Analyzing proofs in analysis*

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1 Introduction

Many theorems in analysis are of the form (or can be transformed into the form):

(1) $\bigwedge x \in X\big(F(x) = 0 \to G(x) = 0\big)$, [1]

where X is a complete separable metric space (CSM–space for short) and $F, G : X \to \mathbb{R}$ are constructively definable (and therefore continuous) functions.
As an example of such a theorem we mention the uniqueness theorem for best Chebycheff approximation of $f \in C[0,1]$ by (algebraic) polynomials $p \in P_n$ (over $\mathbb{R}$) of degree $\le n$. (This example will be studied in detail in section 5 below):

$$\bigwedge f \in C[0,1], p_1, p_2 \in P_n\big(\bigwedge_{i=1}^{2}(\|p_i - f\|_\infty = \mathrm{dist}(f, P_n)) \to \|p_1 - p_2\|_\infty = 0\big);$$

here $\|\cdot\|_\infty$ denotes the sup–norm on $C[0,1]$ and $\mathrm{dist}(f, P_n) := \inf_{p \in P_n} \|p - f\|_\infty$. This theorem has the form (1): take $X := C[0,1] \times P_n \times P_n$, $F(f, p_1, p_2) := \max_{i=1,2}\big(\|p_i - f\|_\infty - \mathrm{dist}(f, P_n)\big)$ and $G(f, p_1, p_2) := \|p_1 - p_2\|_\infty$; then

(1) is equivalent to

(2) $\bigwedge x \in X, k \in \mathbb{N} \bigvee n \in \mathbb{N}\big(|F(x)| \le 2^{-n} \to |G(x)| < 2^{-k}\big)$.

Using a suitable representation of real numbers as Cauchy sequences of rational numbers with fixed rate of convergency (e.g. 2^{-n}) the predicate $\le$ becomes Π^0_1 while $<$ is Σ^0_1. Hence $A :\equiv \big(|F(x)| \le 2^{-n} \to |G(x)| < 2^{-k}\big)$ can be prenexed into a formula $\bigvee l \in \mathbb{N} A_0(x, n, k, l)$ with decidable A_0. [2]Furthermore for a suitable standard representation of X (such that the elements of X are represented by functions $f \in \mathbb{N}^{\mathbb{N}}$ and every function $\in \mathbb{N}^{\mathbb{N}}$ can be conceived as a representative of some element $\in X$)[3]the quantification over X reduces to quantification over $\mathbb{N}^{\mathbb{N}}$. Therefore (2) is essentially a sentence having the form $\bigwedge^1 \bigvee^0 A_0$.
If a sentence $A \equiv \bigwedge f^1, x^0 \bigvee y^0 A_0(f, x, y)$ is proved e.g. in a subsystem $\mathcal{A}$ of classical extensional arithmetic in all finite types E–PA$^\omega$ (from [31]), then one can use (after elimination of extensionality and negative translation) Gödel's method of functional interpretation to extract from the proof a computable functional Ψ, which realizes $\bigvee x^0$, i.e. $\bigwedge f^1, x^0 A_0(f, x, \Psi f x)$. Applied to (2) this yields a realization $\Psi x k$ of $\bigvee n^0$ in k and (a representative of) x.
Since (2) is monotone with respect to $\bigvee n$, i.e.

$$\bigwedge x \in X, k \in \mathbb{N}, n_1, n_2 \in \mathbb{N}\big(A(x, n_1, k) \wedge n_2 \ge n_1 \to A(x, n_2, k)\big),$$

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an upper bound Φ for Ψ suffices for a realization of $\bigvee n$. In many mathematically interesting situations $X := K$ is a compact space and one is interested in constructing a uniform bound Φ for $\bigvee n$ which does not depend on $x \in K$, i.e.

$$(3)\ \bigwedge x \in K, k \in \mathbb{N}\big(|F(x)| \leq 2^{-\Phi k} \rightarrow |G(x)| < 2^{-k}\big).$$

(For a very nice introduction to functional interpretation we refer to Troelstra's introductory notes [33] in [15]. Most parts of the present paper presuppose only information on functional interpretation which can be found in these notes).

In 2 we present a new **monotone** version of Gödel's functional interpretation which directly extracts uniform upper bounds Φ (as described) from given proofs of (2). This interpretation has similar virtues to and is easier to handle than the original functional interpretation: for example functionals defined by cases are not needed. Moreover analytical lemmas of the form

$$(4)\ \bigwedge x \in \tilde{X} \bigvee y \in \tilde{K}\big(\tilde{F}(x,y) = 0\big),$$

(where $\tilde{X}, \tilde{K}$ are CSM–spaces, $\tilde{K}$ is compact and $\tilde{F} : \tilde{X} \times \tilde{K} \rightarrow \mathbb{R}$ is a constructively definable function) have a very simple monotone functional interpretation: realizing terms for the monotone functional interpretation of (4) can be constructed simply from the terms used in the formulation of (4) without analyzing the proof of (4). Thus proofs of lemmas having form (4) do not contribute to the **construction** of the bound Φ in (3) but only to its **verification**. Since many theorems of classical analysis (which mostly have no **usual** functional interpretation by terms of E–PA$^\omega$ at all) – e.g. the attainment of the maximum by $f \in C[0,1]$ on $[0,1]$ – have form (4), our method applies to a large part of classical analysis. (For more specific examples see (5) in 5 below.)

Instead of (4) we may also have more generally arbitrary axioms having the form

$$\bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\tau A_0(x,y,z),$$

where δ, ρ, τ are arbitrary finite types and s is a closed term of $\mathcal{A}$ and $x_1 \leq_{0(\rho_k)\ldots(\rho_1)} x_2 :\equiv \bigwedge y_1^{\rho_1}, \ldots, y_k^{\rho_k}(x_1 y_1 \ldots y_k \leq_0 x_2 y_1 \ldots y_k)$ (where $\leq_0$ is defined primitive recursively as usual). Using the type level 2 we construct a simple sentence of this kind which trivially implies e.g. binary Knig's lemma WKL as well as important analytical theorems such as Dini's theorem. From this we obtain as a special case a new and very perspicuous proof for the conservativity of WKL over $\mathcal{A}$ with respect to sentences $\bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^\tau B_0(u,v,w)$ (γ, τ arbitrary). This was first proved in [21] in a more complicated way.

Interesting mathematical examples of sentences (1) with compact $X := K$ are uniqueness theorems

$$(5)\ \bigwedge x_1, x_2 \in K\big(F(x_1) = 0 = F(x_2) \rightarrow x_1 = x_2\big).$$

Here the uniform bound Φ in (3) provides a quantitative uniqueness result

$$(6)\ \bigwedge x_1, x_2 \in K, k \in \mathbb{N}\Big(\bigwedge_{i=1}^{2} \big(|F(x_i)| \leq 2^{-\Phi k}\big) \rightarrow d(x_1,x_2) < 2^{-k}\Big),$$

where d is the metric on K. We call Φ a **modulus of uniqueness**. Since such moduli do not depend on x_1, x_2 they provide interesting a–priori estimates which can be used in the computation of the

uniquely determined zero of F on K (if it exists classically) with prescribed precision (see theorem 5.1 and the discussion below 5.2 in 5). Moduli of uniqueness appear e.g. in the theory of Chebycheff approximation under the heading 'constants of strong unicity'. A (lower bound for a) constant of strong unicity is a modulus of uniqueness which is linear in $\varepsilon\widehat{=}2^{-k}$. So the concept of strong unicity is a special case of our notion of 'modulus of uniqueness' (similarly a Lipschitz constant is a special case of a modulus of uniform continuity.)
In 5 we give a survey of the numerical results which were obtained by our logical analysis of various proofs of the uniqueness of the best Chebycheff approximation of $f \in C[0,1]$. All these proofs use essentially non–constructive lemmas (4) relative to purely arithmetical reasoning (which can be carried out in subsystems of E-PA$^\omega$). Our numerical results provide new effective moduli of uniqueness and (a–priori estimates for) constants of strong unicity which improve known results significantly. We discuss how the different **numerical data** obtained from these proofs **correspond to the logical form** in which certain key–lemmas (e.g. the alternation theorem) are used in these proofs.

2 A monotone functional interpretation

The usual Gödel functional interpretation (as developed in e.g. [26] or [31]) can be simplified both with respect to the extraction algorithm and with respect to the functionals needed if only the extraction of (good) **bounds** for $\bigwedge x \bigvee y A_0$–sentences is wanted. Such bounds already give exact realizations in many applications in analysis because of monotonicity properties (discussed in the introduction). Furthermore this simplification immediately provides without any additional effort **uniform bounds** if x ranges over a compact domain.

We work within the language of functionals of all finite types. The set of finite types $\mathbf{T}$ is defined inductively as usual by
(i) $0 \in \mathbf{T}$, (ii) $\rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}$. Abbreviation $(1 := 0(0), 2 := 0(0(0)))$
$\tau(\rho)$ denotes the type of functionals which map objects of type ρ into objects of type τ.
Our basic theory is extensional arithmetic in all finite types E–PA$^\omega$ and its intuitionistic version E–HA$^\omega$ (for details see [31]). E–HA$^\omega$ contains only an equality relation $=_0$ for type 0 as primitive notion. Higher type equalities $s =_\rho t$ are abbreviations for $\bigwedge x_1^{\rho_1}, \ldots, x_k^{\rho_k}(sx_1 \ldots x_k =_0 tx_1 \ldots x_k)$ $(\rho = 0\rho_k \ldots \rho_1)$. If the axiom of extensionality is replaced by the weaker quantifier–free rule of extensionality[4]

$$\text{ER–qf } \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]}, \text{ where } A_0 \text{ is quantifier–free,}$$

then the resulting systems are denoted by WE–PA$^\omega$ and WE–HA$^\omega$. The set of all terms of E–PA$^\omega$ is denoted by T. The schema AC$^{\rho,\tau}$–qf of quantifier–free choice for the types ρ, τ is defined as

$$\text{AC}^{\rho,\tau}\text{–qf} \ : \ \bigwedge x^\rho \bigvee y^\tau A_0(x,y) \to \bigvee Y^{\tau(\rho)} \bigwedge x^\rho A_0(x, Yx) \ (A_0 \text{ quantifier–free}).$$

We now carry out our monotone functional interpretation for WE-HA$^\omega$. By doing first elimination of extensionality ([26]) and then negative translation ([13]) this interpretation also applies to classical systems e.g. E–PA$^\omega$+AC^{0,1}–qf.
We stress that our interpretation also works for various subsystems of WE–HA$^\omega$, e.g. the system $\widehat{\text{WE}-\text{HA}}^\omega\!\restriction$ from [11] with quantifier–free induction and elementary recursor constants only and

also to much weaker systems (w.r.t. the growth of provably functionals but not necessarily w.r.t. to proof–theoretic strength). It then yields feasible bounds instead of merely primitive recursive ones. In fact the interpretation below of the logic part of WE–HA$^\omega$ requires only the closure under substitution and λ–abstraction of $M := \{\max_\rho : \rho \in \mathbf{T}\} \cup \{0^0, 1^0\}$ plus majorants (in the sense of 2.3.4) of the terms occuring in quantifier axioms (where the functionals $\max_\rho$ are defined by $\max_{\tau(\rho)}(x_1^{\tau(\rho)}, x_2^{\tau(\rho)}) := \lambda y^\rho. \max_\tau(x_1 y, x_2 y))$ for the construction of the bounds extracted from given proofs. These properties of monotone functional interpretation are used in [25] to show that for significant parts $\mathfrak{A}\!\upharpoonright$ of classical analysis $\mathfrak{A}$ provable sentences $\bigwedge u^1 \bigwedge v \leq_\rho tu \bigvee w^0 F_0(u,v,w)$ have uniform bounds Φ on w, i.e. $\bigwedge u^1 \bigwedge v \leq_\rho tu \bigvee w \leq_0 \Phi u\, A_0$, which are (bounded by) polynomials relatively to u or – if the proof uses terms of exponential growth– polynomials in u and (majorants) of these terms.

