Interrelation between weak fragments of double negation shift and related principles

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Abstract

We investigate two weak fragments of the double negation shift schema, which are motivated respectively from Spector's consistency proof of ACA_0 and from the negative translation of RCA_0, as well as doubly negated variants of logical principles. Their interrelations over both intuitionistic arithmetic and analysis are completely solved.

1 Introduction

Proofs in mathematics which are noneffective by making use of the law of excluded middle schema (LEM) in most concrete cases use only rather restricted forms of LEM, e.g. LEM applied only to formulas of very low complexity. For a fine analysis of the specific uses of classical logic made in proofs, also a number of principles different from LEM (though being derivable from sufficiently strong forms of LEM) have been introduced and studied (see e.g. [1]). To show that (over some intuitionistic base system) one noneffective principle P_1 does not imply another principle P_2 one typically uses so-called proof interpretations (appropriate forms of realizability or functional interpretations) to show that P_1 has a certain semi-constructive interpretation in the sense of the interpretation used which P_2 does not.

In this paper, we study for the first time the interrelation between weak fragments of the double negation shift (DNS):

$$\forall x \neg\neg\varphi \to \neg\neg\forall x\varphi$$

and (mainly double negated versions) of the following three logical principles:

- The law of excluded middle: $\varphi \vee \neg\varphi$;
- De Morgan's law (DML): $\neg(\varphi \wedge \psi) \to \varphi \vee \psi$;
- The double negation elimination (DNE): $\neg\neg\varphi \to \varphi$;

over intuitionistic arithmetic and analysis.

This time the various separation results require rather subtle arguments as none of these principles causes a failure of the usual constructive properties of intuitionistic systems when added to them as they are in a specific sense computationally empty being (equivalent to) negated statements (see e.g. Corollary 5.16 and 5.24 in [12]).

Proof-theoretically, however, DNS (already for number variables x) is very strong as its addition to elementary analysis EL plus the countable choice schema $\mathrm{AC}^{0,0}$ results in a system that allows for an embedding of full classical analysis (see e.g. [12, Chapter 11], and also below) while EL+$\mathrm{AC}^{0,0}$ and even HA^{ω}+AC can be proof-theoretically reduced to HA and - by Goodman's theorem - in fact is conservative over Heyting arithmetic HA.

Below we will give a metamathematical interpretation of DNS restricted to the formula classes Σ^0_1 and Σ^0_2 in terms of consistency proofs to motivate our interest in these principles.

For the rest of this introduction (and mostly also throughout the whole paper), x, y, z denote variables over the natural numbers, so-called type-0 variables, while α, β, γ denote variables for number-theoretic functions of some arity (so-called type-1 variables) where the arity will be clear from the context.

Spector's Consistency Proof of Classical Analysis

In [9], by using the so-called Dialectica (or functional) interpretation together with the so-called negative translation, Gödel gave a consistency proof of Peano arithmetic (in fact, classical finite-type arithmetic) relative to a quantifier-free theory **T** of primitive recursive functionals of finite types. Later in [20], Spector extended Gödel's method to full second-order arithmetic (also called classical analysis). Their method consists of the following steps:

- firstly, by negative translation, one reduces a classical theory to the corresponding (semi-)intuitionistic theory,
- secondly, by the Dialectica interpretation, one reduces the (semi-)intuitionistic theory to the quantifier-free functional theory **T** (+ bar recursion).

Since classical finite-type arithmetic together with the schema of comprehension over numbers

$$\mathrm{CA}^0 : \exists \gamma \forall x\, (\gamma(x) = 0 \leftrightarrow A(x)),$$

contains full second-order arithmetic (in the sense of [19] via the identification of subsets of $\mathbb{N}$ with their characteristic functions) and CA^0 is classically derived from the countable choice schema

$$\mathrm{AC}^{0,0} : \ \forall x\, \exists y A(x,y) \to \exists \gamma \forall x A(x, \gamma(x))$$

(see e.g. [12, Section 11.1]), by Gödel's original work which proceeds by induction on the proof, it is sufficient to give an interpretation of the additional axiom $\mathrm{AC}^{0,0}$. One can easily see that the negative translation of $\mathrm{AC}^{0,0}$ is intuitionistically derived from $\mathrm{AC}^{0,0}$ together with the numerical double negation shift schema

$$\mathrm{DNS}^0 : \ \forall x \neg\neg \exists y A(x,y) \to \neg\neg \forall x\, \exists y A(x,y)$$

(see e.g. [12, Section 11.1]). Since $\mathrm{AC}^{0,0}$ (even full choice schema in all finite-types) is Dialectica interpretable in **T**, the only difficulty is in the interpretation of DNS^0. In fact, the crucial step in [20] was to give the witnessing terms of the Dialectica interpretation of DNS^0 by functionals in **T** extended by bar recursion.

See e.g. [2, 12] for more information.

Motivation for Σ_2^0-DNS^0: Consistency Proof of ACA_0

It is well-known in reverse mathematics (cf. [19]) that the schema of so-called arithmetical comprehension

$$\mathrm{CA}_{\mathrm{ar}}^0 : \exists \gamma \forall x\, (\gamma(x) = 0 \leftrightarrow A_{\mathrm{ar}}(x)),$$

where $A_{\mathrm{ar}}(x)$ is an arbitrary arithmetical formula with function parameters, suffices to carry out a substantial amount of ordinary mathematics. As explained in [12, Section 11.3], it is not hard to see that $\mathrm{CA}_{\mathrm{ar}}^0$ is classically derived from (in fact, equivalent over $\mathsf{EL}+\mathrm{LEM}$ to) the following fragment of the the countable choice schema

$$\Pi_1^0\text{-AC}: \ \forall \alpha\, (\forall x \exists y \forall z\, \alpha(x,y,z) = 0 \to \exists \gamma \forall x, z\, \alpha(x, \gamma(x), z) = 0).$$

In addition, as in the general case, one can straightforwardly see that the negative translation of Π_1^0-AC is intuitionistically derived from Π_1^0-AC and the Σ_2^0 fragment of DNS^0,

$$\Sigma_2^0\text{-DNS}^0 : \forall \alpha\, (\forall x \neg\neg \exists y \forall z\, \alpha(x,y,z) = 0 \to \neg\neg \forall x \exists y \forall z\, \alpha(x,y,z) = 0).$$

Thus Σ_2^0-DNS^0 is the key principle for the consistency proof of ACA_0 by the Gödel-Spector method, where ACA_0 is the subsystem of second-order arithmetic obtained by the schema of arithmetical comprehension (see [19] for details). It is known that witnessing terms of the Dialectica interpretation of Σ_2^0-DNS^0 can be given by functionals in $\mathbf{T}$ extended by bar recursion only for lowest types (see [12, Theorem 11.13]).

Motivation for Σ_1^0-DNS^0: Negative Translation of RCA_0

The most popular base system RCA_0 of (classical) reverse mathematics (cf. [19]) is known to be equivalent to the fragment EL_0 (in the sense of [4]) of elementary analysis EL augmented with the law of excluded middle schema LEM. (Note that EL_0 in [4] is different from that in [15]. In fact, full elementary analysis EL is a definitional extension of EL_0 in [15].) This is also the case for the system RCA, which includes full second-order induction schema, and elementary analysis EL (see [22, 1.9.10] and [4]) augmented with LEM. Then, following [4, 7], we write $\mathsf{EL}+\mathrm{LEM}$ and $\mathsf{EL}_0+\mathrm{LEM}$ as RCA and RCA_0 respectively. Note that EL and EL_0 contain the following weak fragment of quantifier-free axiom of choice

$$\text{QF-AC}^{0,0} : \forall \alpha\, (\forall x \exists y\, \alpha(x,y) = 0 \to \exists \gamma \forall x\, \alpha(x, \gamma(x)) = 0),$$

which is equivalent to QF-AC^{0,0} in [4]. Let us consider the soundness of negative translation for the classical systems RCA and RCA_0. As mentioned in the proof of [7, Lemma 11], the problematic point is only the verification of QF-AC^{0,0}. As in the proof of [12, Proposition 10.6], one can show that the negative translation $(\text{QF-AC}^{0,0})^{\mathrm{N}}$ of QF-AC^{0,0} is derived from QF-AC^{0,0} and the Σ_1^0 fragment of DNS^0,

$$\Sigma_1^0\text{-DNS}^0 : \forall \alpha\, (\forall x \neg\neg \exists y\, \alpha(x,y) = 0 \to \neg\neg \forall x \exists y\, \alpha(x,y) = 0).$$

Thus [7, Lemma 11] can be extended to the following:

Proposition 1. *If* RCA *(resp.* RCA_0*) proves* A*, then* $\mathsf{EL}+\Sigma_1^0$-DNS^0 *(resp.* $\mathsf{EL}_0+\Sigma_1^0$-DNS^0*) proves the negative translation* A^{N} *of* A.

As a corollary, we have the following conservation result:

Corollary 2. *For any $\exists$-free formula* A *(i.e.* A *does not contain* $\exists, \vee$*), if* RCA *(resp.* RCA_0*) proves* A*, then* $\mathsf{EL} + \Sigma^0_1\text{-DNS}^0$ *(resp.* $\mathsf{EL}_0 + \Sigma^0_1\text{-DNS}^0$*) proves* A.

On the other hand, $(\text{QF-AC}^{0,0})^{\text{N}}$ derives $\Sigma^0_1\text{-DNS}^0$. Thus $(\text{QF-AC}^{0,0})^{\text{N}}$ is equivalent to $\Sigma^0_1\text{-DNS}^0$ over EL_0.

Notation. As mentioned already, we basically use the lower-case letters $n, m, x, y, z, \ldots$ for type-0 variables and use the lower-case Greek letters $\alpha, \beta, \gamma \ldots$ for type-1 variables. For finite-type systems, we employ the type notation from [12]. Note that from now on we use superscripts on quantified variables to indicate their types. Throughout this paper, the word "primitive recursive" for functions over natural numbers is used in the ordinary sense unless otherwise stated. In addition, we employ the notation HA^ω in the sense of [22], while [23] uses HA^ω_0 to denote HA^ω in [22].

