On quantitative versions of theorems due to F.E. Browder and R. Wittmann

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Abstract

This paper is another case study in the program of logically analyzing proofs to extract new (typically effective) information ('proof mining'). We extract explicit uniform rates of metastability (in the sense of T. Tao) from two ineffective proofs of a classical theorem of F.E. Browder on the convergence of approximants to fixed points of nonexpansive mappings as well as from a proof of a theorem of R. Wittmann which can be viewed as a nonlinear extension of the mean ergodic theorem. The first rate is extracted from Browder's original proof that is based on an application of weak sequential compactness (in addition to a projection argument). Wittmann's proof follows a similar line of reasoning and we adapt our analysis of Browder's proof to get a quantitative version of Wittmann's theorem as well. In both cases one also obtains totally elementary proofs (even for the strengthened quantitative forms) of these theorems that neither use weak compactness nor the existence of projections anymore. In this way, the present article also discusses general features of extracting effective information from proofs based on weak compactness. We then extract another rate of metastability (of similar nature) from an alternative proof of Browder's theorem essentially due to Halpern that already avoids any use of weak compactness. The paper is concluded by general remarks concerning the logical analysis of proofs based on weak compactness as well as a quantitative form of the so-called demiclosedness principle. In a subsequent paper these results will be utilized in a quantitative analysis of Baillon's nonlinear ergodic theorem.

Keywords: Nonexpansive mappings, metastability, functional interpretation, proof mining.

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1 Introduction

In [11, 5, 14] general logical metatheorems are developed that guarantee for large classes of proofs in nonlinear analysis the extractability of highly uniform effective bounds from the given proofs. By 'highly uniform' we refer to the fact that the bounds are largely independent from parameters ranging over the elements of 'abstract' (e.g. metric, hyperbolic, normed or Hilbert) spaces even in the absence of compactness as long as local metric bounds are given. Adaptations to further classes such as δ-hyperbolic spaces in the sense of Gromov, $\mathbb{R}$-trees in the sense of Tits and uniformly convex hyperbolic spaces are given in [22]. 'Abstract' means that we do not refer to concrete separable spaces but to general classes of spaces treated as atoms which are not assumed to be separable. Even when the theorem for which such a uniform effective bound is extracted is of interest primarily for concrete separable spaces such as L^2 it is crucial to work in a nonseparable setting to achieve such strong uniformity features. This approach has found numerous applications in metric fixed point theory (see e.g. [13] for a survey) as well as in ergodic theory (see [1, 17]).
The last two papers are concerned with effective uniform bounds Φ on the so-called no-counterexample interpretation due to G. Kreisel [19] (recently popularized by T. Tao under the name of 'metastability', see [30]) of the von Neumann Mean Ergodic Theorem:
Assume that X is a real Hilbert space and $T : X \to X$ is a linear operator with $\|Tx\| \leq \|x\|$ for all $x \in X$. Define $(x_n)_{n\geq 0}$ by $x_n := \frac{1}{n+1}\sum_{i=0}^{n} T^i x$. The Mean Ergodic Theorem states that (x_n) converges. As shown in [1], there is – already in rather simple and effective contexts – in general no computable rate of convergence. An example from [21], moreover, shows that the convergence in general is not uniform w.r.t. the starting point x. Nevertheless, as guaranteed by a metatheorem from [11] there exists a computable uniform bound Φ such that for all $b > 0$ and all $x \in X$ with $\|x\| \leq b$

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists P \leq \Phi(\varepsilon, g, b)\, \forall i, j \in [P; P + g(P)] \left(\|x_i - x_j\| < \varepsilon\right).$$

Here $[n; n + m] := \{k \in \mathbb{N} : n \leq k \leq n + m\}$.
Note that (ineffectively)

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists P\, \forall i, j \in [P; P + g(P)] \left(\|x_i - x_j\| < \varepsilon\right)$$

is equivalent to the Cauchy property and hence the convergence of (x_n).

Remark 1.1. *The first such bound Φ was constructed in [1] applying proof mining in the above sense to the standard textbook proof of the mean ergodic theorem (see also [31]). In [17] an analysis of a different proof due to G. Birkhoff led to a better bound as well as a generalization to uniformly convex Banach spaces (where then the bound* *additionally depends on a modulus of uniform convexity for X). For other recent metastability results in nonlinear analysis see [18].*

The existence of a (even effective) uniform such bound which does not depend on T or X and on x only via a norm upper bound is guaranteed already by the aforementioned metatheorems and plays a role in a recent generalization of the mean ergodic theorem to commuting families of operators given in [31]. Note, however, that the proof of the logical metatheorems also contains an algorithm for the actual extraction of an explicit bound from a given proof. That algorithm is based on a monotone variant and extension of Gödel's famous functional ('Dialectica') interpretation ([7, 14]).

So far all these applications only used a small fraction of the power of these metatheorems in that the proofs could be formalized in rather weak fragments of the formal systems allowed which may contain the axiom schema of full dependent choice (and hence countable choice as well as full comprehension over numbers). As shown in [15], that strength makes it possible to formalize very general orthogonal projection arguments as well as the proof of the weak sequential compactness of bounded, closed and convex subset in an abstract Hilbert space. In that paper – as a first application – it is verified that a proof due to F.E. Browder (that is based on weak sequential compactness and a projection argument) of some other convergence result can be formalized so that a logical metatheorem (already from [11]) guarantees a similar uniformity as in the mean ergodic theorem above.

Theorem 1.2 (F. Browder [4])**.** *Let X be a real Hilbert space and $U : X \to X$ be a nonexpansive mapping. Assume that there exists a nonempty bounded closed convex subset $C \subset X$ such that U maps C into itself. For $v_0 \in C$ and $t \in (0,1)$ let $U_t(x) := tU(x) + (1-t)v_0$ and u_t be the unique fixed point of this strict contraction. Then (u_t) converges strongly to a fixed point $p \in C$ of U as $t \to 1$. In fact, it converges to the unique fixed point of U in C that is closest to v_0.*

As shown in [15], a general logical extraction theorem from [11] guarantees the following uniform quantitative version of this result (for simplicity we only consider the closed unit ball $B_1(0)$ instead of C):

Proposition 1.3 ([15])**.** *Under the assumptions of theorem 1.2 with $C := B_1(0)$ there exists a computable functional $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (that is independent from X, U and $v_0 \in B_1(0)$) such that*

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k,g)\ \forall i,j \in [n; n+g(n)]\, (\|x_i - x_j\| < 2^{-k}),$$

where $x_i := u_t$ with $t := 1 - \frac{1}{i+1}$. Similarly, for any sequence (s_n) in $(0,1)$ that converges towards 1 where then the bound depends also on a (majorant of a) rate of metastability of that convergence

$$\forall n \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \forall i \in [\chi(g,n); \chi(g,n) + g(\chi(g,n))]\, (|1 - s_i| \leq \frac{1}{n+1})$$

and a function $h : \mathbb{N} \to \mathbb{N}$ *such that* $\forall n \in \mathbb{N}\,(s_n \le 1 - \frac{1}{h(n)+1})$.

Remark 1.4. *This highly uniform and effective rate of metastability should be contrasted to the lack of an effective rate of convergence (that might even depend on the operator in question): the proof of Theorem 18.4 in [14] shows that there is a computable sequence* $(U_n)_{n\in\mathbb{N}}$ *of nonexpansive operators* $\mathbb{R} \to \mathbb{R}$ *with* $U_n|[0,1] : [0,1] \to [0,1]$ *such that for* $v_0 := 0$ *and* $k := 1$ *there is no computable function* $\chi : \mathbb{N} \to \mathbb{N}$ *such that*

$$\forall m \ge \chi(n) \ \left(|x^n_m - x^n_{\chi(n)}| \le \frac{1}{2} \right)$$

(here $(x^n_k)_{k\in\mathbb{N}}$ *is the sequence as theorem 1.3 for* U_n *instead of* U *starting from* 0*).*

So we have here an example of an elementary convergence theorem established by a proof that uses some amount of weak sequential compactness and projections but for which we have the tools available to extract the finitary combinatorial content of that proof leading to an explicit and quantitative finitary version (in the sense of [30]) of the proven result. In section 2 below we carry out this extraction and even get for general bounded closed convex subsets $C \subset X$ instead of $B_1(0)$ a bound $\Phi(k, g, d)$ that depends on C only via a bound $d \ge diam(C)$ on the diameter of C. The logical analysis leads in the end to an elimination of the use of weak sequential compactness and provides a finitary quantitative analysis of the argument that (u_n) converges to the fixed point of U that us closest to v_0.

Although Browder's theorem has a different proof due to Halpern [8] (which we also analyze in this paper, see below) that already avoids weak compactness, it is Browder's proof technique that is used in many other results so that our analysis can be easily applied to them as well. As an example for this we adapt our analysis to a quantitative version of an important theorem due to Wittmann [32]: Let X be a Hilbert space, $C \subseteq X$ closed and convex and $U : C \to C$ nonexpansive for $\alpha_n \in [0, 1]$ consider the following iteration (due to Halpern [8])

$$u_{n+1} := \alpha_{n+1}\, u_0 + (1 - \alpha_{n+1})\, U(u_n).$$

Under general conditions on (α_n) which for the first time contained the case $\alpha_n := 1/(n + 1)$ (not covered by Halpern's work), Wittmann shows the strong convergence of (u_n) towards the fixed point of U that is closest to u_0 (provided that U has fixed points). For $\alpha_n := 1/(n + 1)$ and **linear** U the above iteration coincides with the Cesàro mean so that Wittmann's result is a nonlinear version of well-known linear ergodic theorems.

After we had carried out our analysis of Browder's proof we learned about another proof of Browder's theorem due to Halpern [8] that is already elementary in the sense that it does not use weak compactness. Although that proof is formulated only for $C := B_1(0)$ and $v_0 := 0$ it is possible to adapt this proof to the situation of general bounded closed and convex subsets C and arbitrary $v_0 \in C$. We carry out the (much simpler) logical analysis of this proof in section 4.

Notation: $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For $f : \mathbb{N} \to \mathbb{N}$ define $f^{(0)}(n) := n,\ f^{(i+1)}(n) := f(f^{(i)}(n))$ and $f^M(n) := \max\{f(i) : i \leq n\}$.
In the following let X be a real Hilbert space, $d \in \mathbb{N}^*$ and $C \subset X$ be a bounded closed convex subset with $d \geq diam(C) := \sup\{\|x - y\| : x, y \in C\}$. Let $U : C \to C$ be a nonexpansive mapping.

Theorem 1 (see theorem 3.3 below): Logical analysis of Wittmann's proof yields the following bound on the metastable version of his theorem: Let $\alpha_n := 1/(n+1)$ and for $u_0 \in C$ define $u_{n+1} := \alpha_{n+1}\, u_0 + (1 - \alpha_{n+1})\, U(u_n) \quad (n \geq 0)$.
Then

$$\forall \varepsilon \in (0,1)\, \forall g : \mathbb{N} \to \mathbb{N}^*\, \exists k \leq \Phi(\varepsilon/2, g^+, d)\, \forall i, j \in [k; k + g(k)]\ \big(\|u_i - u_j\| \leq \varepsilon\big),$$

where

$$\begin{aligned}
&\Phi(\varepsilon, g, d) := \rho(\varepsilon^2/4d^2, \chi_{d,\varepsilon}(N_{\varepsilon,g,d})) \text{ with}\\
&N_{\varepsilon,g,d} := 16d \cdot \big(\max\big\{(\Delta^*_{\varepsilon,g})^{(i)}(1)\ :\ i \leq n_{\varepsilon,d}\big\}\big)^2,\ n_{\varepsilon,d} := \left\lceil \tfrac{d^2}{\varepsilon_d} \right\rceil,\ \varepsilon_d := \tfrac{\varepsilon^4}{8192d^2} \text{ and}\\
&\Delta^*_{\varepsilon,g}(n) := \lceil 1/\Omega_d(\varepsilon/2, \tilde{g}^M, \chi_{d,\varepsilon}(16d \cdot n^2)) \rceil,
\end{aligned}$$

with $\Omega_d(\varepsilon, g, j) := \delta_{\varepsilon, \tilde{g}(\rho(\varepsilon^2/2d^2, j))}$, where $\delta_{\varepsilon,m} := \frac{\varepsilon^2}{16dm}$, $\rho(\varepsilon, n) := \left\lceil \frac{n+1}{\varepsilon} \right\rceil$, $\chi_{d,\varepsilon}(n) := \max\left\{\chi_d(n), \left\lceil \frac{32d^2}{\varepsilon^2} \right\rceil\right\}$, $\chi_d(n) := 4dn(4dn+2)$, $\tilde{g}(n) := \max\{n, g(n)\}$ and $g^+(n) := n + g(n)$.

