

On the equivalence of state transformer semantics and predicate transformer semantics

Dedicated to Viktor Selivanov
at the occasion of his 60th birthday

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G. Plotkin and the author [13] have worked out the equivalence between state transformer semantics and predicate transformer semantics in a domain theoretical setting for programs combining nondeterminism and probability. Works of C. Morgan and co-authors [20], Keimel, Rosenbusch and Streicher [14, 15], already go in the same direction using only discrete state spaces. In fact, Keimel and Plotkin did not restrict to probabilities or subprobabilities, but worked in an extended setting admitting positive measures that may even have infinite values. This extended setting offers technical advantages. It was the intention of the authors to cut down their results to the subprobabilistic case in a subsequent paper. A paper by J. Goubault-Larrecq [7] already goes in this direction. When preparing a first version of the follow-up paper, the author of this paper wanted to clarify for himself the basic ideas. In fact, the paper [13] is technically quite involved, and when one reaches the last section, where the equivalence of predicate and state transformer semantics is finally put together, one is quite exhausted and has difficulties to see the leading ideas. Even the referee of the paper seemed to have given up at that point.

It is the aim of this paper to begin from the other end. In all the situations that the author has been dealing with, the state transformer semantics had been given by a monad \mathcal{T} over the category DCPO of directed complete posets (= dcpos) and Scott-continuous functions (= functions preserving the partial order and suprema of directed subsets). A state transformer interprets the input-output behavior of a program by a Scott-continuous map t from the input domain X to the 'powerdomain' $\mathcal{T}Y$ over the output domain Y . Thus, state transformers live in the Kleisli category associated with the monad \mathcal{T} . If there is an equivalent predicate transformer semantics, predicate transformers have to live in a category (dually) equivalent to the Kleisli category.

In my experience the equivalence between state and predicate transformer semantics is based on a very simple principle derived from the continuation monad. One starts with a dcpo R of 'observations'. The elements of the function space (the exponential) R^X are 'observable predicates' over the dcpo X , and maps $s: R^Y \rightarrow R^X$ are 'predicate transformers'. Assigning to every dcpo X the space R^{R^X} of maps $\varphi: R^X \rightarrow R$ gives rise to a monad, the 'continuation monad'. The maps $t: X \rightarrow R^{R^X}$ are 'state transformers'. It is a simple observation that there is a natural bijection between state transformers and predicate transformers (see Section 1).

Monads are used in denotational semantics to model computational effects. In lots of cases they are obtained by using a dcpo R of observations carrying an additional algebraic structure. This algebraic structure carries over to the function spaces R^X and R^{R^X} . It leads to two kinds of monads 'subordinate' to the continuation monad. One may assign to each dcpo X firstly the dcpo $\mathcal{M}_R X \subseteq R^{R^X}$ of all Scott-continuous algebra homomorphism $\varphi: R^X \rightarrow R$ (see Section 3) and secondly the directed complete subalgebra $\mathcal{F}_R X$ of R^{R^X} generated by the projections $\hat{x} = (f \mapsto f(x)): R^X \rightarrow R, x \in X$ (see Section 4).

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The first monad behaves nice with respect to the natural bijection between state and predicate transformers: The state transformers $t: X \rightarrow \mathcal{M}_R X$ correspond to those predicate transformers that are algebra homomorphisms $s: R^Y \rightarrow R^X$ (see Section 3). But this monad is, in general, uninteresting for semantics. For semantics one uses the second monad $\mathcal{F}_R X$; but it is not clear to me how to characterize the predicate transformers corresponding to the state transformers $t: X \rightarrow \mathcal{F}_R X$. The situation becomes nice when both monads agree.

Problem. Characterize those dcpo algebras R for which the monads \mathcal{F}_R and \mathcal{M}_R agree.

I do not have such a characterization. But I give a sufficient condition for the containment $\mathcal{F}_R X \subseteq \mathcal{M}_R X$; this is the case provided the algebraic structure on R is 'entropic' (see Section 5). This concept borrowed from universal algebra (see, e.g., the monograph [21]) corresponds to commutativity for monads; but I have not pursued this link.

In Section 6 we deal with examples for the entropic situation, powerdomains for nondeterminism (both angelic and demonic) and for (extended) probabilistic choice. Our general approach yields the containment relation $\mathcal{F}_R X \subseteq \mathcal{M}_R X$ and the equivalence between state and predicate transformers. The equality $\mathcal{F}_R X = \mathcal{M}_R X$ has to be proved separately in each special case. In fact, equality does not always hold: often one has to restrict to continuous dcpos X . The proof of equality $\mathcal{F}_R X = \mathcal{M}_R X$ often is the really hard work, and for this we refer to the literature.

The entropic condition does not cover situations combining nondeterministic and probabilistic features (see [13]). Surprisingly there is a relaxed notion of entropicity (see Section 7) that allows to capture these situations (see Section 8). The relaxation consists in replacing certain equalities by inequalities.

Question. Is there a concept for monads over DCPO that corresponds to this relaxed notion of entropicity?

In most presentations, powerdomains are described by collections of certain subsets, for example (convex) Scott-closed sets, (convex) Scott-compact saturated sets, lenses, etc. In this paper the advantage of functional representations is put in evidence. The functional representations may seem less intuitive. But a lot of features become easier to prove and less technical to be handled. In fact, in [13], the set-based representations had to be translated into functional representations in order to prove the equivalence of state and predicate transformer semantics.

There is a long tradition for using functional representations in mathematics as well as in semantics:

- The notion of a 'distribution' has been formalized by L. Schwartz as a linear functional on the space of smooth functions with compact support on \mathbb{R}^n . Here \mathbb{R} corresponds to the space of observations and the smooth functions are the 'predicates'.

- Historically measures have first been introduced as certain set functions on σ -algebras and integrals were defined as a derived concept (see H. Lebesgue [18]). But alternatively the opposite approach has also been pursued: In the Daniell-Stone [5, 24] approach one starts with the abstract notion of an integral, a positive linear functional on a certain function space, and measures are obtained as derived notions. Bourbaki [2] takes the same approach for measures on topological spaces.

- In statistics and decision theory (see Walley [26]) one uses the notion of a 'prevision' in the sense of our 'predicates'. Probabilities and upper/lower probabilities then arise from functionals with certain properties on sets of previsions.

- C. Morgan and co-authors (see [20]) use the notion of an 'expectation' in the sense of our 'predicates' in their investigations on combining nondeterminism and probability. This terminology is a bit misleading; 'prevision' or 'random variable' (as proposed by D. Kozen [16]) seems more appropriate. Indeed in probability theory the term 'expectation' denotes the mean value of a random variable.

- A. Simpson [22] stresses the functional approach in topological domain theory. For the angelic and demonic powerdomain constructions this approach has been carried through by Battenfeld and Schröder [3].

- Labelled Markov processes, simulation and bisimulation are treated in a very appealing way in recent work by Chaput, Danos, Panangaden and Plotkin [4] using a functional approach and a duality that reminds the equivalence between state and predicate transformers. Previously, these constructions needed sophisticated tools from measure theory on Polish and analytic spaces.

In this paper we do not present any new particular case. It is our aim to present a general framework in which one can hope for a canonical equivalence between state and predicate transformer semantics. I do not have a proof, but I conjecture that under fairly general hypotheses the framework presented in this paper is the only one in which such an equivalence may happen:

Claim. If a monad \mathcal{T} over the category DCPO allows a (dually) equivalent predicate transformer semantics, then there is a dcpo R with some additional relaxed entropic structure with the following properties: $\mathcal{T}X$ 'is' a substructure of R^{R^X} consisting of all structure preserving maps. $\mathcal{T}X$ agrees with the substructure of R^{R^X} generated by the projections $\hat{x}, x \in X$. The structure preserving predicate transformers $s: R^Y \rightarrow R^X$ correspond naturally to the state transformers $t: X \rightarrow \mathcal{T}Y$.

Monads yield the free objects for the class of their Eilenberg-Moore algebras. In all the examples that we look at in this paper the powerdomain monads are the free algebras for an (in)equational theory which reflects properties of the choice operators involved. But here we do not elaborate this topic. It is natural to conjecture that the monad \mathcal{F}_R yields the free algebras for the (in)equational theory of the algebra R .

We use some basic background material from universal algebra, from category theory and from domain theory. We refer to Birkhoff's monograph [1] for the background on universal algebra, to Mac Lanes book [19] for monads and Kleisli triples and to [6] for directed complete posets (dcpos) and continuous domains.