2 Fragments of Classical Logic

2.1 Arithmetical Hierarchy of Logical Principles

There are several known results, which are found in the literature[1, 10, 8, 13], on the arithmetical hierarchy of the logical principles. Although there are several motivations for studying the arithmetical hierarchy of the logical principles, one remarkable motivation comes from the intuitionistic version of reverse mathematics (cf. [19]), which investigates the relation between theorems and axioms in mathematics. The logical principles presented at the beginning of this paper are frequently used (sometimes implicitly) in mathematical proofs. However, they are usually refuted to use in so-called constructive mathematics (cf. [23]). On the other hand, it is well-known that a large amount of (Bishop's) constructive mathematics can be formalized over (many-sorted) intuitionistic arithmetic. Based on this fact, so-called constructive reverse mathematics (cf. [11]), which is reverse mathematics over (many-sorted) intuitionistic arithmetic, has been carried out. In fact, some mathematical statements have been established to be equivalent to some fragments of the logical principles over intuitionistic arithmetic ([11, 21, 3]). The study of fragments of the law of excluded middle schema is also relevant for the project of proof mining: if e.g. LEM is only used for Π^0_1-formulas (or - more generally - for existential-free or even for negated formulas) or - alternatively (but not together with it) - in the form of the combination of the Markov principle with the De Morgan law for Σ^0_1-formulas (LLPO), one still can - as in fully intuitionistic proofs - extract effective uniform bounds from proofs of statements of arbitrary logical complexity (see e.g. Chapters 7 and 9 in [12]). In particular, one can extract rates of convergence from proofs of the Cauchy property of a sequence. If LEM is applied to Σ^0_1-formulas, this in general fails, while one still learn a rate of convergence in the sense of limit computability (see e.g. [1]).

The following fragments (or their first-order counterparts) of the logical principles have been studied in the previous literature[1, 10, 8, 13]:

Σ^0_1-LEM: $\forall \alpha \forall x (\exists y\, \alpha(x,y) = 0 \vee \neg \exists y\, \alpha(x,y) = 0)$.

Π^0_1-LEM: $\forall \alpha \forall x (\forall y\, \alpha(x,y) = 0 \vee \neg \forall y\, \alpha(x,y) = 0)$.

Σ^0_1-DML: $\forall \alpha, \beta \forall x (\neg(\exists y\, \alpha(x,y) = 0 \wedge \exists z\, \beta(x,z) = 0) \to \neg \exists y\, \alpha(x,y) = 0 \vee \neg \exists z\, \beta(x,z) = 0)$.

Π^0_1-DML: $\forall \alpha, \beta \forall x (\neg(\forall y\, \alpha(x,y) = 0 \wedge \forall z\, \beta(x,z) = 0) \to \neg \forall y\, \alpha(x,y) = 0 \vee \neg \forall z\, \beta(x,z) = 0)$.

Σ^0_1-DNE (or MP): $\forall \alpha \forall x (\neg\neg \exists y\, \alpha(x,y) = 0 \to \exists y\, \alpha(x,y) = 0)$.

Δ_a-LEM: $\forall \alpha, \beta \forall x((\exists y\, \alpha(x,y) = 0 \leftrightarrow \neg \exists z\, \beta(x,z) = 0) \to \exists y\, \alpha(x,y) = 0 \vee \neg \exists y\, \alpha(x,y) = 0)$.

Δ_b-LEM: $\forall \alpha, \beta \forall x((\neg \exists y\, \alpha(x,y) = 0 \leftrightarrow \exists z\, \beta(x,z) = 0) \to \exists y\, \alpha(x,y) = 0 \vee \neg \exists y\, \alpha(x,y) = 0)$.

Remark 3. *In this paper, we treat the logical principles over many-sorted intuitionistic arithmetic (EL or HA^ω) in parallel with their first-order counterparts, which are studied in Akama et al.[1], over first-order arithmetic* HA. *Since each quantifier-free formula $\mathrm{A}_{\mathrm{qf}}[\underline{x}]$ (in $\mathscr{L}(\mathrm{HA})$) containing only $\underline{x}$ as free variables is (provably in* HA*) equivalent to $t(\underline{x}) = 0$ for some closed primitive recursive t (in the ordinary sense, not in the extended sense of Gödel's* **T***), the first-order counterparts of the principles are equivalent to restricting the second-order principles to all instances where α, β are closed primitive recursive function terms. It is for this reason that always the number parameters "$\forall x$" are inserted after "$\forall \alpha$" or "$\forall \alpha, \beta$" although they are obviously redundant in the second-order formulations (cf. Remark 5). In particular, for the logical principles above, their function-parameter-free fragments (i.e. the restriction to primitive recursive function terms containing at most number-theoretic parameters) and their closed function-parameter-free fragments are the same and we simply say "function-parameter-free fragment" and denote this with a superscript "$-$" in the following. Moreover, in the presence of Church's thesis* CT *in Section 4 the full second-order versions are equivalent to the first-order counterparts.*

Remark 4. *As mentioned in [8], there are several (classically equivalent) ways to define a formula being Δ^0_1 constructively. In this paper, we consider the following two variations:*

(a) $\alpha \in \Delta_a :\equiv \exists \beta \big(\exists x\, \alpha(x) = 0 \leftrightarrow \neg \exists x\, \beta(x) = 0\big)$.

(b) $\alpha \in \Delta_b :\equiv \exists \beta \big(\neg \exists x\, \alpha(x) = 0 \leftrightarrow \exists x\, \beta(x) = 0\big)$.

Note that Δ^0_1-LEM *in [1, 16, 13] is defined in the sense of (a).*

Remark 5. Σ^0_1-LEM, Π^0_1-LEM, Σ^0_1-DML *and* Π^0_1-DML *are equivalent (over e.g.* EL*) to the principles denoted by* LPO, WLPO, LLPO *and* $\mathrm{MP}^\vee$ *respectively in Ishihara's constructive reverse mathematics (cf. [11]). In addition,* Δ_a-LEM *is equivalent (over e.g.* EL*) to so-called* Δ^0_1-LEM *in [13]*:

$$\forall \alpha, \beta (\forall x(\exists n\, \alpha(x,n) = 0 \leftrightarrow \forall n\, \beta(x,n) = 0) \to \forall x(\exists n\, \alpha(x,n) = 0 \vee \neg \exists n\, \alpha(x,n) = 0)),$$

and also to so-called IIIa *in [8]*:

$$\forall \alpha, \beta ((\exists n\, \alpha(n) \neq 0 \leftrightarrow \neg \exists n\, \beta(n) \neq 0) \to \exists n\, \alpha(n) \neq 0 \vee \neg \exists n\, \alpha(n) \neq 0).$$

Note that the definition of Δ^0_1-LEM *in [13] contains a typo. In addition, the use of non-equality in [10, 8] instead of equality is the convention from constructive mathematics, but they are equivalent over standard intuitionistic arithmetic. A similar discussion for* Σ^0_1-DNE*, which is called "Markov's principle* (MP)*" following constructive mathematics, can be found in [18].*

Historically, Ishihara[10] studied Π^0_1-DML and showed that Π^0_1-DML is derivable from either Σ^0_1-DNE or Σ^0_1-DML (in another terminology). Independently, Akama et al.[1] systematically studied an arithmetical hierarchy of logical principles over first-order intuitionistic arithmetic and showed, among other things, that Δ_a-LEM is derivable from either Σ^0_1-DNE or Σ^0_1-DML (in another terminology). Recently, Fujiwara et al.[8] carried out a systematic study of weak logical principles and showed, among other things, that Π^0_1-DML implies Δ_a-LEM. On the other hand, Kohlenbach[13] showed that the converse is not the case. In the end, the existing

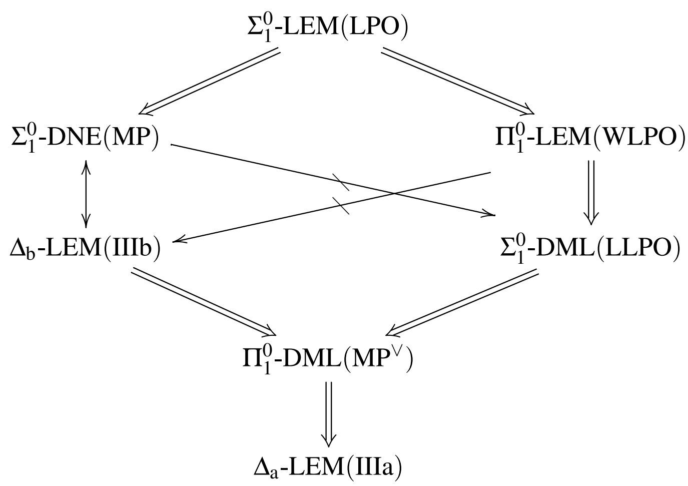

Figure 1: Arithmetical hierarchy of logical principles over EL or HA [1, 10, 8, 13]

results are summarized in Figure 1, where each double arrow $\Rightarrow$ expresses that the converse direction does not hold. Note that the following picture holds over both many-sorted arithmetic and first-order arithmetic respectively.

However, as mentioned in [1, Section 4], the entire hierarchy inside constructivism is still to be discovered.

2.2 Double Negation Shift Scheme and Doubly Negated Variants of Logical Principles

Fact 6 (See e.g. [14])**.** *Over intuitionistic predicate logic* IQC*, the double negation shift schema:*

$$\forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$$

is equivalent to the doubly negated version of the law of excluded middle schema:

$$\neg\neg\forall x(\varphi(x) \vee \neg\varphi(x)).$$

The implication from the latter to the former holds even for each fixed formula $\varphi(x)$. However, to obtain the doubly negated law of excluded middle for $\varphi(x)$ one has to apply the double negation shift to $\forall x \neg\neg(\varphi(x) \vee \neg\varphi(x))$.

Of course, these proofs can be simulated in the setting of (many-sorted) intuitionistic arithmetic. On one hand, as in the case of pure logic, one can see that $\neg\neg\Sigma^0_1$-LEM implies Σ^0_1-DNS^0. On the other hand, since $A \vee \neg A$ is equivalent to

$$\exists k((k = 0 \to A) \wedge (k \neq 0 \to \neg A)),$$

the doubly negated variant of Σ^0_1-LEM,

$$\neg\neg\Sigma^0_1\text{-LEM} : \forall \alpha \neg\neg\forall x(\exists y\, \alpha(x,y) = 0 \vee \neg\exists y\, \alpha(x,y) = 0)$$

follows from Σ^0_2-DNS^0 (see Proposition 9 below). Thus, if one takes the complexity of formulas into account over (many-sorted) intuitionistic arithmetic, the proof of Fact 6 does not give us the equivalence in each level of arithmetical hierarchy anymore. Furthermore, since Markov's principle MP (which implies Σ^0_1-DNS^0) is Dialectica interpretable in Gödel's **T** but Σ^0_2-DNS^0 is not, Σ^0_1-DNS^0 does not imply Σ^0_2-DNS^0. Therefore, a natural question arising from this observation is how the interrelation between Σ^0_2-DNS^0, Σ^0_1-DNS^0 and the doubly negated variant of Σ^0_1-LEM as well as the other doubly negated variants of the logical principles in Figure 1 looks like.