Theorem 2 (see theorem 4.2 below): Logical analysis of Halpern's proof (adapted to general bounded closed and convex C and $v_0 \in C$) yields the following bound on the metastable version of Browder's theorem: Let (s_n) be a sequence in $(0,1)$ that converges to 1 and $h : \mathbb{N} \to \mathbb{N}^*$ be such that $s_n \leq 1 - \frac{1}{h(n)}$ and $h(n) \geq n$ for all $n \in \mathbb{N}$ $\breve{u}_n := u_{s_n}$, where – for $t \in (0,1)$ – u_t is the unique fixed point of $U_t(x) := tU(x) + (1-t)v_0$ in C. Then for all $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$ the following holds:

$$\exists n \leq \Psi(\varepsilon, g, \chi, h, d)\, \forall i, j \in [n; n + g(n)]\ (\|\breve{u}_i - \breve{u}_j\| \leq \varepsilon),$$

where

$$\Psi(\varepsilon, g, \chi_g, h, d) := \chi_g^M\left(g_{h,\chi_g}^{(\lceil 4d^2/\varepsilon^2 \rceil)}(0)\right)$$

with

$$g_{h,\chi_g}(n) := \max\{h(i)\ :\ i \leq \chi_g(n) + g(\chi_g(n))\}$$

and $\chi_g : \mathbb{N} \to \mathbb{N}$ is such that

$$\forall n \in \mathbb{N}\, \forall i \in [\chi_g(n); \chi_g(n) + g(\chi_g(n))]\ (|1 - s_i| \leq \frac{1}{n+1}).$$

Instead of C being bounded it suffices to assume that the sequence $(\tilde{u}_n)$ is bounded and the bound above then holds equally provided that $d \geq \|\tilde{u}_n - v_0\|$ for all $n \in \mathbb{N}$. If (s_n) is an increasing sequence in $(0,1)$ (not necessarily converging to 1), then the bound Ψ can be simplified to $\Psi(\varepsilon, g, d) := \tilde{g}^{(\lceil d^2/\varepsilon^2 \rceil)}(0)$.

The bounds in both theorems are primitive recursive in g (in the sense of Kleene) and – essentially – iterations of g. This is optimal w.r.t. the principles used in the respective proofs as the projection argument in Browder's proof as well as the convergence principle for bounded monotone sequences of real numbers (used in Halpern's proof) imply so-called Σ^0_1-induction which in turn suffices to introduce all primitive recursive functionals.

In the final section of this paper we discuss the situation that results in the elimination of weak compactness in the case of Browder's proof and outline the procedure to be applied in the general case where such an elimination might no longer be possible. This strategy, we actually use in a very recent quantitative analysis of Baillon's famous nonlinear ergodic theorem (see [16]).

2 Logical analysis of Browder's proof

We now give the logical analysis based on a monotone version ([9, 14]) of Gödel's functional interpretation (combined with negative translation the result of which is also called Shoenfield variant) of Browder's proof, i.e. we closely follow the actual extraction algorithm (based on monotone functional interpretation) from the proof of the logical metatheorem from [11] referred to in the introduction (for the logical background on all this see [14]).

Browder's proof (see the proof of his 'Lemma 1') starts by considering the set F of all fixed points of U which – by another theorem of Browder – always is nonempty and which (given the uniform convexity of X) is convex. Hence there exists a (unique) point $u_0 \in F$ which has minimal distance from v_0. Note that we do not have to analyze Browder's proof for the fact that F is nonempty as we simply can **assume** that we have a fixed point and later reduce this assumption to the (trivially true) one stating only the existence of approximate fixed points (see e.g. remark 3.13 in [11]).

Browder's proof continues by noting that for any $v \in F$ also $u_t := (1-t)u_0 + tv \in F$ (by the convexity of F) and so $\|v_0 - u_0\|^2 \leq \|v_0 - u_t\|^2$ by the minimality of u_0. As trivial as this step is, it no longer remains so in the quantitative version as instead of the 'real' minimality of u_0 we only have the 'no-counterexample' version formulated in lemma 2.4 at our disposal. The lemmas 2.1, 2.2, 2.3, 2.4 and 2.6 provide the appropriate quantitative version of the reasoning in Browder's proof (as outlined so far) which closely follows what is suggested by functional interpretation.

In the following, C always is a bounded closed convex subset of a real Hilbert space

X and $d \in \mathbb{N}^*$ with $d \geq diam(C)$.

Lemma 2.1. *Let X be a real Hilbert space. Then $\eta(\varepsilon) = \frac{\varepsilon^2}{8}$ is a modulus of uniform convexity of X, i.e.*

$$\forall x, y \in B_1(0)\, \forall \varepsilon > 0 \left(\left\| \frac{x+y}{2} \right\| > 1 - \eta(\varepsilon) \to \|x - y\| < \varepsilon \right),$$

where $B_1(0)$ denotes the closed unit ball in X.

Proof: For $\varepsilon \in (0, 2]$ and $\eta_X(\varepsilon) := 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ this is well-known (see e.g. [6]). By the mean value theorem one has $\frac{\varepsilon^2}{8} \leq \eta_X(\varepsilon)$. The claim is trivial for $\varepsilon > 2$. $\square$

Lemma 2.2. *Let X be a real Hilbert space. Then the following holds for all $\varepsilon > 0$:*

$$\forall a, x, y \in C \ \left(\left\| a - \frac{x+y}{2} \right\| > K - \frac{\varepsilon^2}{8d} \to \|x - y\| < \varepsilon \right),$$

where $K := \max\{\|a - x\|, \|a - y\|\} \leq d$.

Proof: We may assume that $K > 0$ since, otherwise, $\|x - y\| = 0 < \varepsilon$. Consider $\tilde{x} := \frac{a-x}{K}, \tilde{y} := \frac{a-y}{K}$. Then $\tilde{x}, \tilde{y} \in B_1(0)$. Assume that

$$\left\| a - \frac{x+y}{2} \right\| > K - \frac{\varepsilon^2}{8d} \overset{K \leq d}{\geq} K - \frac{\varepsilon^2}{8K}.$$

Then

$$\left\| \frac{\tilde{x} + \tilde{y}}{2} \right\| = \frac{1}{K} \left\| a - \frac{x+y}{2} \right\| > 1 - \frac{\varepsilon^2}{8K^2}.$$

Since $\frac{\varepsilon^2}{8K^2} = \frac{(\varepsilon/K)^2}{8}$, lemma 2.1 yields that

$$\frac{1}{K} \|x - y\| = \|\tilde{x} - \tilde{y}\| < \frac{\varepsilon}{K}$$

and so $\|x - y\| < \varepsilon$. $\square$

Lemma 2.3. *Let X be a real Hilbert space and $U : C \to C$ be a nonexpansive mapping, i.e.*

$$\forall x, y \in C\, (\|U(x) - U(y)\| \leq \|x - y\|).$$

Then the following holds:

$$\forall t \in [0, 1]\, \forall \varepsilon \in (0, 1]\, \forall p_1, p_2 \in C\, \Big(\bigwedge_{i=1}^{2} \|p_i - U(p_i)\| \leq \frac{\varepsilon^2}{16d} \to \|q_t - U(q_t)\| < \varepsilon \Big),$$

where $q_t := (1 - t)p_1 + tp_2 \in C$.

Proof: Define $K := \max\{\|p_1 - q_t\|, \|p_1 - U(q_t)\|\} \le d$. By assumption we have

$$(1) \bigwedge_{i=1}^{2} \|p_i - U(p_i)\| \le \frac{\varepsilon^2}{16d}.$$

For $i = 2$ this yields

$$(2) \left\{ \begin{array}{l} \left\|p_2 - \frac{q_t+U(q_t)}{2}\right\| \le \frac{1}{2}\|p_2 - q_t\| + \frac{1}{2}\|p_2 - U(q_t)\| \\ \le \frac{1}{2}\|p_2 - q_t\| + \frac{1}{2}\|U(p_2) - U(q_t)\| + \frac{\varepsilon^2}{32d} \\ \le \frac{1}{2}\|p_2 - q_t\| + \frac{1}{2}\|p_2 - q_t\| + \frac{\varepsilon^2}{32d} = \|p_2 - q_t\| + \frac{\varepsilon^2}{32d}. \end{array} \right.$$

(2) implies that

$$(3) \left\|p_1 - \frac{q_t + U(q_t)}{2}\right\| \ge \|p_1 - q_t\| - \frac{\varepsilon^2}{32d}$$

since, otherwise,

$$\begin{array}{rl} \|p_1 - p_2\| \le & \left\|p_1 - \frac{q_t+U(q_t)}{2}\right\| + \left\|p_2 - \frac{q_t+U(q_t)}{2}\right\| \\ \overset{(2)}{<} & \|p_1 - q_t\| - \frac{\varepsilon^2}{32d} + \|p_2 - q_t\| + \frac{\varepsilon^2}{32d} \\ = & t\|p_1 - p_2\| + (1-t)\|p_1 - p_2\| = \|p_1 - p_2\| \end{array}$$

which is a contradiction.
By (1) applied to $i = 1$ we also have

$$(4) \|p_1 - U(q_t)\| \le \|U(p_1) - U(q_t)\| + \frac{\varepsilon^2}{16d} \le \|p_1 - q_t\| + \frac{\varepsilon^2}{16d}.$$

(3) and (4) yield that

$$(5) \left\|p_1 - \frac{q_t + U(q_t)}{2}\right\| \ge \|p_1 - q_t\| - \frac{\varepsilon^2}{32d} \ge K - \frac{\varepsilon^2}{16d} - \frac{\varepsilon^2}{32d} > K - \frac{\varepsilon^2}{8d}.$$

Lemma 2.2 now implies that $\|q_t - U(q_t)\| < \varepsilon$. □

We now come to the lemma stating the existence of a fixed point $u \in C$ of U that has minimal distance to the given point v_0, i.e.

$$(+) \ \exists u \in C\big(U(u) = u \wedge \forall v \in C\, (U(v) = v \to \|v_0 - u\| \le \|v_0 - v\|)\big).$$

We first exhibit the quantifiers hidden in the formulas $U(u) = u, U(v) = v$ and $\|v_0 - u\| \le \|v_0 - v\|$ (switching in the last inequality for later convenience to the squares)

$$\begin{array}{l} \exists u \in C\Big(\forall \delta \in (0,1]\, (\|U(u) - u\| < \delta) \\ \quad \wedge\ \forall v \in C\, (\forall \eta \in (0,1]\, (\|U(v) - v\| < \eta) \to \forall \varepsilon \in (0,1]\, (\|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon))\Big). \end{array}$$

There are still existential resp. universal quantifiers left in $<$ resp. $\leq$ between real numbers. However, as we freely can choose whether to use $<$ or $\leq$ these formulas behave as if they were quantifier-free (which they strictly speaking would become by considering suitable rational approximations which we avoid here for simplicity).
As it will turn out, we only need to analyze quantitatively the weaker 'ε-version' of this statement, where instead of the existence of a $u \in C$ satisfying the conclusion for all $\varepsilon > 0$ we only state that for each $\varepsilon > 0$ a point $u \in C$ exists, i.e.

$$(++)\quad \begin{cases} \forall \varepsilon \in (0,1]\, \exists u \in C \Big(\forall \delta \in (0,1]\, (\|U(u) - u\| < \delta) \\ \quad \wedge\, \forall v \in C\, (\forall \eta \in (0,1]\, (\|U(v) - v\| < \eta) \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon) \Big). \end{cases}$$

Whereas the proof of (+) requires the axiom of countable choice, (++) can be proved using induction only (see [15]).
We now sketch the Gödel functional interpretation (strictly speaking the combination of a negative translation and the actual functional interpretation, see [14]) of (++). The result will be a $\forall\exists$-statement that is equivalent to (++) based only on logic and a form of quantifier-free choice

$$\text{QF-AC: } \forall x \in A\, \exists y \in B\ F_{qf}(x, y) \to \exists Y : A \to B\, \forall x \in A\ F_{qf}(x, Y(x)),$$

where F_{qf} is quantifier-free (or purely existential) and A, B are spaces such that the predicates $x \in A$ and $y \in B$ do not add extra quantifiers (the latter is the case for the treatment of abstract convex sets C as in [14] and for $(0,1]$ if one restricts oneself to rational numbers which we – for convenience – will not do explicitly though).
(++) is logically equivalent to

$$\begin{aligned} &\forall \varepsilon \in (0,1]\, \exists u \in C \Big(\forall \delta \in (0,1]\, (\|U(u) - u\| < \delta) \\ &\quad \wedge\, \forall v \in C\, \exists \eta \in (0,1]\, (\|U(v) - v\| < \eta \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon) \Big). \end{aligned}$$

By QF-AC this is equivalent to

$$\begin{aligned} &\forall \varepsilon \in (0,1]\, \exists u \in C \Big(\forall \delta \in (0,1]\, (\|U(u) - u\| < \delta) \\ &\quad \wedge\, \exists \varphi : C \to (0,1]\, \forall v \in C\, (\|U(v) - v\| < \varphi(v) \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon) \Big) \end{aligned}$$

and – by logic – in turn is equivalent to

$$\begin{aligned} &\forall \varepsilon \in (0,1]\, \exists u \in C\, \exists \varphi : C \to (0,1]\, \forall \delta \in (0,1]\, \forall v \in C \\ &\quad \Big(\|U(u) - u\| < \delta\ \wedge\ (\|U(v) - v\| < \varphi(v) \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon) \Big). \end{aligned}$$