Firstly we need the following

Definition 2.1 *Between functionals of type ρ we define the relations $\geq_\rho$ (greater–or–equal) and s–maj$_\rho$ (strong majorization) by induction on the type:*

1) $$\begin{cases} x_1 \geq_0 x_2 :\equiv x_1 \geq x_2, \text{ where } \geq \text{ is defined primitive recursively as usual,} \\ x_1 \geq_{\tau(\rho)} x_2 :\equiv \bigwedge y^\rho (x_1 y \geq_\tau x_2 y). \end{cases}$$

2) $$\begin{cases} x^* \text{ s–maj}_0\ x :\equiv x^* \geq_0 x, \\ x^* \text{ s–maj}_{\tau(\rho)}\ x :\equiv \bigwedge y^*, y\big(y^* \text{ s–maj}_\rho\ y \to x^* y^* \text{ s–maj}_\tau\ x^* y, xy\big). \end{cases}$$

Remark 2.2 The addition of the clause '$x^* y$' in definition 2.1.2 is a modification of Howard's [16] original relation maj_ρ which is due to Bezem [2]. Although we could use also Howard's notion we prefer Bezem's variant since it has the natural property that x^* s–maj $x \to x^*$ s–maj x^*, which e.g. implies the transitivity of s–maj. (Transitivity does not hold for Howard's maj.)

Lemma 2.3 *1) For $\rho = \tau(\rho_k)\ldots(\rho_1)$ one has*

$$x^* \text{ s–maj}_\rho\ x \leftrightarrow \bigwedge y_1^*, y_1, \ldots, y_k^*, y_k \Big(\bigwedge_{i=1}^{k} (y_i^* \text{ s–maj}_{\rho_i} y_i) \to x^* y_1^* \ldots y_k^* \text{ s–maj}_\tau\ x^* y_1 \ldots y_k, x y_1 \ldots y_k \Big).$$

2) $x^* \text{ s–maj}_\rho\ x \wedge x \geq_\rho y \to x^* \text{ s–maj}_\rho\ y$.

3) $x_1^* \text{ s–maj}_\rho\ x_1 \wedge x_2^* \text{ s–maj}_\rho\ x_2 \to \max_\rho(x_1^*, x_2^*) \text{ s–maj}_\rho\ \max_\rho(x_1, x_2)$.

4) For every term $t[x_1^{\rho_1}, \ldots, x_k^{\rho_k}]^\tau \in T$ containing only $x_1, \ldots, x_k$ free, one can construct a term $t^*[x_1, \ldots, x_k]^\tau \in T$ such that

$$\text{WE-HA}^\omega \vdash \bigwedge x_1^*, x_1, \ldots, x_k^*, x_k \Big(\bigwedge_{i=1}^{k} (x_i^* \text{ s–maj}_{\rho_i} x_i) \to t^*[x_1^*, \ldots, x_k^*] \text{ s–maj}_\tau\ t^*[x_1, \ldots, x_k], t[x_1, \ldots, x_k] \Big).$$

Proof: 1) is proved by induction on k using the fact that y^* s–maj$_\rho$ $y \to y^*$ s–maj$_\rho$ y^*.
2) and 3) are proved by induction on ρ.
4) which essentially is due to Howard [16] is proved e.g. in Bezem [2].

Remark 2.4 In [21] (and also in [23]) we used a pointwise variant maj_ρ of the relation s–maj_ρ with the clause $x^* \ \mathrm{maj}_{\rho 0}\ x :\equiv \bigwedge y^0 (x^* y \ \mathrm{maj}_\rho \ xy)$. This variant which was introduced in [20], [22] (and which is particular useful in the context of bar recursive functionals of finite and infinite types, see [20]) has the advantage of being more closely related to the "mathematical" relation $\geq_\rho$:
(i) $\bigwedge x^*, x(x^* \ \mathrm{maj}_1 \ x \leftrightarrow x^* \geq_1 x)$ in particular $\bigwedge x^1 (x \ \mathrm{maj}_1 x)$,
(ii) $\bigwedge x^*, x(x^* \ \mathrm{maj}_2 \ x \rightarrow x^* \geq_2 x)$.
This is very useful in applications. However s–maj has a better behaviour with respect to substitutions. Because of this we use s–maj for the monotone functional interpretation and modify the **result** pointwise to achieve the properties of maj if they are needed in the concrete mathematical application.

Definition 2.5 *For $x^{\rho 0}$ we define x^M by*

$$\begin{cases} x^M 0 =_\rho x0 \\ x^M (y+1) =_\rho \max_\rho (x^M y, x(y+1)). \end{cases}$$

Definition 2.6 *1) The 'independence–of–premise'–schema IP'_0 is defined as*

$$IP'_0 \ : \ (\bigwedge \underline{x}\, A_0(\underline{x}) \rightarrow \bigvee \underline{y}\, B(\underline{y})) \rightarrow \bigvee \underline{y} (\bigwedge \underline{x}\, A_0(\underline{x}) \rightarrow B(\underline{y})),$$

where $\underline{x}, \underline{y}$ are tuples of variables of arbitrary type, $\underline{y}$ do not occur in A_0 and A_0 is quantifier–free.

2) The Markov schema M'_0 is defined as

$$M'_0 \ : \ \neg\neg \bigvee \underline{x}\, A_0(\underline{x}) \rightarrow \bigvee \underline{x}\, A_0(\underline{x}),$$

where $\underline{x}$ is a tuple of variables of arbitrary types and A_0 is quantifier–free.

Gödel's functional interpretation transforms every formula $A(\underline{a}) \in \mathcal{L}(\text{WE–HA}^\omega)$ (having only $\underline{a}$ as free variables) into a formula $A^D :\equiv \bigvee \underline{x} \bigwedge \underline{y} A_D(\underline{x}, \underline{y}, \underline{a})$ where $\underline{x}, \underline{y}$ are tuples of functionals of finite type and A_D is quantifier–free; the translation has the property that if $\text{WE–HA}^\omega \vdash A(\underline{a})$ then one can extract from the proof a tuple of closed terms $\underline{t} \in T$ such that $\text{WE–HA}^\omega \vdash \bigwedge \underline{y}, \underline{a} A_D(\underline{t}\,\underline{a}, \underline{y}, \underline{a})$ (In fact $A_D(\underline{t}\,\underline{a}, \underline{y}, \underline{a})$ is provable even in the quantifier–free part of WE–HA^ω.)
Our **monotone version** of this **Gödel interpretation** extracts simplified algorithms with simple closed terms $\underline{t}^*$ such that

$$\text{WE–HA}^\omega \vdash \bigvee \underline{x} (\underline{t}^* \ \text{s–maj} \ \underline{x} \wedge \bigwedge \underline{y}, \underline{a} A_D(\underline{x}\,\underline{a}, \underline{y}, \underline{a})).$$

(Here $\underline{x}^*$ s–maj $\underline{x}$ stands for $\bigwedge_{i=1}^{n} (x_i^* \ \text{s–maj} \ x_i)$ where $\underline{x}^* = x_1^*, \ldots, x_n^*; \underline{x} = x_1, \ldots, x_n$.)

Theorem 2.7 (Monotone functional interpretation)

$$\begin{cases} \textit{From a proof } \text{WE–HA}^\omega + AC + M'_0 + IP'_0 \vdash A(\underline{a}) \textit{ one can extract closed terms} \\ \underline{t}^* \in cl(M \cup \{s_1^*, \ldots, s_n^*\} \cup \{r_1^*, \ldots, r_k^*\}) \textit{ such that} \\ \text{WE–HA}^\omega \vdash \bigvee \underline{x} (\underline{t}^* \ \textit{s–maj} \ \underline{x} \wedge \bigwedge \underline{y}, \underline{a} A_D(\underline{x}\,\underline{a}, \underline{y}, \underline{a})), \end{cases}$$

where

(i) $s_1^*, \ldots, s_n^*$ *are majorants in the sense of 2.3.4 of the terms* s_i *which occur in the quantifier axioms* $\bigwedge x\, F(x) \to F(s_i),\ F(s_i) \to \bigvee x\, F(x)$ *used in the given proof,*

(ii) $r_1^*, \ldots, r_k^*$ *are the closed terms needed for the monotone interpretation of the instances of the induction rule (see 7. below) used in the proof of* A*,*

(iii) cl denotes closure under substitution and λ*–abstraction.*

This extends to $WE\text{–}HA^\omega + \Gamma$*, where* Γ *is a set of formulas* $F \in \mathcal{L}(WE\text{–}HA^\omega)$ *such that for each* $F \in \Gamma$

$$(*)\ \ WE\text{–}HA^\omega + \Gamma \vdash \bigvee \underline{x}\big(\underline{q}^*\ s\text{–}maj\ \underline{x} \wedge \bigwedge \underline{y}, \underline{a} F_D(\underline{x}\,\underline{a}, \underline{y}, \underline{a})\big)$$

for suitable closed terms $\underline{q}^* \in T$*. Then the monotone functional interpretation extracts terms* $\underline{t}^* \in cl(M \cup \{s_1^*, \ldots, s_n^*\} \cup \{r_1^*, \ldots, r_k^*\} \cup \{q_1^*, \ldots, q_l^*\})$ *(where* $q_1^*, \ldots, q_l^*$ *are terms satisfying the monotone functional interpretation of those* $F \in \Gamma$ *which are used in the proof) such that*

$$WE\text{–}HA^\omega + \Gamma \vdash \bigvee \underline{x}\big(\underline{t}^*\ s\text{–}maj\ \underline{x} \wedge \bigwedge \underline{y}, \underline{a} A_D(\underline{x}\,\underline{a}, \underline{y}, \underline{a})\big).$$

Remark 2.8 *1) For* $\Gamma = \emptyset$ *such a tuple* $\underline{t}^*$ *can be obtained also by first extracting* $\underline{t}$ *via the usual functional interpretation and then applying lemma 2.3.4 to* $\underline{t}$*. If* Γ *is non–trivial and* $+$ *weakened to* $\oplus$ *(see thm.4.5 below) one has to apply first the deduction theorem to reduce the given proof to a proof of* $\bigwedge \Gamma \to A$ *in* $WE\text{–}HA^\omega$*. This is the method in [21]. However here we construct* $\underline{t}^*$ *directly simplifying both the algorithm and its output.*

2) An advantage – which is essential in mathematical applications – of functional interpretation over cut elimination, ε*–elimination or the no–counterexample interpretation is its* **modularity***: The interpretation of a complex proof can be obtained easily from interpretations of the subproofs of the lemmas which occur within it by substitutions and* λ*–abstraction. This is possible because of the good behaviour of functional interpretation with respect to modus ponens. Our monotone version has the same good behaviour.*

Proof of theorem 2.7 :

Description of the algorithm for extracting uniform bounds by monotone functional interpretation

We use (as in [26] and [31]) the formalization of WE–HA$^\omega$ in Gödel's calculus of intuitionistic logic ([14]).