Moreover, the investigation of such weak logical principles may have importance from a viewpoint of the foundation of constructive mathematics. Although it is our common view that constructive mathematics can be seen "roughly" as mathematics based on intuitionistic logic, it is still debatable whether constructivism can be captured "exactly" by intuitionistic logic. The investigation of weak logical principles will help to develop more detailed discussion for the axiomatization of constructive mathematics.

The following is the list of the doubly negated principles studied in this paper:

$\neg\neg\Sigma^0_1$-LEM: $\forall\alpha\neg\neg\forall x(\exists y\,\alpha(x,y)=0\vee\neg\exists y\,\alpha(x,y)=0)$.

$\neg\neg\Pi^0_1$-LEM: $\forall\alpha\neg\neg\forall x(\forall y\,\alpha(x,y)=0\vee\neg\forall y\,\alpha(x,y)=0)$.

$\neg\neg\Sigma^0_1$-DML: $\forall\alpha,\beta\neg\neg\forall x(\neg(\exists y\,\alpha(x,y)=0\wedge\exists z\,\beta(x,z)=0)\to\neg\exists y\,\alpha(x,y)=0\vee\neg\exists z\,\beta(x,z)=0)$.

$\neg\neg\Pi^0_1$-DML: $\forall\alpha,\beta\neg\neg\forall x(\neg(\forall y\,\alpha(x,y)=0\wedge\forall z\,\beta(x,z)=0)\to\neg\forall y\,\alpha(x,y)=0\vee\neg\forall z\,\beta(x,z)=0)$.

$\neg\neg\Sigma^0_1$-DNE (or $\neg\neg$MP): $\forall\alpha\neg\neg\forall x(\neg\neg\exists y\,\alpha(x,y)=0\to\exists y\,\alpha(x,y)=0)$.

$\neg\neg\Delta_{\mathrm{a}}$-LEM: $\forall\alpha,\beta\neg\neg\forall x((\exists y\,\alpha(x,y)=0\leftrightarrow\neg\exists z\,\beta(x,z)=0)\to\exists y\,\alpha(x,y)=0\vee\neg\exists y\,\alpha(x,y)=0)$.

$\neg\neg\Delta_{\mathrm{b}}$-LEM: $\forall\alpha,\beta\neg\neg\forall x((\neg\exists y\,\alpha(x,y)=0\leftrightarrow\exists z\,\beta(x,z)=0)\to\exists y\,\alpha(x,y)=0\vee\neg\exists y\,\alpha(x,y)=0)$.

Some short discussions on the doubly negated principles can be found in [16]. In addition, Ščedrov and Vesley [18] study Σ^0_1-DNS^0 in connection with Markov's principle MP in some different context.

Remark 7. *In the following, we also consider the closed function-parameter-free fragments in the sense of Remark 3 of the above principles. For example, the closed function-parameter-free fragment* $\neg\neg\Sigma^0_1$-LEM$^-$ *of* $\neg\neg\Sigma^0_1$-LEM *is the following*:

$$\neg\neg\forall x(\exists y\,t(x,y)=0\vee\neg\exists y\,t(x,y)=0),$$

where t is a closed primitive recursive function term. To allow for number parameters in the function-parameter-free fragments of the double negated principles above, i.e. to consider the seemingly more general "$\forall z\neg\neg\forall x$"-form (with number parameters z) is in fact not stronger as it intuitionistically already follows from the "$\neg\neg\forall z,x$"-form which is (coding z,x into a single variable) an instance of our form without "$\forall z$". Therefore, as in Remark 3, the function-parameter-free doubly negated principles above are equivalent to their closed function-parameter-free fragments (so we simply say "function-parameter-free fragment" and denote this with a superscript "$-$" as for the principles without double negations) and - by the same reasoning as in Remark 3 - in turn equivalent to their first-order counterparts. In the presence of CT, *even the second-order versions are equivalent to the first-order counterparts.*

Remark 8. *In contrast to all other principles treated in this paper, for our fragments of* DNS^0, *it makes a difference whether one only has closed instances or allows number parameters to function-parameter-free fragments. On the other hand, the full* DNS^0 *allowing number parameters already follows from the parameter-free version of* DNS^0 *via Fact 6 (in the language of arithmetic) and the discussion in Remark 7.*

3 Derivability Results

To state our results in generality, we do not employ particular intuitionistic arithmetic for the derivability results in this section.

Proposition 9. Σ^0_2-DNS^0 *implies* $\neg\neg\Sigma^0_1$-LEM.

Proof. Fix α. Note that

$$\forall x \neg\neg(\exists y\, \alpha(x,y) = 0 \vee \neg\exists y\, \alpha(x,y) = 0) \tag{1}$$

is intuitionistically provable. Since

$$\begin{aligned}(1) \quad &\Leftrightarrow \quad \forall x \neg\neg\exists m((m = 0 \to \exists y\, \alpha(x,y) = 0) \wedge (m \neq 0 \to \forall z\, \alpha(x,z) \neq 0)) \\ &\Leftrightarrow \quad \forall x \neg\neg\exists m, y \forall z((m = 0 \to \alpha(x,y) = 0) \wedge (m \neq 0 \to \alpha(x,z) \neq 0)),\end{aligned}$$

by applying Σ^0_2-DNS^0, we have

$$\neg\neg\forall x(\exists y\, \alpha(x,y) = 0 \vee \neg\exists y\, \alpha(x,y) \neq 0).$$

□

Proposition 10. *1.* $\neg\neg\Sigma^0_1$-LEM *is equivalent to* $\neg\neg\Sigma^0_1$-DNE $+$ $\neg\neg\Pi^0_1$-LEM.

2. $\neg\neg\Pi^0_1$-LEM *implies* $\neg\neg\Sigma^0_1$-DML.

3. $\neg\neg\Sigma^0_1$-DML *implies* $\neg\neg\Pi^0_1$-DML.

4. $\neg\neg\Sigma^0_1$-DNE *implies* $\neg\neg\Pi^0_1$-DML.

5. $\neg\neg\Pi^0_1$-DML *implies* $\neg\neg\Delta_a$-LEM.

6. $\neg\neg\Sigma^0_1$-DNE *is equivalent to* $\neg\neg\Delta_b$-LEM.

7. $\neg\neg\Sigma^0_1$-DNE *implies* Σ^0_1-DNS^0.

Proof. The proofs for the versions without double negations work. See [1, Theorem 3.1] for (1) and (2). See [10, Proposition 1] for (3) and (4). See [8, Proposition 4] and [8, Proposition 1] for (5) and (6) respectively. (7) is straightforward.

Note that the notation Σ^0_1-LLPO is used for the first-order counterpart of Σ^0_1-DML in [1], the notations MP, SEP and MP$_\vee$ are used for (the equivalents of) Σ^0_1-DNE, Σ^0_1-DML and Π^0_1-DML respectively in [10], and the notations MP, MP$_\vee$, IIIa and IIIb are used for (the equivalents of) Σ^0_1-DNE, Π^0_1-DML, Δ_a-LEM and Δ_b-LEM in [8]. □

Remark 11. *The proofs can be formalized in standard intuitionistic arithmetic with two-sorted variables like* EL *or many-sorted variables as in* HA^ω *in [22]. In fact, all of our derivability results hold even in a stronger sense. Let us write the instance of each principle* $\mathrm{P_i}$ *for a particular* α *(possibly a tuple) as* $\mathrm{P_i}(\alpha)$*. Then, for each derivability result of* $\mathrm{P_2}$ *from* $\mathrm{P_1}$*, one can notice that* HA^ω *(in the sense of [22]) proves*

$$\mathrm{P_1}(\mathrm{t}\alpha) \to \mathrm{P_2}(\alpha)$$

for some closed term $t^{1\to 1}$ in $\mathbf{T_0}$, which is the subset of $\mathbf{T}$ involving only type-0 recursor R_0 (see [12, Chapter 3] for details.). Therefore, since type-1 closed terms in $\mathbf{T_0}$ are just primitive recursive functions over natural numbers and each first-order counterpart is equivalent to the corresponding function-parameter-free fragment (see Remark 3), it follows that all of the derivability results for the first-order counterparts in the sense of [1] also hold over first-order (Heyting) arithmetic using that HA^ω is conservative over HA (see [22, 3.6.2]).

4 Underivability Results

As the system for the underivability results, we employ intuitionistic finite-type arithmetic (with extensionality) HA^ω ($\mathsf{E\text{-}HA}^\omega$) and elementary analysis EL (see [22, 12, 4]). We first recall the following schemata (which may contain parameters of any finite types) in finite-type arithmetic:

AC: $\forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\rho\to\tau} \forall x^\rho A(x, Y(x))$.

$\mathrm{IP}^\omega_{\mathrm{ef}}$: $(A_{\mathrm{ef}} \to \exists x^\rho B(x)) \to \exists x^\rho (A_{\mathrm{ef}} \to B(x^\rho))$, where A_{ef} is $\exists$-free (i.e. A_{ef} does not contain $\exists, \vee$) and does not contain x free.

$\mathrm{IP}^\omega_\forall$: $\left(\forall u^\tau A_{\mathrm{qf}}(u) \to \exists x^\rho B(x)\right) \to \exists x^\rho \left(\forall u^\tau A_{\mathrm{qf}}(u) \to B(x)\right)$, where A_{qf} is quantifier-free and does not contain x free.

M^ω: $\neg\neg\exists x^\rho A_{\mathrm{qf}}(x) \to \exists x^\rho A_{\mathrm{qf}}(x)$, where $A_{\mathrm{qf}}(x)$ is a quantifier-free formula.

The following principle, which states that every function is recursive, is in constructive mathematics sometimes called "Church's thesis".