1 Continuation monads and predicate transformers

We will work in the category DCPO of directed complete partially ordered sets (dcpos) and Scott-continuous functions (maps preserving the partial order and suprema of directed sets).

The category DCPO is Cartesian closed. Finite products, even arbitrary products, are Cartesian products with the pointwise order; suprema of directed sets are formed pointwise. The exponential consists of all Scott-continuous functions $u: X \rightarrow Y$ with the pointwise defined order; suprema of directed sets of Scott-continuous functions are also formed pointwise. We use two notations for the exponential of X and Y in parallel:

$$Y^X = [X \rightarrow Y].$$

The category SET of sets and functions can be considered as a full subcategory of DCPO; just take the discrete order (equality) on every set. Products and exponentials in SET are the same as in DCPO.

We will use notations from simply typed λ -calculus. Since we are in a Cartesian closed category, all maps defined through well-typed λ -calculus expressions are Scott-continuous (see, for example, [17, Part I]). Thus, we never need to prove the continuity of functions. In order to avoid explicit type information, we will fix notations as follows:

- R will be a fixed dcpo, called the dcpo of 'observations';
- X and Y denote arbitrary dcpos;
- x, y and r denote elements of X, Y and R , respectively;
- u denotes Scott-continuous maps $u: X \rightarrow Y$, that is, elements of $Y^X = [X \rightarrow Y]$; 1
- f and g denote Scott-continuous maps $f: X \rightarrow R$ and $g: Y \rightarrow R$, that is, elements of R^X and R^Y ;
- φ and ψ denote Scott-continuous maps $\varphi: R^X \rightarrow R$ and $\psi: R^Y \rightarrow R$.

For a Scott-continuous map $u: X \rightarrow Y$ we denote by R^u the map from R^Y to R^X defined by $R^u = \lambda g. g \circ u$, that is, $R^u(g) = g \circ u$ for all $g \in R^Y$; being even more explicit, $R^u(g)$ is the map from X to R defined by $R^u(g)(x) = g(u(x))$ for all $x \in X$. Note that we use the bracketing convention that $R^u(g)(x)$ has to be read as $(R^u(g))(x)$, a convention that we will use throughout. We obtain a contravariant functor R^- from the category DCPO into itself.

Applying the contravariant functor R^- twice yields a covariant functor $R^{R^-} = [R^- \rightarrow R]$. This is the well-known *continuation monad*. The unit δ of the monad is given by the *projections* or *evaluation maps* $\delta_X: X \rightarrow [R^X \rightarrow R]$ defined by

$$\delta_X = \lambda x. \lambda f. f(x), \text{ that is, } \delta_X(x)(f) = f(x) \quad (1)$$

for $f: X \rightarrow R$ and $x \in X$. It will be convenient to introduce the short notation \hat{x} for $\delta_X(x)$; then the defining identity simply reads $\hat{x}(f) = f(x)$.

For a map $t: X \rightarrow [R^Y \rightarrow R]$ its Kleisli lifting $t^\dagger: [R^X \rightarrow R] \rightarrow [R^Y \rightarrow R]$ is given by

$$t^\dagger = \lambda \varphi. \lambda g. \varphi(\lambda x. t(x)(g)), \text{ that is, } t^\dagger(\varphi)(g) = \varphi(\lambda x. t(x)(g)). \quad (2)$$

For a Scott-continuous map $u: X \rightarrow Y$, the map $R^{R^u}: R^{R^X} \rightarrow R^{R^Y}$ may be considered as a special case of a Kleisli lifting, namely $R^{R^u} = (\delta_Y \circ u)^\dagger$, that is,

$$R^{R^u}(\varphi)(g) = \varphi(\lambda x. (\delta_Y \circ u)(x)(g)) = \varphi(\lambda x. g(u(x))) = \varphi(g \circ u).$$

Lemma 1.1. *The map $\delta_X: X \rightarrow [R^X \rightarrow R]$ is Scott-continuous; if R has at least two elements $r < s$, then δ_X is an order embedding.*

Proof. As δ_X is defined by a λ -expression, it is Scott-continuous. If $x \not\leq x'$, then there is a Scott-continuous function $f: X \rightarrow R$ such that $f(x') = r < s = f(x)$, whence $\delta_X(x) \not\leq \delta_X(x')$. (Just define $f(y) = r$ if $y \leq x'$, else $= s$.) It follows that δ_X is an order embedding. \square

In many situations, objects X occur as 'state spaces'. An 'action', a 'program', acts on states and produces results $\psi \in [R^Y \rightarrow R]$ over a possibly different state space Y . Scott-continuous maps $t: X \rightarrow [R^Y \rightarrow R]$ will be called *state transformers*. The elements $g \in R^Y$ are interpreted as 'predicates', 'previsions', etc. Scott-continuous maps $s: R^Y \rightarrow R^X$ are called *predicate transformers*.

Lemma 1.2. *The dcpos of state and predicate transformers are canonically isomorphic: $[R^Y \rightarrow R]^X \cong [R^Y \rightarrow R^X]$. The mutually inverse natural bijections P and Q are given as follows: The map P assigns to a state transformers $t: X \rightarrow [R^Y \rightarrow R]$ the predicate transformer $P(t): R^Y \rightarrow R^X$ defined by $P(t) = \lambda g. \lambda x. t(x)(g)$, that is,*

$$P = \lambda t. \lambda g. \lambda x. t(x)(g).$$

The other way around, we assign to every predicate transformer $s: R^Y \rightarrow R^X$ the state transformer $Q(s) = (R^s) \circ \delta_X = (\lambda \varphi. \varphi \circ s) \circ (\lambda x. \lambda f. f(x)) = \lambda x. (\lambda f. f(x)) \circ s$, whence $Q(s)(x) = (\lambda f. f(x)) \circ s$. Thus $Q(s)(x)(g) = (\lambda f. f(x))(s(g)) = s(g)(x)$, that is,

$$Q = \lambda s. \lambda x. \lambda g. s(g)(x).$$

Proof. P and Q are mutually inverse bijections between state transformers and predicate transformers. Indeed, for all maps $s: R^Y \rightarrow R^X$, $g \in R^Y$ and $x \in X$, we have $P(Q(s))(g)(x) = Q(s)(x)(g) = s(g)(x)$, that is $P(Q(s)) = s$ for all s . Similarly, for all $t: X \rightarrow [R^Y \rightarrow R]$, $g \in R^Y$ and all $x \in X$, we have $(Q(Pt))(x)(g) = P(t)(g)(x) = t(x)(g)$, that is $Q(P(t)) = t$. \square

In this paper we will consider monads that arise 'inside' the continuation monad in the following sense: We assign to every dcpo X a sub-dcpo $\mathcal{J}X$ of $[R^X \rightarrow X]$ in such a way that the following properties are satisfied:

$$\begin{aligned} \delta_X(x) &\in \mathcal{J}X && \text{for all } x \in X; \\ t^\dagger(\mathcal{J}X) &\subseteq \mathcal{J}Y && \text{for every Scott-continuous } t: X \rightarrow [R^Y \rightarrow R]. \end{aligned}$$

Then R^{R^u} maps $\mathcal{J}X$ into $\mathcal{J}Y$ for every Scott-continuous map $u: X \rightarrow Y$; indeed, R^{R^u} is the Kleisli lifting $(\delta_Y \circ u)^\dagger$ of $\delta_Y \circ u: X \rightarrow [R^Y \rightarrow R]$. Thus R^{R^u} induces a Scott-continuous map $\mathcal{J}u: \mathcal{J}X \rightarrow \mathcal{J}Y$ in such a way that \mathcal{J} becomes a functor, and even a monad with (the corestriction of) δ as unit and the Kleisli lifting $t^\dagger|_{\mathcal{J}X}$ for $t: X \rightarrow \mathcal{J}Y$.

Definition 1.1. We say that \mathcal{J} is a monad *subordinate* to the continuation monad if it arises in the way just described.

2 Adding algebraic structure

We recall a few concepts from universal algebra. Every algebra has a well defined signature which consists of operations symbols of a prescribed arity which will be supposed to be finite in this paper.

Definition 2.1. A *signature* Ω is the disjoint union (sum) of a sequence of sets $\Omega_n, n \in \mathbb{N}$. The members $\omega \in \Omega_n$ are called *operation symbols of arity n* .

