = \bigcup_{t \in \tilde{A}_n} \{|1-t|, |\gamma - t|\}$$

for $n \in \mathbb{N}$ and set

$$\tilde{A}(\theta) := \bigcup_{n=1}^{\infty} \tilde{A}_n.$$

(2) *Define a sequence* $\{A_n\}$ *of subsets of* $[0,\beta]$ *by*

$$A_1 = \{\theta\beta\},$$
$$A_{n+1} = \bigcup_{t\in A_n} \{|\alpha - t|, |\beta - t|\}$$

for $n \in \mathbb{N}$. *Set*

$$A(\theta) := \bigcup_{n=1}^{\infty} A_n.$$

The following quantitative version of the relevant parts of Lemmas 2 and 3 in [18] can be extracted by inspecting the proof in [18].

Lemma 1. *(Quantitative version of (relevant part of) Lemmas 2 and 3 in* [18]*)* *Let* $\{\tilde{A}_n\}$ *be the sequence of subsets of* $[0,1]$ *and* $\{A_n\}$ *be the sequence of subsets of* $[0,\beta]$ *as defined above. Then*
(i) *For* $t \in \mathbb{R}$, *if* $t-[t] \in \tilde{A}_n$ *then* $-t-[-t] \in \tilde{A}_{n+1}$ *in the case* $0 < t-[t] < 1$ *and* $-t-[-t] \in \tilde{A}_n$ *in the case* $t-[t] = 0$.
(ii) *For* $t \in \mathbb{R}$, *if* $t-[t] \in \tilde{A}_n$, *then* $t-\gamma-[t-\gamma] \in \tilde{A}_{n+1}$ *for the case where* $\gamma \le t-[t]$ *and* $t-\gamma-[t-\gamma] \in \tilde{A}_{n+2}$ *for the case* $t-[t] < \gamma$.
(iii) *Define for* $l \in \mathbb{N}$

$$B_l := \{(e\theta + i\gamma - [e\theta + i\gamma])\beta : e \in \{+1,-1\}, i \in \mathbb{Z}, |i| \le l\}$$

and

$$A_n^* := \bigcup_{i\le n} A_i.$$

Then for each $l \in \mathbb{N}$ *we have*

$$B_l \subseteq A^*_{2l+8}$$

i.e. for each $x \in B_l$ *there exists an* $n \le 2l+8$ *with* $x \in A_n$.

Proof. (i) Let $t \in \mathbb{R}$ be with $t-[t] \in \tilde{A}_n$. If $t-[t] = 0$, this is immediate as then $t \in \mathbb{Z}$, thus

$$-t-[-t] = 0 = t-[t].$$

If $0 < t-[t] < 1$, Lemma 1(iii) in [18] gives $-[-t] = [t]+1$ and, therefore,

$$-t-[-t] = -t+[t]+1 = |1-(t-[t])|.$$

By definition $|1-(t-[t])| \in \tilde{A}_{n+1}$ and thus $-t-[-t] \in \tilde{A}_{n+1}$.

We will now show step (ii). Let $t \in \mathbb{R}$ be with $t-[t] \in \tilde{A}_n$. We distinguish two cases: In the case where $\gamma \le t-[t]$, by Lemma 1(iv) from [18], we have $[t-\gamma] = [t]$. Thus

$$t-\gamma-[t-\gamma] = t-[t]-\gamma = |\gamma-(t-[t])|.$$

From the assumption we get $|\gamma-(t-[t])| \in \tilde{A}_{n+1}$ and thus $t-\gamma-[t-\gamma] \in \tilde{A}_{n+1}$. Now consider the other case where $t-[t] < \gamma$. Then Lemma 1(v) in [18] gives:

$$[t-\gamma]+1 = [t].$$

Therefore

$$\gamma - t + [t-\gamma] + 1 = \gamma - t + [t] = |\gamma-(t-[t])| \in \tilde{A}_{n+1}$$

and so, in turn,

$$t-\gamma-[t-\gamma] = |1-(\gamma-t+[t-\gamma]+1)| \in \tilde{A}_{n+2}.$$

We will now show step (iii) : A simple induction on n shows that for all levels n :

$$\forall t \ (t \in \tilde{A}_n \leftrightarrow \beta t \in A_n).$$

Hence it suffices to prove that for $l \in \mathbb{N}$

$$\tilde{B}_l := \{(e\theta + i\gamma - [e\theta + i\gamma]), e \in \{+1, -1\}, i \in \mathbb{Z}, |i| \leq l\}$$

and

$$\tilde{A}^*_n := \bigcup_{i \leq n} \tilde{A}_i$$

we have for each $l \in \mathbb{N}$

$$\tilde{B}_l \subseteq \tilde{A}^*_{2l+8}$$

i.e. for each $x \in \tilde{B}_l$ there exists an $n \leq 2l + 8$ with $x \in \tilde{A}_n$.