CT: $\forall \alpha \exists e \forall x \exists u (T(e,x,u) \wedge \alpha(x) = U(u))$ (see [22, 1.11.7]).

We also recall the principle called "extended Church's thesis".

ECT_0: $\forall x(A \to \exists y B(x,y)) \to \exists z \forall x (A \to \exists u(T(z,x,u) \wedge B(x, U(u))))$, where A is almost negative (see [22, 3.2.14]).

ECT_0^+: ECT_0 with function parameters.

While extended Church's thesis ECT_0 is usually defined in the language of HA, we also treat it in elementary analysis EL which is a second-order system. Note that each instance of ECT_0 and ECT_0^+ does not contain function quantifiers in our setting.

In some proofs, we use the well-known property of Goodstein sequences that Peano arithmetic PA does not prove that every Goodstein sequence terminates.

Theorem 12. $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Pi^0_1\text{-LEM} \nvdash \Sigma^0_1\text{-DNS}^0$.

Proof. Let $\forall n^0 \exists m^0 G_{\mathrm{qf}}(n,m)$ be a sentence which states that for all n, the Goodstein sequence starting from n terminates. Note that $G_{\mathrm{qf}}(n,m)$ is quantifier-free, and hence, it can be represented as $t(n,m) = 0$ for a primitive recursive (in the ordinary sense) t. Suppose that $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Pi^0_1\text{-LEM}$ proves $\Sigma^0_1\text{-DNS}^0$. Then one can show that

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Pi^0_1\text{-LEM} + \forall n \neg\neg\exists m\, G_{\mathrm{qf}}(n,m) + \neg\exists f \forall n\, G_{\mathrm{qf}}(n, f(n)) \vdash \bot$$

since $\forall n\neg\neg\exists m\, G_{\mathrm{qf}}(n,m)$ implies $\neg\neg\exists f\forall n\, G_{\mathrm{qf}}(n,f(n))$ in the presence of Σ^0_1-DNS^0 and AC. Note that $\forall n\neg\neg\exists m$ $G_{\mathrm{qf}}(n,m)$ and $\neg\exists f\forall n\, G_{\mathrm{qf}}(n,f(n))$ are intuitionistically equivalent to $\exists$-free sentences $\forall n\neg\forall m\neg G_{\mathrm{qf}}(n,m)$ and $\forall f\neg\forall n\, G_{\mathrm{qf}}(n,f(n))$ respectively, and also that Π^0_1-LEM is derivable from

$$(\theta): \exists v^{1\to 0} \leq \lambda\beta.1\forall\alpha^1((v(\alpha)=0 \to \forall x\,\alpha(x)=0)\wedge(v(\alpha)\neq 0 \to \neg\forall x\,\alpha(x)=0)).$$

Since (θ) is a sentence of the form $\exists v \leq r\,\mathrm{B}_{\mathrm{ef}}(v)$ where B_{ef} is $\exists$-free and r is a closed term, by the monotone modified realizability interpretation [12, Theorem 7.1(1)], we have that

$$\mathsf{E\text{-}HA}^\omega + (\theta) + \forall n\neg\neg\exists m\, G_{\mathrm{qf}}(n,m) + \neg\exists f\forall n\, G_{\mathrm{qf}}(n,f(n)) \vdash \perp .$$

Now it suffices to show that this system is consistent. Let $\mathscr{S}^\omega$ be the full set-theoretic model (cf. [12, Section 3.6]). Take the functional $g_v \in \{0,1\}^{\mathbb{N}^\mathbb{N}}$ in $\mathscr{S}^\omega$ defined by

$$\forall\alpha \in \mathbb{N}^\mathbb{N}((g_v(\alpha)=0 \to \forall x\,\alpha(x)=0)\wedge(g_v(\alpha)\neq 0 \to \neg\forall x\,\alpha(x)=0)).$$

Let $\mathscr{M}$ be the set of functionals in $\mathscr{S}^\omega$ built up from $g_v \in \{0,1\}^{\mathbb{N}^\mathbb{N}}$ and the functionals that interpret closed terms of Gödel's $\mathbf{T}$. Note that $\forall n\exists m\, G_{\mathrm{qf}}(n,m)$ is true in $\mathscr{M}$ and that $\exists f\forall n\, G_{\mathrm{qf}}(n,f(n))$ is false in $\mathscr{M}$: if $\exists f\forall n\, G_{\mathrm{qf}}(n,f(n))$ would be true in $\mathscr{M}$, since all functionals in $\mathscr{M}$ are majorized by some functional in $\mathbf{T}$ (cf. [12, Proposition 6.6]), one can search primitive recursively (in the sense of $\mathbf{T}$) for the terminating point for each Goodstein sequence, which is a contradiction. Therefore, $\mathscr{M}$ is a model of $\mathsf{N\text{-}HA}^\omega + (\theta) + \forall n\neg\neg\exists m\, G_{\mathrm{qf}}(n,m) + \neg\exists f\forall n\, G_{\mathrm{qf}}(n,f(n))$ (see [22, 1.6.3-1.6.7] for the "neutral" system $\mathsf{N\text{-}HA}^\omega$). Construct $\mathscr{M}^E$ as the extensional collapse of $\mathscr{M}$ as in the proof of [22, 2.4.5]. Note that the sets of functionals of type 0 and 1 in $\mathscr{M}^E$ are identical with those for $\mathscr{M}$ respectively. Then one can easily see that g_v is in $\mathscr{M}^E$, and hence, $\mathscr{M}^E$ is a model of $\mathsf{E\text{-}HA}^\omega + (\theta) + \forall n\neg\neg\exists m\, G_{\mathrm{qf}}(n,m) + \neg\exists f\forall n\, G_{\mathrm{qf}}(n,f(n))$. □

Corollary 13. $\mathsf{EL} + \mathrm{AC}^{0,0} + \neg\neg\Pi^0_1$-LEM $\nvdash \Sigma^0_1$-DNS^0, *where* $\mathrm{AC}^{0,0}$ *is the countable choice schema from the Introduction.*

Remark 14. *In fact, the proof of Theorem 12 shows that* $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Pi^0_1$-LEM *does not prove even the closed function-parameter-free fragment of* Σ^0_1-DNS^0 :

$$\forall x\neg\neg\exists y\, t(x,y)=0 \to \neg\neg\forall x\exists y\, t(x,y)=0 \ (t:\ \textit{primitive recursive}).$$

Therefore it follows that $\mathsf{HA} + \Pi^0_1$-LEM *(in the sense of [1]) does not prove the "closed" (note Remark 8) first-order counterpart of* Σ^0_1-DNS^0 :

$$\forall x\neg\neg\exists y \mathrm{A}_{\mathrm{qf}} \to \neg\neg\forall x\exists y\, \mathrm{A}_{\mathrm{qf}},$$

where A_{qf} *is a quantifier-free* $\mathscr{L}(\mathsf{HA})$*-formula containing only x and y as its free variables.*

Next we show the underivability results for $\neg\neg\Pi^0_1$-DML.

Theorem 15. $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Sigma^0_1$-DNS$^0 \nvdash \neg\neg\Pi^0_1$-DML.

Proof. Suppose that $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Sigma^0_1\text{-DNS}^0 \vdash \neg\neg\Pi^0_1\text{-DML}$. Then one can see that $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Sigma^0_1\text{-DNS}^0$ proves

$$\neg\neg\exists f \forall x (\neg(\forall y\, s(x,y) = 0 \wedge \forall z\, t(x,z) = 0) \to ((f(x) = 0 \to \neg\forall y\, s(x,y) = 0) \wedge (f(x) \neq 0 \to \neg\forall z\, t(x,z) = 0))), \tag{2}$$

where s,t are primitive recursive functions defined in the proof of [13, Lemma 6]. Note that (2) is equivalent to some $\exists$-free sentence, and hence the modified realizability interpretation of (2) is intuitionistically equivalent to (2) itself (see [12, Remark 5.3(1)]). Since the modified realizability interpretation of $\Sigma^0_1\text{-DNS}^0$ is intuitionistically equivalent to

$$\forall \alpha (\forall x \neg\neg\exists y\, \alpha(x,y) = 0 \to \neg\neg\exists Y \forall x\, \alpha(x, Y(x)) = 0),$$

it is derived from $\Sigma^0_1\text{-DNS}^0 + \mathrm{QF\text{-}AC}^{0,0}$. Therefore, by [22, Theorem 3.4.5] (see also [12, Theorem 5.8]), we have that

$$\mathsf{E\text{-}HA}^\omega + \mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-DNS}^0 \vdash (2).$$

This is a contradiction since HEO (see [22, 2.4.11]) is a model of $\mathsf{E\text{-}HA}^\omega + \mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-DNS}^0$ but (2) is not true in HEO as shown in [13, Lemma 6]. □

Corollary 16. $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \Delta_{\mathrm{a}}\text{-LEM} + \Sigma^0_1\text{-DNS}^0 \nvdash \neg\neg\Pi^0_1\text{-DML}$.

Proof. Immediate from Theorem 15 and the fact that $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}}$ proves $\Delta_{\mathrm{a}}\text{-LEM}$ (cf. [13, Proposition 1]). □

Corollary 17. $\mathsf{EL} + \mathrm{AC}^{0,0} + \neg\neg\Delta_{\mathrm{a}}\text{-LEM} + \Sigma^0_1\text{-DNS}^0 \nvdash \neg\neg\Pi^0_1\text{-DML}$.

Remark 18. *In fact, the proof of Theorem 15 shows that* $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Sigma^0_1\text{-DNS}^0 \nvdash \neg\neg\Pi^0_1\text{-DML}^-$, *where* $\neg\neg\Pi^0_1\text{-DML}^-$ *is the function-parameter-free fragment of* $\neg\neg\Pi^0_1\text{-DML}$ *(in the sense of Remark 7). Then it follows from Remark 7 (and the proof of Corollary 16) that the first-order counterpart of* $\neg\neg\Pi^0_1\text{-DML}$ *:*

$$\neg\neg\forall x (\neg(\forall y\, \mathrm{A}_{\mathrm{qf}} \wedge \forall z\, \mathrm{B}_{\mathrm{qf}}) \to \neg\forall y\, \mathrm{A}_{\mathrm{qf}} \vee \neg\forall z\, \mathrm{B}_{\mathrm{qf}}),$$

where A_{qf} *and* B_{qf} *are quantifier-free* $\mathscr{L}(\mathsf{HA})$*-formulas, is not provable in* $\mathsf{HA} + \Delta_{\mathrm{a}}\text{-LEM} + \Sigma^0_1\text{-DNS}^0$ *(in the language of* HA*).*

In the following, we show the underivability results for $\neg\neg\Delta_{\mathrm{a}}\text{-LEM}$ with the use of (the proof of) the underivability results for $\neg\neg\Pi^0_1\text{-DML}$. The following decomposition is helpful in the proofs.