In our examples there will be no operation symbols of arity $n \geq 3$. Thus we can restrict our attention to $\Omega_0, \Omega_1, \Omega_2$, that is, to nullary, unary and binary operation symbols, and in many cases there will be only finitely many operation symbols altogether. The operation symbols of arity 0 are also called constants.

Definition 2.2. An *algebra of signature* Ω consists of a set A together with operations

$$\omega^A: A^n \rightarrow A,$$

one for each $\omega \in \Omega$ of arity n . A map $u: A \rightarrow B$ from one algebra A of signature Ω to another one, B , is a *homomorphism*, if

$$u(\omega^A(a_1, \dots, a_n)) = \omega^B(u(a_1), \dots, u(a_n))$$

for every operation $\omega \in \Omega$ of arity n and all $a_1, \dots, a_n \in A$.

We now replace the category of sets by the category DCPO of dcpos and Scott-continuous functions. We first adapt the notion of a signature Ω to the base category DCPO:

Definition 2.3. A *d-signature* Ω is the disjoint union (sum) of a sequence of dcpos $\Omega_n, n \in \mathbb{N}$.

In all our examples, Ω_n will be empty for $n \geq 3$; $\Omega_0, \Omega_1, \Omega_2$ will consist of finitely many operation symbols in most cases (trivially ordered), and then a d-signature will be the same as a signature. But we will consider some cases in which Ω_1 will be a proper dcpo. By replacing the dcpos Ω_n by their underlying sets $|\Omega_n|$ one retrieves a signature as above.

Definition 2.4. A *directed complete partially ordered algebra* (a *d-algebra*, for short) of d-signature Ω is an algebra A of signature $|\Omega|$ endowed with a structure of a dcpo in such a way that the maps

$$(\omega, (a_1, \dots, a_n)) \mapsto \omega^A(a_1, \dots, a_n): \Omega_n \times A^n \rightarrow A$$

are Scott-continuous for all n . A Scott-continuous algebra homomorphism between two d-algebras of the same d-signature is shortly called *d-homomorphism*.

Note that $\omega^A(a_1, \dots, a_n)$ depends continuously not only on the a_i but also on ω (which is a vacuous requirement, if Ω_n consists of a single operation symbol or of finitely many mutually incomparable operation symbols).

We fix a d-signature Ω for the rest of this paper and all d-algebras are understood to be of this signature. We also fix a d-algebra R of d-signature Ω .

For every dcpo X , the function space R^X is also a d-algebra. For an operation $\omega \in \Omega$ of arity n the natural extension of ω^R to a Scott-continuous operation on the function space R^X is defined pointwise: For all $f_1, \dots, f_n \in R^X$:

$$\omega^{R^X}(f_1, \dots, f_n)(x) = \omega^R(f_1(x), \dots, f_n(x)) \text{ for all } x \in X. \quad (3)$$

In the future, we will often omit the superscripts in $\omega^R, \omega^{R^X}, \dots$ and write simply ω . This simplification does not give rise to misunderstandings.

For every Scott-continuous map $u: X \rightarrow Y$, the induced Scott-continuous map $R^u: R^Y \rightarrow R^X$ is a d-homomorphism: Indeed, $R^u(\omega(g_1, \dots, g_n))(x) = \omega(g_1, \dots, g_n)(u(x)) = \omega(g_1(u(x)), \dots, g_n(u(x))) = \omega(R^u(g_1)(x), \dots, R^u(g_n)(x)) = \omega(R^u(g_1), \dots, R^u(g_n))(x)$ for all $x \in X$. Thus, we may view R^- to be contravariant functor from the category DCPO to the category of d-algebras and d-algebra homomorphisms.

In the same way, the operations ω can be extended to $R^{R^X} = [R^X \rightarrow R]$ so that the latter becomes a d-algebra, too, and the maps R^{R^X} are d-algebra homomorphisms. Since it will be used frequently, let us repeat the definition of the operations ω of arity n on R^{R^X} : For all $\varphi_1, \dots, \varphi_n \in R^X$:

$$\omega(\varphi_1, \dots, \varphi_n)(f) = \omega(\varphi_1(f), \dots, \varphi_n(f)) \text{ for all } f \in R^X. \quad (4)$$

Lemma 2.1. (a) *The projections $\delta_X(x): R^X \rightarrow R$ are d-homomorphisms for every $x \in X$.*

For every state transformer $t: X \rightarrow [R^Y \rightarrow R]$, the Kleisli lifting $t^\dagger: [R^X \rightarrow R] \rightarrow [R^Y \rightarrow R]$

(b) is a d-homomorphism and

(c) maps d-homomorphisms $\varphi: R^X \rightarrow R$ to d-homomorphisms $t^\dagger(\varphi): R^Y \rightarrow R$.

Proof. (a) Let $x \in X$. For $\omega \in \Omega$ of arity n and $f_1, \dots, f_n \in R^X$ we have $\delta_X(x)(\omega(f_1, \dots, f_n)) = \omega(f_1, \dots, f_n)(x) = \omega(f_1(x), \dots, f_n(x)) = \omega(\delta_X(x)(f_1), \dots, \delta_X(x)(f_n))$.

(b) We have to check that, for every $\omega \in \Omega_n$ and all $\varphi_1, \dots, \varphi_n$, we have $t^\dagger(\omega(\varphi_1, \dots, \varphi_n)) = \omega(t^\dagger(\varphi_1), \dots, t^\dagger(\varphi_n))$. For every $g \in R^Y$ we have indeed:

$$\begin{aligned} t^\dagger(\omega(\varphi_1, \dots, \varphi_n))(g) &= \omega(\varphi_1, \dots, \varphi_n)(\lambda x. t(x)(g)) && \text{by the definition (2) of } t^\dagger \\ &= \omega(\varphi_1(\lambda x. t(x)(g)), \dots, \varphi_n(\lambda x. t(x)(g))) && \text{since } \omega \text{ is defined pointwise (4)} \\ &= \omega(t^\dagger(\varphi_1)(g), \dots, t^\dagger(\varphi_n)(g)) && \text{by the definition (2) of } t^\dagger \\ &= \omega(t^\dagger(\varphi_1), \dots, t^\dagger(\varphi_n))(g) && \text{since } \omega \text{ is defined pointwise (4).} \end{aligned}$$

(c) Let $\varphi: R^X \rightarrow R$ be a d-homomorphism. For arbitrary $\omega \in \Omega_n$ and arbitrary $g_1, \dots, g_n \in R^Y$, we have:

$$\begin{aligned} t^\dagger(\varphi)(\omega(g_1, \dots, g_n)) &= \varphi(\lambda x. t(x)(\omega(g_1, \dots, g_n))) && \text{by the definition of } t^\dagger \\ &= \varphi(\lambda x. \omega(t(x)(g_1), \dots, t(x)(g_n))) && \text{since } t(x) \text{ is a homomorphism} \\ &= \varphi(\omega(\lambda x. t(x)(g_1), \dots, \lambda x. t(x)(g_n))) && \text{since } \omega \text{ is defined pointwise} \\ &= \omega(\varphi(\lambda x. t(x)(g_1)), \dots, \varphi(\lambda x. t(x)(g_n))) && \text{since } \varphi \text{ is a homomorphism} \\ &= \omega(t^\dagger(\varphi(g_1)), \dots, t^\dagger(\varphi(g_n))) && \text{by the definition of } t^\dagger. \end{aligned}$$

□

3 Monad I: Homomorphism monads

We continue with a fixed d-algebra R of d-signature Ω .