1) The most complicated axioms for the usual functional interpretation are $A \vee A \to A$ and $A \to A \wedge A$. The later one is even more complicated in requiring the existence of functionals which decide prime formulas.

a) $[A \vee A \to A]^D \equiv \bigvee Y, Y', X'' \bigwedge z^0, x, x', y''$
$$\big\{\big(z =_0 0 \to A_D(x, Yzxx'y'', \underline{a})\big) \wedge \big(z \neq 0 \to A_D(x', Y'zxx'y'', \underline{a})\big) \to A_D(X''zxx', y'', \underline{a})\big\}\ ^5.$$

Define $\begin{cases} t^*_{X''} := \lambda \underline{a}, z, x, x'.\max(x, x'), \\ t^*_Y := t^*_{Y'} := \lambda \underline{a}, z, x, x', y''.y''. \end{cases}$

These terms fulfil our claim: By lemma 2.3.1,2.3.2 $t^*_{X''}, t^*_Y$ and $t^*_{Y'}$ majorize the functionals

$$t_{X''}\underline{a}z^0xx' := \begin{cases} x, \text{ if } z=0 \\ x', \text{ if } z\neq 0, \end{cases} \quad t_Y := t_Y^* \text{ and } t_{Y'} := t_{Y'}^*$$

which realize "$\bigvee Y, Y', X''$".

b) $[A \to A \wedge A]^D \equiv \bigvee Y, X', X'' \bigwedge x, y', y''$

$$\big(A_D(x, Yxy'y'', \underline{a}) \to A_D(X'x, y', \underline{a}) \wedge A_D(X''x, y'', \underline{a})\big).$$

Define $t_Y^* := \lambda\underline{a}, x, y', y''.\max(y', y'')$, $t_{X'}^* := t_{X''}^* := \lambda\underline{a}, x.x$. "$\bigvee Y, X', X''$" is realized by

$t_Y\underline{a}xy'y'' := \begin{cases} y', \text{ if } \neg A_D(x, y', \underline{a}) \\ y'', \text{ if } A_D(x, y', \underline{a}) \end{cases}$ and $t_{X'} := t_{X''} := \lambda\underline{a}, x.x$. Since t_Y^* s–maj t_Y, $t_{X'}^*$ s–maj $t_{X'}$ and $t_{X''}^*$ s–maj $t_{X''}$, the terms $t_Y^*, t_{X'}^*$ and $t_{X''}^*$ fulfil our claim.

2) The interpretation of $A \vee B \to B \vee A$ is also simplified if only a majorizing functional has to be constructed:

$(A \vee B \to B \vee A)^D \equiv \bigvee Z', X', U', Y, V \bigwedge z^0, x, u, y', v'$

$$\{\big(z = 0 \to A_D(x, Yzxuy'v', \underline{a})\big) \wedge \big(z \neq 0 \to B_D(u, Vzxuy'v', \underline{a})\big)$$
$$\to \big(Z'zxu = 0 \to B_D(U'zxu, v', \underline{a})\big) \wedge \big(Z'zxu \neq 0 \to A_D(X'zxu, y', \underline{a})\big)\}.$$

$t_{U'}^* := \lambda\underline{a}, z, x, u.u$, $t_{X'}^* := \lambda\underline{a}, z, x, u.x$, $t_Y^* := \lambda\underline{a}, z, x, u, y', v'.y'$, $t_V^* := \lambda\underline{a}, z, x, u, y', v'.v'$ are defined as in the usual functional interpretation, but $t_{Z'}^*$ is now simply $t_{Z'}^* := \lambda\underline{a}, z, x, u.1^0$ whereas the usual interpretation requires for the realization of "$\bigvee Z'$" the functional $t_{Z'} := \lambda\underline{a}, z, x, u.\overline{sg}(z^0)$, where $\overline{sg}(z^0) := \begin{cases} 0, \text{ if } z \neq 0 \\ 1, \text{ if } z = 0. \end{cases}$

It is clear that $t_{Z'}^*$ majorizes $t_{Z'}$.

3)

$$\big(\bigwedge zA(z) \to A(t)\big)^D \equiv \bigvee Z, Y, X' \bigwedge X, y' \big(A_D(X(ZXy'), YXy', ZXy') \to A_D(X'X, y', t)\big).$$

Define $t_Z^* := \lambda\underline{a}, X, y'.t^*$, $t_Y^* := \lambda\underline{a}, X, y'.y'$, $t_{X'}^* := \lambda\underline{a}, X.X(t^*)$, where $t^*[a_{i_1}, \ldots, a_{i_l}]$ is such that

$\bigwedge a_{i_1}^*, a_{i_1}, \ldots, a_{i_l}^*, a_{i_l}\Big(\bigwedge\limits_{j=1}^{l}(a_{i_j}^* \text{ s–maj } a_{i_j}) \to t^*[a_{i_1}^*, \ldots, a_{i_l}^*] \text{ s–maj } t^*[a_{i_1}, \ldots, a_{i_l}], t[a_{i_1}, \ldots, a_{i_l}]\Big).$

Here $\{a_{i_1}, \ldots, a_{i_l}\} \subset \underline{a}$ is the set of all free variables which occur in t and $\underline{a}$ are all free variables in $A(t)$. t^* can be constructed by 2.3.4. In practise this construction is usually very easy! "$\bigvee Z, Y, X'$" is realized by $t_Z := \lambda\underline{a}, X, y'.t$, $t_Y := \lambda\underline{a}, X, y'.y'$ and $t_{X'} := \lambda\underline{a}, X.X(t)$. Using 2.3.1 it follows that

t_Z^* s–maj t_Z, t_Y^* s–maj t_Y and $t_{X'}^*$ s–maj $t_{X'}$.

The treatment of $\big(A(t) \to \bigvee zA(z)\big)^D$ is similar.

4) The other axioms of the Gödel calculus have almost trivial monotone interpretations: the usual functional interpretations in λ–terms majorize themselves.

5) **Modus ponens:** Let t_1^*, t_2^*, t_3^* be such that

(1) $\bigvee x_1\big(t_1^* \text{ s–maj } x_1 \wedge \bigwedge y, \underline{a}A_D(x_1\underline{a}, y, \underline{a})\big)$ and

(2) $\bigvee x_2, x_3 \Big(t_2^*$ s–maj $x_2 \wedge t_3^*$ s–maj $x_3 \wedge \bigwedge x, v, \underline{a}\big(A_D(x, x_2\underline{a}xv, \underline{a}) \rightarrow B_D(x_3\underline{a}x, v, \underline{a})\big)\Big)$.

Then $t_4^* := \lambda\underline{a}.t_3^*\underline{a}(t_1^*\underline{a})$ s–maj $\lambda\underline{a}.x_3\underline{a}(x_1\underline{a})$ and $\lambda\underline{a}.x_3\underline{a}(x_1\underline{a})$ realizes B^D.

The rule $\frac{A\rightarrow B\ ,\ B\rightarrow C}{A\rightarrow C}$ is treated similarly.

6) The monotone interpretation of the remaining logical rules uses the λ–terms of the usual functional interpretation since they preserve majorizability.

7) **Induction rule:**

$$\frac{B0\ ,\ \bigwedge y^0\big(By \rightarrow B(y+1)\big)}{\bigwedge x Bx}.$$

Let $(By)^D \equiv \bigvee u \bigwedge v B_D(u, v, y, \underline{a})$ and t_0^*, t_1^*, t_2^* be such that

$\bigvee x_0\big(t_0^*$ s–maj $x_0 \wedge \bigwedge v, \underline{a}B_D(x_0\underline{a}, v, 0, \underline{a})\big)$,

$\bigvee x_1, x_2\big(t_1^*$ s–maj $x_1 \wedge t_2^*$ s–maj $x_2 \wedge \bigwedge u, w, y, \underline{a}\big(B_D(u, x_1y\underline{a}uw, y, \underline{a}) \rightarrow B_D(x_2y\underline{a}u, w, y{+}1, \underline{a})\big)\big)$.

Define $t^* := t^M$, where t is defined by recursion

$$\begin{cases} t\underline{a}0 = t_0^*\underline{a} \\ t\underline{a}(y+1) = t_2^*y\underline{a}(t\underline{a}y). \end{cases}$$

One easily verifies (in WE–HA$^\omega$) that

t^* s–maj x, where x is defined by $\begin{cases} x\underline{a}0 = x_0\underline{a} \\ x\underline{a}(y+1) = x_2y\underline{a}(x\underline{a}y) \end{cases}$ and that $\bigwedge y^0, vB_D(x\underline{a}y, v, y, \underline{a})$

(see [31]).

In practice usual mathematical simplifications will be applied to these bounds. If only instances of the schema of quantifier–free induction IA–qf are used in the proof then such terms t^* are not needed.

8) AC, M_0', IP_0' have the same trivial monotone functional interpretation as in the case of the usual interpretation.

3 Uniform bounds by monotone functional interpretation

As we have discussed in the introduction there are important theorems in analysis (e.g. uniqueness theorems) which can be transformed into the logical form

(1) $\bigwedge x \in X \bigwedge y \in K \bigvee n \in \mathbb{N}\, A(x, y, n)$

(where X is a complete separable metric space and K is a compact metric space), and can be transformed further (using a convenient standard representation of X, K) into the form

(2) $\bigwedge x^1 \bigwedge y \leq_1 s \bigvee n^0 \tilde{A}(x, y, n)$,

where $\tilde{A} \in \Sigma_1^0$.

By this transformation the construction of a uniform bound on $\bigvee n$ which is independent of $y \in K$ reduces to the construction of a functional $\Phi^{0(1)}$ such that

(3) $\bigwedge x^1 \bigwedge y \leq_1 s \bigvee n \leq_0 \Phi x\, \tilde{A}(x, y, n)$.

(see [23]) for details on this reduction).
We now show that one can extract such bounds (even in the more general situation where the type of y may be arbitrary, s depends on x and the type of n is ≤ 2) from a given proof by monotone functional interpretation (if the proof is carried out in a system to which the monotone functional interpretation applies).

Let us consider the following situation: $t^{\gamma 1} \in T$ is a closed term and

$$\text{WE–PA}^\omega + \text{AC–qf} \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^0 A_0(u,v,w)$$

(where only u, v, w are free in A_0 and A_0 is quantifier–free). Then by negative translation and monotone functional interpretation one obtains a closed term Φ^* (as in 2) such that

$$\text{WE–HA}^\omega \vdash \bigvee \Phi^{0(\gamma)(1)} \big(\Phi^* \text{ s–maj } \Phi \wedge \bigwedge u^1 \bigwedge v \leq_\gamma tu\ A_0(u,v,\Phi uv)\big)$$

(Note that for $\gamma = 0(\gamma_k)\cdots(\gamma_1):\ \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^0 A_0 \leftrightarrow \bigwedge u^1 \bigwedge v^\gamma \bigvee w^0, z_1^{\gamma_1}, \ldots, z_k^{\gamma_k}(v\underline{z} \leq_0 tu\underline{z} \to A_0)$ i.e. $\bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^0 A_0$ is really a $\bigwedge\bigvee$–sentence).
Define

$$\widehat{\Phi} := \lambda u^1.\Phi^* u^M (t^* u^M),$$

where $t^* \in T$ is such that t^* s–maj t. Then by 2.3.1,2.3.2 and $\bigwedge u^1(u^M \text{ s–maj}_1\ u)$ it follows that

$$\text{WE–HA}^\omega \vdash \bigwedge \Phi(\Phi^* \text{ s–maj } \Phi \to \bigwedge u^1 \bigwedge v \leq_\gamma tu\big(\widehat{\Phi} u \geq_0 \Phi uv)\big).$$

Hence

$$\text{WE–HA}^\omega \vdash \bigwedge u \bigwedge v \leq tu \bigvee w \leq_0 \widehat{\Phi} u A_0(u,v,w).$$

Thus a uniform bound $\widehat{\Phi}$ (which is independent of $v \leq_\gamma tu$) for w has been obtained.
This extraction is also possible if w is of type $\tau \leq 2$ (instead of $\tau = 0$ only). For $\tau = 2$, $\widehat{\Phi}$ must be defined by

$$\widehat{\Phi} := \lambda u^1, y^1.\Phi^* u^M (t^* u^M)(y^M).$$

Remark 3.1 The passage from Φ to $\widehat{\Phi}$ was needed to achieve the property of pointwise majorants that the pointwise majorization u maj_1 u is preserved (see the remark 2.4 above).