Using again QF-AC, the last formula is equivalent to

$$\begin{aligned}&\forall\varepsilon\in(0,1]\,\forall\Delta:C\times(C\to(0,1])\to(0,1]\,\forall V:C\times(C\to(0,1])\to C\\&\quad\exists u\in C\,\exists\varphi:C\to(0,1]\,\Big(\|U(u)-u\|<\Delta(u,\varphi)\\&\qquad\wedge(\|U(V(u,\varphi))-V(u,\varphi)\|<\varphi(V(u,\varphi))\to\|v_0-u\|^2<\|v_0-V(u,\varphi)\|^2+\varepsilon)\Big).\end{aligned}$$

General proof-theoretic results on Gödel's functional interpretation show that '$\exists u\in C\,\exists\varphi:C\to(0,1]$' can be solved explicitly as functionals in ε,Δ,V (and U,v_0 as further parameters) and that these functionals exhibit the correct numerical content of the original statement:

Lemma 2.4. *Let X be a real Hilbert space, $v_0\in C$ and U be as in lemma 2.3. Let $\varepsilon\in(0,1],\Delta:C\times(C\to(0,1])\to(0,1]$ and $V:C\times(C\to(0,1])\to C$. Then one can construct $u\in C$ and $\varphi:C\to(0,1]$ such that*

$$(1)\ \|u-U(u)\|<\Delta(u,\varphi)$$

and

$$(2)\ \begin{cases}\|U(V(u,\varphi))-V(u,\varphi)\|<\varphi(V(u,\varphi))\to\\\|v_0-u\|^2\le\|v_0-V(u,\varphi)\|^2+\varepsilon.\end{cases}$$

In fact, u,φ can be defined explicitly as functionals in ε,Δ,V (as well as in v_0,U and some fixed point $\widehat{u}\in C$ of U which we, however, do not mention as arguments as these are fixed parameters) as follows: for $i<n_\varepsilon:=\left\lceil\frac{d^2}{\varepsilon}\right\rceil$ we define $\varphi_i:C\to(0,1]$ and $u_i\in C$ inductively by

$$\begin{aligned}&\varphi_0(v):=1,\ \varphi_{i+1}(v):=\Delta(v,\varphi_i),\\&u_0:=\widehat{u}\in Fix(U),\ u_{i+1}:=V(u_i,\varphi_{n_\varepsilon-i-1}).\end{aligned}$$

Then for some $i<n_\varepsilon$ (that we may find by bounded search) we have that $u:=u_i,\ \varphi:=\varphi_{n_\varepsilon-i-1}$ satisfy the claim.
Instead of C being d-bounded it suffices to assume that $d\ge\|v_0-\widehat{u}\|$ for some $\widehat{u}\in Fix(U)$.

Remark 2.5. *The bounded search in the construction of u,φ in the proof above can easily be made effective (relative to U,X etc.) by using suitable rational approximations of the norms in (2). However, as we will anyhow only need majorants for these functionals this can be avoided (since $u\in C$, we can take as trivial majorant for u a suitable constant function, whereas for φ one – essentially – takes the minimum of all φ_i for $i<n_\varepsilon$; see definition 2.12 below for a more precise definition of majorization).*

Proof: Case 1:

$$\exists i<n_\varepsilon\ \big(\|U(u_i)-u_i\|\ge\Delta(u_i,\varphi_{n_\varepsilon-i-1})\big).$$

Let i_0 be minimal with this property. Since $\|U(u_0) - u_0\| = 0 < \Delta(u_0, \varphi_{n_\varepsilon - 1})$, we have that $i_0 > 0$. Hence

$$\|U(u_{i_0-1}) - u_{i_0-1}\| < \Delta(u_{i_0-1}, \varphi_{n_\varepsilon-(i_0-1)-1}).$$

If

$$\|U(V(u_{i_0-1}, \varphi_{n_\varepsilon-(i_0-1)-1})) - V(u_{i_0-1}, \varphi_{n_\varepsilon-(i_0-1)-1})\| \geq \varphi_{n_\varepsilon-(i_0-1)-1}(V(u_{i_0-1}, \varphi_{n_\varepsilon-(i_0-1)-1})),$$

then we are done as $u_{i_0-1}, \varphi_{n_\varepsilon-(i_0-1)-1}$ then satisfy the claim. So we may assume that (using that $u_{i_0} = V(u_{i_0-1}, \varphi_{n_\varepsilon-(i_0-1)-1})$)

$$\|U(u_{i_0}) - u_{i_0}\| < \varphi_{n_\varepsilon-(i_0-1)-1}(u_{i_0}) = \Delta(u_{i_0}, \varphi_{n_\varepsilon-i_0-1}).$$

This, however, contradicts the construction of i_0.
Case 2:

$$\forall i < n_\varepsilon \left(\|U(u_i) - u_i\| < \Delta(u_i, \varphi_{n_\varepsilon-i-1})\right).$$

If the claim of the lemma would fail for all $i < n_\varepsilon$, then

$$\forall i < n_\varepsilon \left(\|v_0 - u_i\|^2 - \varepsilon > \|v_0 - \underbrace{V(u_i, \varphi_{n_\varepsilon-i-1})}_{=u_{i+1}}\|^2\right)$$

and so $\|v_0 - u_0\|^2 > n_\varepsilon \cdot \varepsilon \geq d^2$ which contradicts the fact that $\|v_0 - u_0\|^2 \leq d^2$. □

Lemma 2.6. *Let X be a real Hilbert space, $v_0 \in C$ and U be as in lemma 2.3. Let $\varepsilon \in (0,1], t \in [0,1], \Delta : C \times (C \to (0,1]) \to (0,1]$ and $V : C \times (C \to (0,1]) \to C$. Then one can construct $\tilde{u} := \tilde{u}_{v_0,U}(t, \varepsilon, \Delta, V) \in C$ and $\tilde{\varphi} := \tilde{\varphi}_{v_0,U}(t, \varepsilon, \Delta, V) : C \to (0,1]$ such that*

$$(1)\ \|\tilde{u} - U(\tilde{u})\| < \Delta(\tilde{u}, \tilde{\varphi})$$

and

$$(2) \begin{cases} \|U(V(\tilde{u}, \tilde{\varphi})) - V(\tilde{u}, \tilde{\varphi})\| < \tilde{\varphi}(V(\tilde{u}, \tilde{\varphi})) \to \\ \|v_0 - \tilde{u}\|^2 \leq \|v_0 - [(1-t)\tilde{u} + tV(\tilde{u}, \tilde{\varphi})]\|^2 + \varepsilon. \end{cases}$$

Moreover, using the solution operators $u[\varepsilon, \Delta, V], \varphi[\varepsilon, \Delta, V]$ from lemma 2.4 we may take

$$\begin{aligned} &\tilde{u} := \tilde{u}[t, \varepsilon, \Delta, V] := u[\varepsilon, \Delta', V'], \\ &\tilde{\varphi} := \tilde{\varphi}[t, \varepsilon, \Delta, V] := \varphi^*_{t,u} := (\varphi[\varepsilon, \Delta', V'])^*_{t,u}, \end{aligned}$$

where

$$\begin{aligned} &\Delta'(u, \varphi) := \Delta'_t(u, \varphi) := \min\{\Delta(u, \varphi^*_{t,u}), \varphi^*_{t,u}(V(u, \varphi^*_{t,u}))\}, \\ &V'(u, \varphi) := V'_t(u, \varphi) := (1-t)u + tV(u, \varphi^*_{t,u}) \text{ with} \\ &\varphi^*_{t,u}(v) := \tfrac{\varphi((1-t)u+tv)^2}{16d}. \end{aligned}$$

Proof: By lemma 2.4 it follows (with $\tilde{\varphi} := \varphi^*_{t,u}$ for $(\varphi[\varepsilon, \Delta', V'])^*_{t,u}$ and $\tilde{u}$ for $u[\varepsilon, \Delta', V']$)

$$(3)\ \|\tilde{u} - U(\tilde{u})\| < \min\{\Delta(\tilde{u}, \tilde{\varphi}), \tilde{\varphi}(V(\tilde{u}, \tilde{\varphi}))\} \leq \Delta(\tilde{u}, \tilde{\varphi}), \frac{\varphi((1-t)\tilde{u} + tV(\tilde{u}, \tilde{\varphi}))^2}{16d}.$$

Now assume that

$$(4)\ \|U(V(\tilde{u}, \tilde{\varphi})) - V(\tilde{u}, \tilde{\varphi})\| < \tilde{\varphi}(V(\tilde{u}, \tilde{\varphi})) = \frac{\varphi((1-t)\tilde{u} + tV(\tilde{u}, \tilde{\varphi}))^2}{16d}.$$

Since

$$(5)\ V'(\tilde{u}, \varphi) = (1-t)\tilde{u} + tV(\tilde{u}, \tilde{\varphi})$$

$(3), (4)$ and lemma 2.3 yield that

$$(6)\ \|U(V'(\tilde{u}, \varphi)) - V'(\tilde{u}, \varphi)\| < \varphi(V'(\tilde{u}, \varphi)).$$

Using again that $\tilde{u}, \varphi := \varphi[\varepsilon, \Delta', V']$ solve lemma 2.4 for Δ', V', (6) implies

$$\|v_0 - \tilde{u}\|^2 \leq \|v_0 - V'(\tilde{u}, \varphi)\|^2 + \varepsilon = \|v_0 - ((1-t)\tilde{u} + tV(\tilde{u}, \tilde{\varphi}))\|^2 + \varepsilon.$$

□

The aforementioned fact that $\|v_0 - u_0\|^2 \leq \|v_0 - u_t\|^2$ is used by Browder (still in the proof of his 'Lemma 1') to show that $\langle v_0 - u_0, u_0 - v\rangle \geq 0$. We now give a quantitative version of this step:

Lemma 2.7. *Let X be a real Hilbert space and $v_0, v, u \in C$, $t \in [0,1]$ and define $w_t := (1-t)u + tv \in C$. Then the following holds:*

$$\forall \varepsilon \in (0,1] \left(\|v_0 - u\|^2 \leq \frac{\varepsilon^2}{2d^2} + \|v_0 - w_{\frac{\varepsilon}{3d^2}}\|^2 \to \langle v_0 - u, u - v\rangle > -\varepsilon\right).$$

Instead of C being d-bounded is suffices that $d \geq \|u - v\|$.

Proof: Assume

$$(1)\ \|v_0 - u\|^2 \leq \frac{\varepsilon^2}{2d^2} + \|v_0 - w_{\frac{\varepsilon}{3d^2}}\|^2.$$

We have

$$(2) \begin{cases} \|v_0 - w_{\frac{\varepsilon}{3d^2}}\|^2 \\ = \langle v_0 - w_{\frac{\varepsilon}{3d^2}}, v_0 - w_{\frac{\varepsilon}{3d^2}}\rangle = \langle v_0 - \left((1 - \frac{\varepsilon}{3d^2})u + \frac{\varepsilon}{3d^2}v\right), v_0 - \left((1 - \frac{\varepsilon}{3d^2})u + \frac{\varepsilon}{3d^2}v\right)\rangle \\ = \langle v_0 - u, v_0 - u + \frac{\varepsilon}{3d^2}u - \frac{\varepsilon}{3d^2}v\rangle + \langle \frac{\varepsilon}{3d^2}u - \frac{\varepsilon}{3d^2}v, v_0 - u + \frac{\varepsilon}{3d^2}u - \frac{\varepsilon}{3d^2}v\rangle \\ = \|v_0 - u\|^2 + \langle v_0 - u, \frac{\varepsilon}{3d^2}(u - v)\rangle + \langle \frac{\varepsilon}{3d^2}(u - v), v_0 - u\rangle + \langle \frac{\varepsilon}{3d^2}(u - v), \frac{\varepsilon}{3d^2}(u - v)\rangle. \end{cases}$$

Hence by (1) and (2)

$$-\frac{\varepsilon^2}{2d^2} \leq \frac{2}{3d^2}\varepsilon\langle v_0 - u, u - v\rangle + \frac{\varepsilon^2}{9d^4}\langle u - v, u - v\rangle$$

and so multiplying through with $3d^2/2\varepsilon$ and using that $\|u-v\|^2 \le d^2$

$$-\frac{3}{4}\varepsilon \le \langle v_0 - u, u - v\rangle + \frac{\varepsilon}{6d^2}\|u-v\|^2 \le \langle v_0 - u, u - v\rangle + \frac{1}{6}\varepsilon.$$

Hence

$$-\varepsilon < -\frac{3}{4}\varepsilon - \frac{1}{6}\varepsilon \le \langle v_0 - u, u - v\rangle.$$

□

Lemma 2.8. *Let X be a real Hilbert space and $U : C \to C$ as in lemma 2.3. Define $T := Id - U$. Then for all $u, v \in C$*

$$\langle T(u) - T(v), u - v\rangle \ge 0.$$

Proof: Using Cauchy-Schwarz we get $0 \le \|u-v\|^2 - \langle U(u) - U(v), u - v\rangle = \langle T(u) - T(v), u - v\rangle$. □

In Browder's proof the convergence of $(\breve{u}_n)$ towards u_0, where $\breve{u}_n := u_{1-\frac{1}{n}}$, is derived from the fact that for each $k_n \to \infty$ the sequence $(\breve{u}_{k_n})$ has a subsequence converging to u_0 (see the beginning of his 'Proof of Theorem 1'). In our 'metastable' rendering this argument gives rise to:

Lemma 2.9. *Let X be a normed linear space. Then the following holds:*

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall u \in X\, \forall (v_n) \subset X\, \forall m \in \mathbb{N} \\ \big(\|v_{g_{u,\varepsilon}(m)} - u\| \le \tfrac{\varepsilon}{2} \to \|v_{g(m)} - v_m\| \le \varepsilon\big),$$

where

$$g_{u,\varepsilon}(m) := \begin{cases} g(m), & \text{if } \|v_{g(m)} - u\| > \frac{\varepsilon}{2} \\ m, & \text{otherwise} \end{cases}$$

(more precisely one should write $g_{u,\varepsilon,(v_n)}$ instead of $g_{u,\varepsilon}$ as the function also depends on (v_n)).