A subset C of a dcpo X is called a *sub-dcpo* if the supremum $\bigvee_i x_i$ of any directed family $(x_i)_i$ of elements in C stays in C . For two d-algebras A and B of the same d-signature, we denote by

$$[A \circ \rightarrow B]$$

the set of all d-homomorphisms $u: A \rightarrow B$. Since the pointwise supremum of a directed family of d-homomorphisms is again a d-homomorphism, we have:

Lemma 3.1. *For any two d-algebras A and B of the same d-signature, the d-homomorphisms $u: A \rightarrow B$ form a sub-dcpo $[A \circ \rightarrow B]$ of the dcpo $[A \rightarrow B]$ of all Scott-continuous maps from A to B .*

In particular $[R^X \circ \rightarrow R]$, the set of all d-homomorphisms $\varphi: R^X \rightarrow R$, is a sub-dcpo of $[R^X \rightarrow R]$ by Lemma 3.1. By Lemma 2.1(a), $\delta_X(x) \in [R^X \circ \rightarrow R]$ for all $x \in X$. By Lemma 2.1(c), for every state transformer $t: X \rightarrow [R^Y \rightarrow R]$, its Kleisli lifting maps $[R^X \circ \rightarrow R]$ into $[R^Y \circ \rightarrow R]$. We are in the situation described in Definition 1.1):

Proposition 3.1. *For a d-algebra R , the assignment $X \mapsto [R^X \circ \rightarrow R]$ yields a monad subordinate to the continuation monad. The unit is (the corestriction of δ and the Kleisli lifting of a Scott-continuous map $t: X \rightarrow [R^Y \circ \rightarrow R]$ is (the restriction-corestriction) $t^\dagger: [R^X \circ \rightarrow R] \rightarrow [R^Y \circ \rightarrow R]$*

The 'homomorphism monad' $([R^- \circ \rightarrow R], \delta, \dagger)$ exhibited in the previous proposition has the remarkable property that it behaves well with respect to the one-to-one correspondence between state and predicate transformers in the sense that there is a simple characterization of the predicate transformers corresponding to the state transformers $t: X \rightarrow [R^X \circ \rightarrow R]$:

Proposition 3.2. *Let R be d-algebra. The maps P and Q (see Lemma 1.2) induce a one-to-one correspondence between predicate transformers $s: R^Y \rightarrow R^X$ that are d-homomorphisms and those state transformers $t: X \rightarrow [R^Y \rightarrow R]$ for which each $t(x), x \in X$, is a d-homomorphism:*

$$[R^Y \circ \rightarrow R]^X \cong [R^Y \circ \rightarrow R^X]$$

Proof. Let $t(x)$ be a d-algebra homomorphism. For all $\omega \in \Omega$ of arity n and for all $g_1, \dots, g_n \in R^Y$ we have:

$$\begin{aligned} P(t)(\omega(g_1, \dots, g_n))(x) &= t(x)(\omega(g_1, \dots, g_n)) && \text{by Lemma 1.2} \\ &= \omega(t(x)(g_1), \dots, t(x)(g_n)) && \text{since } t(x) \text{ is a homomorphism} \\ &= \omega(P(t)(g_1)(x), \dots, P(t)(g_n)(x)) && \text{again by Lemma 1.2} \\ &= \omega(P(t)(g_1), \dots, P(t)(g_n))(x) && \text{since } \omega \text{ is defined pointwise on } R^X \end{aligned}$$

If this holds for all $x \in X$, then $P(t)(\omega(g_1, \dots, g_n)) = \omega(P(t)(g_1), \dots, P(t)(g_n))$ which shows that $P(t)$ is a homomorphism. If conversely $s: R^Y \rightarrow R^X$ is a d-homomorphism, then $Q(s)(x)(\omega(g_1, \dots, g_n)) = s(\omega(g_1, \dots, g_n)(x)) = \omega(s(g_1), \dots, s(g_n))(x) = \omega(s(g_1)(x), \dots, s(g_n)(x)) = \omega(Q(s)(x)(g_1), \dots, Q(s)(x)(g_n))$ which shows that $Q(s)(x)$ is a homomorphism. \square

For later use let us record the following: Let us replace the dcpo X by a d-algebra A . The map $\delta_A: A \rightarrow [R^A \rightarrow R]$ is by no means a homomorphism. But after replacing R^A by $[A \circ \rightarrow R]$ the situation changes: For $a \in A$, the map $\delta_A(a)$ from R^A to R is restricted to a map from $[A \circ \rightarrow R]$ to R ; we still use the same notation $\delta_A(a)$ for the restricted map.

Lemma 3.2. *For a d-algebra A , the unit $\delta_A: A \rightarrow R^{[A \circ \rightarrow R]}$ is a d-homomorphism.*

Proof. We just have to show that δ_A is a homomorphism, that is, for every operation ω of arity n and all $a_1, \dots, a_n \in A$, we have $\delta_A(\omega(a_1, \dots, a_n)) = \omega(\delta_A(a_1), \dots, \delta_A(a_n))$. Indeed, for every d-homomorphism $h: A \rightarrow R$ we have $\delta_A(\omega(a_1, \dots, a_n))(h) = h(\omega(a_1, \dots, a_n)) = \omega(h(a_1), \dots, h(a_n)) = \omega(\delta_A(a_1)(h), \dots, \delta_A(a_n)(h)) = \omega(\delta_A(a_1), \dots, \delta_A(a_n))(h)$. \square

4 Monad II: Free algebras

We keep the setting of the previous section and consider a fixed d-algebra R of d-signature Ω . In the previous section we exhibited a monad $[R^- \circ \rightarrow R]$ over the category of dcpos subordinate to the continuation monad by restricting to d-homomorphisms $R^X \rightarrow R$ instead of arbitrary Scott-continuous maps. I doubt that this monad is of any intrinsic interest. Of real interest for semantics and otherwise are free algebras. We consider a second monad subordinate to the continuation monad and we will investigate in which sense this is a free construction.

The intersection of any family of sub-dcpos in a dcpo X is a sub-dcpo, in fact, the sub-dcpos are the closed sets of a topology, called the *d-topology* (see, e.g., [11, Section 5]).

A subalgebra of a d-algebra A which is a sub-dcpo, too, is called a *d-subalgebra*. The intersection of any family of d-subalgebras is again a d-subalgebra.

Lemma 4.1. [12, Corollary 5.7] *In any d-algebra A , the d-closure of a subalgebra B , that is, the smallest sub-dcpo containing B , is a d-subalgebra.*

For every dcpo X we consider the d-subalgebra $\mathcal{F}_R X$ of $[R^X \rightarrow R]$ generated by the projections $\hat{x} = \delta_X(x), x \in X$. Indeed, since the intersection of any family of d-subalgebras is again a d-subalgebra, there is a smallest d-subalgebra $\mathcal{F}_R X$ in $[R^X \rightarrow R]$ containing the projections $\hat{x}, x \in X$.

For a map $t: X \rightarrow \mathcal{F}_R Y \subseteq [R^Y \rightarrow R]$, the Kleisli lifting $t^\dagger: [R^X \rightarrow R] \rightarrow [R^Y \rightarrow R]$ maps $\mathcal{F}_R X$ into $\mathcal{F}_R Y$, since t^\dagger is a d-homomorphism by Lemma 2.1(b). This shows that $(\mathcal{F}_R, \delta, \dagger)$ is a monad subordinate to the continuation

monad in the sense of Definition 1.1, the Kleisli lifting of a map $t: X \rightarrow \mathcal{F}_R Y$ being the restriction and corestriction of the Kleisli lifting \dagger for the continuation monad $[R^- \rightarrow R]$.

Proposition 4.1. $(\mathcal{F}_R, \delta, \dagger)$ is a monad over the category DCPO subordinate to the continuation monad.

Since we have a monad, the d-algebras $\mathcal{F}_R X$ are free for the class of its Eilenberg-Moore algebras. It is a challenge to determine these Eilenberg-Moore algebras more concretely. The natural conjecture is that the $\mathcal{F}_R X$ are free over X for the class of d-algebras determined by the (in)equational theory of the d-algebra R . But in this paper we will not discuss this question.

Of course, we can consider the predicate transformers $s: R^Y \rightarrow R^X$ that correspond to state transformers $t: X \rightarrow \mathcal{F}_R Y$ under the mutually inverse bijections P and Q according to Lemma 1.2. But I do not know of any intrinsic characterization of these state transformers. This is in contrast to the situation that we encountered in the previous section with the homomorphism monad. Thus, with respect to a characterization of the predicate transformers we are in an excellent position in those cases where the monads \mathcal{F}_R and $[R^- \circ \rightarrow R]$ agree. It would already be an advantage to be in the position, where the free d-algebra $\mathcal{F}_R X$ is contained in the dcpo $[R^X \circ \rightarrow R]$.

Clearly, the generators of $\mathcal{F}_R X$, the projections $\hat{x}, x \in X$, are homomorphisms and thus belong to $[R^X \circ \rightarrow R]$. But no other element of $\mathcal{F}_R X$ need to be a d-algebra homomorphism. For example, if we choose for R the d-semiring $\overline{\mathbb{R}}_+$ (with two constants 0 and 1 and the two binary operations addition and multiplication) and for X the unordered two element set, then the two projections $(x_1, x_2) \mapsto x_i, i = 1, 2$, are the only Scott-continuous homomorphisms from $\overline{\mathbb{R}}_+^2$ to $\overline{\mathbb{R}}_+$. But the free d-algebra with two generators is quite big, containing for example all polynomials in two variables x_i, x_2 with nonnegative integer coefficients.

