The Extended Probabilistic Powerdomain Monad over Stably Compact Spaces

(Extended Abstract)

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Abstract. For the semantics of probabilistic features in programming mainly two approaches are used for building models. One is the Giry monad of Borel probability measures over metric spaces, and the other is Jones' probabilistic powerdomain monad [6] over dcpos (directed complete partial orders). This paper places itself in the second domain theoretical tradition. The probabilistic powerdomain monad is well understood over continuous domains. In this case the algebras of the monad can be described by an equational theory [6, 9, 5]. It is the aim of this work to obtain similar results for the (extended) probabilistic powerdomain monad over stably compact spaces. We mainly want to determine the algebras of this powerdomain monad and the algebra homomorphisms.

1 Introduction

We introduce the extended probabilistic powerdomain monad over the category of all T_0-spaces. For this, we have to deal with objects that we call *topological cones*. Topological cones are a kind of asymmetric variant of topological vector spaces over the reals. The term 'asymmetric' refers to the features that scalar multiplication is only defined for nonnegative reals, no element has an additive inverse, and topologies are never Hausdorff, but only T_0. In order to deal with these structures one has to adapt functional analytic tools to this non-Hausdorff setting extending the work [15]. As far as necessary, these notions and methods are presented in the first sections. These tools are also close to classical Choquet theory over compact convex sets (see [2]).

Our main concern are the algebras and the algebra homomorphisms of this extended probabilistic powerdomain monad. The algebras have to be topological cones and the algebra homomorphisms have to be continuous linear maps. But these properties are not strong enough to determine algebras and the algebra homomorphisms, in general. But if we restrict our attention to stably compact spaces, we conjecture that these properties together with local convexity are sufficient for being algebras and algebra homomorphisms. In the general case, the existence of averaging operators seems to be crucial for the algebras. Such operators exist for the dual cones of topological cones. Our results are contained in the last two sections.

The extended probabilistic powerdomain monad has been studied in detail over the category of continuous domains and Scott-continuous functions. In this case the algebras have been characterized as the cones which are continuous domains at the same time, and with addition and scalar multiplication being Scott-continuous. And the algebra homomorphisms are Scott-continuous linear maps (see [6, 9, 5]). In topological measure theory there is a similar result (see [4, Theorem 2.14]): Over compact Hausdorff spaces, the algebras of the monad of probability measures are the compact convex sets embeddable in locally convex topological vector spaces and the homomorphisms are the continuous affine maps. It is our goal to obtain analogous results over the category of stably compact spaces.

For background material on continuous domains and stably compact spaces we refer to [5]. Basic concepts on topological cones are collected in [7]. No proofs are included in this extended abstract.

Alex Simpson has communicated to us that the main result by himself and M. Schröder announced in a talk at MFPS 21 in May 2005 (see [13]) implies the unicity property that we leave open in our Lemma 2 below and that it also implies property (1) in our Corollary 2.

2 Ordered Cones and Topological Cones

Elementary topological cones are $\mathbb{R}_+$, the set of nonnegative real numbers, and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, the set of nonnegative real numbers extended by an infinite element, both with the upper topology ν the only proper nonempty open sets of which are the infinite upper intervals $]r, +\infty] = \{s \mid r < s\}$, $r \in \mathbb{R}_+$. This upper topology is T_0 but far from being Hausdorff. If not specified otherwise, we will use this topology on the (extended) reals and not the usual open interval topology λ.

We want to consider structures that are close to vector spaces but asymmetric in the sense that elements do not have additive inverses. Accordingly, scalar multiplication is restricted to nonnegative real numbers.