Case I : $\theta \neq 1$. We have $[\theta] = 0$, therefore $\theta - [\theta] = \theta \in \tilde{A}_1$. Now we apply step (ii) l times which results in an increase by at most 2 in each step. Hence we obtain

$$\theta - l\gamma - [\theta - l\gamma] \in \tilde{A}^*_{1+2l}.$$

Now we apply step (i) which increases the level at most by 1. Hence we have

$$-\theta + l\gamma - [-\theta + l\gamma] \in \tilde{A}^*_{2+2l}.$$

This, in particular, holds for $l = 1$ and so

$$-\theta + \gamma - [-\theta + \gamma] \in \tilde{A}^*_4.$$

We now apply again step (ii) $l + 1$ times resulting in at most

$$-\theta - l\gamma - [-\theta - l\gamma] \in \tilde{A}^*_{4+2(l+1)} = \tilde{A}^*_{6+2l}.$$

At this point we have covered the case $e = -1$ for both positive and negative $l \in \mathbb{Z}$. We now apply step (i) which gives a shift by at most 1 and therefore obtain

$$\theta + l\gamma - [\theta + l\gamma] \in \tilde{A}^*_{7+2l}.$$

So we have now covered the $e = +1$ case for both positive and negative $l \in \mathbb{Z}$.

Case II : $\theta = 1$. Here $\theta - [\theta] = 0 \in \tilde{A}_2$, therefore there is a shift by 1 on all the above.

Combining Cases I and II, we obtain at most

$$\theta + l\gamma - [\theta + l\gamma] \in \tilde{A}^*_{8+2l}$$

where $l \in \mathbb{Z}$. □

The proof of the following lemma was omitted in [18] because it originates from well-known classical results. However, we give a proof here because we will later make use of it so as to extract our quantitative version of this lemma that will be needed for the proof of our Theorem 2.

Lemma 2. *(Lemma 4 in* [18] *) Consider the sequence* $\{A_n\}$ *of subsets of* $[0, \beta]$ *and the set* $A(\theta)$ *as defined above. Suppose that* $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$*. Then*

$$\overline{A(\theta)} = [0, \beta],$$

where $\overline{A(\theta)}$ *denotes the closure of* $A(\theta)$*.*

Proof. By Lemma 3 in [18]

$$A(\theta) \setminus \{\beta\} = \{(e\theta + l\gamma - [e\theta + l\gamma])\beta : e \in \{+1, -1\}, l \in \mathbb{Z}\}.$$

Notice that it is always true, by the definition of the floor function '$[\cdot]$', that

$$\forall l \in \mathbb{Z}\ (e\theta + l\gamma - [e\theta + l\gamma] \in [0, 1)).$$

We will show that $A(\theta)$ is dense in $[0, \beta]$. It is enough to show that $A(\theta) \setminus \{\beta\}$ is dense in $[0, \beta]$. For that we will first show that the set $\{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}\}$ is dense in $[0, 1]$. [1] We argue as follows. Fix $k \in \mathbb{N}$. Cut $[0, 1]$ into pieces of length $\frac{1}{k+1}$ each. Then by the pigeonhole principle there must exist $i, j \in \mathbb{Z}$ so that $i \neq j$ and

$$0 \leq j, i \leq k + 1$$

such that $i\gamma - [i\gamma]$ and $j\gamma - [j\gamma]$ belong to the same piece so that

$$|i\gamma - [i\gamma] - (j\gamma - [j\gamma])| \leq 1/(k + 1) < 1/k.$$

Notice that because $\gamma \notin \mathbb{Q}$ and since $i \neq j$ we have

$$i\gamma - [i\gamma] \neq j\gamma - [j\gamma],$$

for $i\gamma - [i\gamma] = j\gamma - [j\gamma]$ would give

$$\gamma = \frac{[i\gamma] - [j\gamma]}{i - j} \in \mathbb{Q}$$

which is a contradiction. W.l.o.g. assume that $i\gamma - [i\gamma] > j\gamma - [j\gamma]$.
We now define

$$X := \max\{x \in \mathbb{Z}^+\ :\ x(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1\}.$$

Now notice that for all $p \in \mathbb{N}$ we have

$$|p(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) - (p + 1)(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))|$$
$$= |i\gamma - [i\gamma] - (j\gamma - [j\gamma]))| < 1/k.$$

Therefore, for any $m \in [0, 1, ..., k - 1]$ we can find an $\tilde{m} \in [1, ..., X]$ so that

$$\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) \in [\frac{m}{k}, \frac{m+1}{k}].$$

Moreover notice that, because of

$$0 < \tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1,$$

we have

$$[\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))] = 0$$

and, therefore,

$$\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))$$
$$= \tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) - [\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))]$$

(by (II))

$$= \tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) - \tilde{m}(-[i\gamma] + [j\gamma]) - [\tilde{m}(i - j)\gamma]$$
$$= \tilde{m}(i - j)\gamma - [\tilde{m}(i - j)\gamma].$$

[1] This fact is classical. Our proof is inspired by a proof given at 'http://math.stackexchange.com/questions/272545/multiples-of-an-irrational-number-forming-a-dense-subset' but we replaced the use of Bolzano-Weierstraß by the finitary pigeonhole principle.

Therefore

$$\tilde{m}(i-j)\gamma - [\tilde{m}(i-j)\gamma] \in [\frac{m}{k}, \frac{m+1}{k}] \cap \{l\gamma - [l\gamma] \ : \ l \in \mathbb{Z}\}$$

and, because $k \in \mathbb{N}$ was arbitrary, we conclude that $\{l\gamma - [l\gamma] \ : \ l \in \mathbb{Z}\}$ is dense in $[0,1]$. Therefore, by (II) for $\theta \in \{0,1\}$ the set

$$\{e\theta + l\gamma - [e\theta + l\gamma] \ : \ l \in \mathbb{Z}, e \in \{+1,-1\}\}$$

is dense in $[0,1]$. To show the density of the set

$$\{e\theta + l\gamma - [e\theta + l\gamma] \ : \ l \in \mathbb{Z}, e \in \{+1,-1\}\}$$

in $[0,1]$ where $\theta \in (0,1)$ it is enough to show the density of

$$\{\theta + l\gamma - [\theta + l\gamma] \ : \ l \in \mathbb{Z}\} \subset \{e\theta + l\gamma - [e\theta + l\gamma] \ : \ l \in \mathbb{Z}, e \in \{+1,-1\}\}$$

in $[0,1]$. Fix $k \in \mathbb{N}$.