Proof: Assume that $\|v_{g_{u,\varepsilon}(m)} - u\| \le \frac{\varepsilon}{2}$. Then, by the construction of $g_{u,\varepsilon}$,

$$\|v_{g(m)} - u\| \le \frac{\varepsilon}{2} \text{ and } g_{u,\varepsilon}(m) = m.$$

Hence

$$\|v_{g(m)} - v_m\| \le \|v_{g(m)} - u\| + \|v_m - u\| \le \varepsilon.$$

□

Browder continues by showing that $\breve{u}_n$ is an approximate fixed point sequence. The next lemma gives a rate of convergence:

Lemma 2.10. *Let X be a Banach space and U as in lemma 2.3. Let $v_0 \in C$ and $U_t(x) := tU(x) + (1-t)v_0$ for $t \in (0,1)$.*
Then $U_t : C \to C$ has a unique fixed point $u_t \in C$. Moreover, the following holds:

$$\forall \varepsilon \in (0,1]\, \forall t \in (1 - \frac{\varepsilon}{d}, 1)\, (\|u_t - U(u_t)\| < \varepsilon).$$

Proof: The first part of the lemma easily follows from the Banach fixed point theorem since U_t is a strict contraction. For the last part we argue as follows:

$$\begin{aligned}
&\|u_t - U(u_t)\| \le \|u_t - U_t(u_t)\| + \|U_t(u_t) - U(u_t)\| = \|U_t(u_t) - U(u_t)\| \\
&\le \|tU(u_t) + (1-t)v_0 - tU(u_t) - (1-t)U(u_t)\| \\
&\le (1-t)\|v_0 - U(u_t)\| \le (1-t)d < \varepsilon.
\end{aligned}$$

□

The next lemma is a quantitative version of the main combinatorial core of Browder's proof where it is applied to the weak limit v of a suitable subsequence of $(\check{u}_n)$.

Lemma 2.11. *Let X be a real Hilbert space and U as in lemma 2.3. Let (s_n) be a sequence in $(0,1)$ and $h : \mathbb{N} \to \mathbb{N}^*$ be such that $s_n \le 1 - \frac{1}{h(n)}$ for all $n \in \mathbb{N}$ Furthermore, let $v_0, u, v \in C$, $\check{u}_n := u_{s_n}$ $(n \in \mathbb{N})$, where u_t is defined as in lemma 2.10, $\varepsilon \in (0,1], j \in \mathbb{N}^*$. Then the following holds:*
If

$$(1)\ \|u - U(u)\| \le \frac{\varepsilon^2}{3d \cdot h(j)},\ (2)\ \langle v_0 - u, \check{u}_j - v\rangle \le \frac{\varepsilon^2}{3}\ \textit{and}\ (3)\ \langle v_0 - u, v - u\rangle \le \frac{\varepsilon^2}{3},$$

then $\|\check{u}_j - u\| \le \varepsilon$.

Proof: One easily verifies using the definition of $\check{u}_j$ that

$$(4)\ (1 - s_j)\check{u}_j + s_j(\check{u}_j - U(\check{u}_j)) = (1 - s_j)v_0.$$

By (1) we have that

$$(5)\ \|s_j(U(u) - u)\| = s_j \cdot \|U(u) - u\| \le \|U(u) - u\| \le \frac{\varepsilon^2}{3d \cdot h(j)}.$$

With $T := \mathrm{Id} - U$, (4) and (5) yield

$$(6)\ \|(1 - s_j)(\check{u}_j - u) + s_j(T(\check{u}_j) - T(u)) - (1 - s_j)(v_0 - u)\| \le \frac{\varepsilon^2}{3d \cdot h(j)}.$$

Hence (since $\|\check{u}_j - u\| \le d$)

$$(7)\ \begin{cases} (1-s_j)\langle \check{u}_j - u, \check{u}_j - u\rangle + s_j\langle T(\check{u}_j) - T(u), \check{u}_j - u\rangle \\ \le (1-s_j)\langle v_0 - u, \check{u}_j - u\rangle + \frac{\varepsilon^2}{3h(j)} \end{cases}$$

and, therefore,

$$(8)\ (1-s_j)\|\check{u}_j - u\|^2 \le (1-s_j)\langle v_0 - u, \check{u}_j - u\rangle + \frac{\varepsilon^2}{3h(j)}$$

since by lemma 2.8 $\langle T(\check{u}_j) - T(u), \check{u}_j - u\rangle \ge 0$.
Hence (using that $1/(h(j)(1-s_j)) \le 1$)

$$\begin{aligned} &\|\check{u}_j - u\|^2 \le \langle v_0 - u, \check{u}_j - u\rangle + \tfrac{\varepsilon^2}{3} \\ &= \langle v_0 - u, v - u\rangle + \langle v_0 - u, \check{u}_j - v\rangle + \tfrac{\varepsilon^2}{3} \overset{(2),(3)}{\le} \varepsilon^2. \end{aligned}$$

So, finally, $\|\check{u}_j - u\| \le \varepsilon$. □

Corollary to the proof of lemma 2.11: By instantiating $v := \check{u}_j$, which makes '(2)' trivially true with '$= 0$' instead of '$\le \frac{\varepsilon^2}{3}$', one gets

$$\|u - U(u)\| \le \frac{\varepsilon^2}{2d\cdot h(j)} \ \wedge\ \langle v_0 - u, \check{u}_j - u\rangle \le \frac{\varepsilon^2}{2} \ \to\ \|\check{u}_j - u\| \le \varepsilon.$$

Definition 2.12. *For $\varphi : C \to (0,1]$ and $k \in \mathbb{N}^*$ we define a notion of majorization as follows:*

$$k \gtrsim \varphi :\equiv \forall v \in C\,(1/k \le \varphi(v)).$$

Lemma 2.13. *Let $X, U, v_0, \check{u}_n$ be as before and $\chi : \mathbb{N} \to \mathbb{N}$ be a rate of convergence of (s_n) towards 1, i.e. $\forall n \in \mathbb{N}\, \forall i \ge \chi(n)\, (|1 - s_i| \le \frac{1}{n+1})$. Then the following holds:*

$$\begin{aligned} &\forall \varepsilon \in (0,1]\, \forall g : \mathbb{N} \to \mathbb{N}^*\, \forall \varphi : C \to (0,1]\, \forall u \in C\, \forall k \gtrsim \varphi \\ &\qquad \Big(\|U(\check{u}_{\tilde{g}_{u,\varepsilon}(\chi(d\cdot k))}) - \check{u}_{\tilde{g}_{u,\varepsilon}(\chi(d\cdot k))}\| < \varphi(\check{u}_{\tilde{g}_{u,\varepsilon}(\chi(d\cdot k))})\Big), \end{aligned}$$

where $\tilde{g}_{u,\varepsilon}$ is defined as in lemma 2.9 (but with $\tilde{g}(n) := \max\{n, g(n)\}$ instead of g and with $v_n := \check{u}_n$).

Proof: Use lemma 2.10 and $\tilde{h}(j) \ge j$ and that $h = \tilde{h}$ for $h(n) := \tilde{g}_{u,\varepsilon}(n)$. □

Definition 2.14. *We say that a function $f^* : \mathbb{N} \to \mathbb{N}$ majorizes a function $f : \mathbb{N} \to \mathbb{N}$ (short: $f^* \gtrsim f$), if*

$$\forall n, m \in \mathbb{N}(m \ge n \to f^*(m) \ge f^*(n), f(n)).$$

In the following, we again use the construction $f^M(n) := \max\{f(i) : i \leq n\}$ to construct a majorant for $f : \mathbb{N} \to \mathbb{N}$.

Theorem 2.15. *Let X be a real Hilbert space, $d \in \mathbb{N}^*$ and $C \subset X$ be a bounded closed convex subset with $d \geq diam(C)$. Let $U : C \to C$ be a nonexpansive mapping, $v_0 \in C$. Let (s_n) be a sequence in $(0,1)$ that converges towards 1 and $h : \mathbb{N} \to \mathbb{N}^*$ such that $s_n \leq 1 - \frac{1}{h(n)}$ and $h(n) \geq n$ for all $n \in \mathbb{N}$. $\check{u}_n := u_{s_n}$ where – for $t \in (0,1)$ – u_t is the unique fixed point of $U_t(x) := tU(x)+(1-t)v_0$ in C. Then for all $\varepsilon \in (0,1]$, $g : \mathbb{N} \to \mathbb{N}^*$*

$$\exists j \leq \Phi(\varepsilon, g, \chi_g, h, d) \left(\|\check{u}_j - \check{u}_{\tilde{g}(j)}\| \leq \varepsilon\right),$$

where $\tilde{g}(n) := \max\{n, g(n)\}$, $\varepsilon_d := \frac{(\varepsilon/2)^4}{8d^2}$, $n_{\varepsilon,d} := \left\lceil \frac{d^2}{\varepsilon_d} \right\rceil$,

$$\Phi(\varepsilon, g, \chi_g, h, d) := \chi_g^M \left(16d^2 \cdot \left(\max\{(\Delta^*_{\varepsilon,g})^{(i)}(1) : i < n_{\varepsilon,d}\}\right)^2\right)$$

with

$$\Delta^*_{\varepsilon,g}(n) := \left\lceil \frac{8d \cdot h^M\left(\tilde{g}^M\left(\chi_g^M(16d^2 \cdot n^2)\right)\right)}{\varepsilon^2} \right\rceil$$

and $\chi_g : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}\, \forall i \in [\chi_g(n); \tilde{g}^M(\chi_g(n))]\, (|1 - s_i| \leq \frac{1}{n+1})$.

Before we prove the theorem we adapt it to get a bound on the so-called metastability of $(\check{u}_n)$ in the sense of [30]:

Corollary 2.16.

$$\forall \varepsilon \in (0,1], g : \mathbb{N} \to \mathbb{N}^*\, \exists n \leq \Phi(\varepsilon/2, g^+, \chi_{g^+}, h, d)\, \forall i, j \in [n; n + g(n)]\ (\|\check{u}_i - \check{u}_j\| \leq \varepsilon),$$

where $g^+(n) := n + g(n)$.