4 Lemmas whose term–structure only contributes to the bound but not their proofs

Our monotone functional interpretation makes it possible to treat lemmas having the form $F :\equiv \bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\tau F_0(x,y,z)$ (where F_0 is quantifier–free and contains only x, y, z free, $\delta, \rho, \tau \in \mathbf{T}$ are arbitrary and $s \in T$ is closed) in given proofs simply as axioms:
If

$$\text{WE–PA}^\omega + \text{AC–qf} + F \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^0 A_0(u,v,w),$$

then a fortiori

$$\text{WE–PA}^\omega + \text{AC–qf} + F^D \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^0 A_0(u,v,w),$$

where $F^D :\equiv \bigvee Y \leq_{\rho(\delta)} s \bigwedge x, z F_0(x, Yx, z)$. Since the negative translation of F^D follows (intuitionistically) from F^D, this yields

$$(I)\ \text{WE–HA}^\omega + M_0' + \text{AC–qf} + F^D \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^0 A_0(u,v,w).$$

F^D is the usual functional interpretation of (F and) itself. Let $s^* \in T$ be such that

$$\text{WE–HA}^\omega \vdash s^* \text{ s–maj } s.$$

Then (by lemma 2.3.2)

$$\text{WE–HA}^\omega \vdash \bigwedge Y (Y \leq s \to s^* \text{ s–maj } Y).$$

Thus s^* fulfils the **monotone** functional interpretation of F^D and therefore sentences F^D satisfy the properties of Γ in thm. 2.7. Hence by the results of 3 the monotone functional interpretation applied to (I) extracts a bound $\Phi \in T$ from the proof such that

$$(+)\ \text{WE–HA}^\omega + \bigvee Y \leq_{\rho\delta} s \bigwedge x, z F_0(x, Yx, z) \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w \leq_0 \Phi u A_0(u,v,w)$$

and therefore

$$(*)\ \text{WE–HA}^\omega + F + \text{b–AC}^{\delta,\rho} \vdash \bigwedge u \bigwedge v \leq tu \bigvee w \leq \Phi u\ A_0(u,v,w),\ \text{where}$$

b–AC$^{\delta,\rho}$ is the axiom schema

$$\bigwedge x^\delta \bigvee y \leq_\rho Zx A(x,y,Z) \to \bigvee Y \leq_{\rho\delta} Z \bigwedge x A(x, Yx, Z)$$

(See [22] for a discussion of b–AC).

The construction of the bound Φ depends on F only by s^* and possibly the terms occurring in F_0 but not on a proof of F. The **verification** of Φ requires only the truth of F. If F holds in the full type–structure $\mathcal{S}^\omega$ of all set–theoretic (in the sense of ZFC) functionals or in the type–structure $\mathcal{M}^\omega$ of all strongly majorizable functionals (as defined in [2]), then $\bigwedge u \bigwedge v \leq_\gamma tu \bigvee w \leq \Phi u\ A_0$ is valid in $\mathcal{S}^\omega$ respectively in $\mathcal{M}^\omega$ since both type–structures satisfy b–AC. (For $\mathcal{S}^\omega$ this holds by definition and for $\mathcal{M}^\omega$ a proof is given in [22] (3.12.1).)
Concerning a constructive verification of $(*)$: In the special case where $\delta, \rho, \tau \leq 1$ in F it is possible to eliminate b–AC from the verification of Φ by additional work. Furthermore
$F \equiv \bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^0 F_0(x,y,z)$ can be weakened to its "ε–version" $\bigwedge x^1, n^0 \bigvee y \leq_1 sx \bigwedge_{i=0}^{n} F_0(x,y,i)$ which is usually provable in WE–HA$^\omega$ (while the passage from this ε–weakening to F itself requires in general binary Knig's lemma WKL, see below). Moreover even in the general case it is possible to restrict b–AC in $(*)$ to quantifier–free formulas. All this and more general results in this direction can be found in [21] .
On the other hand it is extremely rewarding to use such sentences F even in higher types (and thus avoid coding of finite sequences etc.) for a classical justification. This is demonstrated by the following

Application

We consider the sentence $\tilde{F} :\equiv \bigwedge x^2, y^1 \bigvee y_0 \leq_1 y \bigwedge z \leq_1 y (xz \leq_0 xy_0)$ which holds e.g. in the type–structure $\mathcal{M}^\omega$ and also in the extensional continuous functionals ECF:=ECF(ω^ω) (see [31]) but not in $\mathcal{S}^\omega$. Using extensionality one can show in E–PA$^\omega$ that

$$\tilde{F} \leftrightarrow F :\equiv \bigwedge x^2, y^1 \bigvee y_0 \leq_1 y \bigwedge z^1 \big(x(\min_1(z,y)) \leq_0 xy_0\big)$$

(Only the implication $F \to \tilde{F}$ needs extensionality).

F has the logical form of such axioms which have a simple monotone functional interpretation, namely $s^* := s := \lambda x^2, y^1.y$.

If $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$ and

$$\text{E–PA}^\omega + \text{AC}^{\alpha,\beta}\text{–qf} + \tilde{F} \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w^0 \; A_0(u,v,w), \text{ then}$$

$$\text{E–PA}^\omega + \text{AC}^{\alpha,\beta}\text{–qf} + F \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w^0 \; A_0(u,v,w).$$

By the elimination of extensionality procedure from [26] this implies

$$\text{WE–PA}^\omega + \text{AC}^{\alpha,\beta}\text{–qf} + (F)_e \vdash \big(\bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w^0 \; A_0(u,v,w)\big)_e,$$

where $()_e$ is the result of restricting all quantifiers to hereditarily extensional functionals. Since functionals of type 1 are provably extensional in WE–PA$^\omega$ we have
$\big(\bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w^0 \; A_0(u,v,w)\big)_e \leftrightarrow \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w^0 \; A_0(u,v,w)$ and $F \to (F)_e$. Hence

$$\text{WE–PA}^\omega + \text{AC}^{\alpha,\beta}\text{–qf} + F \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w^0 \; A_0(u,v,w).$$

By the reasoning before we can extract (using monotone functional interpretation) a closed term $\Phi^{0(1)} \in T$ such that

$$\mathcal{M}^\omega \models \bigwedge u^1, v \leq_1 tu \bigvee w \leq_0 \Phi u \; A_0(u,v,w)$$

and therefore (since the set of type–1–objects in $\mathcal{M}^\omega$ is $\mathbb{N}^\mathbb{N}$)

$$(**) \bigwedge u, v \in \mathbb{N}^\mathbb{N} \big(\bigwedge k \in \mathbb{N}(vk \leq tuk) \to \bigvee w \leq \Phi u \; A_0(u,v,w)\big).$$

This allows a particular simple analysis of proofs involving Knig's lemma for 0,1–trees, so–called weak Knig's lemma WKL:

$$\text{WKL} \;: \bigwedge f^1 \big(T(f) \wedge \bigwedge x^0 \bigvee n^0 (lth\; n = x \wedge fn = 0) \to \bigvee b \leq_1 \lambda k.1 \bigwedge x^0 (f(\overline{b}x) = 0)\big),$$
$$\text{where } T(f) :\equiv \bigwedge n^0, m^0 (f(n * m) = 0 \to fn = 0) \wedge \bigwedge n, x (f(n * \langle x \rangle) = 0 \to x \leq 1)$$

(Here lth, $*$, $\langle\rangle$, $\Phi(b,x) := \overline{b}x$ denote the primitive recursive function(al)s used in [31] for the primitive recursively coding of finite sequences).

Lemma 4.1 *WE–PA$^\omega$+AC^{1,0}–qf+F $\vdash$ WKL. But WE–PA$^\omega$+AC^{1,0}–qf+WKL$\nvdash$ F.*

Proof: Assume $\bigwedge b \leq_1 \lambda k.1 \bigvee x^0 (f(\overline{b}x) \neq 0)$. Then $\bigwedge b \bigvee x^0, z^0 \big(bz \leq_0 1 \to f(\overline{b}x) \neq 0\big)$. By AC^{1,0}–qf there exists a functional $\chi^{0(1)}$ such that $\bigwedge b \leq_1 \lambda k.1 \big(f(\overline{b}(\chi b) \neq 0\big)$. F implies that χ is bounded on the set of all functions having the form $\min_1(b, \lambda k.1)$. Let x_0 denote such a bound. Then $\neg \bigvee n(lth\; n = x_0 \wedge fn = 0)$. The second part of the lemma follows from the fact that
$\mathcal{S}^\omega \models$ WE–PA$^\omega$+AC^{1,0}–qf+WKL but $\mathcal{S}^\omega \not\models F$.

Remark 4.2 The proof of WKL in WE–PA$^\omega$+AC^{1,0}–qf+F uses only the consequence of F that every x^2 is bounded on $\{y^1 : y \leq_1 \lambda k.1\}$ (more precisely on $\{\min_1(y, \lambda k.1)\}$). This boundedness property $\bigwedge x^2 \bigvee n \bigwedge y \leq_1 \lambda k.1(xy \leq_0 n)$ has, in contrast to F, not the form $\bigwedge x \bigvee y \leq sx \bigwedge z\ F_0$ (since $\bigvee n$ is not bounded) and can therefore not treated directly as an axiom by our monotone functional interpretation.