Definition 1. *A* cone *is defined to be a commutative monoid C together with a scalar multiplication by nonnegative real numbers satisfying the same axioms as for vector spaces; that is, C is endowed with an addition $(x, y) \mapsto x + y: C \times C \to C$ which is associative, commutative and admits a neutral element 0, and with a scalar multiplication $(r, x) \mapsto r \cdot x: \mathbb{R}_+ \times C \to C$ satisfying the following axioms for all $x, y \in C$ and all $r, s \in \mathbb{R}_+$:*

$$r \cdot (x + y) = r \cdot x + r \cdot y \qquad (rs) \cdot x = r \cdot (s \cdot x) \qquad 1 \cdot x = x$$
$$(r + s) \cdot x = r \cdot x + s \cdot x \qquad\qquad 0 \cdot x = 0$$

An ordered cone *is a cone C endowed with a partial order $\leq$ such that, for all $x, y, z \in C$ and all $r, s \in \mathbb{R}_+$,*

$$x \leq y \text{ and } r \leq s \Longrightarrow x + z \leq y + z \text{ and } r \cdot x \leq s \cdot y\,.$$

Cones may occur as subsets of real vector spaces: such a subset C is a cone if it satisfies (1) $0 \in C$, (2) $a, b \in C \Rightarrow a + b \in C$ and (3) $a \in C, r \in \mathbb{R}_+ \Rightarrow ra \in C$. But unlike for cones in vector spaces, addition need not satisfy the cancellation property, in general, and cones need not be embeddable in vector spaces. For example $\overline{\mathbb{R}}_+$ and its powers $\overline{\mathbb{R}}_+^I$ are ordered cones that are not embeddable in vector spaces. Thus, our notion of a cone is more general than that used in classical functional analysis. On the other hand, our concept of an ordered cone is more restrictive as the one used in in functional analysis, where our ordered cones would be called *pointed ordered cones*. In an ordered cone C in our sense, one has $a \geq 0$ for every element a.

As in real vector spaces, there is a notion of convexity in cones. Because of the possible existence of *infinite* elements in cones, convex sets may look unusual.

Definition 2. *A subset A of a cone C is* convex *if, for all $a, b \in A$, the convex combination $ra + (1-r)b$ belongs to A for every real number $0 \leq r \leq 1$.*

Recall that a topological vector space is a vector space endowed with a Hausdorff topology in such a way that addition and scalar multiplication are jointly continuous. The scalars are endowed with the usual (Hausdorff) topology λ. For cones we continue our programme by using *asymmetric* topologies.

On $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ we use the *upper topology* ν. Then, for any topological space X, there are fewer continuous functions $f : \overline{\mathbb{R}}_+ \to X$ than those which are continuous with respect to the usual topology λ. This fact has striking consequences for topological cones in our sense. On the other hand, there are more continuous functions $f : X \to \overline{\mathbb{R}}$ than those which are continuous with respect to the usual topology λ on the reals. The functions $f : X \to \overline{\mathbb{R}}_+$ which are continuous with respect to the upper topology on $\overline{\mathbb{R}}_+$ are called *lower semicontinuous* in classical analysis. We shall adopt this terminology also for this paper and we use the abbreviation *lsc* for *lower semicontinuous*.[3]

Any T_0-space X comes with an intrinsic order, the *specialisation order* defined by $x \leq y$ if the closure of the singleton $\{y\}$ contains x or, equivalently, if every open neighbourhood of x is also a neighbourhood of y. In the remainder of the paper, references to order will always be with respect to the specialisation order of the space under consideration. Open sets are upper sets and closed sets are lower sets. Upper sets are also called *saturated*, and coincide with those sets that are the intersection of their neighbourhoods. In Hausdorff spaces, the specialisation order is the identity and hence trivial. But on $\overline{\mathbb{R}}_+$ with the upper topology, the specialisation order is just the usual linear order.