Proof: Apply theorem 2.15 to

$$g^-(n) := n + \min i \leq g(n)\, [\forall j \leq g(n)\, (\|\check{u}_n - \check{u}_{n+j}\| \leq \|\check{u}_n - \check{u}_{n+i}\|)]$$

and $\varepsilon/2$. Note that $\tilde{g^-} = g^-$ and $(g^+)^M \gtrsim (g^-)^M$. Hence

$$\exists n \leq \Phi(\varepsilon/2, g^-, \chi_{g^+}, h, d) \leq \Phi(\varepsilon/2, g^+, \chi_{g^+}, h, d)\, (\|\check{u}_n - \check{u}_{g^-(n)}\| \leq \varepsilon/2).$$

From

$$\|\check{u}_n - \check{u}_{n+j}\| \leq \|\check{u}_n - \check{u}_{g^-(n)}\| \leq \varepsilon/2$$

for all $j \leq g(n)$ one now gets

$$\forall i, j \in [n; n + g(n)]\ (\|\check{u}_i - \check{u}_j\| \leq \|\check{u}_n - \check{u}_i\| + \|\check{u}_n - \check{u}_j\| \leq \varepsilon).$$

$\square$

Proof of theorem 2.15: For $\varepsilon \in (0,1]$ and $g : \mathbb{N} \to \mathbb{N}^*$ we define functionals

$$J_{\varepsilon,g} : C \times (C \to (0,1]) \to \mathbb{N} \text{ and } V_{\varepsilon,g} : C \times (C \to (0,1]) \to C$$

as follows: for given $u \in C$ and $\varphi : C \to (0,1]$ let $J_{\varepsilon,g}(u,\varphi)$ be the least j such that

$$(0)\ \|U(\check{u}_{\tilde{g}_{u,\varepsilon}(j)}) - \check{u}_{\tilde{g}_{u,\varepsilon}(j)}\| < \varphi(\check{u}_{\tilde{g}_{u,\varepsilon}(j)})$$

if such a j exists (which by lemma 2.13 always is the case if φ is majorizable in the sense that there exists a k with $k \gtrsim \varphi$) and $:= 0$ otherwise. Let $V_{\varepsilon,g}(u,\varphi) := \check{u}_{\tilde{g}_{u,\varepsilon}(J_{\varepsilon,g}(u,\varphi))}$. Define $\Delta_{\varepsilon,g}(u,\varphi) := \frac{(\varepsilon/2)^2}{2d\cdot h(\tilde{g}_{u,\varepsilon}(J_{\varepsilon,g}(u,\varphi)))}$ and find $\tilde{u}, \tilde{\varphi}$ as in lemma 2.6 with $\varepsilon_d := \frac{(\varepsilon/2)^4}{8d^2}$ instead of ε and $t := \xi^2/(6d^2)$, where $\xi := \varepsilon/2$. In (10) we will see that this $\tilde{\varphi}$ is majorizable. Hence for $j := J_{\varepsilon,g}(\tilde{u},\tilde{\varphi})$ and for $v := V_{\varepsilon,g}(\tilde{u},\tilde{\varphi})$ one has from lemmas 2.6 (applied to ε_d) and 2.13 that

$$(1)\ \|U(\tilde{u}) - \tilde{u}\| < \frac{\xi^2}{2d \cdot h(\tilde{g}_{\tilde{u},\varepsilon}(j))}$$

and

$$(2)\ \|v_0 - \tilde{u}\|^2 \le \|v_0 - [(1 - \frac{\xi^2}{6d^2})\tilde{u} + \frac{\xi^2}{6d^2}v]\|^2 + \varepsilon_d.$$

Because of

$$\frac{\left(\frac{\xi^2}{2}\right)^2}{2d^2} = \frac{\xi^4}{8d^2} = \varepsilon_d,$$

(2) together with lemma 2.7 (applied to $\xi^2/2$ instead of ε) yields that

$$(3)\ \langle v_0 - \tilde{u}, \tilde{u} - v\rangle \ge -\frac{\xi^2}{2}.$$

(1) and (3) together with the corollary to the proof of lemma 2.11 imply

$$\|\check{u}_{\tilde{g}_{\tilde{u},\varepsilon}(j)} - \tilde{u}\| \le \xi = \frac{\varepsilon}{2}$$

and so, by lemma 2.9,

$$(4)\ \|\check{u}_{\tilde{g}(j)} - \check{u}_j\| \le \varepsilon.$$

It remains to show that

$$\Phi(\varepsilon, g, \chi_g, h, d) \ge J_{\varepsilon,g}(\tilde{u},\tilde{\varphi}).$$

One easily verifies that

$$(5)\ \Delta^*_{\varepsilon,g}(n) \ge 16d \cdot n^2 \ge n$$

and

$$(6)\ k_1 \le k_2 \to \Delta^*_{\varepsilon,g}(k_1) \le \Delta^*_{\varepsilon,g}(k_2).$$

From lemma 2.13 (using that $\tilde{g}^M \gtrsim \tilde{g}_{u,\varepsilon}$) we have that for all $n \in \mathbb{N}$

$$(7)\ n \gtrsim \varphi \to J_{\varepsilon,g}(u,\varphi) \le \chi_g(d\cdot n) \qquad \forall u \in C, \varphi : C \to (0,1]$$

and so (again using $\tilde{g}^M \gtrsim \tilde{g}_{u,\varepsilon}$ and $h^M \gtrsim h, \chi_g^M \gtrsim \chi_g, \Delta^*_{\varepsilon,g}(n) \ge 16d\cdot n^2$)

$$(8)\ n \gtrsim \varphi \to\ \ 16d\cdot n^2 \gtrsim \varphi^*_{t,u} \to \frac{1}{\Delta^*_{\varepsilon,g}(n)} \le \Delta'_{\varepsilon,g}(u,\varphi)$$

for all $u \in C, \varphi : C \to (0,1]$, where $\Delta'_{\varepsilon,g}$ is defined from $\Delta_{\varepsilon,g}$ and $V_{\varepsilon,g}$ as in lemma 2.6 and

$$\varphi^*_{t,u}(v) := \frac{\varphi((1-t)u+tv)^2}{16d}.$$

(8) implies that for all $n \in \mathbb{N}$ and all $\varphi : C \to (0,1]$

$$(9)\ n \gtrsim \varphi \to \Delta^*_{\varepsilon,g}(n) \gtrsim \varphi',$$

where $\varphi'(u) := \Delta'_{\varepsilon,g}(u,\varphi)$ for $u \in C$.
Using these properties one shows (for the $\tilde{\varphi}$ at hand and using the construction on $\tilde{\varphi}$ from lemmas 2.6 and 2.4) that

$$(10)\ \ 16d\cdot\left(\max\{(\Delta^*_{\varepsilon,g})^{(i)}(1) : i < n_{\varepsilon,d}\}\right)^2 \gtrsim \tilde{\varphi}.$$

Hence (by (7))

$$\Phi(\varepsilon, g, \chi_g, h, d) \ge J_{\varepsilon,g}(\tilde{u}, \tilde{\varphi}).$$

□

Remark 2.17. *1. The proofs of theorem 2.15 and lemmas 2.6 and 2.4 yield that the assumption on $\widehat{u}$ being a fixed point of U (made in lemma 2.4) can be weakened to*

$$\|U(\widehat{u}) - \widehat{u}\| < \frac{1}{\Delta^*_{\varepsilon,g}(k)},$$

where

$$k := 16d\cdot\Big(\max\{(\Delta^*_{\varepsilon,g})^{(i)}(1) : i < n_{\varepsilon,d}\}\Big)^2.$$

Since that bound does not depend on $\widehat{u}$ it follows that the whole proof of theorem 2.15 only uses the (trivial) fact that U has approximate fixed points and so, in *fact, establishes the (nontrivial) existence of a fixed point: since the Herbrand normal form*

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N}^* \, \exists j \in \mathbb{N} \; (\|\check{u}_j - \check{u}_{\tilde{g}(j)}\| \leq \varepsilon)$$

of the Cauchy property of $(\check{u}_n)$ *is (ineffectively) equivalent to the Cauchy property itself and hence to the convergence of* $(\check{u}_n)$*, the existence of a limit (which obviously must be a fixed point of* U*) follows.*

2. *Given the quantitative analysis in the proof of theorem 2.15 one can notice in retrospect that the ineffective definition of* $J_{\varepsilon,g}$ *can be replaced first by (dropping for readability the arguments* χ_g, h*)*

$$J_{\varepsilon,g}(u,\varphi) := \begin{cases} \min j \leq \Phi(\varepsilon, g, d) \; (\|U(\check{u}_{\tilde{g}_{u,\varepsilon}(j)}) - \check{u}_{\tilde{g}_{u,\varepsilon}(j)}\| < \varphi(\check{u}_{\tilde{g}_{u,\varepsilon}(j)}) \text{ if existent,} \\ 0, \text{ otherwise} \end{cases}$$

as (7) *in the proof above is not used for all* φ *but only for* φ*'s that are majorized by the bound in* (10).

To make this definition fully effective (relative to $\|\cdot\|, U, v_0$*) we replace the condition*

$$\|U(\check{u}_{\tilde{g}_{u,\varepsilon}(j)}) - \check{u}_{\tilde{g}_{u,\varepsilon}(j)}\| < \varphi(\check{u}_{\tilde{g}_{u,\varepsilon}(j)})$$

by

$$\|U(\check{u}_{\tilde{g}_{u,\varepsilon}(j)}) - \check{u}_{\tilde{g}_{u,\varepsilon}(j)}\|_r <_{\mathbb{Q}} \frac{2}{3}\varphi(\check{u}_{\tilde{g}_{u,\varepsilon}(j)}),$$

where $\|\ldots\|_r$ *is a* $\frac{1}{3}\varphi(\check{u}_{\tilde{g}_{u,\varepsilon}(j)})$*-good rational approximation to* $\|\ldots\|$.[1]

Then, however, (7) *only holds for the* φ*'s in question with* $d \cdot n$ *replaced by* $3d \cdot n$ *so that the bound in '*$\min j \leq \Phi(\varepsilon, g, d)$*' above must be replaced by* $3\Phi'(\varepsilon, g, d)$*, where* Φ' *is defined as* Φ *but with in* $\Delta^*_{\varepsilon,g}$ *the constant '*16*' being replaced by '*$3 \cdot 16$*'.*

3. *By the previous two items in this remark we have in the end obtained a fully elementary proof of the metastable-version of Browder's theorem and hence also of Browder's theorem itself: as discussed in 1. above, the metastable version implies the strong convergence of* $(\check{u}_n)$ *towards a point* $v \in C$*. By lemma 2.10 and the continuity of* U *this limit* v *trivially is a fixed point of* U*. From lemma 2.11 it follows that* $u = v$*, where* u *is the unique fixed point of* U *in* C *that is closest to* v_0*, since – by lemma 2.7 – this condition implies*

$$\langle v_0 - u, v - u \rangle \leq 0.$$

[1]Note that for rational ε the φ's in question are all rational valued.

In fact, by lemmas 2.7 (applied to $\varepsilon^2/2$) and the corollary to the proof of 2.11 the following 'metastable'-version of the existence of an approximately-closest (to v_0) approximate fixed point u is sufficient:

$$\|v_0 - u\|^2 \leq \|v_0 - \Phi(u)\|^2 + \frac{\varepsilon^4}{8d^2},$$

where $\Phi(u) := w_{\varepsilon^2/6d^2} = (1-\varepsilon^2/6d^2)u + (\varepsilon^2/6d^2)v$ suffices. Such a point can be elementary constructed (see lemma 2.6 and '1.' of this remark).

4. *It is clear that in the main result one can replace the condition that $\breve{u}_n$ is a fixed point of U_{s_n} by being a suitable approximate ('δ'-)fixed point (replacing also ε in the bound Φ by $\varepsilon/2$), where δ depends on ε, d, g only. Then the conditions on X being complete and C being closed can be dropped.*

3 A quantitative version of Wittmann's theorem

Let X be a real Hilbert space, $C \subseteq X$ a bounded closed and convex subset with $diam(C) \leq d \in \mathbb{N}^*$, $U : C \to C$ a nonexpansive mapping and $\alpha_n := \frac{1}{n+1}$. For $u_0 \in C$ define

$$u_{n+1} := \alpha_{n+1}\, u_0 + (1-\alpha_{n+1})\, U(u_n) \quad (n \geq 0).$$

In [32] it is shown that (u_n) strongly converges to a fixed point of U (namely the fixed point closest to u_0). In fact, as mentioned in the introduction, Wittmann considered more general sequences (α_n) of which the above one, though, is the most important instance, where it reduces to the well-known Cesàro mean for linear U. The general type of iteration is due to Halpern [8] whose results, however, do not cover the case $\alpha_n := \frac{1}{n+1}$. We confine ourselves here to this case to keep things less technical, but the general iterations can be treated in a similar way. Wittmann does not assume C to be bounded but stipulates that the fixed point set of U is nonempty (which is the case when C is bounded). We will comment later on the fact that this situation can also be covered by our analysis but, for the time being, assume that C is bounded.

In this section we adapt the analysis of Browder's proof from the previous section to an analysis of the proof of Wittmann's theorem. The parts of Wittmann's proof that are hard to analyze because of their ineffective nature are almost identical to the reasoning in Browder's proof: a projection to the fixed point closest to the starting point, the characterizing property of such a fixed point as well as the use of sequential weak compactness to the iteration sequence are as before. As a consequence of this we can re-use lemmas 2.1, 2.2, 2.3, 2.4, 2.6, 2.7 and lemma 2.9 unchanged. In fact, only the reasoning that resulted in the (corollary to the proof of the) quantitative lemma 2.11 is significantly different now, giving rise to a new lemma 3.2 below. Moreover, the trivial bound on asymptotic regularity (lemma 2.10) is more involved but, fortunately, has been extracted already from Wittmann's proof by L. Leuştean in [23] and – slightly improved and generalized to hyperbolic spaces – in [24] (pp.172-176), again using logical 'proof-mining'-techniques. Moreover, for the case $\alpha_n := 1/(n+1)$ that bound can be significantly improved:

Lemma 3.1. *Let $\chi_d(n) := 4dn(4dn+2)$. Then*

$$\forall n \in \mathbb{N}^* \, \forall k \geq \chi_d(n) \, \big(\|u_k - U(u_k)\| \leq \frac{1}{n}\big).$$

Instead of C being d-bounded only $d \geq \|u_n - u_m\|, \|u_0 - U(u_n)\|$ for all $n, m \in \mathbb{N}$ is needed.