Case A : Let $x \in [\theta, 1-\frac{1}{k}]$. Then $x' := x - \theta \in [0,1]$. Hence there exists an $i \in \mathbb{Z}$ so that

$$|x' - (i\gamma - [i\gamma])| < \frac{1}{k}(!).$$

Then

$$i\gamma - [i\gamma] + \theta < x' + \theta + \frac{1}{k} = x + \frac{1}{k} \le 1.$$

Notice that $i\gamma - [i\gamma] + \theta < 1$, by Lemma 1(v) in [18], gives us $[i\gamma - (1-\theta)] = [i\gamma] - 1$ and by (II) we have

$$[i\gamma - (1-\theta)] = [i\gamma + \theta] - 1.$$

Therefore

$$\theta + i\gamma - [\theta + i\gamma] = \theta + i\gamma - [i\gamma].$$

By (!) we have (since $x = x' + \theta$)

$$|x - (i\gamma - [i\gamma] + \theta)| < \frac{1}{k}$$

and so

$$|x - (\theta + i\gamma - [\theta + i\gamma])| < \frac{1}{k}.$$

Case B : Let $x \in [\frac{1}{k}, \theta)$. Then $x' := x - \theta + 1 \in [0,1)$. Again there exists an $i \in \mathbb{Z}$ so that

$$|x' - (i\gamma - [i\gamma])| < \frac{1}{k}(!).$$

Then

$$i\gamma - [i\gamma] + \theta \ge 1.$$

Therefore, by Lemma 1(iv) in [18], we have $[i\gamma - (1-\theta)] = [i\gamma]$. Moreover, by (II),

$$[i\gamma - (1-\theta)] = [i\gamma + \theta] - 1.$$

Therefore

$$\theta + i\gamma - [\theta + i\gamma] = \theta + i\gamma - ([i\gamma] + 1) = i\gamma - [i\gamma] + \theta - 1.$$

By (!) (since $x = x' + \theta - 1$)

$$|x - (i\gamma - [i\gamma] + \theta - 1)| < \frac{1}{k}$$

and so

$$|x - (\theta + i\gamma - [\theta + i\gamma])| < \frac{1}{k}.$$

By combining Cases A and B, we have that

$$\forall x \in [1/k, 1 - 1/k]\ \exists i \in \mathbb{Z}\ (|x - (\theta + i\gamma - [\theta + i\gamma])| < \frac{1}{k})$$

and hence

$$\forall x' \in [0,1]\ \exists i \in \mathbb{Z}\ (|x' - (\theta + i\gamma - [\theta + i\gamma])| < \frac{2}{k}).$$

Therefore

$$\forall \tilde{x} \in [0,\beta]\, \exists x' := \tilde{x}/\beta \in [0,1]\ \exists i \in \mathbb{Z}$$

$$(|\tilde{x} - (\theta + i\gamma - [\theta + i\gamma])\beta| = |x'\beta - (\theta + i\gamma - [\theta + i\gamma])\beta| = \beta|x' - (\theta + i\gamma - [\theta + i\gamma])| < \frac{2\beta}{k}).$$

Hence for $\mathbb{N} \ni D \geq \beta$ we have

$$\forall \tilde{x} \in [0,\beta]\ \exists i \in \mathbb{Z}\ (|\tilde{x} - (\theta + i\gamma - [\theta + i\gamma])\beta| < \frac{2D}{k}).$$

Since $k \in \mathbb{N}$ was arbitrary, the claim follows. □

We will show a quantitative version of the above lemma.

Lemma 3. *(Quantitative version of Lemma 4 in* [18]*) Define the sequence* $\{A_n\}$ *of subsets of* $[0,\beta]$ *and the set* $A(\theta)$ *as before. Let* $\mathbb{N} \ni D \geq \beta$*. Suppose that* $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ *is irrational and has an effective irrationality measure* f_γ*. Then*

$$\forall k \in \mathbb{N}\, \forall s \in [0,\beta]\, \exists s' \in A^*_{\phi(k,f_\gamma)}\ (|s - s'| < \frac{2D}{k})\quad (*),$$

where

$$\phi(k,f) := \max\{2f(i-j) + 6\ :\ 0 \leq j < i \leq k+1\}.$$

Proof. First recall that, for $i, j \in \mathbb{Z}$ with $i \neq j$ and $0 \leq j, i \leq k+1$, X was defined as the maximal nonnegative integer s.t.

$$X(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1,$$

where we assumed w.l.o.g. that $i\gamma - [i\gamma] - (j\gamma - [j\gamma]) > 0$.
Since γ is assumed to be irrational with an effective irrationality measure f_γ,

$$|\gamma - \frac{[i\gamma] - [j\gamma]}{i-j}| \geq \frac{1}{f_\gamma(|i-j|)}.$$

(taking in (I)

$$p' := [i\gamma] - [j\gamma]$$

and

$$p := i - j$$

if $i > j$ and $-p', -p$ otherwise using Remark 1).
From the proof of the previous lemma we have

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in A^*_n\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$

$$\left(\left|\gamma - \frac{[i\gamma] - [j\gamma]}{i-j}\right| \geq \frac{1}{f_\gamma(|i-j|)} \to |s - s'| < \frac{2D}{k}\right)$$

i.e.