But we observe:

Remark 4.1. If the d-homomorphisms $\varphi: R^X \rightarrow R$ form a subalgebra of $[R^X \rightarrow R]$, then $\mathcal{F}_R X \subseteq [R^X \circ \rightarrow R]$ for every dcpo X . Indeed, if $[R^X \circ \rightarrow R]$ is a subalgebra, then it is a d-subalgebra of $[R^X \rightarrow R]$. As the projections $\hat{x}, x \in X$, are homomorphisms, they belong to $[R^X \circ \rightarrow R]$. Hence, $[R^X \circ \rightarrow R]$ contains the d-subalgebra $\mathcal{F}_R X$ generated by the projections.

We are led to ask the question under which hypothesis the d-homomorphisms $\varphi: R^X \rightarrow R$ form a subalgebra of $[R^X \rightarrow R]$. Classical universal algebra offers an answer to that question.

5 Entropic algebras

Let us begin with classical universal algebra (over the category of sets) and consider algebras B and R of the same signature Ω . The set $\text{Hom}(B, R)$ of all algebra homomorphisms $\varphi: B \rightarrow R$ is a subset of the product algebra R^B . We ask the question, whether $\text{Hom}(B, R)$ is a subalgebra of R^B .

In order to answer this question, consider an operation $\sigma \in \Omega$ of arity n . For $\text{Hom}(B, R)$ to be a sub-algebra we have to show that, for all $\varphi_1, \dots, \varphi_n \in \text{Hom}(B, R)$, also $\sigma(\varphi_1, \dots, \varphi_n)$ is an algebra homomorphism, that is, for every operation $\omega \in \Omega$ of arity m and for all $f_1, \dots, f_m \in B$, we have

$$\sigma(\varphi_1, \dots, \varphi_n)(\omega(f_1, \dots, f_m)) = \omega(\sigma(\varphi_1, \dots, \varphi_n)(f_1), \dots, \sigma(\varphi_1, \dots, \varphi_n)(f_m)). \quad (5)$$

Definition 5.1. We will say that an operation σ of arity n and an operation ω of arity m on an algebra R commute if, for all $x_{ij} \in R, i = 1, \dots, n, j = 1, \dots, m$:

$$\sigma(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})) = \omega(\sigma(x_{11}, \dots, x_{n1}), \dots, \sigma(x_{1m}, \dots, x_{nm})). \quad (6)$$

Such an equational law is also called an *entropic law*. It can also be expressed by the commutativity of the following

diagram:

$$\begin{array}{ccc}
 (A^m)^n \cong (A^n)^m & \xrightarrow{\sigma^m} & A^m \\
 \omega^n \downarrow & & \downarrow \omega \\
 A^n & \xrightarrow{\sigma} & A
 \end{array}$$

If this entropic law holds in R , it also holds in any power R^I , in R^{R^I} and in all subalgebras thereof. As a consequence, equation (5) holds if σ commutes with ω in R . Indeed,

$$\begin{aligned}
 \sigma(\varphi_1, \dots, \varphi_n)(\omega(f_1, \dots, f_m)) &= \sigma(\varphi_1(\omega(f_1, \dots, f_m)), \dots, \varphi_n(\omega(f_1, \dots, f_m))) \\
 &\quad \text{(the operation } \sigma \text{ being defined pointwise)} \\
 &= \sigma(\omega(\varphi_1(f_1), \dots, \varphi_1(f_m)), \dots, \omega(\varphi_n(f_1), \dots, \varphi_n(f_m))) \\
 &\quad \text{(the } \varphi_i \text{ being homomorphisms)} \\
 &= \omega(\sigma(\varphi_1(f_1), \dots, \varphi_n(f_1)), \dots, \sigma(\varphi_1(f_m), \dots, \varphi_n(f_m))) \\
 &\quad \text{(since } \sigma \text{ commutes with } \omega \text{ in } R, \text{ equation (6))}
 \end{aligned}$$

Definition 5.2. An algebra of signature Ω is called entropic if any two operations $\sigma, \omega \in \Omega$ commute.

We have to be careful with the nullary operations: If c is a constant, then the entropic law says that $\omega(c, \dots, c) = c$ and that two constants have to agree. Thus, for an entropic algebra, we can suppose that there is at most one nullary operation c and, if there is one, the constant c is a subalgebra, in fact, the smallest subalgebra.

Example . (a) We have to be careful with nullary operations: If c is a constant, then the entropic law says that $\omega(c, \dots, c) = c$ and that two constants have to agree. Thus, for an entropic algebra, we can suppose that there is at most one nullary operation c and, if there is one, the constant c is a subalgebra, in fact, the smallest subalgebra.

(b) A unary operation ρ commutes with a binary operation $+$ if

$$\rho(x + y) = \rho(x) + \rho(y).$$

(c) A binary relation $*$ commutes with itself if ¹

$$(x_1 * x_2) * (x_3 * x_4) = (x_1 * x_3) * (x_2 * x_4).$$

In particular, every commutative, associative binary operation commutes with itself. Thus, commutative semigroups, commutative monoids, commutative groups and semilattices are entropic.

(d) Two binary operation $+$ and $*$ commute iff

$$(x_1 * x_2) + (x_3 * x_4) = (x_1 + x_3) * (x_2 + x_4)$$

As this identity does not hold for addition and multiplication in semirings and rings, these are not entropic. Similarly lattices, even distributive lattices are not entropic.

¹This law has been called the *entropic law* by I.M.H. Etherington, *Groupoids with additive endomorphisms*, American Mathematical Monthly **65** (1958), pages 596–601. Etherington used this law in order to characterize those groupoids, in which the pointwise product of two endomorphisms is again an endomorphism. For quasi-groups, the entropic law had already been considered under another name by D. C. Murdoch, *Quasi-groups which satisfy certain generalized associativity laws*, American Journal of Mathematics **61** (1939), pages 508–522. O. Frink Jr., *Symmetric and self-distributive systems*, American Mathematical Monthly **62** (1955), pages 697–707, used the notion *symmetric* for a binary operation $+$ satisfying nothing but the entropic law. He notes that all means (arithmetic, geometric, harmonic) are symmetric, that barycentric operations are entropic. For a symmetric groupoid $(S, +)$ he proves that the powerset with the induced operation $+$ is symmetric, that the pointwise defined sum of two endomorphisms is an endomorphism and that the endomorphisms form a symmetric groupoid under this operation; further, if α, β are two commuting endomorphisms, then $x * y = \alpha(x) + \beta(y)$ is a symmetric operation, too. A lot more on entropic algebras one can find in the monograph [21] by A.B. Romanowska and J.D.H. Smith.

From the considerations preceding the definition we have:

Proposition 5.1. (see, e.g., [21, Proposition 5.1]) *If the algebra R is entropic and B any algebra of the same signature, the algebra homomorphisms $\varphi: B \rightarrow R$ form a subalgebra of the product algebra R^B .*

We now turn to a d-algebra R of d-signature Ω . In this section, all d-algebras are supposed to be of the same d-signature Ω .

Since entropicity is defined by equational laws, every homomorphic image of a subalgebra of a product of entropic algebras is entropic. Thus, if R is an entropic d-algebra, then the function spaces R^X and R^{R^X} are entropic, too, as well as all d-subalgebras thereof. From the previous proposition we deduce:

Corollary 5.1. *For an entropic d-algebra R and any dcpo X , the collection $[R^X \circ \rightarrow R]$ of all d-homomorphisms $\varphi: R^X \rightarrow R$ is a d-subalgebra of $[R^X \rightarrow R]$.*

As a subalgebra of $[R^X \rightarrow R]$, the algebra $[R^X \circ \rightarrow R]$ is again entropic.

We now can state the first main result in this section. It follows from the corollary above and Remark 4.1:

Proposition 5.2. *If R is an entropic d-algebra, then $\mathcal{F}_R X \subseteq [R^X \circ \rightarrow R]$ for any dcpo X .*

We have seen in Proposition 3.2 that the state transformers $t: X \rightarrow [R^Y \circ \rightarrow R]$ correspond bijectively to the predicate transformers $s: R^Y \rightarrow R^X$ which are d-homomorphisms. Is there a characterization of those predicate transformers that correspond to the state transformers $t: X \rightarrow \mathcal{F}_R X$? Of course, this is not a problem in case $\mathcal{F}_R X = [R^X \circ \rightarrow R]$.