Proof: In the proof of lemma 3.3 in [23] one can replace (for $\lambda_j := 1/(j+1)$) the expression '$\delta(N + \lceil \ln(2D/\varepsilon) \rceil)$' by '$\frac{2D\cdot(N+1)}{\varepsilon}$' (where $N := \gamma(\varepsilon/2)+1$) since

$$a_N \prod_{j=N}^{N+m-1} (1 - \lambda_{j+1}) = a_N \prod_{j=N}^{N+m-1} \left(1 - \frac{1}{j+2}\right) = a_N \frac{N+1}{N+m+1} \leq \frac{\varepsilon}{2}$$

for $m \geq \frac{2D\cdot(N+1)}{\varepsilon} - N$. From the proofs of Theorem 2.1 and Corollary 2.3 in [23] it then follows that (for $\varepsilon \in (0,2)$)

$$k \geq \max\left(\frac{4d\big(\alpha(\varepsilon/4d)+2\big)}{\varepsilon}, \alpha(\frac{\varepsilon}{2d})\right) \;\rightarrow\; \|u_k - U(u_k)\| < \varepsilon,$$

where $\alpha(\varepsilon) := \lceil 1/\varepsilon \rceil$ is a rate of convergence for $(\lambda_n)_n$. □

Lemma 3.2. *For $\varepsilon \in (0,1), j \in \mathbb{N}^*, g : \mathbb{N} \to \mathbb{N}^*, \tilde{g}(n) := \max\{n, g(n)\}$ define*

$$\Omega_d(\varepsilon, g, j) := \delta_{\varepsilon, \tilde{g}(\rho(\varepsilon^2/2d^2, j))}, \;\; \textit{where } \delta_{\varepsilon,m} := \min\left\{\sqrt{\frac{\varepsilon^2}{8m}}, \frac{\varepsilon^2}{16dm}\right\}$$

and $\rho(\varepsilon, n) := \lceil \frac{n+1}{\varepsilon} \rceil > n$ so that

$$\forall \varepsilon > 0 \, \forall n \in \mathbb{N} \, \forall m \geq \rho(\varepsilon, n) \left(\prod_{i=n+1}^{m} (1 - \alpha_i) \leq \varepsilon\right).$$

Then the following holds for all $u \in C, g : \mathbb{N} \to \mathbb{N}^*$:

$$\begin{cases} (1)\; j \geq \left\lceil \frac{8d^2}{\varepsilon^2} \right\rceil \;\wedge\; (2)\; \|u - U(u)\| \leq \Omega_d(\varepsilon, g, j) \;\wedge \\ (3)\; \forall \tilde{j} \in [j, \tilde{g}(\rho(\varepsilon^2/2d^2, j)) - 1] \; \left(\langle u_0 - u, U(u_{\tilde{j}}) - u\rangle \leq \frac{\varepsilon^2}{16}\right) \\ \qquad \rightarrow \|u_{\tilde{g}(\rho(\varepsilon^2/2d^2,j))} - u\| \leq \varepsilon. \end{cases}$$

Instead of C being d-bounded only $d \geq \|u_n - u\|$ for all $n \in \mathbb{N}$ is needed. (Note that for $\varepsilon \in (0,1), d \in \mathbb{N}^*$ one has $\delta_{\varepsilon,m} = \varepsilon^2/16dm$).

Proof: Since $\|u_0 - u\|^2 \le d^2$, (1) implies

$$\forall \tilde{j} \ge j \ (\alpha_{\tilde{j}}\|u_0 - u\|^2 \le \frac{\varepsilon^2}{8}).$$

Hence for all $\tilde{j} \in [j, \tilde{g}(\rho(\varepsilon^2/2d^2, j)) - 1]$ we get using (3)

$$\begin{aligned}
&\|u_{\tilde{j}+1} - u\|^2 = \\
&\alpha^2_{\tilde{j}+1}\|u_0 - u\|^2 + 2\alpha_{\tilde{j}+1}(1-\alpha_{\tilde{j}+1})\langle u_0 - u, U(u_{\tilde{j}}) - u\rangle + (1-\alpha_{\tilde{j}+1})^2\|U(u_{\tilde{j}}) - u\|^2 \\
&\le \alpha_{\tilde{j}+1}\cdot\tfrac{\varepsilon^2}{8} + \alpha_{\tilde{j}+1}\cdot\tfrac{\varepsilon^2}{8} + (1-\alpha_{\tilde{j}+1})\|U(u_{\tilde{j}}) - u\|^2 \\
&\le \alpha_{\tilde{j}+1}\tfrac{\varepsilon^2}{4} + (1-\alpha_{\tilde{j}+1})\|U(u_{\tilde{j}}) - u\|^2 \\
&\le \alpha_{\tilde{j}+1}\tfrac{\varepsilon^2}{4} + (1-\alpha_{\tilde{j}+1})\big(\|U(u_{\tilde{j}}) - U(u)\| + \|u - U(u)\|\big)^2 \\
&\stackrel{(2)}{\le} \alpha_{\tilde{j}+1}\tfrac{\varepsilon^2}{4} + (1-\alpha_{\tilde{j}+1})\big(\|u_{\tilde{j}} - u\|^2 + 2d\Omega_d(\varepsilon,g,j) + \Omega_d(\varepsilon,g,j)^2\big).
\end{aligned}$$

Hence for $\tilde{j} := \tilde{g}(\rho(\varepsilon^2/2d^2, j))$ one gets

$$\begin{aligned}
&\|u_{\tilde{j}} - u\|^2 \stackrel{\tilde{g}(n)\ge n}{\le} \tfrac{\varepsilon^2}{4} + \|u_j - u\|^2 \cdot \prod_{i=j+1}^{\tilde{j}}(1-\alpha_i) + \tilde{j}\cdot\big(2d\Omega_d(\varepsilon,g,j) + \Omega_d(\varepsilon,g,j)^2\big) \\
&\le \tfrac{\varepsilon^2}{4} + \|u_j - u\|^2\cdot\tfrac{\varepsilon^2}{2d^2} + \tilde{j}\left(\tfrac{\varepsilon^2}{8\tilde{j}} + \tfrac{\varepsilon^2}{8\tilde{j}}\right) \le \tfrac{\varepsilon^2}{4} + \tfrac{\varepsilon^2}{2} + \tfrac{\varepsilon^2}{4} = \varepsilon^2
\end{aligned}$$

and so $\|u_{\tilde{j}} - u\| \le \varepsilon$. □

Theorem 3.3. *Let $X, C, U, (u_n), \alpha_n$ be as above and $diam(C) \le d \in \mathbb{N}^*$. Then*

$$\forall \varepsilon \in (0,1)\,\forall g : \mathbb{N} \to \mathbb{N}^*\,\exists k \le \Phi(\varepsilon/2, g^+, d)\,\forall i, j \in [k; k+g(k)]\ \big(\|u_i - u_j\| \le \varepsilon\big),$$

where

$$\begin{aligned}
&\Phi(\varepsilon, g, d) := \rho(\varepsilon^2/8d^2, \chi_{d,\varepsilon}(N_{\varepsilon,g,d})) \text{ with} \\
&N_{\varepsilon,g,d} := 16d\cdot\big(\max\big\{(\Delta^*_{\varepsilon,g})^{(i)}(1) : i \le n_{\varepsilon,d}\big\}\big)^2,\ n_{\varepsilon,d} := \left\lceil \tfrac{d^2}{\varepsilon_d}\right\rceil,\ \varepsilon_d := \tfrac{\varepsilon^4}{8192d^2} \text{ and} \\
&\Delta^*_{\varepsilon,g}(n) := \lceil 1/\Omega_d(\varepsilon/2, \tilde{g}^M, \chi_{d,\varepsilon}(16d\cdot n^2))\rceil,\ \chi_{d,\varepsilon}(n) := \max\left\{\chi_d(n), \left\lceil\tfrac{32d^2}{\varepsilon^2}\right\rceil\right\}
\end{aligned}$$

with Ω_d, χ_d from the previous two lemmas and $g^+(n) := n + g(n)$.

Proof: The proof is very similar to that of theorem 2.15: define

$$(0)\ J_\varepsilon(\varphi) := \min l \ge \left\lceil\frac{4\cdot 8d^2}{\varepsilon^2}\right\rceil\ \left[\forall v \in C\,\forall \tilde{l} \ge l\,\big(\|U(u_{\tilde{l}}) - u_{\tilde{l}}\| \le \varphi(v)\big)\right]$$

if existent (which – by lemma 3.1 – is the case for majorizable φ in the sense of definition 2.12) and := 0, otherwise.

Put $j := J_\varepsilon(\varphi)$ and define $V_{\varepsilon,g}(u,\varphi) := U(u_{i_0})$, where $i_0 \in [j, \tilde{g}_{u,\varepsilon}(\rho(\varepsilon^2/8d^2, j)) - 1]$ is the least index i s.t.

$$\forall k \in [j, \tilde{g}_{u,\varepsilon}(\rho(\varepsilon^2/8d^2, j)) - 1]\big(\|v_0 - [(1-t)u - tU(u_i)]\| \leq \|v_0 - [(1-t)u - tU(u_k)]\|\big),$$

where $t := \varepsilon^2/192d^2$. Define $\Delta_{\varepsilon,g}(u,\varphi) := \Omega_d(\varepsilon/2, \tilde{g}_{u,\varepsilon}, j)$. Find $\tilde{u}, \tilde{\varphi}$ as in lemma 2.6 (applied to $\varepsilon_d := \frac{(\varepsilon/8)^4}{2d^2} = \frac{\varepsilon^4}{8192d^2}$). Then for $\tilde{j} := J_\varepsilon(\tilde{\varphi})$ and $v := V_{\varepsilon,g}(\tilde{u},\tilde{\varphi})$ one obtains (using (0) under the assumption that $\tilde{\varphi}$ is majorizable which we will verify below) and the fact that with u_i also $U(u_i)$ as a $\varphi(v)$-good approximate fixed point for all $v \in C$) that

$$(1)\ \|U(\tilde{u}) - \tilde{u}\| < \Omega_d(\varepsilon/2, \tilde{g}_{\tilde{u},\varepsilon}, \tilde{j})$$

and

$$\|u_0 - \tilde{u}\|^2 \leq \|u_0 - [(1 - \frac{\varepsilon^2}{192d^2})\tilde{u} + \frac{\varepsilon^2}{192d^2}v]\|^2 + \varepsilon_d$$

and so (by the v-definition)

$$(2)\ \forall k \in [\tilde{j}, \tilde{g}_{\tilde{u},\varepsilon}(\rho(\varepsilon^2/8d^2, \tilde{j}))-1]\ \big(\|u_0-\tilde{u}\|^2 \leq \|u_0-[(1-\frac{\varepsilon^2}{192d^2})\tilde{u}+\frac{\varepsilon^2}{192d^2}U(u_k)]\|^2+\varepsilon_d\big).$$

(2) together with lemma 2.7 (applied to $(\varepsilon/8)^2$) yields (noticing that $\frac{(\varepsilon/8)^2}{3d^2} = \frac{\varepsilon^2}{192d^2}$) that

$$(3)\ \forall k \in [\tilde{j}, \tilde{g}_{\tilde{u},\varepsilon}(\rho(\varepsilon^2/8d^2, \tilde{j})) - 1]\ \big(\langle u_0 - \tilde{u}, U(u_k) - \tilde{u}\rangle < \Big(\frac{\varepsilon}{8}\Big)^2 = \frac{(\varepsilon/2)^2}{16}\big).$$

Hence by lemma 3.2 (for $\varepsilon/2$), (1) and (3) imply

$$\|\tilde{u} - u_{\tilde{g}_{\tilde{u},\varepsilon}(\rho(\varepsilon^2/8d^2,\tilde{j}))}\| \leq \frac{\varepsilon}{2}$$

and so by lemma 2.9

$$(4)\ \|u_k - u_{\tilde{g}(k)}\| \leq \varepsilon \text{ for } k := \rho(\varepsilon^2/8d^2, \tilde{j}).$$

It remains to show that $\tilde{\varphi}$ is majorizable and that $k \leq \Phi(\varepsilon, g, d)$: from lemma 3.1 one has

$$(5)\ \forall n \in \mathbb{N}^* \,\forall u \in C\, \forall \varphi : C \to (0,1]\ (n \gtrsim \varphi \to J_\varepsilon(\varphi) \leq \chi_{d,\varepsilon}(n)).$$

Since $\tilde{g}^M \gtrsim \tilde{g}_{u,\varepsilon}$ and $\chi_{d,\varepsilon} \gtrsim \chi_{d,\varepsilon}$ one gets

$$n \gtrsim \varphi \to 16d \cdot n^2 \gtrsim \varphi^*_{t,u} \to \frac{1}{\Delta^*_{\varepsilon,g}(n)} \leq \Delta'_{\varepsilon,g}(u,\varphi),$$

where $\Delta'_{\varepsilon,g}, \varphi^*_{t,u}$ are as in the proof of theorem 2.15. As in the proof of theorem 2.15 it now follows that

$$N_{\varepsilon,g,d} \gtrsim \tilde{\varphi}.$$

Hence – by (5) – $\chi_{d,\varepsilon}(N_{\varepsilon,g,d}) \geq J_\varepsilon(\tilde{\varphi})$ and so $k \leq \Phi(\varepsilon, g, d)$. The theorem now follows from this and (4) precisely as in the proof of corollary 2.16.