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in A^*_n\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$
$$\left(\left|\frac{\gamma(i-j) - ([i\gamma]-[j\gamma])}{i-j}\right| \geq \frac{1}{f_\gamma(|i-j|)} \to |s-s'| < \frac{2D}{k}\right)$$

thus

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in A^*_n\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$
$$(|\gamma(i-j) - ([i\gamma]-[j\gamma])| \geq \frac{|i-j|}{f_\gamma(|i-j|)} \to |s-s'| < \frac{2D}{k})$$

and, therefore,

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in A^*_n\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$
$$(|\gamma(i-j) - ([i\gamma]-[j\gamma])| \geq \frac{|i-j|}{f_\gamma(|i-j|)} \to 1/X > \frac{|i-j|}{f_\gamma(|i-j|)} \wedge |s-s'| < \frac{2D}{k}).$$

So

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in A^*_n\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$
$$(|\gamma(i-j) - ([i\gamma]-[j\gamma])| \geq \frac{|i-j|}{f_\gamma(|i-j|)} \to X < \frac{f_\gamma(|i-j|)}{|i-j|} \wedge |s-s'| < \frac{2D}{k}).$$

Having bounded X means having bounded $\tilde{m}$, (where $X, \tilde{m}$ are as in the previous lemma) and recall that in the previous lemma our conclusion that gave the density of the set $\{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}\}$ in $[0,1]$, thus (replacing k by $2D/k$) also the density of the set $\{(e\theta + l\gamma - [e\theta + l\gamma])\beta\ :\ e \in \{+1,-1\},\ l \in \mathbb{Z}\}$ in $[0,\beta]$ for a fixed $\theta \in [0,1]$, was

$$\tilde{m}(i-j)\gamma - [\tilde{m}(i-j)\gamma] \in [\frac{m}{k}, \frac{m+1}{k}] \cap \{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}\}$$

for arbitrary $k \in \mathbb{N}$. Note that the proof of the previous lemma shows that in order to construct an $l \in \mathbb{Z}$ such that for a given $x \in [0,\beta]$ one has $|x - (\theta + l\gamma - [\theta + l\gamma])\beta| < \frac{2D}{k}$ it suffices to construct for a suitable $x' \in [0,1]$ an $l \in \mathbb{Z}$ such that $|x' - (l\gamma - [l\gamma])| < \frac{1}{k}$. Hence a bound on $|l|$ for the latter problem gives also a bound on $|l|$ for the former problem.

We have

$$|\tilde{m}(i-j)| = \tilde{m}|i-j| \leq X|i-j| < \frac{f_\gamma(|i-j|)}{|i-j|}|i-j| = f_\gamma(|i-j|)$$

and so

$$|\tilde{m}(i-j)| \leq f_\gamma(|i-j|) - 1.$$

Recall now that by Lemma 1(iii)

$$A^*_{2\tilde{m}|i-j|+8} \supseteq \{(e\theta + \tilde{m}(i-j)\gamma - [e\theta + \tilde{m}(i-j)\gamma])\beta\ :\ e \in \{+1,-1\}\}.$$

Thus we may set

$$\phi(k,f) := \max\{2f(i-j)+6\ :\ 0 \leq j < i \leq k+1\}.$$

□

Lemma 4. *(Lemma 6 in [18]) Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on a subset C of a Banach space E. Assume that there exist $q \in C$, $\lambda \in (0,1)$ such that*

$$\lambda T(\alpha)q + (1-\lambda)T(\beta)q = q$$

and let $\tau \in A(\theta)$, where $A(\theta)$ is as defined previously, so that

$$\|T(\tau)q - q\| = \max\{\|T(t)q - q\| \ : \ t \in A(\theta)\}.$$

Define a sequence $\{H_n\}$ of subsets of $[0,\beta]$ by $H_1 = \{\tau\}$ and

$$H_{n+1} = \bigcup_{t \in H_n} \{|\alpha - t|, |\beta - t|\}.$$

Then

$$\forall n \in \mathbb{N}\ \forall t \in H_n \ \ (\|T(\tau)q - q\| = \|T(t)q - q\|)\,.$$

We show the following:

Lemma 5. *(Quantitative version of Lemma 6 in [18]) Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on a subset C of a Banach space E. Let $\Lambda \in \mathbb{N}$ be such that $1/\Lambda \leq \lambda, 1-\lambda$. Let $\delta > 0$ and $q \in C$ be such that*

$$\|(\lambda T(\alpha) + (1-\lambda)T(\beta))q - q\| \leq \delta.$$

Let $\tau \in A(\theta)$, where $A(\theta)$ is defined as previously so that

$$\|T(\tau)q - q\| = \max\{\|T(t)q - q\| \ : \ t \in A(\theta)\}.$$

Define a sequence $\{H_n\}$ of subsets of $[0,\beta]$ as in the lemma above. Then

$$(**)\ \forall n \in \mathbb{N}\ \forall t \in H_n \ \left(\|T(\tau)q - q\| \leq \|T(t)q - q\| + \delta \sum_{i=1}^{n-1} \Lambda^i\right).$$