I do not know a general criterion for the equality $\mathcal{F}_R X = [R^X \circ \rightarrow R]$ to hold, even in the entropic setting. It does not hold in general:

Example . The nonnegative extended reals with the constant 0 and the binary operation $+$ form an entropic d-monoid $R = (\overline{\mathbb{R}}_+, +, 0)$. For any dcpo X , $\mathcal{F}_R X$ is the d-closure of the set of all finite sums $\sum_{i=1}^n n_i \hat{x}_i$ where the n_i range over positive integers and $x_i \in X$. But for every $r \in \mathbb{R}_+$ and every d-monoid homomorphism $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ also $r\varphi$ is a d-monoid homomorphism. Thus $[R^X \circ \rightarrow R]$ contains all scalar multiples $r\hat{x}$ where r ranges over positive reals, hence, all linear combinations $\sum_{i=1}^n r_i \hat{x}_i$, where the r_i range over positive reals. But clearly, if r is not an integer, then $r\hat{x}$ is not a member of $\mathcal{F}_R X$.

If we still want the equality $\mathcal{F}_R X = [R^X \circ \rightarrow R]$ in the previous example, we have at least to enrich \mathcal{F}_R by allowing multiplication with scalars $r \in \mathbb{R}_+$. In order to do this in the general setting, we observe that the maps $x \mapsto rx: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ for $r \in \mathbb{R}_+$ are precisely the d-endomorphisms of the d-monoid $R = (\overline{\mathbb{R}}_+, +, 0)$. In the general situation, every d-homomorphism $\varphi \in [R^X \circ \rightarrow R]$ composed with a d-endomorphism ρ of R yields again a d-homomorphism $\rho \circ \varphi \in [R^X \circ \rightarrow R]$.

The following corollary arises as a special case of Corollary 5.1, where X consists of one element only:

Corollary 5.2. *For an entropic d-algebra R , the Scott-continuous endomorphisms $\rho: R \rightarrow R$ form an entropic d-algebra, the operations $\omega \in \Omega$ being defined pointwise.*

For the d-endomorphisms of R we have an additional Scott-continuous binary operation, the composition $\rho_1 \circ \rho_2$. We denote by $\text{End}(R) \subseteq [R \rightarrow R]$ the d-algebra of all Scott-continuous endomorphisms of R with the operations $\omega \in \Omega$ and composition as an additional binary operation. The identity map on R is denoted by $\mathbf{1}_R$. Thus, the d-signature of the d-algebra $\text{End}(R)$ is Ω augmented by composition and a constant for the identity. Note that the operation \circ destroys entropicity.

The d-algebra $\text{End}(R)$ acts not only on R but also on R^X : For $f \in R^X$ and $\rho \in \text{End}(R)$, $\rho \circ f$ is again an element of R^X . In a similar way, $\text{End}(R)$ acts on $[R^X \rightarrow R]$. Composing a d-algebra homomorphism $\varphi \in [R^X \circ \rightarrow R]$ with an endomorphism $\rho \in \text{End}(R)$ yields again a d-algebra homomorphism $\rho \circ \varphi \in [R^X \circ \rightarrow R]$. Thus R , R^X , $[R^X \rightarrow R]$ and $[R^X \circ \rightarrow R]$ are $\text{End}(R)$ -modules in the sense of the following definition:

Definition 5.3. Let R be an entropic d-algebra of d-signature Ω . An $\text{End}(R)$ -d-module is a d-algebra A of d-signature Ω together with a Scott-continuous map $(\rho, x) \mapsto \rho \cdot x: \text{End}(R) \times A \rightarrow A$ satisfying the following axioms for all $\rho, \rho_1, \dots, \rho_n \in \text{End}(R)$, all $\omega \in \Omega$ and all $x, x_1, \dots, x_n \in A$:

$$\mathbf{1}_R \cdot x = x \quad (7)$$

$$(\rho_1 \circ \rho_2) \cdot x = \rho_1 \cdot (\rho_2 \cdot x) \quad (8)$$

$$\omega(\rho_1, \dots, \rho_n) \cdot x = \omega(\rho_1 \cdot x, \dots, \rho_n \cdot x) \quad (9)$$

$$\rho \cdot \omega(x_1, \dots, x_n) = \omega(\rho \cdot x_1, \dots, \rho \cdot x_n) \quad (10)$$

A map u from an $\text{End}(R)$ -d-module A to an $\text{End}(R)$ -d-module A' is said to be an $\text{End}(R)$ -d-module homomorphism, if

$$u(\omega(x_1, \dots, x_n)) = \omega(u(x_1), \dots, u(x_n)) \quad \text{for all } \omega \in \Omega \quad (11)$$

$$u(\rho \cdot x) = \rho \cdot f(x) \quad \text{for all } \rho \in \text{End}(R). \quad (12)$$

Axiom (10) says that $\rho \mapsto \rho \cdot x$ is an Ω -algebra homomorphism from $\text{End}(R)$ into A for every fixed $x \in A$, and equation (9) says that $x \mapsto \rho \cdot x$ is an endomorphism of A for every fixed ρ . We can subsume these two statement under the slogan that $(\rho, x) \mapsto \rho \cdot x: \text{End}(R) \times A \rightarrow A$ is an Ω -bimorphism.

On an $\text{End}(R)$ -d-module A , we may interpret each endomorphism ρ to be a unary operation on A . In this way, A becomes a d-algebra of d-signature $\Omega \cup \text{End}(R)$. The defining axioms (7) – (10) become equational laws. Axiom (10) shows that the unary operations ρ commute with the operations $\omega \in \Omega$. We have:

Proposition 5.3. *For an entropic d-algebra R of d-signature Ω , every $\text{End}(R)$ -d-module A is an entropic d-algebra of d-signature $\Omega \cup \text{End}(R)$ provided that A is entropic for the signature Ω .*

We are now in a position to reinterpret the material of this section in the following way: We fix an entropic d-algebra R of d-signature Ω and we regard it as an $\text{End}(R)$ -d-module of d-signature $\Omega \cup \text{End}(R)$. It stays entropic by Proposition 5.3. For any dcpo X , the function spaces R^X and $[R^X \rightarrow R]$ are entropic $\text{End}(R)$ -d-modules, too; the module operation is given by $\rho \circ f$ and $\rho \circ \varphi$ for $\rho \in \text{End}(R)$, $f \in \overline{\mathbb{R}}_+^X$ and $\varphi \in R^{R^X}$, respectively. The subset $[R^X \circ \rightarrow_{\text{mod}} R]$ of all $\text{End}(R)$ -d-module homomorphisms $\varphi: R^X \rightarrow R$ is an $\text{End}(R)$ -d-submodule of $[R^X \rightarrow R]$ by Proposition 4.1.

Note that $[R^X \circ \rightarrow_{\text{mod}} R]$ might be properly smaller than the set $[R^X \circ \rightarrow R]$ of all d-algebra homomorphisms $\varphi: R^X \rightarrow R$. Now $\mathcal{F}_{\text{mod}}X$ will be the $\text{End}(R)$ -d-submodule of $[R^X \rightarrow R]$ generated by the projections \hat{x} , $x \in X$. These projections are not only d-algebra but also $\text{End}(R)$ -d-module homomorphisms, since $\hat{x}(\rho \circ f) = \rho(f(x)) = \rho(\hat{x}(f)) = (\rho \circ \hat{x})(f)$ for $\rho \in \text{End}(R)$ and $f \in R^X$. Thus, $\hat{x} \in [R^X \circ \rightarrow_{\text{mod}} R]$. It follows that the $\text{End}(R)$ -d-submodule $\mathcal{F}_{\text{mod}}X$ generated by the projections in $[R^X \rightarrow R]$ is contained in $[R^X \circ \rightarrow_{\text{mod}} R]$.

Although the $\text{End}(R)$ -d-module $\mathcal{F}_{\text{mod}}X$ is bigger than the d-algebra $\mathcal{F}_R X$ the question remains open whether $\mathcal{F}_{\text{mod}}X = [R^X \circ \rightarrow_{\text{mod}} R]$. Maybe that this question has to be decided in every special case separately.

6 Examples: Powerdomains

We want to illustrate that some standard powerdomain constructions (see e.g. M.B. Smyth [23]) fit under the framework developed until now. Powerdomains are used for interpreting programs involving nondeterministic or probabilistic choice.