Definition 3. *A* topological cone *is a cone C endowed with a T_0-topology such that addition and scalar multiplication are jointly continuous,*

As we use the upper topology on $\mathbb{R}_+$, the continuity of $r \mapsto ra : \mathbb{R}_+ \to C$ has the striking consequence that the topology on a topological cone C cannot satisfy the Hausdorff separation property: Because continuous maps preserve the respective specialisation orders, the map $r \mapsto ra : \mathbb{R}_+ \to C$ is order preserving, that is, the rays $\mathbb{R}_+ \cdot a$ in the cone are nontrivially ordered (except for the singleton ray $\{0\}$). As continuous maps between topological spaces preserve the specialisation order, a topological cone is an ordered cone. The cone $\overline{\mathbb{R}}_+$ and arbitrary powers $\overline{\mathbb{R}}_+^I$ with the upper product

[3] It is somewhat unfortunate that those functions are *lower* semicontinuous which are continuous with respect to the *upper* topology. But we do not wish to deviate from the terminology adopted in analysis and in [5].

topology are topological cones. As for topological vector spaces, we have notions of linearity for maps and local convexity for cones.

Definition 4. *A topological cone C is called* locally convex*, if each point has a neighbourhood basis of open convex neighbourhoods.*

Definition 5. *Let C and D be cones. A function $f: C \to D$ is called* linear *if $f(r \cdot a) = r \cdot f(a)$ and $f(a+b) = f(a) + f(b)$ for all $a, b \in C$ and all $r \in \mathbb{R}_+$.*

Maps from a cone C into $\overline{\mathbb{R}}_+$ are called *functionals*. Notice that we allow the value $+\infty$. As for topological vector spaces, local convexity for cones allows to prove Hahn-Banach type separation theorems.

Lemma 1. ([7]) *For any two elements of a locally convex topological cone there is a lsc linear functional separating these two elements.*

Arbitrary powers $(\overline{\mathbb{R}}_+, \nu)^I$ of the extended nonnegative reals with the upper topology ν are examples of locally convex topological cones. The same holds for subcones of such powers with the subspace topology induced from the product topology ν^I.

3 The Extended Probabilistic Powermonad

TOP will denote the category of T_0-spaces X and continuous maps $g: X \to Y$, TOPCONE will denote the category of topological cones C and continuous linear maps $\psi: C \to D$, and LCCONE the subcategory of locally convex cones.

We firstly define a contravariant functor

$$\mathcal{L}: \mathsf{TOP}^{\mathrm{op}} \to \mathsf{LCCONE}$$

from the category of T_0-spaces into the category of locally convex topological cones in the following way: For every topological space X, let $\mathcal{L}X$ be the cone of all lsc functions $f: X \to \overline{\mathbb{R}}_+$ with pointwise addition and scalar multiplication. We endow $\mathcal{L}X$ with the coarsest topology such that, for all $x \in X$, the point evaluations

$$\eta_X(x) = \big(f \mapsto f(x)\big): \mathcal{L}X \to \overline{\mathbb{R}}_+$$

become lsc. Notice that these functionals $\eta_X(x)$ are linear. The sets

$$W_{x,r} = \{f \in \mathcal{L}X \mid f(x) > r\},\ x \in X, r \in \mathbb{R}_+\,,$$

form a subbasis for the open sets of this topology. As this topology is nothing but the subspace topology induced by the product topology on $(\overline{\mathbb{R}}_+, \nu)^X$, the cone $\mathcal{L}X$ becomes indeed a locally convex topological cone. For a continuous map $g: X \to Y$, the map

$$\mathcal{L}g = \big(f \mapsto f \circ g\big): \mathcal{L}Y \to \mathcal{L}X$$

is linear and continuous.

We secondly define a contravariant endofunctor

$$^*: \mathsf{TOPCONE}^{\mathrm{op}} \to \mathsf{LCCONE}$$

as follows: For every topological cone C, let C^* be the *dual cone* of all lsc linear functionals $\varphi: C \to \overline{\mathbb{R}}_+$ with pointwise addition and scalar multiplication. We endow C^* with the *upper weak*topology*, that is, the coarsest topology making lsc the functions

$$\eta_C(f) = \big(\varphi \mapsto \varphi(f)\big): C^* \to \overline{\mathbb{R}}_+$$

for all $f \in C$. Notice that these functionals are linear on C^*. The sets

$$W_{f,r} = \{\varphi \in C^* \mid \varphi(f) > r\},\ f \in C, r \in \overline{\mathbb{R}}_+ \,,$$