Proof. Note that by Lemma 3 in [18], our assumption that $\tau \in A(\theta)$ gives us that

$$A(\theta) = \bigcup_{n=1}^{\infty} H_n.$$

We now proceed by induction. Let $n = 1$. Then by definition $H_1 = \{\tau\}$. Notice that it is true that

$$\|T(\tau)q - q\| \leq \|T(\tau)q - q\| + \delta \sum_{i=1}^{0} \Lambda^i = \|T(\tau)q - q\|$$

and so we see that for $n = 1$, $(**)$ holds. Assume that $(**)$ holds for some fixed n. Let

$$|\alpha - t|, |\beta - t| \in H_{n+1}$$

with $t \in H_n$. We then have:

$$\|T(\tau)q - q\| \leq \|T(t)q - q\| + \delta \sum_{i=1}^{n-1} \Lambda^i$$

$$= \|T(t)q - q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q + \lambda T(\alpha)q + (1-\lambda)T(\beta)q\| + \delta \sum_{i=1}^{n-1} \Lambda^i$$

$$\leq \|T(t)q - (\lambda T(\alpha)q + (1-\lambda)T(\beta)q)\| + \|\lambda T(\alpha)q + (1-\lambda)T(\beta)q - q\| + \delta \sum_{i=1}^{n-1} \Lambda^i$$

$$\leq \|T(t)q - (\lambda T(\alpha)q + (1-\lambda)T(\beta)q)\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

$$= \|T(t)q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q + \lambda T(t)q - \lambda T(t)q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

$$\leq \lambda\|T(t)q - T(\alpha)q\| + (1-\lambda)\|T(t)q - T(\beta)q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

$$= \lambda\|T(t)q - T(\alpha)q\| + (1-\lambda)\|T(t)q - T(\beta - t + t)q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

$$= \lambda\|T(t)q - T(\alpha)q\| + (1-\lambda)\|T(t)q - T(t)T(\beta - t)q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

$$\leq \lambda\|T(t)q - T(\alpha)q\| + (1-\lambda)\|q - T(\beta - t)q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

and similarly, by replacing, in $\|T(t)q - T(\alpha)q\|$, t with $t-\alpha+\alpha$ in the case where $t > \alpha$ or α by $\alpha - t + t$ in the case where $t \leq \alpha$, (notice that we always have $t \leq \beta$) again by Properties (2) and (1) of Definition 2 the above gives

$$\leq \lambda\|T(|t-\alpha|)q - q\| + (1-\lambda)\|T(|t-\beta|)q - q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i.$$

Therefore (since by $\bigcup_{n=1}^{\infty} H_n = A(\theta)$ we have $\|T(|t-\alpha|)q - q\| \leq \|T(\tau)q - q\|$)

$$\|T(\tau)q - q\| \leq \lambda\|T(\tau)q - q\| + (1-\lambda)\|T(|t-\beta|)q - q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

i.e.

$$(1-\lambda)\|T(\tau)q - q\| \leq (1-\lambda)\|T(|t-\beta|)q - q\| + \delta + \delta\sum_{i=1}^{n-1}\Lambda^i$$

i.e.

$$\|T(\tau)q - q\| \leq \|T(|t-\beta|)q - q\| + \frac{1}{1-\lambda}(\delta + \delta\sum_{i=1}^{n-1}\Lambda^i)$$

$$\leq \|T(|t-\beta|)q - q\| + \Lambda(\delta + \delta\sum_{i=1}^{n-1}\Lambda^i)$$

$$= \|T(|t-\beta|)q - q\| + \delta\sum_{i=1}^{n}\Lambda^i,$$

and similarly:

$$\|T(\tau)q - q\| \leq \|T(|t-\alpha|)q - q\| + \delta\sum_{i=1}^{n}\Lambda^i.$$

We have thus shown that for all $s \in H_{n+1}$

$$\|T(\tau)q - q\| \leq \|T(s)q - q\| + \delta\sum_{i=1}^{n}\Lambda^i$$

and so that $(**)$ holds for $n+1$.
This concludes the inductive proof of $(**)$ for all n.

□

3. Proof of Theorem 2

We can now proceed to show Theorem 2 which is a quantitative version of Theorem 1 in [18]:

Proof. As explained in Section 1, we will obtain a quantitative version of

$$\bigcap_{t\geq 0} F(T(t)) \supseteq F(\lambda T(\alpha)+(1-\lambda)T(\beta))$$

by proof mining on the proof of Theorem 1 in [18]. We will follow the same pattern as in [18] but use our quantitative versions of the corresponding lemmas in [18] that we have obtained in the previous section.