Our basic domain of observations is the two element dcpo $\mathbf{2} = \{0, 1\}$ with $0 < 1$. Here 1 denotes termination of a program and is observable, while 0 denotes nontermination which is not observable.

For a dcpo X , an observable predicate will be a Scott-continuous map $p: X \rightarrow \mathbf{2}$, and $\mathbf{2}^X$ will be the domain of observable predicates. The Scott-continuous functions from a dcpo X to $\mathbf{2}$ are the characteristic functions of Scott-open subsets. Thus, the domain $\mathbf{2}^X$ of observable predicates can be identified with the complete lattice $\mathcal{O}X$ of Scott-open subsets of X ordered by inclusion.

A predicate transformer will be a Scott-continuous map $s: \mathbf{2}^Y \rightarrow \mathbf{2}^X$ or, equivalently, a Scott-continuous map $s: \mathcal{O}Y \rightarrow \mathcal{O}X$. A state transformer will be a Scott-continuous map $t: X \rightarrow [\mathbf{2}^Y \rightarrow \mathbf{2}]$, equivalently, $t: X \rightarrow \mathcal{O}\mathcal{O}Y$,

where $\mathcal{O}\mathcal{O}Y$ denotes the complete lattice of all Scott-open subsets of the complete lattice $\mathcal{O}Y$. According to Lemma 1.2, state and predicate transformers are in a canonical one-to-one correspondence.

6.1 The deterministic case

For deterministic programs the state transformers t will be Scott-continuous maps from the input domain to the output domain:

$$X \longrightarrow Y \xrightarrow{\delta_Y} [\mathbf{2}^Y \rightarrow \mathbf{2}]$$

We can reason about properties of such programs using the connectives 'and' and 'or' as usual: If we can observe each of the predicates p and q , then we can also observe their conjunction $p \wedge q$ and their disjunction $p \vee q$. Thus, we consider our two element dcpo $\mathbf{2}$ as a d-algebra with two binary operations \wedge ($= \max$) and \vee ($= \max$) and we add 0 and 1 as constants. The algebra $(\mathbf{2}, \wedge, \vee, 0, 1)$ is not entropic, so that our previous developments do not apply. Let us describe this situation:

On $\mathbf{2}^X$ and $[\mathbf{2}^X \rightarrow \mathbf{2}]$ the operations \wedge and \vee are pointwise binary inf and sup, the constants being interpreted by the constant functions 0 and 1. On $\mathcal{O}X$ and $\mathcal{O}\mathcal{O}X$, the operations \wedge and \vee are interpreted by \cap and \cup , the constant by the empty set and the whole space, respectively. Since we have directed suprema in dcpos anyway, the algebraic structure is that of a frame: We have arbitrary suprema and finite infima connected by meet-distributivity. The dcpo $[\mathbf{2}^X \circ \rightarrow \mathbf{2}]$ of frame homomorphisms $\varphi: \mathbf{2}^X \rightarrow \mathbf{2}$ is the sobrification of X^s of X . The state transformers $t: X \rightarrow Y^s$ are in bijective correspondence with the predicate transformers $s: \mathbf{2}^Y \rightarrow \mathbf{2}^X$ which are frame homomorphisms according to Proposition 3.2. We see:

Only if the dcpo Y is a sober space in its Scott topology, the state transformers $t: X \rightarrow Y$ are in bijective correspondence with the frame homomorphisms $h: \mathbf{2}^Y \rightarrow \mathbf{2}^X$. But notice that the frame generated in $[\mathbf{2}^X \rightarrow \mathbf{2}]$ by the projections $\hat{x}, x \in X$, is much bigger than X^s .

6.2 The nondeterministic case

We now suppose that we interpret programs that admit a nondeterministic choice operator \sqcup . The effect is that a program, if it terminates, may lead to several results. There are two basic ways for interpreting such a choice operator, the *angelic* and the *demonic* interpretation. In the first case we are happy if at least one of the possible outcomes has the desired property, in the second case we demand that all of the possible outcomes have the desired property. This boils down to interpret the nondeterministic choice operator on our domain $\mathbf{2}$ of observations by the binary operation \vee in the first case, but by the binary operation \wedge in the second case. Thus our algebras of observations are

$$\mathbf{2}_{ang} = (\mathbf{2}, \vee, 0) \quad \text{and} \quad \mathbf{2}_{dem} = (\mathbf{2}, \wedge, 1)$$

for angelic and demonic nondeterminism, respectively.

In both cases we have a semilattice with unit, hence an entropic d-algebra according to Example (c). Accordingly, $\mathbf{2}_{ang}^X$ and $[\mathbf{2}_{ang}^X \rightarrow \mathbf{2}_{ang}]$ will be unital \vee -semilattices and $\mathbf{2}_{dem}^X$ and $[\mathbf{2}_{dem}^X \rightarrow \mathbf{2}_{dem}]$ are unital \wedge -semilattices. In the equivalent presentation through Scott-open sets, these function spaces correspond to $\mathcal{O}X$ and $\mathcal{O}\mathcal{O}X$ with binary union and \emptyset as a constant in the angelic case and binary intersection and the whole space as a constant in the demonic case.

For any dcpo X we define

$$\mathcal{H}X = [\mathbf{2}_{ang}^X \circ \rightarrow \mathbf{2}_{ang}]$$

to be the dcpo of all d- \vee -semilattice homomorphisms $\varphi: \mathbf{2}_{ang}^X \rightarrow \mathbf{2}_{ang}$. By Corollary 5.1, $\mathcal{H}X$ is a d- \vee -subsemilattice with a bottom element. It is called the *angelic* or *lower* or *Hoare powerdomain* over X . We have the following well-known result (see e.g. [23]):

Proposition 6.1. (a) The angelic powerdomain $\mathcal{H}X$ can be identified with the complete lattice of all Scott-closed subsets of X .²

(b) The predicate transformers corresponding to the state transformers $t: X \rightarrow \mathcal{H}Y$ are those Scott-continuous maps $s: \mathcal{O}Y \rightarrow \mathcal{O}X$ preserving binary unions and \emptyset (hence those maps preserving arbitrary unions because of Scott continuity).

(c) The unital join d -subsemilattice $\mathcal{F}_{\mathbf{2}_{ang}}X$ generated by the projections $whx, x \in X$, equals $\mathcal{H}X$.

Proof. (a) For every Scott-closed subset C of X , the open sets U contained in $X \setminus C$ form a Scott-closed ideal of the lattice $\mathcal{O}X$; hence, the map defined by $\varphi(U) = 0$, if $U \cap C = \emptyset$, else $= 1$, is a Scott-continuous unital \vee -semilattice homomorphism and every such homomorphism is of this form.

(b) follows from Proposition 3.2.

(c) follows from the fact that a Scott closed subset C is the union of the collection of principal ideals $\downarrow x, x \in C$, and that these principal ideals $\downarrow x$ correspond to the projections \hat{x} under the correspondence given in (a). \square

We see that our general developments yield the claims (a) and (b) of the previous proposition. For the claim (c), our general developments only tell us that the unital d - \vee -semilattice generated by the projections is contained in $\mathcal{H}X$. For the equality we have to use the special situation.

For any dcpo X we define

$$\mathcal{S}X = [\mathbf{2}_{dem}^X \circ \rightarrow \mathbf{2}_{dem}]$$

to be the dcpo of all d - \wedge -semilattice homomorphisms $\varphi: \mathbf{2}_{dem}^X \rightarrow \mathbf{2}_{dem}$. By Corollary 5.1, $\mathcal{S}X$ is a d - \wedge -subsemilattice with a top element. It is called the *demonic* or *upper* or *Smyth powerdomain* over X . We have the following well-known result (see e.g. [23]):

Proposition 6.2. (a) The demonic powerdomain $\mathcal{S}X$ can be identified with the \cap -semilattice of all Scott-open filters of $\mathcal{O}X$. If X is sober for its Scott topology, $\mathcal{S}X$ can be identified with the \cup -semilattice of Scott-compact saturated subsets of X ordered by reverse inclusion.³

(b) The predicate transformers corresponding to the state transformers $t: X \rightarrow \mathcal{S}Y$ are those Scott-continuous maps $s: \mathcal{O}Y \rightarrow \mathcal{O}X$ preserving binary intersections and the top (hence those maps preserving finite intersections).