form an open subbasis for this topology. Notice again that the upper weak*topology on C^* is nothing but the topology induced by the product topology on $(\overline{\mathbb{R}}_+, \nu)^C$, whence C^* becomes indeed a locally convex topological cone, in fact, a subcone of the locally convex topological cone $\mathcal{L}C$. For every continuous linear map $\psi: C \to D$ of topological cones, the map

$$\psi^* = \big(\varphi \mapsto \varphi \circ \psi\big): D^* \to C^*$$

is linear and upper weak*continuous.

Composing the contravariant functors $\mathcal{L}$ and * we obtain a (covariant) functor

$$\mathcal{V}: \mathsf{TOP} \to \mathsf{LCCONE}$$

from the category of T_0-spaces to the category of locally convex topological cones. For a T_0-space X, we have $\mathcal{V}X = (\mathcal{L}X)^*$ and for a continuous map $g: X \to Y$, the map $\mathcal{V}g: \mathcal{V}X \to \mathcal{V}Y$ is defined by

$$\mathcal{V}g = (f \mapsto f \circ g)^* = \big(\varphi \mapsto \varphi \circ (f \mapsto f \circ g)\big)\,.$$

We will see that the functor $\mathcal{V}$ defines a monad over the category TOP.

The unit η : For a topological space X we have the function

$$\eta_X: X \to \mathcal{V}X\,,$$

which assigns the point evaluation $\eta_X(x): f \mapsto f(x)$ to every $x \in X$. We see that η_X is an embedding; indeed, $\eta_X(x)(f) > r$ iff $f(x) > r$. It is readily verified that η is a natural transformation.

The multiplication μ : For every topological cone C with dual cone C^*, we define

$$m_{C^*}: \mathcal{V}C^* \to C^*$$

in the following way: For every $f \in C$, the map $\eta_C(f) = \big(\varphi \mapsto \varphi(f)\big) : C^* \to \overline{\mathbb{R}}_+$ is lsc (and linear), hence an element of the double dual C^{**} which is a subcone of the cone $\mathcal{L}C^*$. Thus, every $\Phi \in \mathcal{V}(C^*) = (\mathcal{L}C^*)^*$ may be applied to $\eta_C(f)$. One checks that $f \mapsto \Phi(\eta_C(f)) \colon C \to \overline{\mathbb{R}}_+$ is linear and lsc, hence an element of the dual cone C^*. We thus may define m_{C^*} by

$$m_{C^*}(\Phi) = \big(f \mapsto \Phi(\eta_C(f))\big)\,.$$

Clearly, m_{C^*} is linear and continuous. It also is surjective as

$$m_{C^*}(\eta_{C^*}(\varphi)) = m_{C^*}(F \mapsto F(\varphi)) = \varphi\,,$$

whence

(A1) $$m_{C^*} \circ \eta_{C^*} = \mathrm{id}\,_{C^*}\,.$$

We now apply this construction to the case $C = \mathcal{L}X$ and $C^* = \mathcal{V}X$. We define μ_X to be $m_{\mathcal{V}X}$ as above, i.e.,

$$\mu_X \colon \mathcal{V}^2X \to \mathcal{V}X \text{ is defined by } \mu_X(\Phi) = \big(f \mapsto \Phi(\eta_{\mathcal{L}X}(f))\big)\,.$$

It is readily seen that μ is a natural transformation. The following requires a lengthy but straightforward verification:

Proposition 1. *The functor $\mathcal{V}$ defines a monad over the category* TOP *of T_0-spaces, with unit η und multiplication μ.*

For a space X, the topological cone $\mathcal{V}X$ is called the *extended probabilistic powerdomain* over X. This denomination has its justification in a domain theoretical analogue of a measure which is called a continuous valuation.