Recall that in general by assumption we have

$$\forall b\in\mathbb{N}\ \forall q\in C_b\ \forall K\in\mathbb{N}\ \forall m\in\mathbb{N}\ \forall s,s'\in[0,K]$$
$$(|s-s'|<2^{-\omega_{K,b}(m)}\rightarrow \|T(s)q-T(s')q\|<2^{-m}).$$

Let $b\in\mathbb{N}$, $q\in C_b$, $\lambda\in(0,1)$ and assume that

$$\|(\lambda T(\alpha)+(1-\lambda)T(\beta))q-q\|\leq\delta.$$

The map $t\mapsto T(t)q$ is by assumption continuous, hence the map $h(t):=||T(t)q-q||$ is continuous. Because $[0,\beta]$ is compact, h attains its maximum on $[0,\beta]$ at a point $\tau\in[0,\beta]$, i.e.

$$\exists\tau\in[0,\beta]\,\forall t\in[0,\beta]\ (\|T(\tau)q-q\|\geq\|T(t)q-q\|).$$

Let $\gamma:=\alpha/\beta\in(0,1)$ and $\theta:=\tau/\beta\in[0,1]$, let $A(\theta)$ be as in the previous lemmas. Then, by definition, $\tau=\theta\beta\in\{\theta\beta\}=A_1\subseteq A(\theta)\subseteq[0,\beta]$. So

$$\|T(\tau)q-q\|=\max\{\|T(t)q-q\|\,:\,t\in A(\theta)\}.$$

We set

$$K:=D$$

where $\mathbb{N}\ni D\geq\beta$ i.e. here we have

$$\forall m\in\mathbb{N}\ \forall s,s'\in[0,D]$$
$$(|s-s'|<2^{-\omega_{D,b}(m)}\rightarrow\|T(s)q-T(s')q\|<2^{-m})\ (***).$$

From now on we recall the assumption that γ is irrational with an effective irrationality measure f_γ.
Now recall $(*)$ shown in Lemma 3:

$$\forall k\in\mathbb{N}\,\forall s\in[0,\beta]\,\exists s'\in A^*_{\phi(k,f_\gamma)}\ (|s-s'|<\frac{2D}{k})\quad(*)$$

and notice that the premise of $(***)$ is fulfilled for

$$\frac{2D}{k}\leq 2^{-\omega_{D,b}(m)}$$

i.e. for

$$k\geq D2^{\omega_{D,b}(m)+1}.$$

We therefore set $k := D2^{\omega_{D,b}(m)+1}$ in $(*)$ and we get

$$\forall m \in \mathbb{N}\, \forall s \in [0,\beta]\, \exists s' \in A^*_{\phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)} \subseteq [0,\beta]\ (|s-s'| < 2^{-\omega_{D,b}(m)}).$$

By $(***)$ the above gives

$$\forall m \in \mathbb{N}\ \forall s \in [0,\beta] \subseteq [0,D]\ \exists s' \in A^*_{\phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)}\ (\|T(s)q - T(s')q\| < 2^{-m}).$$

By the triangle inequality :

$$\|T(s')q - q\| \leq \|T(s)q - T(s')q\| + \|T(s)q - q\|,$$

therefore

$$\forall m \in \mathbb{N}\ \forall s \in [0,\beta]\ \exists s' \in A^*_{\phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)}\ (\|T(s')q - q\| < \|T(s)q - q\| + 2^{-m}).$$

Thus by $(**)$ shown in Lemma 5 (putting $\Lambda^*_{\gamma,D,b,m} := \sum_{i=1}^{\phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)-1} \Lambda^i$) and using that $H_n = A^*_n$ (since $\tau = \theta\beta$)

$$\forall m \in \mathbb{N}\ \forall s \in [0,\beta]\ \exists s' \in A^*_{\phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)}$$

$$(\|T(\tau)q - q\| \leq \|T(s')q - q\| + \delta\Lambda^*_{\gamma,D,b,m} < \|T(s)q - q\| + 2^{-m} + \delta\Lambda^*_{\gamma,D,b,m}).$$

Now, applying the above to both $s = \alpha, \beta \in [0,\beta]$, we have, for all $m \in \mathbb{N}$

$$\begin{aligned}
2\|T(\tau)q - q\| &< \|T(\alpha)q - q\| + \|T(\beta)q - q\| + 2\delta\Lambda^*_{\gamma,D,b,m} + 2\cdot 2^{-m}\\
&= \|T(\alpha)q - q + \lambda T(\alpha)q + (1-\lambda)T(\beta)q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q\|\\
&+\|T(\beta)q - q + \lambda T(\alpha)q + (1-\lambda)T(\beta)q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q\| + 2\delta\Lambda^*_{\gamma,D,b,m} + 2^{-m+1}\\
&\leq \|\lambda T(\alpha)q + (1-\lambda)T(\beta)q - q\| + (1-\lambda)\|T(\alpha)q - T(\beta)q\|\\
&+\|\lambda T(\alpha)q + (1-\lambda)T(\beta)q - q\| + \lambda\|T(\beta)q - T(\alpha)q\| + 2\delta\Lambda^*_{\gamma,D,b,m} + 2^{-m+1}\\
&\leq 2\delta + \|T(\alpha)q - T(\beta)q\| + 2\delta\Lambda^*_{\gamma,D,b,m} + 2^{-m+1}\\
&= \|T(\alpha)q - T(\alpha)T(\beta-\alpha)q\| + 2\delta(\Lambda^*_{\gamma,D,b,m} + 1) + 2^{-m+1}\\
&\leq \|q - T(\beta-\alpha)q\| + 2\delta(\Lambda^*_{\gamma,D,b,m} + 1) + 2^{-m+1}\\
&\leq \|T(\tau)q - q\| + 2\delta(\Lambda^*_{\gamma,D,b,m} + 1) + 2^{-m+1}
\end{aligned}$$