Lemma 19. $\mathsf{EL} \vdash \mathrm{ECT}^+_0 \leftrightarrow \mathrm{ECT}_0 + \mathrm{CT}$. *Thus each instance of* ECT^+_0 *is provable in* $\mathsf{EL} + \mathrm{CT} + \mathrm{ECT}_0$ *(in the language of* HA*).*

Proof. See [6, Lemma 6.3.7]. □

Before we come to the underivability results for $\neg\neg\Delta_{\mathrm{a}}\text{-LEM}$, we first provide an underivability result for $\Delta_{\mathrm{a}}\text{-LEM}$. In fact, the key idea to obtain the underivability results for $\neg\neg\Delta_{\mathrm{a}}\text{-LEM}$, which is originally from [8], appears in the proof of the following theorem:

Theorem 20. $\mathsf{EL} + \mathrm{ECT}^+_0 + \neg\neg\Sigma^0_1\text{-DNE} \nvdash \Delta_{\mathrm{a}}\text{-LEM}$.

Proof. Suppose that $\mathsf{EL} + \mathrm{ECT}_0^+ + \neg\neg\Sigma_1^0\text{-DNE} \vdash \Delta_a\text{-LEM}$. Then, by [8, Proposition 9] and [10, Proposition 1(1), Proposition 2], we have that $\mathsf{EL} + \mathrm{ECT}_0^+ + \neg\neg\Sigma_1^0\text{-DNE}$ proves Σ_1^0-DNE (MP), and hence, $\forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z))$. Since $\neg\neg\Sigma_1^0$-DNE is equivalent to its function-parameter-free form $\neg\neg\Sigma_1^0\text{-DNE}^-$:

$$\neg\neg\forall x(\neg\neg\exists y\, s(x,y) = 0 \to \exists y\, s(x,y) = 0) \ (s : \text{primitive recursive})$$

in the presence of CT (see also Remark 7), by Lemma 19, we have that

$$\mathsf{EL} + \mathrm{CT} + \mathrm{ECT}_0 + \neg\neg\Sigma_1^0\text{-DNE}^- \vdash \forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z)).$$

Since $\mathsf{EL} + \mathrm{CT}$ is conservative over HA for $\mathscr{L}(\mathsf{HA})$-formulas (cf. [22, Theorem 3.6.2]), we have that

$$\mathsf{HA} + \mathrm{ECT}_0 + \neg\neg\Sigma_1^0\text{-DNE}^- \vdash \forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z)).$$

Note that the instances of $\neg\neg\Sigma_1^0\text{-DNE}^-$ and $\forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z))$ are almost negative. Then, by using Kleene realizability as in [8, Remark 6], we have that

$$\mathsf{HA} + \neg\neg\Sigma_1^0\text{-DNE}^- \vdash \forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z)).$$

Since $\neg\neg\Sigma_1^0\text{-DNE}^-$ is modified realizable in Gödel's $\mathbf{T}$ (in fact, the realizer of $\neg\neg\Sigma_1^0\text{-DNE}^-$ is empty) verifiably in $\mathscr{S}^\omega$, it follows that $\forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z))$ is modified realizable in $\mathbf{T}$ verifiably in $\mathscr{S}^\omega$, which is a contradiction as any *mr*-realizer could be used to decide the special halting problem $\{x : \exists z T(x,x,z)\}$. $\square$

Remark 21. *By a careful inspection, one can even show that* $\mathsf{EL} + \mathrm{ECT}_0^+ + \neg\neg\Sigma_1^0\text{-DNE} \nvdash \Delta_a\text{-LEM}^-$, *where* $\Delta_a\text{-LEM}^-$ *is the function-parameter-free fragment of* $\Delta_a\text{-LEM}^-$:

$$\forall x((\exists y\, s(x,y) = 0 \leftrightarrow \forall z\, t(x,z) = 0) \to \exists y\, s(x,y) = 0 \vee \neg\exists y\, s(x,y) = 0) \ (s,t : \textit{primitive recursive}).$$

At first, one can observe that the proof of [8, Proposition 9] shows that $\mathsf{EL} + \mathrm{ECT}_0^+ + \Delta_a\text{-LEM}^-$ *proves* $\Pi_1^0\text{-DML}^-$, *where* $\Pi_1^0\text{-DML}^-$ *is the function-parameter-free fragment of* Π_1^0-DML :

$$\forall x(\neg(\forall y\, s(x,y) = 0 \wedge \forall z\, t(x,z) = 0) \to \neg\forall y\, s(x,y) = 0 \vee \neg\forall z\, t(x,z) = 0) \ (s,t : \textit{primitive recursive}).$$

On the other hand, the proof of [10, Proposition 2] shows that $\mathsf{EL} + \mathrm{ECT}_0^+$ *proves, more than* WMP *(Weak Markov's principle [10]), even a stronger variant* WMP^+ :

$$\forall \alpha(\forall x(\neg\neg\exists n T^*(x,x,n) \vee \neg\neg\exists n(\alpha(n) \neq 0 \wedge \neg T^*(x,x,n))) \to \exists x\, \alpha(x) \neq 0),$$

and the proof of [10, Proposition 1(1)] shows that $\mathsf{EL} + \mathrm{WMP}^+ + \Pi_1^0\text{-DML}^-$ *proves the function-parameter-free fragment of* MP :

$$\forall x(\neg\neg\exists y\, s(x,y) = 0 \to \exists y\, s(x,y) = 0) \ (s : \textit{primitive recursive}).$$

Therefore, as long as $\mathsf{EL} + \mathrm{ECT}_0^+ + \neg\neg\Sigma_1^0\text{-DNE}$ *proves just* $\Delta_a\text{-LEM}^-$, *it follows that*

$$\mathsf{HA} + \neg\neg\Sigma_1^0\text{-DNE}^- \vdash \forall x(\neg\neg\exists z T(x,x,z) \to \exists z T(x,x,z)),$$

which is a contradiction from the proof of Theorem 20.

By the above argument, it follows that the first-order Δ_a-LEM *in [1] is not provable in* $\mathsf{HA} + \mathrm{ECT}_0$ *augmented with the first-order counterpart of* $\neg\neg\Sigma_1^0$-DNE.

On the other hand, the proof of [22, Theorem 3.4.14] shows that $\mathsf{HA} + \mathrm{ECT}_0 + \neg\neg\Sigma_1^0\text{-LEM}$ *(in the language of* HA*) is inconsistent. In this sense, our underivability result is optimal.*

Theorem 22. $\mathsf{EL}+\mathrm{ECT}_0^+ + \Sigma_1^0\text{-DNS}^0 \nvdash \neg\neg\Delta_{\mathrm{a}}\text{-LEM}$.

Proof. Suppose that $\mathsf{EL}+\mathrm{ECT}_0^+ + \Sigma_1^0\text{-DNS}^0 \vdash \neg\neg\Delta_{\mathrm{a}}\text{-LEM}$. Then, as in the proof of [8, Proposition 9], one can show that $\mathsf{EL}+\mathrm{ECT}_0^+ + \Sigma_1^0\text{-DNS}^0$ proves $\neg\neg\Pi_1^0\text{-DML}$, in particular,

$$\neg\neg\forall x(\neg(\forall y\, s(x,y)=0 \wedge \forall z\, t(x,z)=0) \to \neg\forall y\, s(x,y)=0 \vee \neg\forall z\, t(x,z)=0), \tag{3}$$

where s,t are the primitive recursive functions defined in the proof of [13, Lemma 6]. Since $\Sigma_1^0\text{-DNS}^0$ is of almost negative form and (3) is intuitionistically derived from its (Kleene) realizability interpretation, as in the proof of Theorem 20, one can drop ECT_0^+ and obtain $\mathsf{E\text{-}HA}^\omega + \Sigma_1^0\text{-DNS}^0 \vdash (3)$. Then we have that

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}_{\mathrm{ef}}^\omega + \Sigma_1^0\text{-DNS}^0 \vdash (2) \text{ in the proof of Theorem 15,}$$

which is a contradiction as shown in (the proof of) Theorem 15. □

Corollary 23. $\mathsf{EL}+\mathrm{AC}^{0,0} + \Sigma_1^0\text{-DNS}^0 \nvdash \neg\neg\Delta_{\mathrm{a}}\text{-LEM}$.

Proof. By Theorem 22, it suffices to show that $\mathrm{AC}^{0,0}$ is provable in in $\mathsf{EL}+\mathrm{ECT}_0^+$. One can easily see that $\mathsf{EL}+\mathrm{ECT}_0^+$ proves the countable choice for all arithmetical formulas. On the other hand, it is straightforward to show that each second-order formula is equivalent to some arithmetical formula in the presence of CT and $\mathrm{QF\text{-}AC}^{0,0}$ (which is contained in EL). Therefore, by Lemma 19, the full countable choice schema $\mathrm{AC}^{0,0}$ is provable in in $\mathsf{EL}+\mathrm{ECT}_0^+$. □

Remark 24. *By inspecting the proofs of [8, Proposition 9], we have that*

$$\mathsf{EL}+\mathrm{ECT}_0^+ + \neg\neg\Delta_{\mathrm{a}}\text{-LEM}^- \vdash \neg\neg\Pi_1^0\text{-DML}^-,$$

where $\neg\neg\Delta_{\mathrm{a}}\text{-LEM}^-$ *and* $\neg\neg\Pi_1^0\text{-DML}^-$ *are function-parameter-free fragments (in the sense of Remark 7) of* $\neg\neg\Delta_{\mathrm{a}}\text{-LEM}$ *and* $\neg\neg\Pi_1^0\text{-DML}$ *respectively. Therefore, by the proof of Theorem 22, we have that*

$$\mathsf{EL}+\mathrm{ECT}_0^+ + \Sigma_1^0\text{-DNS}^0 \nvdash \neg\neg\Delta_{\mathrm{a}}\text{-LEM}^-.$$

Since $\Sigma_1^0\text{-DNS}^0$ *is equivalent to its function-parameters-free fragments with number parameters*:

$$\forall z(\forall x\neg\neg\exists y\, t(x,y,z)=0 \to \neg\neg\ \forall x\exists y\, t(x,y,z)=0)\ (t: \textit{primitive recursive})$$

in the presence of CT *(see Remark 7 and Remark 39 below), it follows (using Lemma 19) that the first-order counterpart of* $\neg\neg\Delta_{\mathrm{a}}\text{-LEM}$ *is not provable in* $\mathsf{HA}+\mathrm{ECT}_0$ *augmented with the first-order counterpart of* $\Sigma_1^0\text{-DNS}^0$ *with number parameters.*

Next we show the underivability results for $\neg\neg\Sigma_1^0\text{-DML}$.