The reasoning above yielding the truth of $(**)$ can be formalized in say ZFC but not in e.g. E–PA$^\omega$. We now show that after the extraction of Φ as above one can provide a verification $(**)$ of this bound Φ even in WE–HA$^\omega$. This **constructive** proof requires much more effort than the extraction of Φ but need not be carried out in mathematical applications where only the bound Φ itself (and the classical truth of $(**)$) is of interest:
Define

$$\widehat{F} :\equiv \bigvee Y_0 \leq \lambda x^2, y^1.y \bigwedge x^2, y^1, z^1 \big(x(\min(z,y)) \leq_0 x(Y_0 xy)\big).$$

The extraction of Φ yields (see $(+)$ above):

$$\text{WE–HA}^\omega + \widehat{F} \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w \leq_0 \Phi u\ A_0(u,v,w).$$

Define (as in [31])

$$MUC :\equiv \bigvee \Omega^3 \bigwedge x^2 \bigwedge y_1, y_2 \leq_1 \lambda n^0.1^0 \big(\overline{y}_1(\Omega x) =_0 \overline{y}_2(\Omega x) \to xy_1 =_0 xy_2\big).$$

One easily verifies (applying MUC to x^2 and using the fact that the fan $\{z : z \leq_1 y\}$ can be transformed into a sub–fan of $\{z : z \leq_1 \lambda n^0.1^0\}$, (see [31], 1.9.24) that E–HA$^\omega + MUC \vdash \widehat{F}$. Hence

$$\text{E–HA}^\omega + MUC \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w \leq_0 \Phi u\ A_0(u,v,w).$$

Since E–HA$^\omega \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w \leq_0 \Phi u A_0(u,v,w) \leftrightarrow \bigwedge u^1, v^1(\chi uv =_0 0)$ where $\chi \in T$ is such that $\chi uv =_0 0 \leftrightarrow \bigvee w \leq_0 \Phi u A_0(u, \min(v, tu), w)$, it follows by [32] (Thm.4(a)) that

$$\text{EL+FAN+AC}^{0,1} \vdash \bigwedge u^1, v^1[\chi uv =_0 0]_{\text{ECF}},$$

where $\bigwedge u^1, v^1[\chi uv =_0 0]_{\text{ECF}} \in \mathcal{L}(\text{EL})$ is prenex.
Thus by [32] (Thm.2)

$$\text{EL+AC}^{0,1} \vdash \bigwedge u^1, v^1[\chi uv = 0]_{\text{ECF}}.$$

Provably (in WE–HA$^\omega$+AC^{0,0}) the interpretation in ECF agrees with the unrestricted interpretation (see [31], 2.6.12) and so

$$\text{WE–HA}^\omega + \text{AC}^{0,1} \vdash \bigwedge u^1, v^1(\chi uv = 0)$$

which yields (by functional interpretation)

$$\text{WE–HA}^\omega \vdash \bigwedge u^1, v^1(\chi uv = 0), \text{i.e. WE–HA}^\omega \vdash \bigwedge u^1 \bigwedge v \leq_1 tu \bigvee w \leq_0 \Phi u A_0.$$

Remark 4.3 The **extraction** of Φ also works for many subsystems of E–PA$^\omega$ even very weak ones since no coding of finite sequences of natural numbers is needed but only $\max_\rho$ (besides the functionals needed for the monotone functional interpretation of the non–logical axioms and rules of the specific subsystem). In contrast to this the constructive verification requires the full primitve recursive coding machinary e.g. in order to formalize ECF.

Let us consider now the following weakening F^- of F:

$$F^- :\equiv \bigwedge x^2, y^1 \bigvee y_0 \leq_1 y \bigwedge k^0 \big(x(\min(y, \lambda n.(k)_n)) \leq_0 xy_0\big).$$

The proof of 4.1 yields

Lemma 4.4 *WE–PA$^\omega$ + F^-+$AC^{1,0}$–qf $\vdash WKL$.*

We now consider the situation where only F^- instead of F is used. We show that, using only monotone functional interpretation, we can extract a bound Φ together with an easy verification in WE–HA$^\omega$ even when γ is an arbitrary finite type and $\tau \leq 2$ (instead of $\gamma = 1, \tau = 0$ as above).

Theorem 4.5 *Let τ be ≤ 2. If WE–PA$^\omega \oplus AC$–qf$\oplus F^- \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^\tau A_0(u,v,w)$. Then one can extract by monotone functional interpretation a closed term $\Phi \in T$*
such that WE–HA$^\omega \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w \leq_\tau \Phi u \; A_0(u,v,w)$
(Here "$\oplus$" means that F^- and AC–qf must not be used in the proof of the premise of an application of ER–qf. WE–PA$^\omega$ fulfils the deduction theorem only for $\oplus$ but not for $+$).

Proof: By the assumption we have

$$\text{WE–PA} \oplus \text{AC–qf} \vdash F^- \to \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^\tau A_0(u,v,w).$$

By [21] (2.13.2 and 2.14.2) one can extract $\Phi, \Psi \in T$ such that

$$\text{WE–HA}^\omega \vdash \bigwedge u^1 \Big(\bigvee Y_0 \leq \lambda x^2, y^1.y \bigwedge x^2, y^1 \bigwedge k \leq_0 \Psi u \big(x(\min(y, \lambda n.(k)_n)) \leq x(Y_0 xy)\big) \\ \to \bigwedge v \leq_\gamma tu \bigvee w \leq_\tau \Phi u \; A_0(u,v,w)\Big)$$

(The proof of 2.13.2 in [21] uses the usual functional interpretation followed by pointwise majorization. However instead of this one can also use our monotone functional interpretation.)
One easily shows that

$$\text{WE–HA}^\omega \vdash \bigwedge m^0 \bigvee Y_0 \leq \lambda x, y.y \bigwedge x^2, y^1 \bigwedge k \leq_0 m \big(x(\min(y, \lambda n.(k)_n)) \leq x(Y_0 xy)\big).$$

Hence WE–HA$^\omega \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w \leq_\tau \Phi u \; A_0(u,v,w)$.

In [21] we used a different strategy to capture WKL:
WKL can be shown to be equivalent to a sentence having the form $\bigwedge x^1 \bigvee y \leq_1 \lambda k.1 \bigwedge z^0 \; A_0^K$ (where A_0^K is quantifier–free) and such that WE–HA$^\omega \vdash \bigwedge x^1, n^0 \bigvee y \leq_1 \lambda k.1 \bigwedge_{i=0}^{n} A_0^K$ (see [21] ,4.7). From this we obtained the conservativity of WKL for $\bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w^\tau \; A_0$–sentences relative to WE–PA$^\omega$+AC–qf and the extractability of bounds for w (if $\tau \leq 2$) which depend on u only (see [21] ,4, for a variety of results in this direction). However if one is only interested in obtaining such uniform bounds and not in eliminating WKL from the verification proof for this bound, then our method using F is easier since F is not as complicated as $\bigwedge x^1 \bigvee y \leq_1 \lambda k.1 \bigwedge z^0 \; A_0^K$ and can be formulated without any coding technique for finite sequences (thus this method also works for very weak subsystems of WE–PA$^\omega$). Furthermore F almost trivially implies important theorems in analysis, e.g. **Dini's theorem**:

Proposition 4.6 *E–PA$^\omega$+AC^{1,0}–qf+F proves*

$$\bigwedge \Phi^{1(1)}, \Phi^{1(1)(0)}_{(\cdot)}\Big(\bigwedge x, y \in [0,1], n^0 (x =_{\mathbb{R}} y \to \Phi x =_{\mathbb{R}} \Phi y \land \Phi_n x =_{\mathbb{R}} \Phi_n y)$$
$$\land \bigwedge k^0 \bigwedge x \in [0,1] \bigvee n^0 \bigwedge l \geq_0 n\big(\Phi x -_{\mathbb{R}} \Phi_l x <_{\mathbb{R}} 2^{-k}\big)$$
$$\land \bigwedge m^0, n^0 \bigwedge x \in [0,1]\big(n \geq m \to \Phi x \geq_{\mathbb{R}} \Phi_n x \geq_{\mathbb{R}} \Phi_m x\big)$$
$$\to \bigwedge k^0 \bigvee n^0 \bigwedge x \in [0,1] \bigwedge l \geq_0 n\big(\Phi x -_{\mathbb{R}} \Phi_l x <_{\mathbb{R}} 2^{-k}\big)\Big).$$

In words: If Φ, Φ_n represent functions $[0,1] \to \mathbb{R}$ in WE–PA$^\omega$ and if $(\Phi_n)_{n\in\mathbb{N}}$ is increasing and converges pointwise to Φ then this convergence is uniform on [0,1].

Proof: Using the standard representation of $\mathbb{R}$ and $[0,1]$ from [23] the assumption $\bigwedge k^0 \bigwedge x \in [0,1] \bigvee n^0 \big(\Phi x -_{\mathbb{R}} \Phi_n x <_{\mathbb{R}} 2^{-k}\big)$ has the form $\bigwedge k^0 \bigwedge x \leq_1 t \bigvee n^0 A_1(k,x,n)$, where $A_1 \in \Sigma^0_1$ and $t^1 \in T$ is a suitable closed term. Hence by AC^{1,0}–qf:

$$\bigwedge k^0 \bigvee \chi^{0(1)} \bigwedge x \leq_1 t\ A_1(k,x,\chi x).$$

By $\tilde{F}$ every χ^2 is bounded on $\{x^1 : x \leq_1 t\}$. Hence

$$\bigwedge k^0 \bigvee n^0 \bigwedge x \in [0,1] \bigvee l \leq_0 n\big(\Phi x -_{\mathbb{R}} \Phi_l x <_{\mathbb{R}} 2^{-k}\big),$$

which implies – using the assumption that $(\Phi_n)_{n\in\mathbb{N}}$ is pointwise increasing to Φ – that

$$\bigwedge k \bigvee n \bigwedge x \in [0,1] \bigwedge l \geq_0 n\big(\Phi x -_{\mathbb{R}} \Phi_l x <_{\mathbb{R}} 2^{-k}\big).$$

At first sight it seems surprizing that in our formulation of Dini's theorem it is not assumed that Φ and Φ_n are uniformly continuous on [0,1] (as in the usual formulation of this theorem). However as we show next, F together with AC^{1,0}–qf and extensionality already implies that all functional $x^{0(1)}$ are uniformly continuous on every fan $\{z^1 : z \leq_1 y\}$. This implies that all functionals $x^{1(1)}$ are continuous on $\{z^1 : z \leq_1 y\}$ and hence – using our standard representation and the proof of prop. 3.21 from [23] – every functional $x^{1(1)}$ which represents a function $[0,1] \to \mathbb{R}$ (i.e. which is extensional with respect to $=_{\mathbb{R}}$) is uniformly continuous (in the usual sense).

Proposition 4.7 *E-PA$^\omega$+AC^{1,0}–qf* $\vdash \bigwedge x^2, y^1 (x^2$ *is bounded on* $\{z^1 : z \leq_1 y\})$
$\leftrightarrow \bigwedge x^2, y^1 (x^2$ *is uniformly continuous on* $\{z^1 : z \leq_1 y\})$.

Proof: $\leftarrow$ is clear.
$\to$: Using extensionality it follows that

$\bigwedge x^2, y^1_1, y^1_2 \bigvee z^0 (\overline{y_1}z =_0 \overline{y_2}z \to xy_1 =_0 xy_2)$ and hence

$$\bigwedge x^2, y^1 \bigvee z^0 \big(\overline{(j^1_1 y)}z =_0 \overline{(j^1_2 y)}z \to x(j^1_1 y) =_0 x(j^1_2 y)\big),$$

where $j^0(n,m) := 2^n(2m+1) - 1$ with the projections j^0_1, j^0_2 and $j^1(y^1_1, y^1_2) := \lambda n.j^0(y_1 n, y_2 n)$, $j^1_i y^1 := \lambda n.j^0_i(yn)$ $(i = 1,2)$.
By AC^{1,0}–qf one concludes that

$$\bigwedge x^2 \bigvee \omega^{0(1)} \bigwedge y\big(\overline{j^1_1 y}(\omega y) =_0 \overline{j^1_2 y}(\omega y) \to x(j^1_1 y) =_0 x(j^1_2 y)\big).$$

For all y_1, y_2 such that $y_1, y_2 \leq_1 y$ it follows that $j^1(y_1, y_2) \leq_1 j^1(y, y)$. By the assumption we have that ω is bounded on $\{z : z \leq_1 j^1(y, y)\}$ and thus $\widehat{\omega}(y_1, y_2) := \omega(j^1(y_1, y_2))$ is bounded by a number n_y for all $y_1, y_2 \leq_1 y$. Hence

$$\bigwedge x^2, y^1 \bigvee n \bigwedge y_1, y_2 \leq_1 y(\overline{y}_1 n =_0 \overline{y}_2 n \to xy_1 =_0 xy_2).$$

Corollary 4.8 *E–PA^ω+$AC^{1,0}$–qf+$F \vdash \bigwedge x^2, y^1(x$ is uniformly continuous on $\{z : z \leq_1 y\})$.*

The case $F \equiv \bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^{0/1} F_0$ however is of great importance as the next section shows.