Therefore

$$\forall m \in \mathbb{N}\ (\|T(\tau)q - q\| < 2\delta(\Lambda^*_{\gamma,D,b,m} + 1) + 2^{-m+1})$$

and, because for all $t \in [0,\beta]$, by the definition of $\tau \in [0,\beta]$, we have

$$\|T(t)q - q\| \leq \|T(\tau)q - q\|$$

and so

$$\forall m \in \mathbb{N}\ \forall t \in [0,\beta](\|T(t)q - q\| < 2\delta(\Lambda^*_{\gamma,D,b,m} + 1) + 2^{-m+1}).$$

Now for all $\mathbb{R}^+ \ni t > \beta$ there exist $r \in \mathbb{N}$, $s \in [0,\beta]$ such that

$$t = r\beta + s.$$

Therefore

$$\begin{aligned}
\|T(t)q - q\| = \|T(r\beta + s)q - q\| &= \|T(r\beta)T(s)q - q\| = \|T^r(\beta)T(s)q - q\|\\
&= \|T^r(\beta)T(s)q - q + T^r(\beta)q - T^r(\beta)q\|\\
&\leq \|T^r(\beta)T(s)q - T^r(\beta)q\| + \|T^r(\beta)q - q\|\\
&\leq \|T(s)q - q\| + \|T^r(\beta)q - q\|\\
&= \|T(s)q - q\| + \|T^r(\beta)q - q + T(\beta)q - T(\beta)q\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|T(s)q-q\| + \|T(\beta)T^{r-1}(\beta)q - T(\beta)q\| + \|T(\beta)q-q\| \\
&\leq \|T(s)q-q\| + \|T^{r-1}(\beta)q-q\| + \|T(\beta)q-q\| \\
&= \|T(s)q-q\| + \|T^{r-1}(\beta)q - q + T(\beta)q - T(\beta)q\| + \|T(\beta)q-q\| \\
&\leq \|T(s)q-q\| + \|T(\beta)T^{r-2}(\beta)q - T(\beta)q\| + 2\|T(\beta)q-q\| \\
&\leq \|T(s)q-q\| + \|T^{r-2}(\beta)q-q\| + 2\|T(\beta)q-q\| \\
&\leq .... \leq \\
&\leq \|T(s)q-q\| + r\|T(\beta)q-q\| \\
&\leq \|T(\tau)q-q\|(1+r) \\
&< (2\delta(\Lambda^*_{\gamma,D,b,m}+1)+2^{-m+1})(1+r).
\end{aligned}$$

Let $M \in \mathbb{N}$ so that $t \leq M$ and $N \in \mathbb{N}$ so that $\beta \geq 1/N$. We may then estimate:

$$M \geq t = r\beta + s \geq r/N \ + s \geq r/N$$

thus we have

$$r \leq MN.$$

Therefore

$$\begin{aligned}
&\forall M \in \mathbb{N}\ \forall t \in [0,M]\ \forall m \in \mathbb{N} \\
&\|T(t)q-q\| < (2\delta(\Lambda^*_{\gamma,D,b,m}+1)+2^{-m+1})(1+r) \\
&\leq (2\delta(\Lambda^*_{\gamma,D,b,m}+1)+2^{-m+1})(1+MN)).
\end{aligned}$$

For a yet to be determined $m \in \mathbb{N}$, we set $\delta > 0$ to be so small so that

$$2\delta(\Lambda^*_{\gamma,D,b,m}+1) \leq 2^{-m+1}$$

i.e.

$$(2\delta(\Lambda^*_{\gamma,D,b,m}+1)+2^{-m+1})(1+MN) \leq 2\cdot 2^{-m+1}(1+MN) = 4\cdot 2^{-m}(1+MN).$$

Now let $\tilde{m} \in \mathbb{N}$ be given. We have to achieve

$$4\cdot 2^{-m}(1+MN) \leq 2^{-\tilde{m}},$$

i.e.

$$\log_2(4\cdot 2^{-m}(1+MN)) \leq \log_2(2^{-\tilde{m}})$$

and thus

$$2 - m + \log_2(1+MN) \leq -\tilde{m},$$

i.e.

$$m \geq 2 + \log_2(1+MN) + \tilde{m}.$$

Hence we may choose

$$m := 3 + [\log_2(1+MN)] + \tilde{m}.$$

To fulfill

$$\delta \leq \frac{2^{-m}}{\Lambda^*_{\gamma,D,b,m}+1}$$

we choose $\delta > 0$ such that

$$\delta \leq \frac{2^{-\tilde{m}}}{8(\Lambda^*_{\gamma,D,b,3+[\log_2(1+MN)]+\tilde{m}}+1)(1+MN)}.$$

Hence (renaming $\tilde{m}$ as m) we have shown:

$$\forall b \in \mathbb{N}\ \forall q \in C_b\ \forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall m \in \mathbb{N}$$
$$(\|(\lambda T(\alpha)+(1-\lambda)T(\beta))q-q\| \leq \frac{2^{-m}}{8(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+\lceil\log_2(1+MN)\rceil+m)+1},f_\gamma)-1}\Lambda^i+1)(1+MN)}$$
$$\to \|T(t)q-q\| < 2^{-m}).$$

□

Corollary to the proof: If the semigroup $\{T(t) : t \geq 0\}$ is just a strongly continuous semigroup of nonexpansive mappings without the equicontinuity condition, then the bound holds with $\omega_{D,b}$ being replaced by a modulus $\omega_{D,q}$ of uniform continuity for $t \in [0, D] \mapsto T(t)q$. Then, however, the bound no longer is independent of q.