□

Remark 3.4. *1. The statements 1.-3. from remark 2.17 also hold (with obvious adaptations) for our analysis of Wittmann's proof.*

2. As in the case of Browder's theorem, the extractability of an effective bound Φ that as in theorem 3.3 only depends on ε, g, d (but not on X, C, T, u_0) follows from the formalizability of the proof in the formal context of [11] and the general logical metatheorems proved in that paper.

3. Instead of assuming C to be bounded with $d \geq diam(C)$ it actually suffices that $d \geq 2\|u_0 - \widehat{u}\|$ for some fixed point $\widehat{u} \in C$ of U (in fact it suffices that U has in the $d/2$-ball around u_0 arbitrarily good approximate fixed points). Then also $\|u_n - \widehat{u}\|, \|U(u_n) - \widehat{u}\| \leq d/2$ and so $\|u_n - u_m\|, \|u_n - U(u_m)\| \leq d$. Note that, in particular, for the $V_{\varepsilon,g}$ used in the proof above the construction of $\tilde{u}$ from lemma 2.4 via lemma 2.6 yields that $\|\widehat{u} - \tilde{u}\| \leq d/2$ and so $\|u_n - \tilde{u}\|, \|U(u_n) - \tilde{u}\| \leq \frac{d}{2} + \frac{d}{2} < d$. Finally note that lemma 2.7 is only applied to $u := \tilde{u}, v := V_{\varepsilon,g}(\tilde{u}, \tilde{\varphi}) = U(u_i)$ (for some i), lemma 3.2 only to $u := \tilde{u}$ and lemma 2.3 only to $p_1 := \tilde{u}, p_2 := V'_{\varepsilon,g}(\tilde{u}, \tilde{\varphi})$, where $V'_{\varepsilon,g}$ is defined from $V_{\varepsilon,g}$ as in lemma 2.4.

Because of this, theorem 3.3 gives another quantitative version of the usual mean ergodic theorem (treated in [1] and [17]) with $C := X$ and U being linear (so that we can take $\widehat{u} := 0$). Our bound for Wittmann's theorem has – despite of its more involved details – a complexity similar to that from [17] (for the Hilbert case): in both cases a simple transformation (using only functions of polynomial growth) of g (in Wittmann's case $\Delta^*_{\varepsilon,g}$) involving only a single use of g is being iterated. In the bound on Wittmann's theorem, however, the number of iterations essentially is given by d^4/ε^4 whereas in the bound on the mean ergodic theorem from [17] it is d^2/ε^2. The bound in [1] on the mean ergodic theorem is more complex as it involves an iteration of (essentially) $(g(n))^2$.

4 An elementary proof due to Halpern [8]

Halpern's paper [8], which is frequently cited as the paper that introduced the iteration used in Wittmann's theorem from the previous section, contains, moreover, a new elementary proof of Browder's theorem for the case $C := B_1(0)$ and $v_0 := 0$ that seems to have remained rather unnoticed. E.g. both Wittmann's paper [32] (though referring to Halpern's paper) as well as the 2000 textbook treatment of Browder's theorem in [29] follow weak compactness arguments similar to the one used in Browder's proof. Also, Xu and Yin in 1995 ([33]) use a weak compactness argument to show that in Browder's theorem the boundedness of C can be replaced by that of the sequence (u_t) (as $t \in (0,1)$ tends to infinity) although this immediately follows from Halpern's proof. We now first adapt Halpern's proof to general bounded and convex C and general $v_0 \in C$. The proof has the feature that it separates the issue of the convergence of (u_t) for $t \in (0,1)$ tending to 1 from the fact that this sequence converges to the fixed point of U that is closest to v_0. The whole proof does not use any weak compactness argument and establishes in the course of the proof the existence of a fixed point of U (and so does not rely on that existence in contrast to Browder's proof). So the proof pretty much shows already the features Browder's proof only displays after the nontrivial logical analysis carried out in the previous section. Halpern's proof (also when adapted to general C, v_0) even shows the convergence of (u_{t_n}) as long as (t_n) is any increasing sequence in $(0,1)$ irrespectively of whether it converges to 1. If, however, the sequence increases towards 1 the limit of of (u_{t_n}) always is the same point p which (by a subsequent argument) is shown to be the fixed point of U that is closest to v_0. Now let (t_n) be an **arbitrary** (i.e. not necessarily increasing) sequence in $(0,1)$ that converges to 1. Then (t_n) has an increasing subsequence (t_{n_k}). Hence for any subsequence of a sequence (t_n) in $(0,1)$ that converges towards 1 (let that subsequence be denoted again by (t_n)) we can find a subsequence of (u_{t_n}) that converges towards p. So already the original sequence (u_{t_n}) converges towards p.

We now present this proof together with its quantitative analysis. The latter is based on the no-counterexample interpretation of the convergence of bounded monotone sequences in $\mathbb{R}$ that has been first treated in [20] and was recently re-invented by Tao under the name of 'finite convergence principle'. Our precise quantitative version is taken from [10] (see also proposition 2.27 and remark 2.29 in [14] to which we refer for a detailed discussion):

Lemma 4.1. *Let $D \in \mathbb{R}_+$ be a real number and (a_n) be an increasing sequence in the interval $[0, D]$, i.e. $0 \le a_n \le a_{n+1} \le D$. Then the following holds*

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \le \tilde{g}^{(\lceil D/\varepsilon \rceil)}(0)\, \forall i, j \in [n; n + g(n)]\, (|a_i - a_j| \le \varepsilon),$$

where $\tilde{g}(n) := n + g(n)$. Moreover, n can be taken as $\tilde{g}^{(i)}(0)$ for some suitable $i \le \lceil D/\varepsilon \rceil$.

Now let (t_n) be a sequence in $(0,1)$ with $t_n < t_{n+1}$ for all $n \in \mathbb{N}$. Let U, C, d and $v_0 \in C$ be as in the previous section. Let u_n denote the unique fixed point of the contraction $U_{t_n}(x) := t_n U(x) + (1-t_n)v_0$ on C. Then

$$u_n = t_n U(u_n) + (1-t_n)v_0, \text{ i.e. } U(u_n) - v_0 = t_n^{-1}(u_n - v_0).$$

Let $l, k \in \mathbb{N}$ be with $k < l$ and define $D := u_l - u_k$.
Then (using that U is nonexpansive on C) the following holds:

$$\begin{aligned}
&\|D\|^2 \geq \|U(u_l) - U(u_k)\|^2 = \|(U(u_l) - v_0) - (U(u_k) - v_0)\|^2 \\
&= \|t_l^{-1}(u_l - v_0) - t_k^{-1}(u_k - v_0)\|^2 = \|t_l^{-1}((u_k - v_0) + D) - t_k^{-1}(u_k - v_0)\|^2 \\
&= \langle t_l^{-1}((u_k - v_0) + D) - t_k^{-1}(u_k - v_0), t_l^{-1}((u_k - v_0) + D) - t_k^{-1}(u_k - v_0)\rangle \\
&= \langle (t_l^{-1} - t_k^{-1})(u_k - v_0) + t_l^{-1}D, (t_l^{-1} - t_k^{-1})(u_k - v_0) + t_l^{-1}D\rangle \\
&= (t_l^{-1} - t_k^{-1})^2\|u_k - v_0\|^2 + t_l^{-2}\|D\|^2 + 2(t_l^{-1} - t_k^{-1})t_l^{-1}\langle u_k - v_0, D\rangle.
\end{aligned}$$

Hence

$$0 \geq (t_l^{-1} - t_k^{-1})^2\|u_k - v_0\|^2 + (t_l^{-2} - 1)\|D\|^2 + 2(t_l^{-1} - t_k^{-1})t_l^{-1}\langle u_k - v_0, D\rangle,$$

i.e.

$$(t_l^{-1} - t_k^{-1})^2\|u_k - v_0\|^2 + (t_l^{-2} - 1)\|D\|^2 \leq 2(t_k^{-1} - t_l^{-1})t_l^{-1}\langle u_k - v_0, D\rangle.$$

$t_k < t_l \in (0,1)$ implies that $(t_k^{-1} - t_l^{-1})t_l^{-1} > 0$ and $t_l^{-2} - 1 > 0$.
Hence $\langle u_k - v_0, D\rangle \geq 0$.
Also

$$\|u_l - v_0\|^2 = \langle (u_k - v_0) + D, (u_k - v_0) + D\rangle = \|u_k - v_0\|^2 + \|D\|^2 + 2\langle u_k - v_0, D\rangle$$

and so

$$\|u_l - v_0\|^2 \geq \|u_k - v_0\|^2 + \|u_l - u_k\|^2.$$

Hence the sequence $(\|u_l - v_0\|^2)_{l\in\mathbb{N}}$ is increasing and bounded by d^2 and

$$(*)\ \|u_l - u_k\|^2 \leq \big|\|u_l - v_0\|^2 - \|u_k - v_0\|^2\big|$$

for all $k, l \in \mathbb{N}$.
Now let $g : \mathbb{N} \to \mathbb{N}, \varepsilon > 0$ be arbitrary and $n \in \mathbb{N}$ be such that

$$\forall i, j \in [n; n + g(n)]\ \left(\big|\|u_i - v_0\|^2 - \|u_j - v_0\|^2\big| \leq \varepsilon^2\right).$$

Then $\|u_i - u_j\|^2 \leq \varepsilon^2$, i.e. $\|u_i - u_j\| \leq \varepsilon$ for all $i, j \in [n; n + g(n)]$.
By the above lemma,

$$\Phi(\varepsilon, g, d) := \tilde{g}^{(\lceil d^2/\varepsilon^2\rceil)}(0)$$

with $\tilde{g}(n) := n + g(n)$ provides an upper bound for such an n.

Now let (s_n) be a sequence in $(0,1)$ that no longer is assumed to be increasing but which converges to 1 with a rate of convergence $\chi : \mathbb{N} \to \mathbb{N}$, i.e.

$$(1)\ \forall n \in \mathbb{N}\, \forall i \geq \chi(n)\ (|1-s_i| \leq \frac{1}{n+1}).$$

In fact, all we need below is that χ satisfies the metastable version (i.e. the no-counterexample interpretation) of the convergence towards 1 w.r.t. the counterfunction $\tilde{g}$, i.e.

$$(1)^*\ \forall n \in \mathbb{N}\, \forall i \in [\chi(n); \tilde{g}(\chi(n))]\ (|1-s_i| \leq \frac{1}{n+1}).$$

Let $h : \mathbb{N} \to \mathbb{N}$ be such that

$$(2)\ \forall n \in \mathbb{N} (s_n \leq 1 - \frac{1}{h(n)+1}),$$

e.g. $h(n) := \left\lfloor \frac{1}{1-s_n} \right\rfloor$. Define $t_n := 1 - \frac{1}{n+1}$ and

$$g_{h,\chi}(n) := \max\{h(i) \,:\, i \leq \tilde{g}(\chi(n))\} \geq n.$$

Let u_n be defined as before and $\breve{u}_n$ be the unique fixed point of $s_n U(x) + (1-s_n)v_0$. Define

$$a_n := \|\breve{u}_n - v_0\|^2 \text{ and } b_n := \|u_n - v_0\|^2.$$

By the lemma, let $n \leq g_{h,\chi}^{(\lceil 4d^2/\varepsilon^2 \rceil)}(0)$ be such that

$$(3)\ \forall k, l \in [n; g_{h,\chi}(n)]\ \left(|b_k - b_l| \leq \left(\frac{\varepsilon}{2}\right)^2 \right).$$

Let $i \in [\chi(n); \chi(n) + g(\chi(n))] = [\chi(n); \tilde{g}(\chi(n))]$. Then

$$t_{g_{h,\chi}(n)} = 1 - \frac{1}{g_{h,\chi}(n)+1} \geq 1 - \frac{1}{h(i)+1} \overset{(2)}{\geq} s_i \overset{(1)^*}{\geq} t_n.$$

Hence – for $i, j \in [\chi(n); \chi(n) + g(\chi(n))]$ – the monotonicity stated before $(*)$ above implies that

$$b_{g_{h,\chi}(n)} \geq a_i, a_j \geq b_n.$$

Together with $(*)$ and (3) (applied to $k := n$ and $l := g_{h,\chi}(n)$) this yields

$$\|u_n - \breve{u}_i\|^2 \leq a_i - b_n \leq \left(\tfrac{\varepsilon}{2}\right)^2, \text{ and}$$
$$\|u_n - \breve{u}_j\|^2 \leq a_j - b_n \leq \left(\tfrac{\varepsilon}{2}\right)^2.$$

Hence

$$\|\breve{u}_i - \breve{u}_j\| \leq \|\breve{u}_i - u_n\| + \|u_n - \breve{u}_j\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Put together, we have established the following

Theorem 4.2. *Logical analysis of Halpern's proof (adapted to general bounded closed and convex C and $v_0 \in C$) yields the following bound on the metastable version of Browder's theorem: Let X be a real Hilbert space, $d \in \mathbb{N}^*$ and $C \subset X$ be a bounded closed and convex subset with $d \geq diam(C)$. Let $U : C \to C$ be a nonexpansive mapping and $v_0 \in C$. Let (s_n) be a sequence in $(0,1)$ that converges towards 1 and $h : \mathbb{N} \to \mathbb{N}$ be such that $s_n \leq 1 - \frac{1}{h(n)+1}$ for all $n \in \mathbb{N}$. Let $\breve{u}_n$ be the unique fixed point of $U_{s_n}(x) := s_n U(x) + (1 - s_n)v_0$. Then for all $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$ the following holds:*

$$\exists n \leq \Psi(\varepsilon, g, \chi_g, h, d)\, \forall i, j \in [n; n + g(n)]\ (\|\breve{u}_i - \breve{u}_j\| \leq \varepsilon),$$

where

$$\Psi(\varepsilon, g, \chi_g, h, d) := \chi_g^M \left(g_{h,\chi_g}^{(\lceil 4d^2/\varepsilon^2 \rceil)}(0) \right)$$

with

$$g_{h,\chi_g}(n) := \max\{h(i) \,:\, i \leq \chi_g(n) + g(\chi_g(n))\} \text{ and } \chi_g^M(n) := \max\{\chi_g(i) : i \leq n\}$$

and χ_g is a quasi-rate of convergence for (s_n) in the weak sense of $(1)^*$ above.
Instead of C being bounded it suffices to assume that the sequence $(\breve{u}_n)$ is bounded and the bound above then holds equally provided that $d \geq \|\breve{u}_n - v_0\|$ for all $n \in \mathbb{N}$.
If (s_n) is an increasing sequence in $(0,1)$ (not necessarily converging to 1), then the bound Ψ can be simplified to $\Psi(\varepsilon, g, d) := \tilde{g}^{(\lceil d^2/\varepsilon^2 \rceil)}(0)$.