5 Applications to uniqueness proofs in approximation theory

In this section we give a survey of our proof–theoretic applications to analysis from [23] and [24] and analyze them from the perspective of the present paper.
In [23] and [24] we applied a combination of functional interpretation and majorization which was developed in [21] to concrete proofs from the theory of Chebycheff approximation. In analyzing these applications we made the observation that we never had to use functionals defined by cases or functionals depending on prime formulas. Moreover in sentences

$$\bigwedge x \in X, k \in \mathbb{N} \bigvee n \in \mathbb{N} \big(|F(x)| \leq 2^{-n} \to |G(x)| < 2^{-k}\big),$$

which we discussed in the introduction, we did not have to take care of the functional interpretation of the quantifiers hidden in $\leq \in \Pi^0_1$ and $< \in \Sigma^0_1$ (and therefore we had not to go back to the level of coding real numbers as sequences of natural numbers). It was an investigation of this phenomenon which led us to the development of the monotone functional interpretation in the present paper since it turned out that it was just this monotone simplification of functional interpretation which we actually had carried out: The monotone functional interpretation needs no functionals defined by cases and no decision of prime formulas. Because of this '$|F(x)| \leq 2^{-n}$' and '$|G(x)| < 2^{-k}$' can be treated as prime formulas.

Modulo a suitable standard representation of CSM–spaces and compact CSM–spaces every sentence $\mathcal{F} \equiv \bigwedge x \in X \bigvee y \in Y_x \big(F(x, y) =_{\mathbb{R}} 0\big)$, where X, Y are constructively definable CSM–spaces, $Y_x \subset Y$ is a constructively definable family of compact sets and $F : X \times Y \to \mathbb{R}$ is a constructive (and therefore continuous) function, has the logical form $\bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^0 F_0(x, y, z)$ ($s \in T$ is closed and F_0 is a quantifier–free formula which contains only x, y, z as free variables) (see [23] for details)[6].
Sentences having the form $\mathcal{F}$ are central in analysis. Examples are

1) The attainment of the maximum of $f \in C[0, 1]$ on $[0, 1]$.

2) The mean value theorem for integration.

3) The intermediate value theorem for $f \in C[0, 1]$.

4) The existence of a best Chebycheff approximation together with an extremal alternant; see (5) below.

We now apply our results in 4 to uniqueness theorems in best approximation theory which are mostly of the logical form $A \equiv \bigwedge u^1 \bigwedge v_1, v_2 \leq_1 tu \bigvee w^0 A_0$ and whose proofs usually make essential use of lemmas $\mathcal{F}$.
Let us consider the following general situation: $G : U \times V \to \mathbb{R}$ is a continuous function, U, V are CSM–spaces and we are interested in points v_u in a compact set $V_u \subset V$ where $G(u, \cdot)$ assumes its infimum on V_u. To be more specific, let U be a (real) normed space and $V := E$ a finite dimensional linear subspace of U. Then a best approximation of $u \in U$ in E is an element $v_{b,u} \in E$ such that $\|u - v_{b,u}\| = \text{dist}(u, E)$. Since $0 \in E$ it follows that $\|u - v_{b,u}\| \leq \|u\|$ and thus $\|v_{b,u}\| \leq 2\|u\|$. Hence $\text{dist}(u, E) = \text{dist}(u, E_u)$, where $E_u := \{v \in E : \|v\| \leq 2\|u\|\}$ is compact in E. So best approximations are exactly those points $\in E_u$ where $G(u, \cdot) := \|u - \cdot\|$ assumes its infimum on E_u. Non–constructively best approximations always exist since E_u is compact. In many important situations the uniqueness of the best approximation can be proved (from lemmas $\mathcal{F}$ relative to subsystems of E–PA$^\omega$). Such uniqueness theorems are of the form

$$\bigwedge u \in U, v_1, v_2 \in V_u \Big(\bigwedge_{i=1}^{2} \big(G(u, v_i) = \inf_{v \in V_u} G(u, v)\big) \to v_1 = v_2 \Big), \text{ i.e.}$$

$$\bigwedge u \in U, v_1, v_2 \in V_u, k \in \mathbb{N} \bigvee n \in \mathbb{N} \Big(\bigwedge_{i=1}^{2} \big(G(u, v_i) - \inf_{v \in V_u} G(u, v) \leq 2^{-n}\big) \to d_V(v_1, v_2) < 2^{-k} \Big),$$

which has (modulo standard representation of U, V_u, G) the form A.

In the following let X, Y, U, V be constructively definable CSM–spaces, $Y_x \subset Y$, $V_u \subset V$ constructively definable families of compact sets in Y, V and $F : X \times Y \to \mathbb{R}$, $G : U \times V \to \mathbb{R}$ constructive functions, $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$:

Theorem 5.1 ([23]) *Let $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$.*

$$\begin{cases} E\text{–}PA^\omega + AC^{\alpha,\beta}\text{–}qf \vdash \bigwedge x \in X \bigvee y \in Y_x \big(F(x, y) = 0\big) \to \\ \qquad \bigwedge u \in U, v_1, v_2 \in V_u \Big(\bigwedge_{i=1}^{2} \big(G(u, v_i) = \inf_{v \in V_u} G(u, v)\big) \to v_1 = v_2 \Big). \\ \text{Then one can extract closed terms } \Phi, \Psi \in T \text{ such that} \\ WE\text{–}HA^\omega \vdash \bigwedge u \in U, k \Big(\bigwedge x \in X \bigvee y \in Y_x \big(|F(x, y)| \leq 2^{-\Psi uk}\big) \to \\ \qquad \bigwedge v_1, v_2 \in V_u \Big(\bigwedge_{i=1}^{2} \big(G(u, v_i) - \inf_{v \in V_u} G(u, v) \leq 2^{-\Phi uk}\big) \to d_V(v_1, v_2) \leq 2^{-k} \Big)\Big). \end{cases}$$

We call such an operation Φ a **modulus of uniqueness**.

Remark 5.2 *1)* Φ and Ψ are defined on the representatives of the elements of U under the standard representation of U. Thus Φ, Ψ are in general not extensional with respect to $=_U$. However they are extensional (in our examples) in enriched data as e.g. $f \in C[0, 1]$ endowed with a modulus of uniform continuity or an estimate $M \geq \|f\|_\infty$.

2) In [23] , 5.1 is proved using first the usual functional interpretation and then (pointwise) majorization of the functionals extracted. Now it is possible in a more simple way to use the monotone functional interpretation as developed 2,3 above. In fact this has been done implicitly in the actual analysis of three concrete uniqueness proofs for best Chebycheff approximation in [23] ,[24] . If one is not interested in the weakening of $\mathcal{F} \equiv \bigwedge x \in X \bigvee y \in Y_x \big(F(x, y) = 0\big)$

to its ε–version by constructing Ψ then $\mathcal{F}$ can be treated simply as an axiom (as was shown in 4). Only for the extraction of Ψ it is necessary to consider the proof of the whole implication $\mathcal{F} \to$ uniqueness. In this case a generalization to the situation where $\mathcal{F}$ depends on u, v_1, v_2 is possible (see [23] for this and other generalizations).

The modulus of uniqueness Φ plays a role in applications namely for the computation of best approximations: [7]
Let χ be an arbitrary algorithm which computes ε–best approximations, i.e.

$$\bigwedge u \in U, k \in \mathbb{N}\big(G(u,\chi uk) - \inf_{v\in V_u} G(u,v) \leq 2^{-k} \wedge \chi uk \in V_u\big)$$

(Such a $\chi \in T$ can be constructed by searching through a finite $2^{-\omega_u(k)}$–net for V_u where ω_u is a modulus of uniform continuity of $G(u,\cdot)$ on V_u. In concrete applications one uses of course better algorithms χ which are adapted to the special situation.)
Φ provides an a priori rate of convergence of $(\chi uk)_{k\in\mathbb{N}}$ to the uniquely determined $v_u \in V_u$ with $G(u,v_u) = \inf\limits_{v\in V_u} G(u,v)$, i.e. Φu tells us how large the input l of χul must be in order to guarantee that the 2^{-l}–best approximation of u computed by χul has distance $\leq 2^{-k}$ from the best approximation v_u: take $l := \Phi uk$, i.e.

$$\bigwedge u \in U, k \in \mathbb{N}\big(d_V(\chi u(\Phi uk), v_u) \leq 2^{-k}\big).$$

The proof that $\chi u(\Phi uk)$ converges with modulus 2^{-k} to the best approximation can be proved even in WE–HA$^\omega$ without assuming the existence of the best approximation if $\chi \in T$. Since such a χ always exists we obtain a constructive existence proof for v_u together with an algorithm $\tilde{\Phi} \in T$ for v_u:

$$\begin{cases} \text{E–PA}^\omega + \text{AC}^{\alpha,\beta}\text{–qf} \vdash \bigwedge x \in X \bigvee y \in Y_x\big(F(x,y) = 0\big) \to \\ \qquad\qquad \bigwedge u \in U, v_1, v_2 \in V_u\Big(\bigwedge\limits_{i=1}^{2} \big(G(u,v_i) = \inf\limits_{v\in V_u} G(u,v)\big) \to v_1 = v_2\Big). \\ \Rightarrow \exists \tilde{\Phi} \in T \text{ such that} \\ \text{WE–HA}^\omega \vdash \bigwedge x \in X, m \in \mathbb{N} \bigvee y \in Y_x\big(|F(x,y)| \leq 2^{-m}\big) \to \\ \qquad\qquad \bigwedge u \in U\big(G(u,\tilde{\Phi}u) = \inf\limits_{v\in V_u} G(u,v) \wedge \tilde{\Phi}u \in V_u\big) \end{cases}$$

(see Kohlenbach [23] for a proof of this and more general results).

In the following situations in best approximation theory there are classical proofs for the uniqueness of the best approximation which are formalizable in subsystems of E–PA$^\omega$+AC^{0,1}–qf plus lemmas $\mathcal{F}$:

1) Best Chebycheff approximation (i.e. approximation w.r.t. $\|\cdot\|_\infty$) of $f \in C[0,1]$ by elements of a Haar space $H \subset C[0,1]$. In particular for $H := P_n$ where P_n denotes the set of (algebraic) polynomials over $\mathbb{R}$ having degree $\leq n$.

2) Best approximation in strictly convex (and in particular uniformly convex) spaces U (e.g. $U = L_p$ $(1 < p < \infty)$) by elements from a finite dimensional subspace $E \subset U$.

3) Best approximation of $f \in C[0,1]$ with respect to the norm $\|f\|_1 := \int\limits_0^1 |fx|dx$ by elements of a Haar space e.g. P_n. [8]

4) Best uniform approximation of $f \in C[0,1]$ by polynomials having bounded coefficients.

5.1 is applicable to 1)–4). Let us indicate this for 1). (For more details see [20],[23] and [24] .)
The uniqueness of best Chebycheff approximation of $f \in C[0,1]$ by polynomials in P_n reads as follows

$$(*) \ \bigwedge f \in C[0,1] \bigwedge n \in \mathbb{N} \bigwedge p_1, p_2 \in P_n \Big(\bigwedge_{i=1}^{2} (\|p_i - f\|_\infty = \mathrm{dist}(f, P_n)) \to \|p_1 - p_2\|_\infty = 0 \Big).$$

As we have already noticed, P_n may be replaced by $K_{f,n} := \{p \in P_n : \|p\|_\infty \le 2\|f\|_\infty\}$ without a restriction of the uniqueness theorem. The usual proofs for $(*)$ can be easily formalized in
$\mathrm{E\text{–}PA}^\omega + \bigwedge f \in C[0,1] \bigvee x_0 \in [0,1](fx_0 = \sup_{x\in[0,1]} fx)$.