Remark 2. *Note that the statement that a general continuous function $h : [a, b] \to \mathbb{R}$ attains its maximum at some $\tau \in [0, \beta]$ used in the proof above is noneffective as even for computable h such a point τ will in general not be computable (see Theorem I.10.3(6) in [17] where this principle is shown to be equivalent to the noncomputational so-called weak König's lemma WKL). The reason why this does not cause a problem in the quantitative analysis is that τ is only used via θ and that the bound obtained in Lemma 3 is independent of θ where the latter is obtained by a majorization argument applied to $\theta \in [0, 1]$ (see [9] for a general logical discussion of this point).*

4. Asymptotic Regularity

Definition 6 ([12])**.** *Let C be a convex subset of a Banach space E and let $S : C \to C$. The sequence*

$$x_{n+1} := \frac{1}{2}x_n + \frac{1}{2}Sx_n$$

is called the Krasnoselskii iteration of S starting at x_0.

If

$$\|x_n - Sx_n\| \overset{n\to\infty}{\to} 0$$

for all $x_0 \in C$, then S (or - more precisely - $S_{1/2} := \frac{1}{2}I + \frac{1}{2}S$) is called asymptotically regular ([5]).
A rate of convergence for the above convergence is called a rate of asymptotic regularity for S.

Considering the following classical result by Ishikawa [7], we will apply our main result Theorem 2 to obtain Corollary 1.

Theorem 3 ([7])**.** *Let $(E, \|\cdot\|)$ be a normed space , $C \subseteq E$ convex and $S : C \to C$ nonexpansive. If the Krasnoselskii iteration $\{x_n\}_{n\in\mathbb{N}}$ of S is bounded, then*

$$\|x_n - Sx_n\| \overset{n\to\infty}{\to} 0.$$

Corollary 1. *Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on a convex subset $C \subseteq E$ for some Banach space E. Let $\alpha, \beta \in \mathbb{R}^+$ with $0 < \alpha < \beta$ and let $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with an effective irrationality measure f_γ. Let $S : C \to C$ be defined as*

$$S := \lambda T(\alpha) + (1 - \lambda)T(\beta)$$

with $\lambda \in (0,1)$. *Let* $\Lambda \in \mathbb{N}$ *be such that* $1/\Lambda \leq \lambda, 1-\lambda$, $N \in \mathbb{N}$ *so that* $\beta \geq 1/N$, $\mathbb{N} \ni D \geq \beta$. *Moreover, assume that* $\{T(t) : t \geq 0\}$ *is uniformly equicontinuous with a modulus of uniform equicontinuity* ω. *Then for the Krasnoselskii iteration* $\{x_n\}_{n\in\mathbb{N}}$ *of* S, *starting at* x_0, *if* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded by* $b \in \mathbb{N}$, *we have*

$$\forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall m \in \mathbb{N}\ \forall n \geq \Phi\ (\|T(t)x_n - x_n\| < 2^{-m})$$

with a rate of asymptotic regularity

$$\Phi(m, M, N, \Lambda, D, b, f_\gamma, \omega, d) = \frac{2^{2m+8}d^2((\sum_{i=1}^{\phi(k,f_\gamma)-1}\Lambda^i + 1)(1+MN))^2}{\pi},$$

where $d > 0$ *is such that*

$$d \geq \|x_0 - Sx_n\|$$

for all $n \in \mathbb{N}$,

$$k := D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1} \in \mathbb{N}$$

and

$$\phi(k, f) := \max\{2f(i-j) + 6\ :\ 0 \leq j < i \leq k+1\} \in \mathbb{N}.$$

Proof. By a well-known deep result due to Baillon and Bruck [1], for the nonexpansive mapping $S : C \to C$ and for its Krasnoselskii iteration x_n we have

$$\forall \epsilon > 0\ \forall n \geq \theta(\epsilon, d)\ (\|x_n - Sx_n\| < \epsilon)$$

with a rate of asymptotic regularity (using that $\|x_n - Sx_n\| = 2\|x_{n+1} - x_n\|$)

$$\theta(\epsilon, d) := \frac{4d^2}{\pi\epsilon^2}$$

where $d > 0$ is such that

$$d \geq \|x_0 - Sx_n\|$$

for all $n \in \mathbb{N}$.
In Theorem 2 we showed that

$$\forall b \in \mathbb{N}\ \forall q \in C_b\ \forall M \in \mathbb{N}\ \forall m \in \mathbb{N}$$
$$(\|Sq - q\| \leq \Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) \to \forall t \in [0, M]\ (\|T(t)q - q\| < 2^{-m}))$$

with

$$\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) = \frac{2^{-m}}{8(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1}, f_\gamma)-1}\Lambda^i + 1)(1+MN)}.$$

Thus, by the above it directly follows that (having substituted ϵ with Ψ)

$$\forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall m \in \mathbb{N}\ \forall n \geq \Phi\ (\|T(t)x_n - x_n\| < 2^{-m})$$

with a rate of asymptotic regularity

$$\begin{aligned}\Phi(m, M, N, \Lambda, D, b, f_\gamma, \omega, d) &:= \theta(\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega), d)\\ &= \frac{4d^2}{\pi(\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega))^2}.\end{aligned}$$

□

We emphasize that the rate above is very uniform as it depends on the semigroup only via the modulus ω.

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