Lemma 25. $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega \nvdash \neg\mathrm{CT}$.

Proof. Suppose that $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega \vdash \neg\mathrm{CT}$. Then $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega + \mathrm{CT} \vdash \perp$. By [22, 3.6.6(v)], we have $\mathsf{HA} \vdash \perp$, which is a contradiction. □

Theorem 26. $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega \nvdash \neg\neg\Sigma_1^0\text{-DML}$.

Proof. It is straightforward to see that $\mathrm{AC}^{0,0} + \neg\neg\Sigma^0_1$-DML implies

$$\neg\neg\exists\alpha \leq 1 \forall x((\alpha(x) = 0 \to \neg\exists u(T(x,x,u) \wedge U(u) = 0) \wedge (\alpha(x) \neq 0 \to \neg\exists u(T(x,x,u) \wedge U(u) = 1)). \tag{4}$$

By using this, we claim that

$$\neg\neg\exists\alpha\neg\exists e\forall x\exists u(T(e,x,u) \wedge \alpha(x) = U(u)). \tag{5}$$

Fix α in (4) and let e satisfy

$$\forall x\exists u(T(e,x,u) \wedge \alpha(x) = U(u)). \tag{6}$$

Now $\alpha(e) = 0 \vee \alpha(e) \neq 0$ holds. If $\alpha(e) = 0$, then we have $\neg\exists u(T(e,e,u) \wedge U(u) = 0)$ by (4). On the other hand, we have $\exists u(T(e,e,u) \wedge U(u) = \alpha(e) = 0)$ by (6). Thus we have a contradiction. If $\alpha(e) \neq 0$, then $\alpha(e) = 1$. Therefore we obtain a contradiction using (4) and (6) in the same manner.

Since (5) implies $\neg$CT, our theorem follows from Lemma 25. $\square$

Corollary 27. $\mathsf{EL} + \mathrm{AC}^{0,0} + \neg\neg\Sigma^0_1$-DNE $(\neg\neg\mathrm{MP}) \nvdash \neg\neg\Sigma^0_1$-DML.

Remark 28. *In fact, the proof of Theorem 26 shows that* $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega \nvdash \neg\neg\Sigma^0_1\text{-DML}^-$, *where* $\neg\neg\Sigma^0_1\text{-DML}^-$ *is the function-parameter-free fragment of* $\neg\neg\Sigma^0_1$-DML *(in the sense of Remark 7). Then it follows from Remark 7 that* $\mathsf{HA} + \Sigma^0_1$-DNE *(in the sense of [1]) does not prove the first-order counterpart of* $\neg\neg\Sigma^0_1$-DML :

$$\neg\neg\forall x(\neg(\exists y\, \mathrm{A}_{\mathrm{qf}} \wedge \exists z\, \mathrm{B}_{\mathrm{qf}}) \to \neg\exists y\, \mathrm{A}_{\mathrm{qf}} \vee \neg\exists z\, \mathrm{B}_{\mathrm{qf}}),$$

where A_{qf} *and* B_{qf} *are quantifier-free* $\mathscr{L}(\mathsf{HA})$*-formulas.*

Next we show the underivability results for $\neg\neg\Pi^0_1$-LEM.

Theorem 29. $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \Sigma^0_1\text{-DML} \nvdash \neg\neg\Pi^0_1$-LEM.

Proof. Suppose that $\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \Sigma^0_1\text{-DML} \vdash \neg\neg\Pi^0_1$-LEM. As in the proof of Theorem 12, let again $\forall n\exists m\, G_{\mathrm{qf}}(n,m)$ be the sentence which states that for all n, the Goodstein sequence starting from n terminates. Since $\forall n\exists m\, G_{\mathrm{qf}}(n,m)$ is provable in $\mathsf{PA}^\omega + \Pi^0_1$-CA (the termination of the Goodstein sequences is equivalent to the 1-consistency of PA which can be proved in the system at hand, see e.g. [5, P. 941]), where Π^0_1-CA:

$$\forall\alpha\exists\delta\forall x(\delta(x) = 0 \leftrightarrow \forall y\, \alpha(x,y) = 0),$$

by negative translation (see e.g. [12, Section 10.1]), we have that $\forall n\neg\neg\exists m\, G_{\mathrm{qf}}(n,m)$ is provable in $\mathsf{HA}^\omega + (\Pi^0_1\text{-CA})^{\mathrm{N}}$, where $(\Pi^0_1\text{-CA})^{\mathrm{N}}$ is the negative translation of Π^0_1-CA:

$$\forall\alpha\neg\neg\exists\delta\forall x\neg\neg(\delta(x) = 0 \leftrightarrow \forall y\, \alpha(x,y) = 0).$$

Note that $(\Pi^0_1\text{-CA})^{\mathrm{N}}$ is intuitionistically derived from $\neg\neg\Pi^0_1\text{-LEM} + \mathrm{AC}$. Therefore, by our assumption (with the use of M^ω), we have that

$$\mathsf{HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \Sigma^0_1\text{-DML} \vdash \forall n\exists m\, G_{\mathrm{qf}}(n,m).$$

Then, by the monotone Dialectica interpretation (see [12, Chapter 9]), there exists a closed term t in Gödel's **T** such that $\forall n\exists m \leq t(n)\, G_{\mathrm{qf}}(n,m)$ is verifiable in the full set-theoretic model $\mathscr{S}^\omega$, and hence, a PA-provably recursive function h_t such that $\forall n \in \mathbb{N}\exists m \leq h_t(n)\, G_{\mathrm{qf}}(n,m)$ holds, which is a contradiction to the fact that any such h_t would have to grow faster than any provably recursive function of PA. $\square$

Corollary 30. $\mathsf{EL}+\mathrm{AC}^{0,0}+\neg\neg\Sigma^0_1\text{-DML}\nvdash\neg\neg\Pi^0_1\text{-LEM}$.

In contrast to the previous underivability proofs, the proof of Theorem 29 does not show that $\mathsf{HA}^\omega+\mathrm{AC}+\mathrm{IP}^\omega_\forall+\mathrm{M}^\omega+\Sigma^0_1\text{-DML}\nvdash\neg\neg\Pi^0_1\text{-LEM}^-$ unfortunately. Nevertheless, one can establish the corresponding first-order underivability by using further elaborated arguments from proof theory. Recall that $\mathbf{T_n}$ is the subset of closed terms in $\mathbf{T}$ involving only recursors R_ρ with $deg(\rho)\leq n$ (see [12, Chapter 3]).

Theorem 31. $\mathsf{HA}^\omega+\mathrm{AC}+\mathrm{IP}^\omega_\forall+\mathrm{M}^\omega+\Sigma^0_1\text{-DML}\nvdash\neg\neg\Pi^0_1\text{-LEM}^-$*, where* $\neg\neg\Pi^0_1\text{-LEM}^-$ *is the function-parameter-free fragment of* $\neg\neg\Pi^0_1\text{-LEM}$:

$$\neg\neg\forall x(\forall y\, t(x,y)=0\vee\neg\forall y\, t(x,y)=0)\ (t: \textit{primitive recursive}).$$

Proof. Suppose that $\mathsf{HA}^\omega+\mathrm{AC}+\mathrm{IP}^\omega_\forall+\mathrm{M}^\omega+\Sigma^0_1\text{-DML}\vdash\neg\neg\Pi^0_1\text{-LEM}^-$. Note that one can write $\neg\neg\Pi^0_1\text{-LEM}^-$ (which officially is an infinite schema) as a sentence

$$\neg\neg\forall e,x(\neg\exists z T(e,x,z)\vee\neg\neg\exists z T(e,x,z)).$$

Then there exists $n\in\mathbb{N}$ and a fragment $\mathsf{HA}^{\omega-}\supseteq\mathsf{HA}_0$ of HA^ω such that $\neg\neg\Pi^0_1\text{-LEM}^-$ is provable in $\mathscr{T}:=\mathsf{HA}^{\omega-}+\mathrm{AC}+\mathrm{IP}^\omega_\forall+\mathrm{M}^\omega+\Sigma^0_1\text{-DML}$ and $\mathscr{T}$ has a monotone Dialectica interpretation in $\mathbf{T_n}$. Here HA_0 is the fragment of HA where the induction schema is restricted to quantifier-free formulas (which we indentify with its obvious embedding into HA^ω). Then, by Parsons' result [17, Theorem 4], $\mathscr{T}^+:=\mathscr{T}+(\Pi^0_{\mathrm{n+1}}\text{-IA})^{\mathrm{N}}$, where $(\Pi^0_{n+1}\text{-IA})^{\mathrm{N}}$ is the negative translation of the $\Pi^0_{\mathrm{n+1}}$-induction schema, also has a monotone Dialectica interpretation in $\mathbf{T_n}$ (here we use that closed terms in $\mathbf{T_n}$ can be majorized by closed terms in $\mathbf{T_n}$). By our assumption (with the use of M^ω), we have that $\mathscr{T}^+$ proves $\neg\neg\Sigma^0_1\text{-LEM}^-$, where $\neg\neg\Sigma^0_1\text{-LEM}^-$ is the function-parameter-free fragment of $\neg\neg\Sigma^0_1\text{-LEM}$ (in the sense of Remark 7). Note that $\neg\neg\Sigma^0_1\text{-LEM}^-$ and AC derive the doubly negated comprehension schema $\neg\neg\Sigma^0_1\text{-CA}^-$ for Σ^{0-}_1-formulas (without loss of generality we may assume the instances of $\neg\neg\Sigma^0_1\text{-CA}^-$ to be closed). In addition, $\Sigma^0_1\text{-CA}^-$ and $(\Pi^0_{\mathrm{n+1}}\text{-IA})^{\mathrm{N}}$ derive the negative translation $(\Pi^0_{\mathrm{n+2}}\text{-IA}^-)^{\mathrm{N}}$ of the function-parameter-free $\Pi^0_{\mathrm{n+2}}$-induction schema. Therefore, since $\neg\neg(\Pi^0_{\mathrm{n+2}}\text{-IA}^-)^{\mathrm{N}}$ is intuitionistically equivalent to $(\Pi^0_{\mathrm{n+2}}\text{-IA}^-)^{\mathrm{N}}$, we have that