A *continuous valuation* on a space X is a function v that associates to every open set U in X an element $v(U) \in \overline{\mathbb{R}}_+$ such that the following properties are satisfied:

$$v(\emptyset) = 0 \tag{1}$$
$$v(U) + v(U') = v(U \cup U') + v(U \cap U') \tag{2}$$
$$v(U) \le v(U') \text{ whenever } U \subseteq U' \tag{3}$$
$$v\big(\bigcup_i U_i\big) = \sup_i v(U_i) \text{ for every directed family of open sets } U_i \tag{4}$$

Property (2) corresponds to finite additivity and, together with (4), it replaces and strengthens countable additivity.

Every lsc function $f \colon X \to \overline{\mathbb{R}}_+$ has a Choquet type *integral* with respect to a continuous valuation (see [14]) defined by

(Ch) $$\int f\,dv = \int_0^\infty v\big(f^{-1}(]r, +\infty])\big)\,dr\,.$$

There is a *Riesz Representation Theorem* (see [9]) which tells us that integrating with respect to a continuous valuation v defines a lsc linear functional $f \mapsto \int f\,dv$ on $\mathcal{L}X$ and that, for every lsc linear functional φ on $\mathcal{L}X$, there is a unique continuous valuation v representing φ in the sense that $\varphi(f) = \int f\,dv$. Thus, we may identify the the linear functionals $\varphi \in \mathcal{V}X = (\mathcal{L}X)^*$ with the continuous valuations v on X representing them.

Consider now any topological cone C and its dual C^*. If we rewrite the definition of the the map $m_{C^*}: \mathcal{V}C^* \to C^*$ using the representation of the elements of $\mathcal{V}(C^*)$ by continuous valuations Φ on C^*, then

$$\text{(V)} \qquad m_{C^*}(\Phi)(f) = \int \eta_C(f)\, d\Phi \quad \text{for all } f \in C\,.$$

This should be read as follows: $m_{C^*}(\Phi)$ is the linear functional φ on C given by $\varphi(f) = \int \eta_C(f)\, d\Phi$ for all $f \in C$, i.e., by integrating the functional $\varphi \mapsto \varphi(f): C^* \to \overline{\mathbb{R}}_+$ with respect to the valuation Φ.

This formula may be better understood by introducing the notion of a cone-valued integral for functions with values in a cone, in analogy to vector-valued integrals in the classical theory of topological vector spaces: Let Φ be a valuation on a space Y and D a locally convex cone. For a continuous function $F: Y \to D$, we say that an element $a \in D$ is the integral of F with respect to Φ, and we write $a = \int F\, d\Phi$ if, for every lsc linear functional g on D, we have

$$\text{(I)} \qquad g(a) = \int g \circ F\, d\Phi\,.$$

As $g \circ F$ is a lsc real-valued function, the latter integral is well defined by (Ch). As the lsc linear functionals on locally convex cones separate the points by Lemma 1, the cone-valued integral is uniquely determined, if it exists. It even suffices to require that (I) holds for a separating family of lsc linear functionals g.

We apply the notation of the previous paragraph to the special situation where $Y = D = C^*$ and $F = \mathrm{id}_{C^*}$, and where Φ is a continuous valuation on C^*. As the functionals $\eta_C(f), f \in C$, separate the points of C^*, we can rewrite (V) in the form:

$$m_{C^*}(\Phi) = \int \mathrm{id}_{C^*}\, d\Phi\,.$$

If Φ is a probability valuation (i.e., of total mass 1) on the cone C^*, we may view $m_{C^*}(\Phi)$ to be the barycenter of the mass distribution given by Φ. In the general case we may view $m_{C^*}(\Phi)$ as the average of Φ on C^* weighted by the total mass $\Phi(X)$. One may compare these concepts with the corresponding ones in the classical theory of compact convex sets (see e.g. [2]).