Now 5.1 applies by taking

$$X := C[0,1],\ Y := \mathbb{R},\ Y_x := [0,1],\ U := C[0,1] \times \mathbb{N},\ V := C[0,1],\ V_u := K_{f,n},$$
$$F(f, x_0) := \sup_{x\in[0,1]} fx - fx_0 \text{ and } G(f,p) := \|f - p\|_\infty.$$

In fact in 1)–3) one can obtain moduli of uniqueness which are valid on the whole space E and not only on E_u: Replace E_u by $\tilde{E}_u := \{v \in E : \|v\| \le \|\frac{5}{2}u\|\}$ which is also compact and extract a modulus of uniqueness Φ on $\tilde{E}_u$. Define $\tilde{\Phi}uk := 3 + \max(\Phi uk, k)$ and assume that
$\|u - v_1\|, \|u - v_2\| \le \mathrm{dist}(u, E) + 2^{-\tilde{\Phi}uk}$.
Case 1: $2^{-k} \le 4\mathrm{dist}(u,E)$:
$\mathrm{dist}(u,E) + 2^{-\tilde{\Phi}uk} \le \frac{3}{2}\mathrm{dist}(u,E) \le \frac{3}{2}\|u\|_\infty \Rightarrow v_1, v_2 \in \tilde{E}_u$.
Case 2: $2^{-k} > 4\mathrm{dist}(u,E)$: $\|u - v_1\|, \|u - v_2\| \le \mathrm{dist}(u,E) + 2^{-\tilde{\Phi}uk} \le 2^{-k-2} + 2^{-k-3}$ implies
$\|v_1 - v_2\| \le 2 \cdot (2^{-k-2} + 2^{-k-3}) < 2^{-k}$.
(Note that if Φuk is linear in k then also $\tilde{\Phi}uk$ is linear.)
Furthermore $\tilde{\Phi}$ can be easily extended to a modulus $\widehat{\Phi} : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$, i.e.

$$(*) \ \bigwedge u \in U, v_1, v_2 \in E, q \in \mathbb{Q}^*_+ \big(\|u - v_1\|, \|u - v_2\| \le \ \mathrm{dist}(u,E) + \widehat{\Phi}uq \to \|v_1 - v_2\| \le q\big).$$

In the following by a modulus of uniqueness we always mean an operation which satifies $(*)$.

We now summarize our results on moduli of uniqueness for best Chebycheff approximation obtained in [20] ,[23] and [24] . For simplicity we consider mainly the case $H := P_n$ (For the generalization to arbitrary constructively given Haar spaces see [24] .) Firstly we present a general result which further illustrates the importance of the notion 'modulus of uniqueness':

Proposition 5.3 ([23]) *Let U be a normed space, $E \subset U$ a finite dimensional subspace and assume that Φ is a modulus of uniqueness for the best approximation of $u \in U$ in E. (It suffices that Φ is such a modulus for the special case that v_2 is taken to be the best approximation.) Then*

1) *$\frac{1}{2}\Phi$ is a modulus of pointwise continuity for the projection $\mathcal{P} : U \to E$ which maps $u \in U$ to its best approximation in E, i.e.*

$$\bigwedge u, u_0 \in U, q \in \mathbb{Q}^*_+ \big(\|u - u_0\| \le \frac{1}{2}\Phi u_0 q \to \|\mathcal{P}(u) - \mathcal{P}(u_0)\| \le q\big).$$

If in addition Φ is linear in q, i.e. $\Phi uq = q \cdot \gamma(u)$ (where $\gamma(u) \in \mathbb{Q}^*_+$), then

2) $\gamma(u)$ is a (lower estimate of a) constant of strong unicity, i.e.

$$\bigwedge u \in U, v \in E\big(\|u-v\| \geq \|u-v_b\| + \gamma(u)\cdot\|v-v_b\|\big),$$

where v_b is the best approximation of u.

3) $\lambda(u) := \frac{2}{\gamma(u)}$ is a (pointwise) Lipschitz constant of $\mathcal{P}$ (in u).

Remark 5.4 The notion 'constant of strong unicity' is common in the context of Chebycheff approximation and refers (for $f \in C[0,1]$) to the greatest $\gamma \in \mathbb{R}^*_+$ such that
$(*)$ $\bigwedge p \in H(\|f-p\|_\infty \geq \|f-p_b\|_\infty + \gamma\cdot\|p-p_b\|_\infty)$ holds.
(Here $H \subset C[0,1]$ denotes a Haar space.)
The existence of a γ satisfying $(*)$ was proved (ineffectively) first in [28] (see also [10]). A proof of this fact is already implicit in [12] (see [5]). For more information on strong unicity see [24] .

5.3.2 shows that the concept 'modulus of uniqueness' generalizes the concept of strong unicity.

In [23],[24] we analyze three different proofs of the uniqueness of the best Chebycheff approximation of $f \in C[0,1]$ by polynomials $\in P_n$ (the third one also for general Haar spaces):

1) the most common proof from de La Vallée Poussin [29] (56) (as presented with all details e.g. in [27]),
2) a proof due to Kirchberger [17] and Borel [6] and
3) a simplification of a proof sketched by Young [34] (and worked out in Rice [30]).

From all three proofs $i = 1,2,3$ we obtained moduli of uniqueness Φ_i which are linear in q if the data $f \in C[0,1],\ n \in \mathbb{N}$ are enriched by a lower estimate $0 < l_{f,n} \leq \mathrm{dist}(f,P_n)$ ($l_{f,n} \in \mathbb{Q}^*_+$), i.e. for all $f \in C[0,1], n \in \mathbb{N}$ and $l_{f,n} \in \mathbb{Q}^*_+$ such that $l_{f,n} \leq \mathrm{dist}(f,P_n)$ we have

$$(+)\left\{\begin{array}{l} \bigwedge p_1,p_2 \in P_n, q \in \mathbb{Q}^*_+\Big(\bigwedge\limits_{i=1}^{2}\big(\|p_i-f\|_\infty - \mathrm{dist}(f,P_n) \leq (\Phi_i fnl_{f,n})\cdot q\big) \\ \hfill \to \|p_1-p_2\|_\infty \leq q\Big). \end{array}\right.$$

Φ_i can be modified to a modulus $\tilde{\Phi}_i$ which no longer depends on an estimate $l_{f,n} \leq \mathrm{dist}(f,P_n)$ (but which is not linear in q in contrast to Φ_i):
Claim: $\tilde{\Phi}_i fnq := \min\Big(\frac{q}{4}, \Phi_i fn\big(\frac{q}{4}\big)\cdot q\Big)$ is also a modulus of uniqueness.
Proof: Case 1: $\mathrm{dist}(f,P_n) \geq \frac{q}{4}$. In this case the proposition follows immediately from (+).
Case 2: $\mathrm{dist}(f,P_n) < \frac{q}{4}$. Then
$\|p_1-f\|_\infty, \|p_2-f\|_\infty \leq \mathrm{dist}(f,P_n) + \tilde{\Phi}_i fnq < \frac{q}{4}+\frac{q}{4} = \frac{q}{2}$ implies
$\|p_1-p_2\|_\infty \leq \|p_1-f\|_\infty + \|f-p_2\|_\infty < q$.

In [23],[24] we obtain the following results for $i = 1,2,3$ (writing l instead of $l_{f,n}$ for notational simplicity):

$$\Phi_1 fnl = \frac{1}{10(n+1)}\prod_{i=1}^{\lfloor\frac{n\dot{-}1}{2}\rfloor}(2i-\frac{1}{2})\cdot\prod_{i=1}^{\lceil\frac{n\dot{-}1}{2}\rceil}(2i-\frac{3}{2})\cdot\left\lfloor\frac{n}{2}\right\rfloor!\left\lceil\frac{n}{2}\right\rceil!\cdot\left(\omega_{f,n}\left(\frac{l}{2}\right)\right)^{n\dot{-}1}\cdot\left(\omega_{f,n}\left(\frac{3l}{2}\right)\right)^{n}.$$

$$\Phi_2 fnl = \frac{1}{n^n} \cdot \Phi_3 fnl, \text{ where}$$

$$\Phi_3 fnl = \frac{\lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !}{2(n+1)} \cdot (\omega_{f,n}(2l))^n ,$$

where $\omega_{f,n}$ is always defined by

$$\omega_{f,n}(q) := \begin{cases} \min\left(\omega_f\left(\frac{q}{2}\right), \frac{q}{10n^2M}\right), & \text{if } n \geq 1 \\ 1 & , \text{ if } n = 0, \end{cases}$$

where ω_f is a modulus of uniform continuity for f and $M \in \mathbb{Q}_+^*$ such that $M \geq \|f\|_\infty$.

Remark 5.5 *1)* The moduli Φ_i depend – strictly speaking – also on the estimate $M \geq \|f\|_\infty$. However such an estimate can be easily computed from f since f is given with a modulus of uniform continuity. We formulate Φ_i using M in order to make explicit that **any** upper bound of $\|f\|_\infty$ can be used.

2) In Φ_2, Φ_3 one can improve $\omega_{f,n}$ by replacing $10n^2$ by $8n^2$. Furthermore the factor $\frac{1}{2}$ can be omitted if Φ_2, Φ_3 are only applied to the special case where $v_2 := v_b$ (as is sufficient for 5.3).

It is clear that Φ_3 is the best one of these three moduli. It is roughly $\sqrt[2]{\Phi_1}$ (Note that $\left(\omega_{f,n}(2l)\right)^n \leq 1$ is very close to 0 in practice). Φ_2 is less good than Φ_3 (but better than Φ_1) because of the factor $\frac{1}{n^n}$.
The extraction of Φ_2 and Φ_3 is much easier than the extraction of Φ_1. From the logical point of view it is interesting that this great difference in the complexity of the proof analysis and also the numerical improvement from Φ_1 to Φ_3 (and from Φ_1 to Φ_2) corresponds to the different logical forms in which certain key lemmas – mainly the so-called alternation theorem – are used in the proofs (1)–(3). All three proofs use essentially this alternation theorem:

$$(1) \begin{cases} \bigwedge f \in C[0,1], p_b \in P_n \Big(\|f - p_b\|_\infty = \text{dist}(f, P_n) \to \bigvee j \in \{0,1\}, (x_1, \ldots, x_{n+2}) \in [0,1]^{n+2} \\ \qquad \Big(\bigwedge_{i=1}^{n+1} (x_{i+1} \geq x_i) \wedge \bigwedge_{i=1}^{n+2} \big((-1)^{i+j} (p_b(x_i) - f(x_i)) = \text{dist}(f, P_n) \big) \Big) \Big). \end{cases}$$

(1) has the logical form

$$(2) \ \bigwedge x \in X \Big(\bigwedge k \in \mathbb{N} A_0(k) \to \bigvee y \in K (F(x,y) =_{\mathbb{R}} 0) \Big),$$

where A_0 is quantifier–free, $X = C[0,1] \times P_n$ is a CSM–space, $K = \{0,1\} \times [0,1]^{n+2}$ is a compact CSM–space and $F : X \times K \to \mathbb{R}$ is a constructive function. The fomula $\bigwedge k \in \mathbb{N} A_0(k)$ expresses the equality $\|f - p_b\|$ =dist(f, P_n). Because of this universal premise (2) does not have the form

$$(3) \ \bigwedge x \in X \bigvee y \in K \big(F(x,y) =_{\mathbb{R}} 0 \big)$$

which would allow us to treat this lemma as an axiom in the proof–analysis. In contrast to lemmas (3) sentences having the form (2) do contribute (in general) to the numerical data, namely by an operation χ such that

$$(4) \ \bigwedge x \in X, q \in \mathbb{Q}_+^* \Big(\bigwedge_{k=0}^{\chi x q} A_0(k) \to \bigvee y \in K \big(|F(x,y)| \leq q \big) \Big) \text{ see [21].}$$