$$\mathscr{T}^+\vdash(\Pi^0_{\mathrm{n+2}}\text{-IA}^-)^{\mathrm{N}}.$$

Now let $\mathrm{G}:=\forall m\exists k\,\mathrm{G_{qf}}(m,k)$ (with $\mathrm{G_{qf}}$ being a quantifier-free formula in $\mathscr{L}(\mathsf{HA})$) be a sentence stating the totality of some function which is $\mathbf{T_{n+1}}$-definable but not $\mathbf{T_n}$-definable (when interpreted in the full set-theoretic model $\mathscr{S}^\omega$). Since the totality of $\mathbf{T_{n+1}}$-definable functions is provable in $\mathsf{PA}_0+\Pi^0_{\mathrm{n+2}}\text{-IA}^-$ (see [17, Lemma 4]), where PA_0 is the fragment of PA where the induction schema is restricted to quantifier-free formulas, there is such a sentence G with

$$\mathsf{HA}_0+(\Pi^0_{\mathrm{n+2}}\text{-IA}^-)^{\mathrm{N}}\vdash\mathrm{G}^{\mathrm{N}}.$$

Therefore, by using M^ω, we have that $\mathscr{T}^+\vdash\mathrm{G}$. By the monotone Dialectica interpretation (see [12, Chapter 9]), there exists a closed term $t\in\mathbf{T_n}$ such that

$$\mathscr{S}^\omega\models\forall m\exists k\leq t(m)\mathrm{G_{qf}}(m,k).$$

Then the function $\tilde{t}\in\mathbb{N}^\mathbb{N}$ defined as $\tilde{t}(m):=\min k\leq t(m)\,\mathrm{G_{qf}}(m,k)$ in $\mathbf{T_n}$ satisfy $\forall m\,\mathrm{G_{qf}}(m,\tilde{t}(m))$, which is a contradiction. □

Corollary 32. $\mathsf{HA}+\Sigma^0_1$-DNE$+\Sigma^0_1$-DML *does not prove the first-order counterpart of* $\neg\neg\Pi^0_1$-LEM :

$$\neg\neg\forall x(\forall y \mathrm{A}_{\mathrm{qf}} \vee \neg\forall y \mathrm{A}_{\mathrm{qf}}),$$

where A_{qf} *is a quantifier-free* $\mathscr{L}(\mathsf{HA})$*-formula.*

We now show underivability results for Σ^0_2-DNS^0 by using a relativized version of the proof of Theorem 12. For this we first need the following:

Lemma 33. *Let* $\alpha : \mathbb{N}^2 \to \mathbb{N}$ *be defined by the property*

$$\forall e,x\big(\forall z(T(e,x,z) \to T(e,x,\alpha(e,x))) \wedge (\neg T(e,x,\alpha(e,x)) \to \alpha(e,x)=0)\big).$$

Then there exists a true Π^0_3*-sentence* $\forall x\exists y\forall z \mathrm{A}_{\mathrm{qf}}(x,y,z)$ *in* $\mathscr{L}(\mathsf{HA})$ *such that for no* $f \in \mathbb{N}^{\mathbb{N}}$ *that is definable by a closed term of* $\mathbf{T}+\alpha$ *(when interpreted in the full set-theoretic model* $\mathscr{S}^{\omega}$*) one would have that*

$$\mathscr{S}^{\omega} \models \forall x\exists y \leq f(x)\forall z \mathrm{A}_{\mathrm{qf}}(x,y,z).$$

Proof. In the following we identity α with the unary function $\alpha'(x) := \alpha(j_1(x), j_2(x))$, where j_i are the projections of the Cantor pairing function j.
Let $\{F_n\}_{n\in\mathbb{N}}$ be an effective enumeration of all closed terms $t^{1\to 1}$ of type $1 \to 1$ of $\mathbf{T}$. Then

$$F(n) := \max\{F_i(\alpha, j)+1 : i, j \leq n\}$$

defines a function $\mathbb{N} \to \mathbb{N}$ which is oracle computable in α (here we identify F_n with its unique interpretation in $\mathscr{S}^{\omega}$). Clearly, F eventually dominates every function definable in $\mathbf{T}+\alpha$ since any such function has a definition of the form $F_n\alpha$ for some $n \in \mathbb{N}$, but $F(m) > F_n(\alpha, m)$ for all $m \geq n$. Since F is oracle computable in α, by the Kleene normal form theorem for oracle computations, there exists an $e \in \mathbb{N}$ such that

$$F(x) = U(\min m \tilde{T}(e,x,\overline{\alpha}(m),m))$$

for suitable primitive recursive $\tilde{T}, U$, where we may assume that $U(n) \leq n$ since U extracts the value from the code of the terminating computation n.

Consider now (here we identify e with its numeral $\overline{e}$) the $\mathscr{L}(\mathsf{HA})$-sentence

$$\forall x\exists y \tilde{T}(e,x,\overline{\alpha}(y),y). \tag{7}$$

Clearly, by the above, no function definable in $\mathbf{T}+\alpha$ can satisfy

$$\mathscr{S}^{\omega} \models \forall x\exists y \leq f(x)\tilde{T}(e,x,\overline{\alpha}(y),y).$$

We now show that '$\tilde{T}(e,x,\overline{\alpha}(y),y)$' can be written in Σ^0_2-form in $\mathscr{L}(\mathsf{HA})$: this follows by, firstly, noting that (here we may assume that our sequence coding is surjective as in [22])

$$\tilde{T}(e,x,\overline{\alpha}(y),y) \leftrightarrow \exists k\,(lth(k)=y \wedge \forall i<y((k)_i=\alpha(i)) \wedge \tilde{T}(e,x,k,y))$$

and, secondly, that (using that $T(x,y,z)$ can hold for at most one z)

$$l=\alpha(i) \leftrightarrow \forall z\,\big(T(j_1(i),j_2(i),z) \to T(j_1(i),j_2(i),l)\big) \wedge (\neg T(j_1(i),j_2(i),l) \to l=0).$$

So (7) can be written in the form $\forall x \exists y, k \forall z\, \mathrm{A}'_{\mathrm{qf}}(x,y,k,z)$ in $\mathscr{L}(\mathsf{HA})$ for some quantifier-free $\mathrm{A}'_{\mathrm{qf}}$ so that no $(\mathbf{T}+\alpha)$-definable function f can bound '$\exists y$' and so also not '$\exists y$' in

$$\forall x \exists y \forall z\, \underbrace{\mathrm{A}'_{\mathrm{qf}}(x, j_1(y), j_2(y), z)}_{\mathrm{A}_{\mathrm{qf}}(x,y,z):\equiv}$$

(here we use that $y \geq j_1(y), j_2(y)$) which finishes the proof. □

Theorem 34. $\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Pi^0_2\text{-LEM}^- \nvdash (\Sigma^0_2\text{-DNS}^0)^-$, *where* $\Pi^0_2\text{-LEM}^-$ *is the function-parameter-free fragment of* Π^0_2-LEM :

$$\forall x (\forall y \exists z\, s(x,y,z) = 0 \vee \neg \forall y \exists z\, s(x,y,z) = 0) \ (s : \textit{primitive recursive})$$

and $(\Sigma^0_2\text{-DNS}^0)^-$ *is the "closed" (note Remark 8) function-parameter-free fragment of* $\Sigma^0_2\text{-DNS}^0$:

$$\forall x \neg\neg \exists y \forall z\, t(x,y,z) = 0 \to \neg\neg \forall x \exists y \forall z\, t(x,y,z) = 0 \ (t : \textit{primitive recursive}).$$

Proof. Suppose that $\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Pi^0_2\text{-LEM}^- \vdash (\Sigma^0_2\text{-DNS}^0)^-$. Since every Σ^0_1-predicate $\exists z\, s(x,y,z) = 0$ for a primitive recursive s is (provably in HA) of the form $\exists z\, T(\bar{e}, j(x,y), z)$ for some $e \in \mathbb{N}$, $\Pi^0_2\text{-LEM}^-$ can be written as a single sentence

$$\forall e, x (\forall y \exists z\, T(e, j(x,y), z) \vee \neg \forall y \exists z\, T(e, j(x,y), z)).$$

Add a new function symbol α and its defining axiom (α) :

$$\forall e, x \big(\forall z (T(e,x,z) \to T(e,x,\alpha(e,x))) \wedge (\neg T(e,x,\alpha(e,x)) \to \alpha(e,x) = 0)\big).$$

Then one can easily see that $\Pi^0_2\text{-LEM}^-$ is derivable from Π^0_1-LEM and (α). Therefore we have that

$$\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Pi^0_1\text{-LEM} + (\alpha) \vdash (\Sigma^0_2\text{-DNS}^0)^-.$$

Take a true Π^0_3-sentence $\forall x \exists y \forall z \mathrm{A}_{\mathrm{qf}}(x,y,z)$ (for this α) from Lemma 33. By our assumption, we have that

$$\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + (\theta) + (\alpha) + \forall x \neg\neg \exists y \forall z\, \mathrm{A}_{\mathrm{qf}}(x,y,z) + \neg \exists f \forall x, z\, \mathrm{A}_{\mathrm{qf}}(x, f(x), z) \vdash \perp,$$

where (θ) is as in the proof of Theorem 12. Since (α) is $\exists$-free, by the monotone modified realizability [12, Theorem 7.1(1)], we have that

$$\mathsf{E\text{-}HA}^{\omega} + (\theta) + (\alpha) + \forall x \neg\neg \exists y \forall z\, \mathrm{A}_{\mathrm{qf}}(x,y,z) + \neg \exists f \forall x, z\, \mathrm{A}_{\mathrm{qf}}(x, f(x), z) \vdash \perp$$

as before. However, by constructing its model as in the proof of Theorem 12 with "closed term of $\mathbf{T}$" replaced by "closed term of $\mathbf{T}+\alpha$" etc., we have that $\mathsf{E\text{-}HA}^{\omega} + (\theta) + (\alpha) + \forall x \neg\neg \exists y \forall z\, \mathrm{A}_{\mathrm{qf}}(x,y,z) + \neg \exists f \forall x, z\, \mathrm{A}_{\mathrm{qf}}(x, f(x), z)$ is consistent. Here we use that by majorization every closed term $t[g_v, \alpha]$ of $\mathbf{T} + g_v + \alpha$ (with g_v being some constant representing the equally denoted functional in the proof of Theorem 12) can be majorized (valid in $\mathscr{S}^{\omega}$) by a term $t^*[\alpha^M, 1^2]$ in $\mathbf{T}$+(α), where $\alpha^M(x) := \max\{\alpha(i) : i \leq x\}$ majorizes α. □