As a particular case we may think of $\mu_X: \mathcal{V}^2 X \to \mathcal{V}X$ as the weighted averaging operator defined by

$$\mu_X(\Phi) = \int \mathrm{id}_{\mathcal{V}X}\, d\Phi\,.$$

4 Algebras of the Extended Probabilistic Powerdomain Monad

One would like to know the algebras of the monad $\mathcal{V}$ and the algebra homomorphisms in an explicit way. Recall that the Eilenberg-Moore algebras of the extended probabilistic powerdomain monad $(\mathcal{V}, \eta, \mu)$ are pairs (A, α), where A is a topological space and

$\alpha: \mathcal{V}A \to A$ a continuous map such that $\alpha \circ \eta_A = \mathrm{id}_A$ and $\alpha \circ \mu_A = \alpha \circ \mathcal{V}\alpha$. A homomorphism of two $\mathcal{V}$ algebras (A, α) and (B, β) is a continuous function $h: A \to B$ such that $h \circ \alpha = \beta \circ \mathcal{V}h$.

On every $\mathcal{V}$-algebra (A, α) we may define an addition and a scalar multiplication by on A by $a + b = \alpha(\eta_A(a) + \eta_A(b))$ and $r \cdot a = \alpha(r\eta_A(a))$, which gives the first part of the following proposition; the converse holds if one assumes local convexity:

Proposition 2. *Every $\mathcal{V}$-algebra A carries the structure of a topological cone, and $\mathcal{V}$-algebra homomorphisms become linear and continuous. Conversely, if A is a locally convex topological cone such that $m_A(\varphi) = \int \mathrm{id}_A \, d\varphi$ exists for all $\varphi \in \mathcal{V}A$, then (A, m_A) is a $\mathcal{V}$-algebra.*

The question is whether all $\mathcal{V}$-algebras are of this type. As the hypotheses of the second part of the previous proposition hold in every dual cone, we have:

Corollary 1. *The dual cone $A = C^*$ of every topological cone with the weighted averaging map $m_A: \mathcal{V}A \to A$ is a $\mathcal{V}$-algebra.*

As $\overline{\mathbb{R}}_+$ is isomorphic to its own dual, we can specialize the above in the following way: $\overline{\mathbb{R}}_+$ together with the weighted averaging operator $m_{\overline{\mathbb{R}}_+}: \mathcal{V}\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ given by

$$m_{\overline{\mathbb{R}}_+}(\Phi) = \int \mathrm{id}_{\,\overline{\mathbb{R}}_+} \, d\Phi = \int x \, d\Phi(x)$$

is a $\mathcal{V}$-algebra.

For dual cones $A = C^*$ and $B = D^*$, a map $h: A \to B$ is a $\mathcal{V}$-algebra homomorphism if it preserves weighted averages of valuations, in the sense that, for all continuous valuations Φ on A,

$$h\Big(\int \mathrm{id}_{\,A} \, d\Phi\Big) = \int \mathrm{id}_{\,B} \, d(h\Phi)\,,$$

where $h\Phi$ is the image of the valuation Φ under the map h given by $(h\Phi)(U) = \Phi(h^{-1}U)$ for every open subset U of A.

The question is whether there is a simpler description of the algebras and their homomorphisms. One can try to guess such a description and prove the required universal property:

Lemma 2. *Let X be a T_0-space and A a locally convex topological cone for which the weighted averaging operator m_A is well defined. Then, for every continuous function $g: X \to A$ there is a continuous linear map $\hat{g}: \mathcal{V}(X) \to A$ such that $\hat{g} \circ \eta_X = g$.*

Proof. We compose the continuous linear maps $\mathcal{V}g: \mathcal{V}X \to \mathcal{V}A$ and $\mu_C: \mathcal{V}A \to A$ and we obtain a continuous linear map $\hat{g} = \mu_C \circ \mathcal{V}g: \mathcal{V}X \to A$ which satisfies the required equation. □

The problem is that, in general, we cannot assert the uniqueness of the continuous linear map $\hat{g}$ in the lemma above. In the special case where X is a continuous domain with its Scott topology and C a d-cone, then the uniqueness of $\hat{g}$ has been proved (see [6, 9, 5]). Thus, over the category of continuous domains, the algebras of the extended probabilistic powermonad $\mathcal{V}$ are the continuous d-cones, and the algebra homomorphisms are the Scott-continuous linear maps. We conjecture that an analogous result holds over the category of stably compact spaces.