In fact it follows from the proof of theorem 4.17 in [21] that only a majorant χ^* for χ is needed. Such a χ^* can be extracted from the proof of (1) by our monotone functional interpretation since this proof can be carried out in $\text{E–PA}^\omega + (A)$ where

$$(A) \ : \bigwedge f \in C[0,1] \bigvee x_0 \in [0,1](fx_0 = \sup_{x\in[0,1]} fx)$$

has the form (3). Such an extraction of χ^* is carried out in [23] yielding a new quantitative version of the usual alternation theorem. However it is just this passage through the whole non–constructive proof of the alternation theorem which makes the extraction of Φ_1 quite complicated and causes the factor

$$\prod_{i=1}^{\lfloor \frac{n \dot{-} 1}{2} \rfloor} (2i - \frac{1}{2}) \cdot \prod_{i=1}^{\lceil \frac{n \dot{-} 1}{2} \rceil} (2i - \frac{3}{2}) \cdot \left(\omega_{f,n}\left(\frac{l}{2}\right)\right)^{n \dot{-} 1}$$

in Φ_1 which makes Φ_1 less good than Φ_3.
On the other hand a mathematically slight but logically decisive modification of the proofs 3) (due to Young/Rice) and 2) (due to Borel) (see [24]) use the alternation theorem only in the following form:

$$(5) \begin{cases} \bigwedge f \in C[0,1] \bigvee p_b \in K_{f,n}, (x_1, \cdots, x_{n+2}) \in [0,1]^{n+2}, j \in \{0,1\} \Big(\|p_b - f\|_\infty = \text{dist}(f, P_n) \\ \qquad \wedge \bigwedge_{i=1}^{n+1} (x_{i+1} \geq x_i) \wedge \bigwedge_{i=1}^{n+2} \big((-1)^{i+j} (p_b(x_i) - f(x_i)) = \text{dist}(f, P_n) \big) \Big), \end{cases}$$

which has the form (3) (Here $K_{f,n} := \{p \in P_n : \|p\|_\infty \leq 2\|f\|_\infty\}$).
(5) follows immediately from (1) plus the existence of a best approximation p_b. In the other direction (5) implies (1) **if one assumes** already the uniqueness of p_b. Thus although (5) is even more non–constructive than (1) in asserting the existence of a best approximation it can be conceived simply as an axiom whose proof doesn't matter for the extraction of the modulus of uniqueness. Besides (5) also the intermediate value theorem is used in (our simplification of) the proof 3) by Young/Rice but only to derive a purely universal lemma which can be treated also as an axiom. Hence despite its non–constructivity the proof by Young/Rice plus a logical improvement to (5) is much easier to unwind than the proof 1) and yields a significantly better result (Φ_3).
Although the analysis of the proof 1) is very complicated and provides an effective estimate Φ_1 for strong unicity which is less good than Φ_3 it is interesting in providing an estimate for strong unicity at all since this proof - which is presented in most text books because of its shortness and elegance - has never been used to prove any **quantitative** version of uniqueness as strong unicity.[9]

Since the proof 3) works also for arbitrary Haar spaces instead of P_n it is possible (by our analysis of this proof) to construct explicit moduli of uniqueness also for other (constructively definable) Haar spaces (see [24]). In particular we can improve significantly estimates for general Haar spaces obtained by D. Bridges in [7], [8] (who works entirely within the framework of Bishop's constructive analysis [4]):

Definition 5.6 (D. Bridges) *Let* $\phi := \{\phi_1, \ldots, \phi_n\}$ *be a Chebycheff system over* $[0,1]$*,* $\underline{\phi}(x) := \big(\phi_1(x), \ldots, \phi_n(x)\big) \in \mathbb{R}^n$, $\|\underline{\phi}\| := \sup_{x\in[0,1]} \|\underline{\phi}(x)\|_2$*, where* $\|\cdot\|_2$ *denotes the Euclidean norm on* $\mathbb{R}^n$*.*

1) $\beta, \gamma, \kappa : (0, \frac{1}{n}] \to \mathbb{R}^*_+$ are defined by

$$\beta(\alpha) := \begin{cases} \inf\limits_{x\in[0,1]} |\phi_1(x)|, \ \text{if } n = 1 \\ \inf \left\{ |\det(\phi_j(x_i))| : 0 \le x_1, \ldots, x_n \le 1, \bigwedge\limits_{i=1}^{n-1} (x_{i+1} - x_i \ge \alpha) \right\}, \ \text{if } n > 1 \end{cases}$$

and

$$\gamma(\alpha) := \min \left(\|\underline{\phi}\|, \frac{\beta(\alpha)}{n^{\frac{1}{2}}(n-1)! \prod\limits_{i=1}^{n}(1 + \|\phi_i\|_\infty)} \right), \ \kappa(\alpha) := \gamma(\alpha)^{-1} \cdot \|\underline{\phi}\|$$

for $\alpha \in (0, \frac{1}{n}]$. Since ϕ is a Chebycheff system it follows that $\beta(\alpha) > 0$. $H :=$Lin$_{\mathbb{R}}(\phi_1, \ldots, \phi_n)$.

2) Suppose that $A \subset C[0,1]$ is totally bounded, ω_A is a common modulus of uniform continuity for all $f \in A$ and $M > 0$ is a common bound $M \ge \|f\|_\infty$ for all $f \in A$, $\omega_{\underline{\phi}}$ a modulus of uniform continuity for $\underline{\phi}$. Then

$$\omega_{A,H}(\varepsilon) := \min \left(\omega_A(\frac{\varepsilon}{2}), \omega_{\underline{\phi}} \left(\frac{\varepsilon \cdot \beta(\frac{1}{n})}{4Mn^{\frac{3}{2}}(n-1)! \prod\limits_{i=1}^{n}(1 + \|\phi_i\|_\infty)} \right) \right).$$

Theorem 5.7 ([24]) *Let H, $A \subset C[0,1]$ as in 5.6 and $E_{H,A} := \inf\limits_{f\in A} dist(f, H)$. Then the following holds:*
For $l_{H,A} \in \mathbb{Q}^*_+$ such that $l_{H,A} < E_{H,A}$ and $0 < \alpha \le \min\left(\frac{1}{n}, \omega_{A,H}(2 \cdot l_{H,A})\right)$ we have $\frac{\gamma(\alpha)}{n\cdot\|\underline{\phi}\|}$ as a common constant of strong unicity for all $f \in A$, i.e.

$$\bigwedge f \in A, \varphi \in H \Big(\|f - \varphi\|_\infty \ge \|f - \varphi_b\|_\infty + \left(\frac{\gamma(\alpha)}{n \cdot \|\underline{\phi}\|} \right) \cdot \|\varphi - \varphi_b\|_\infty \Big),$$

where φ_b is the best approximation of f in H. Furthermore $2n \cdot \kappa(\alpha)$ is a common Lipschitz constant for the Chebycheff projection for all points $f_0 \in A$.

Bridges obtains in [7],[8] the following estimates: $n^{-2}\left(\frac{\gamma(\alpha)}{\|\underline{\phi}\|}\right)^{2n+1}$ as a common constant of strong unicity and $2n\kappa(\alpha) \cdot \left(\sum\limits_{i=1}^{n+1} \kappa(\alpha)^{n+i-1} - 1 \right)$ as a common Lipschitz constant for all $f \in A$, where $0 < \alpha \le \min\left(\frac{1}{n}, \omega_{A,H}(l_{H,A})\right)$ and $0 < l_{H,A} \le E_{H,A}$. These estimates are much weaker than ours since $\frac{\gamma(\alpha)}{\|\underline{\phi}\|}$ (≤ 1) is very close to 0 in practice. The moduli of uniqueness and pointwise continuity from Bridges [8] and [9] allow a similar improvement.
Furthermore our modulus Φ_3 improves a modulus of uniqueness for P_n which is implicit in Ko [18],[19] (see [24] for details).

Notes

1) Here and in the following we may also have tuples $\bigwedge x_1 \in X_1, \ldots, \bigwedge x_n \in X_n(\ldots)$.

2) By $A_0, B_0, C_0, \ldots$ we denote always quantifier–free formulas.

3) Let us motivate this for $X := \mathbb{R}$: Since rational numbers can be coded in $\mathbb{N}$ (as pairs of natural numbers), every number theoretic function f^1 can be conceived as a sequence of rational numbers. Thus quantification over $\mathbb{R}$ reduces to quantification over those functions f^1 which represent Cauchy sequences with fixed Cauchy modulus (say 2^{-k}). One can define a construction $f^1 \mapsto \widehat{f}^1$ such that $\widehat{f}$ always represents a Cauchy sequences with this modulus and –if already f represents such a sequence– we have $f =_{\mathbb{R}} \widehat{f}$. Using this construction, quantification over such Cauchy sequences reduces to $\bigwedge f^1 A(\widehat{f})$ for $=_{\mathbb{R}}$–extensional properties A. Thus the quantifiers hidden in the implicative premise 'f represents a Cauchy sequence of rationals with modulus 2^{-k}' are eliminated by the use of $f \mapsto \widehat{f}$. See Kohlenbach [23] 3 (and also [1]) for details on this.

4) In [3] it is shown that this form of ER–qf is in fact derivable from the simpler one without A_0. However for the formalization of given proofs our version is more convenient.

5) Instead of Y, Y', X'', x, x', y'' we have in fact to consider tuples $\underline{Y}, \underline{Y}', \underline{X}'', \ldots$ of these variables. However for notational simplicity we formulate only the special case where all tuples have length 1 since the (correct) treatment with arbitrary tuples is than routine.

6) In fact $\bigwedge x \in X \bigvee y \in Y_x \bigwedge z \in Z\big(F(x, y, z) = 0\big)$–assumptions ($Z$ also a CSM–space, $F : X \times Y \times Z \to \mathbb{R}$) are admissible.

7) The relevance of the information provided by a (slight modification of our notion of) modulus of uniqueness for a finite computation of the best approximation of $f \in C[0, 1]$ in P_n seems to be firstly noticed by de La Vallée Poussin (see [29] 66).

8) Since $\big(C[0, 1], \|\cdot\|_1\big)$ is not complete we have to represent $C[0, 1]$ with respect to $\|\cdot\|_\infty$. This means that the modulus of uniqueness and the algorithm are only effective in f together with a modulus of uniform continuity of f (instead of a – weaker – modulus of integration).

9) In fact it is interesting that de La Vallée Poussin proves in 65 of [29] the existence of (a slight modification of) a modulus of uniqueness for $f \in C[0, 1], p_b \in P_n$ but uses instead of his uniqueness proof in 56 a completely different argument which is more closely related to the proofs by Borel and Young since it also uses the alternation theorem only via (5). The proof however gives (also without analyzing the proof of (5)) an estimate which is less good than Φ_2, Φ_3. Roughly it is of order Φ_3^2 and so similar to Φ_1 (but for different reasons). Thus bypassing the proof of the alternation theorem does not always guarantee a good result.

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