Corollary 35. $\mathsf{EL}+\mathrm{CT}+\Pi^0_2\text{-LEM}\nvdash(\Sigma^0_2\text{-DNS}^0)^-$, *where* Π^0_2-LEM *is*

$$\forall\alpha\forall x(\forall y\exists z\,\alpha(x,y,z)=0\vee\neg\forall y\exists z\,\alpha(x,y,z)=0).$$

Proof. Suppose that $\mathsf{EL}+\mathrm{CT}+\Pi^0_2\text{-LEM}\vdash(\Sigma^0_2\text{-DNS}^0)^-$. Note that Π^0_2-LEM is equivalent to its function-parameter-free fragment $\Pi^0_2\text{-LEM}^-$ in the presence of CT (see also Remark 3). Since $\mathsf{EL}+\mathrm{CT}$ is conservative over HA for $\mathscr{L}(\mathsf{HA})$-formulas (cf. [22, Theorem 3.6.2]), we have that

$$\mathsf{HA}+\Pi^0_2\text{-LEM}^-\vdash(\Sigma^0_2\text{-DNS}^0)^-,$$

which contradicts Theorem 34. □

Since Π^0_2-LEM implies Σ^0_1-LEM, we have the following:

Corollary 36. $\mathsf{EL}+\neg\neg\Sigma^0_1\text{-LEM}\nvdash\Sigma^0_2\text{-DNS}^0$. *The analogous result for the first-order counterparts also holds.*

Remark 37 (Warning). *The situation dramatically changes in the presence of the combination of countable choice schema and* $\neg\neg\Sigma^0_1$-LEM *(where function parameters are allowed). In fact, one can show without difficulty that* $\mathsf{EL}+\neg\neg\Sigma^0_1\text{-LEM}+\Pi^0_1\text{-AC}$ *proves* $\neg\neg\Sigma^0_n$-LEM *for any n, thus the hierarchy above* $\neg\neg\Sigma^0_1$-LEM *collapses. Then, of course,* $\mathsf{EL}+\Pi^0_1\text{-AC}+\neg\neg\Sigma^0_1\text{-LEM}$ *proves* $\Sigma^0_2\text{-DNS}^0$. *Theorem 34 and Corollary 35 state that if* Π^0_1-AC *or the function-parameter condition for* $\neg\neg\Sigma^0_1$-LEM *is lacking,* $\Sigma^0_2\text{-DNS}^0$ *is already underivable. In this sense, our results are optimal.*

5 Some Additional Comments

As mentioned in Remark 5, there are essentially three ways of formalization of Δ_a-LEM and those three are pairwise equivalent over EL. This is also the case for Σ^0_1-DNE, Σ^0_1-DML, Π^0_1-DML and Δ_b-LEM. In particular, the following three formalizations of Σ^0_1-DNE (Markov's principle) are pairwise equivalent over EL:

1. $\forall\alpha\forall x(\neg\neg\exists y\,\alpha(x,y)=0\rightarrow\exists y\,\alpha(x,y)=0)$,
2. $\forall\alpha(\forall x\neg\neg\exists y\,\alpha(x,y)=0\rightarrow\forall x\exists y\,\alpha(x,y)=0)$.
3. $\forall\alpha(\neg\neg\exists y\,\alpha(y)=0\rightarrow\exists y\,\alpha(y)=0)$,

However, this is not the case for the doubly negated variants. In particular, the definitions of doubly negated variants based on the third-type of formalizations do not make sense. This is just because

$$\forall\alpha\neg\neg(\exists n\,\alpha(n)=0\vee\neg\exists n\,\alpha(n)=0)$$

is provable intuitionistically. In this paper, our definitions of doubly negated variants are based on the first-type of formalizations. On the other hand, if we consider the doubly negated variant of the second formalization of Σ^0_1-DNE, we have

$$\forall\alpha\neg\neg(\forall x\neg\neg\exists y\,\alpha(x,y)=0\rightarrow\forall x\exists y\,\alpha(x,y)=0),$$

which is intuitionistically equivalent to $\Sigma^0_1\text{-DNS}^0$. Thus $\Sigma^0_1\text{-DNS}^0$ can be seen as a different formalization of a doubly negated variant of Σ^0_1-DNE. Therefore it seems to be interesting to study the doubly negated variants of $\Sigma^0_1\text{-DNE},\Sigma^0_1\text{-DML},\Pi^0_1\text{-DML},\Delta_a\text{-LEM},\Delta_b\text{-LEM}$ in this sense. However, here we leave it as a future work except for the equivalence between $\Sigma^0_1\text{-DNS}^0$ and

$$\neg\neg\Delta_b\text{-LEM}':\ \forall\alpha,\beta\neg\neg(\forall x(\neg\exists y\,\alpha(x,y)=0\leftrightarrow\exists z\,\beta(x,z)=0)\rightarrow\forall x(\exists y\,\alpha(x,y)=0\vee\neg\exists y\,\alpha(x,y)=0)).$$

Proposition 38. Σ^0_1-DNS^0 *is intuitionistically equivalent to* $\neg\neg\Delta_b$-LEM$'$.

Proof. We first show that Σ^0_1-DNS^0 implies $\neg\neg\Delta_b$-LEM$'$. Fix α, β such that

$$\forall x(\neg\exists y\, \alpha(x,y) = 0 \leftrightarrow \exists z\, \beta(x,z) = 0). \tag{8}$$

Note that

$$\forall x \neg\neg(\exists y\, \alpha(x,y) = 0 \vee \neg\exists y\, \alpha(x,y) = 0) \tag{9}$$

is intuitionistically provable. By our assumption (8), (9) is equivalent to

$$\forall x \neg\neg(\exists y\, \alpha(x,y) = 0 \vee \exists z\, \beta(x,z) = 0),$$

which is equivalent to

$$\forall x \neg\neg\exists m,y,z((m = 0 \to \alpha(x,y) = 0 \wedge (m \neq 0 \to \beta(x,z) = 0)).$$

By Σ^0_1-DNS^0, we have

$$\neg\neg\forall x \exists m,y,z((m = 0 \to \alpha(x,y) = 0 \wedge (m \neq 0 \to \beta(x,z) = 0)),$$

which is shown to be equivalent to

$$\neg\neg\forall x(\exists y\, \alpha(x,y) = 0 \vee \neg\exists y\, \alpha(x,y) = 0)$$

again by using (8).

In the following, we show that $\neg\neg\Delta_b$-LEM$'$ implies Σ^0_1-DNS^0. Fix α such that $\forall x\neg\neg\exists y\, \alpha(x,y) = 0$. By $\neg\neg\Delta_b$-LEM$'$, we have

$$\forall \beta \big(\forall x(\neg\exists y\, \alpha(x,y) = 0 \leftrightarrow \exists z\, \beta(x,z) = 0) \to \neg\neg\forall x(\neg\neg\exists y\, \alpha(x,y) = 0 \to \exists y\, \alpha(x,y) = 0)\big). \tag{10}$$

One can easily show the hypothesis of (10) by taking β as constant-1 function. Then we have

$$\neg\neg\forall x(\neg\neg\exists y\, \alpha(x,y) = 0 \to \exists y\, \alpha(x,y) = 0),$$

which implies $\neg\neg\forall x\exists y\, \alpha(x,y) = 0$ by our assumption. ☐

Remark 39. *From this perspective, it is also possible to see* Σ^0_1-DNS^0 *as a third-type formalization of*

$$\forall \alpha \forall z(\forall x \neg\neg\exists y\, \alpha(x,y,z) = 0 \to \neg\neg\forall x\exists y\, \alpha(x,y,z) = 0). \tag{11}$$

However, one can easily show that all of three formalizations of (11) and three formalizations of the doubly negated variants of (11) are pairwise equivalent.

6 Summary

We have shown all the underivability results in their strongest form, namely,

$$\mathsf{EL} + \mathrm{P_i} \nvdash \mathrm{P_j^-},$$

and so they hold, in particular, over both of EL and HA. On the other hand, as mentioned in Remark 11, all the the derivability results also hold over both of EL and HA. Thus our results are summarized in Figure 2. Note that whenever an implication does not follow by transitivity from Figure 2 then it actually does not hold.

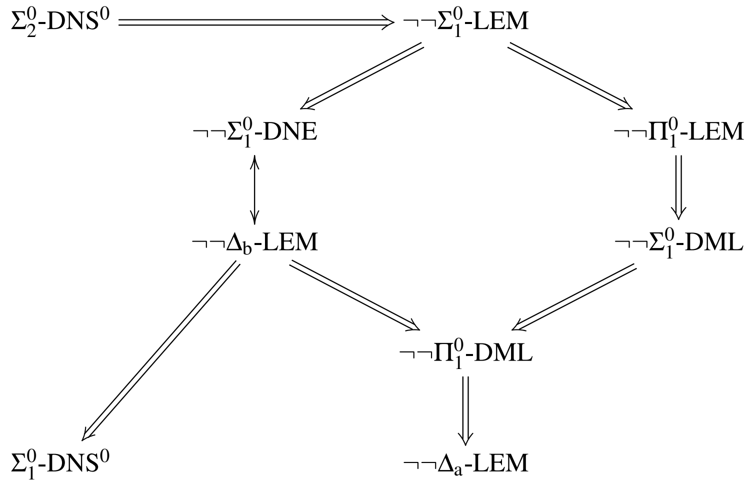

Figure 2: Interrelation between our logical principles over EL or HA

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