5 Stably Compact Spaces

Recall that a space is *stably compact* if it is compact, locally compact, sober and coherent. The coherence property says that the intersection of any two compact saturated sets is compact. The extended probabilistic powercone $\mathcal{V}X$ over a stably compact space X is stably compact, too (see [3]). Thus, we may restrict the extended probabilistic powerdomain monad to the category SCTOP of stably compact spaces and continuous maps between them.

Conjecture 1. The algebras (A, α) of the extended probabilistic powerdomain monad $\mathcal{V}$ over the category SCTOP of stably compact spaces are the locally convex stably compact cones and the algebra homomorphisms are the continuous linear maps.

The following universal property (U) for a locally convex topological cone C is crucial for proving the conjecture:

(U) For every stably compact space X and every continuous map $g: X \to C$, there is a unique continuous linear map $\hat{g}: \mathcal{V}X \to C$ such that $\hat{g} \circ \eta_X = g$.

The following Theorem is our main tool. It may look quite innocent, but its proof requires quite some work. We essentially use [8, II-3.7/8/9] and [12, Theorem 5.1].

Theorem 1. *For a stably compact space X, every lsc linear functional G on $\mathcal{V}X$ can be represented by some $g \in \mathcal{L}X$ in the sense that $G(\varphi) = \int g\, d\varphi$ for all $\varphi \in \mathcal{V}X$.*

This result is equivalent to the statement that $\mathcal{L}X \cong (\mathcal{V}X)^*$, and hence is a kind of dual to the Riesz Representation Theorem, which is equivalent to $\mathcal{V}X \cong (\mathcal{L}X)^*$. Thus, $\mathcal{L}X$ and $\mathcal{V}X$ are reflexive topological cones, where a topological cone C is called *reflexive* if it is naturally isomorphic to C^{**}. We have the following consequences, which in fact can be easily proved to be equivalent to the statement of the theorem, without knowledge of its truth.

Corollary 2. *1. The cone $C = \overline{\mathbb{R}}_+$ satisfies property* (U).
2. *The dual $C = D^*$ of every topological cone D has property* (U).
3. *For every stably compact space Y, the cone $C = \mathcal{L}Y$ has property* (U).
4. *For every stably compact space X and every lsc linear functional $h: \mathcal{V}X \to \overline{\mathbb{R}}_+$,*

$$h(\mu_X(\Phi)) = \int h\, d\Phi \quad \textit{for all} \quad \Phi \in \mathcal{V}^2 X\,,$$

i.e., the integral of h is equal to the image under h of the weighted average of Φ.
5. *For every stably compact space Y, the cone $C = \mathcal{V}Y$ has property* (U).

Notice that the statement (4) of the previous corollary is a variant of a theorem of Choquet for a compact convex sets X: every lsc affine functional h on X and every probability measure Φ on X, one has $h(\beta(\Phi)) = \int h\, d\Phi$, where $\beta(\Phi)$ denotes the barycenter of Φ (see [2]).

The previous results strongly support our conjecture. They allow the following characterization of the Eilenberg-Moore algebras of the monad $\mathcal{V}$ over the category of stably compact spaces:

Theorem 2. *Let (A, α) be an Eilenberg-Moore algebra of the extended probabilistic powerdomain monad $\mathcal{V}$ over the category of stably compact spaces. Then A is a stably compact topological cone. If A is locally convex, then $\alpha = m_A$, i.e., $\alpha(\varphi) = \int \mathrm{id}_A \, d\varphi$, and A has property* (U).

That is, the *locally convex* algebras coincide with the locally convex stably compact cones, and their homomorphisms are the continuous linear maps. Thus, to prove our conjecture, it remains to show that every algebra is locally convex.

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