5 General comments on the logical analysis of proofs based on weak sequential compactness

In this section we for simplicity only consider the case $C := B_1(0)$.
Lemma 2.13 actually resulted in the course of the logical analysis of Browder's use of weak compactness via the next lemma. This lemma – in its non-quantitative version – certainly is satisfied by any weak cluster point v of $(\breve{u}_{\tilde{g}(j)})_j$ as such a point (by Browder's demiclosedness principle, see below) must be a fixed point of U.

Lemma 5.1. *Let X be a real Hilbert space, U as in lemma 2.3, $v_0 \in B_1(0), (\breve{u}_n)$ as in lemma 2.11. There is a computable $\chi^* : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that the following holds:*

$$\begin{aligned}&\forall \varepsilon \in (0,1]\, \forall g : \mathbb{N} \to \mathbb{N}^*\, \forall \varphi : B_1(0) \times \mathbb{N} \to (0,1] \forall u \in B_1(0)\, \forall f \gtrsim \varphi \\ &\qquad \exists v \in B_1(0)\, \exists j \leq \chi^*(f, \lceil \tfrac{1}{\varepsilon} \rceil)\ \big(|\langle v_0 - u, v - \breve{u}_{\tilde{g}(j)} \rangle| < \varepsilon \wedge \|U(v) - v\| < \varphi(v,j) \big),\end{aligned}$$

where $\tilde{g}(n) := \max\{n, g(n)\}$.
Here $f \gtrsim \varphi :\equiv \forall j \in \mathbb{N}\, \forall v \in B_1(0)\, (\frac{1}{f(j)} \leq \varphi(v,j))$ for $f : \mathbb{N} \to \mathbb{N}^*$.

In the analysis of Browder's proof it turned out to be sufficient to use this lemma for φ's that are not allowed to depend on j (but only on v). If then $k \gtrsim \varphi$ as in definition 2.12, the lemma can be trivially solved by taking $j := \chi(2k)$ and $v := \check{u}_{\tilde{g}(j)}$ which is our lemma 2.13. This, however, is blocked in the general form of lemma 5.1 above, where φ may depend on j (and v).

Remark 5.2. *One would have actually ended up in this situation if in the transformation leading to the formulation of lemma 2.4 we had transformed the formula*

$$\forall \varepsilon \in (0,1]\, \exists u \in C \Big(\forall \delta \in (0,1]\, (\|U(u) - u\| < \delta) \\ \wedge\ \forall v \in C\, \exists \eta \in (0,1]\, (\|U(v) - v\| < \eta \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon)\Big)$$

first (logically equivalent) to

$$\forall \varepsilon \in (0,1]\, \exists u \in C\, \forall \delta \in (0,1]\, \forall v \in C\, \exists \eta \in (0,1] \\ \Big(\|U(u) - u\| < \delta \wedge\ (\|U(v) - v\| < \eta \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon)\Big)$$

and only **then** *had applied* QF-AC *to obtain*

$$\forall \varepsilon \in (0,1]\, \exists u \in C\, \exists \varphi : C \times (0,1] \to (0,1]\, \forall \delta \in (0,1]\, \forall v \in C \\ \Big(\|U(u) - u\| < \delta \wedge\ (\|U(v) - v\| < \varphi(v,\delta) \to \|v_0 - u\|^2 < \|v_0 - v\|^2 + \varepsilon)\Big).$$

As a result, in '(2)' in lemma 2.4 one then would have $\varphi(V(u,\varphi), \Delta(u,\varphi))$. Now note that in the proof of theorem 2.15 the Δ to which lemma 2.4 is applied is defined in terms of the index $J_{\varepsilon,g}(u,\varphi)$ that solves lemma 2.13.
This alternative way of arriving at a functional interpretation of (++) *actually coincides with the so-called Shoenfield interpretation which – by [28] – is the combination of a negative translation due to Krivine and Gödel's functional interpretation whereas our approach has been based on the combination of a negative translation due to Kuroda with Gödel's interpretation (see [14]). Of course, as both interpretations are equivalent over a weak base system plus* QF-AC *one could have transformed any solution of lemma 2.4 based on the Krivine-Gödel interpretation into one for our version of lemma 2.4.*

One then has to proceed via a quantitative version of weak sequential compactness (given in lemma 5.4 below) as well as a quantitative version of the so-called demiclosedness principle (Lemma 2 in Browder's proof) which we present first:
Browder shows that a weak limit v of a sequence (v_n) of approximate fixed points in $B_1(0)$ must be a fixed point. This fact, called the demiclosedness principle for nonexpansive functions in the literature, plays a crucial role in fixed point theory. The next lemma gives the appropriate quantitative version of this:

Lemma 5.3 (Quantitative demiclosedness principle)**.** *Let X be a real Hilbert space and $U : B_1(0) \to B_1(0)$ be as in lemma 2.3. Define $T := \mathrm{Id} - U$. Let $v \in B_1(0)$, (v_n) be a sequence in $B_1(0)$ and $j \in \mathbb{N}$ be such that for $\varepsilon \in (0,1]$*

$$|\langle T(u), v_j\rangle - \langle T(u), v\rangle| < \frac{\varepsilon^4}{96} \wedge \|U(v_j) - v_j\| < \frac{\varepsilon^4}{96},$$

where $u := u_\varepsilon := v - \frac{\varepsilon^2}{16}T(v) \in B_1(0)$. Then $\|T(v)\| < \varepsilon$.

Proof:

$$(1) \begin{cases} 0 \leq \|u - v_j\|^2 - \langle U(u) - U(v_j), u - v_j\rangle = \langle T(u) - T(v_j), u - v_j\rangle \\ \quad = \langle T(u), u\rangle - \langle T(v_j), u\rangle - \langle T(u), v_j\rangle + \langle T(v_j), v_j\rangle. \end{cases}$$

Since $\|v_j\| \leq 1$ and

$$\|u\| = \|v - \frac{\varepsilon^2}{16}(v - U(v))\| = \|(1 - \frac{\varepsilon^2}{16})v + \frac{\varepsilon^2}{16}U(v)\| \leq (1 - \frac{\varepsilon^2}{16})\|v\| + \frac{\varepsilon^2}{16}\|U(v)\| \leq 1,$$

we have that

$$\|U(v_j) - v_j\| = \|T(v_j)\| \geq \begin{cases} \|T(v_j)\| \cdot \|v_j\| \geq |\langle T(v_j), v_j\rangle| \\ \|T(v_j)\| \cdot \|u\| \geq |\langle T(v_j), u\rangle|. \end{cases}$$

Hence (1) and the assumption yield

$$\begin{aligned} -\tfrac{\varepsilon^4}{32} < \langle T(u), u - v\rangle &= \langle T(v - \tfrac{\varepsilon^2}{16}T(v)), -\tfrac{\varepsilon^2}{16}T(v)\rangle \\ &= -\tfrac{\varepsilon^2}{16}\langle T(v - \tfrac{\varepsilon^2}{16}T(v)), T(v)\rangle. \end{aligned}$$

Thus

$$\frac{\varepsilon^2}{2} > \langle T(v - \frac{\varepsilon^2}{16}T(v)), T(v)\rangle$$

and so (!)

$$\varepsilon^2 > \langle T(v), T(v)\rangle = \|T(v)\|^2, \text{ i.e. } \|T(v)\| < \varepsilon.$$

'!' holds since $T \in Lip(2)$ (i.e. T is Lipschitz continuous with Lipschitz constant 2) and $\|T(v)\| \leq \|v\| + \|U(v)\| \leq 2$ and so

$$\|T(v) - T(v - \frac{\varepsilon^2}{16}T(v))\| \leq 2\|\frac{\varepsilon^2}{16}T(v)\| \leq \frac{\varepsilon^2}{4},$$

which implies

$$|\langle T(v), T(v)\rangle - \langle T(v - \frac{\varepsilon^2}{16}T(v)), T(v)\rangle| \leq \|T(v) - T(v - \frac{\varepsilon^2}{16}T(v))\| \cdot \|T(v)\| \leq \frac{\varepsilon^2}{2}.$$

□

Based on a complicated logical analysis of the standard proof of the sequential weak compactness of $B_1(0)$ ([16]) and of the usual sequential compactness of a suitable compact Polish space by means of bar recursion in the sense of Spector [27] though only of lowest types (see [25]) one can extract a computable functional χ (in fact definable by primitive recursion in the sense of Gödel's calculus T,) satisfying the following lemma:

Lemma 5.4 ([16]). *There is a computable function $\chi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $\varepsilon > 0$ and $f : \mathbb{N} \to \mathbb{N}^*$*

$$\begin{cases} \forall (x_n) \subset B_1(0)\, \forall \varphi : B_1(0) \times \mathbb{N} \to (0,1]\, \forall w \in B_2(0)\, \forall \tilde{T} : B_1(0) \times \mathbb{N} \to B_3(0) \\ (f \gtrsim \varphi \to \exists v \in B_1(0)\, \exists j \leq \chi(f, \lceil 1/\varepsilon \rceil))\, \exists \tilde{j} \geq \frac{2}{\varphi(v,j)} \\ \quad \big(|\langle w, v - x_j \rangle| < \varepsilon \wedge |\langle \tilde{T}(v,j), v - x_{\tilde{j}} \rangle| < \varphi(v,j)\big). \end{cases}$$

Proof of lemma 5.1: Apply lemma 5.4 to $w := v_0 - u, x_n := \breve{u}_{\tilde{g}(n)}$, $\widehat{\varphi}(v,j) := \frac{\varphi(v,j)^4}{96}$ instead of φ and

$$\tilde{T}(v,j) := T(v - \frac{\varphi(v,j)^2}{16} T(v)).$$

$\tilde{T}(v) \in B_3(0)$ since

$$\begin{aligned} &\left\| T(v - \tfrac{\varphi(v,j)^2}{16} T(v)) \right\| \leq \|T(v)\| + \left\| T(v) - T(v - \tfrac{\varphi(v,j)^2}{16} T(v)) \right\| \\ &\overset{T \in Lip(2)}{\leq} \|T(v)\| + \tfrac{\varphi(v,j)^2}{8} \|T(v)\| \leq \|T(v)\| + \tfrac{1}{2} \|T(v)\| \leq 3. \end{aligned}$$

Then use lemmas 5.3 and 2.10 (with $d = 2$ and note that $\tilde{g}(j) \geq j$). Now define

$$\chi^*(f,k) := \chi(\widehat{f}, k), \quad \text{where } \widehat{f}(j) := 96 \cdot f(j)^4.$$

Note that $f \gtrsim \varphi \to \widehat{f} \gtrsim \widehat{\varphi}$. □

Only future research can show whether the fact that in the end weak sequential compactness could be by-passed in the proof-theoretic analysis of the proof by Browder (since lemma 2.13 was only used for a φ that did not depend on j) – and similarly in Wittmann's proof – points to a general phenomenon in proofs of strong convergence results that are based on weak compactness or is just an accident in the proof at hand. However, it is plausible to expect that weak compactness arguments cannot be by-passed in proofs that itself establish a weak convergence result such as in the case of nonlinear ergodic theorems as the famous Baillon ergodic theorem (see [2] and [29]). In fact, very recently we succeeded in a quantitative analysis of a proof due to Brézis and Browder [3] of Baillon's theorem following the path of reasoning outlined in this section (making crucial use of our quantitative demiclosedness principle [16]).

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