(c) If X is a continuous dcpo, the unital d - \wedge -subsemilattice $\mathcal{F}_{\mathbf{2}_{ang}}X$ generated by the projections $\hat{x}, x \in X$, equals $\mathcal{S}X$.

Proof. (a) Clearly a map $\varphi: \mathbf{2}_{dem}^X \cong \mathcal{O}X \rightarrow \mathbf{2}_{dem}$ is Scott-continuous and preserves finite meets if and only if $\varphi^{-1}(1)$ is a Scott-open filter of the complete lattice $\mathcal{O}X$. Thus $\mathcal{S}X$ can be identified with the collection of all Scott open filters of $\mathcal{O}X$ (including $\mathcal{O}X$ as a filter). In a sober space, the Scott open filters \mathcal{F} of $\mathcal{O}X$ correspond bijectively to the Scott-compact saturated sets, the bijection being given by $\mathcal{F} \mapsto \bigcap \mathcal{F}$ (see e.g. [6, Theorem II-1.20]).

(b) follows from Proposition 3.2.

(c) For a proof we refer to [6, Theorem IV-8.10]. \square

We see that our general developments yield the claims (a) and (b) of the previous proposition. Concerning (c), we cannot use any general principle. In general the unital d - \wedge -subsemilattice of $[\mathbf{2}_{dem}^X \rightarrow \mathbf{2}_{dem}]$ can be strictly smaller than $\mathcal{S}X$. As often, one has to restrict here to continuous dcpos, where one can use approximations from way-below.

6.3 The extended probabilistic powerdomain

In order to catch probabilistic choice in programming, some kind of measure theory had to be introduced for domains. Measures take non-negative real values and possibly the value $+\infty$. In defining measures one needs addition of nonnegative extended reals and suprema of increasing sequences.

²Most authors exclude the empty set, the bottom element, from the Hoare powerdomain; then the predicate transformers are just supposed to preserve unions of nonempty families of closed sets.

³We have included \emptyset as the top element of $\mathcal{S}X$. Most authors exclude the empty set from the Smyth powerdomain.

Thus, let $\overline{\mathbb{R}}_+$ denote the dcpo of nonnegative real numbers augmented by $+\infty$ with the usual linear order. The algebraic structure will be given by the usual addition ($x + \infty = +\infty$) and the constant 0 which yield a commutative d-monoid. Every commutative monoid is entropic.

The d-monoid $(\overline{\mathbb{R}}_+, +, 0)$ has endomorphisms: For every $r \in \overline{\mathbb{R}}_+$, the map $x \mapsto rx: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ is a Scott-continuous endomorphism of the d-monoid $\overline{\mathbb{R}}_+$ (for $r = +\infty$ one agrees on $0 \cdot (+\infty) = 0$ and $r \cdot (+\infty) = +\infty$ for $r > 0$ as usually in measure theory); and every Scott continuous endomorphism of $\overline{\mathbb{R}}_+$ is of this form. The composition of two endomorphisms given by r and r' is the endomorphisms given by rr' . Since $(\overline{\mathbb{R}}_+, +, 0)$ is entropic, $\text{End}(R)$ is also a commutative monoid with respect to addition. Altogether, the algebra $\text{End}(R)$ is canonically isomorphic to semiring $(\overline{\mathbb{R}}_+, +, \cdot, 0, 1)$. The $\text{End}(\overline{\mathbb{R}}_+)$ -d-modules and module homomorphisms are precisely the d-cones and the linear maps as introduced for example by [1]. A cone is a commutative monoid C together with a scalar multiplication by nonnegative real numbers extended by $+\infty$ satisfying the same axioms as for vector spaces; in detail:

Definition 6.1. We take a signature consisting of a constant 0, unary operations $r \in \overline{\mathbb{R}}_+$ and a binary operation $+$. A *cone* is an algebra of this signature, that is, a set C endowed with a distinguished element 0, an addition $(x, y) \mapsto x + y: C \times C \rightarrow C$ and with a scalar multiplication $(r, x) \mapsto r \cdot x: \overline{\mathbb{R}}_+ \times C \rightarrow C$ satisfying for all $x, y, z \in C$ and all $r, s \in \overline{\mathbb{R}}_+$:

$$\begin{aligned} x + (y + z) &= (x + y) + z \\ x + y &= y + x \\ x + 0 &= x \end{aligned}$$

and

$$\begin{aligned} 1 \cdot x &= x \\ (rs) \cdot x &= r \cdot (s \cdot x) \\ r \cdot (x + y) &= r \cdot x + r \cdot y \\ (r + s) \cdot x &= r \cdot x + s \cdot x \\ 0 \cdot x &= 0 \end{aligned}$$

A map $f: C \rightarrow C'$ between cones is called *linear*, if it is additive and positively homogeneous, that is, if

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(rx) = rf(x)$$

for all $x, y \in C$ and all $f \in \overline{\mathbb{R}}_+$.

If a cone C is endowed with a directed complete partial order such that addition and scalar multiplication are Scott-continuous, we have a d-cone.

For a dcpo X , the function space $\overline{\mathbb{R}}_+^X$ is a d-cone, too. We denote by $\mathcal{V}X$ the set of Scott-continuous linear maps $\mu: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$. Note that Scott-continuous additive maps between d-cones are easily shown to be positively homogeneous, hence linear. Cones are entropic algebras. We infer that $\mathcal{V}X$ is a d-cone, too, the order and the algebraic operations being defined pointwise.

Definition 6.2. The d-cone $\mathcal{V}X$ is called the extended probabilistic powerdomain over X .

As a special case of 3.2 we obtain:

Proposition 6.3. There is a canonical one-to-one correspondence between state transformers $t: X \rightarrow \mathcal{V}Y$ and linear predicate transformers $s: \overline{\mathbb{R}}_+^Y \rightarrow \overline{\mathbb{R}}_+^X$.

By Corollary 5.1, $\mathcal{V}X$ contains the d-subcone of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$ generated by the projections $\delta_X(x), x \in X$, which are the classical Dirac measures. But we do not have equality, in general, but we have equality for an important subclass of dcpos.

Lemma 6.1. [1, 2] If X is a continuous dcpo, the d-cone $\mathcal{V}X$ is continuous, too. In fact, every $\varphi \in \mathcal{V}X$ is the join of a directed family of 'simple' valuations, that is, of finite linear combinations of projections $\sigma = \sum_{i=1}^n r_i \delta_X x_i$ with $\sigma \ll \varphi$.

Corollary 6.1. If X is a continuous dcpo, the d-cone $\mathcal{V}X$ of Scott-continuous linear functionals $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ equals the d-cone generated by the projections.

7 Relaxed morphisms and relaxed entropic algebras

It is our aim to combine probability with nondeterminism. For this we have to combine the semilattice structure for nondeterminism with the additive structure for extended probability, that is, our algebra of observations should be the extended reals $\overline{\mathbb{R}}_+$ with two binary operations $+$ and \vee (or \wedge). As $+$ and \vee do not commute, we no longer have an entropic algebra. The framework developed in the previous sections is too narrow. Surprisingly, one can deal with this situation by relaxing the previous setting in replacing equalities by inequalities.

Definition 7.1. Let ω be an operation of arity n defined on dcpos A and A' . A Scott-continuous map $h: A \rightarrow A'$ is called an ω -submorphism if

$$h(\omega(x_1, \dots, x_n)) \leq \omega(h(x_1), \dots, h(x_n)) \text{ for all } x_1, \dots, x_n \in A.$$

An ω -supermorphism is defined in the same way replacing the inequality \leq by its opposite \geq .

For d-algebras of d-signature Ω , we want to distinguish some operations $\omega \in \Omega$ for which we would like to consider relaxed morphisms. For this, we suppose that each Ω_n is the union of two sub-dcpo Ω_n^{\leq} and Ω_n^{\geq} which need not be disjoint. The subsets Ω_n^{\leq} and Ω_n^{\geq} will be kept fixed, and we let $\Omega^{\leq} = \bigcup_n \Omega_n^{\leq}$ and $\Omega^{\geq} = \bigcup_n \Omega_n^{\geq}$.

Definition 7.2. A Scott-continuous map $h: A \rightarrow A'$ between d-algebras of d-signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$ is said to be a *relaxed d-morphism* if h is an ω -submorphism for all $\omega \in \Omega^{\leq}$, but an ω -supermorphism for $\omega \in \Omega^{\geq}$. (For ω in both Ω^{\leq} and Ω^{\geq} , h will be an ω -homomorphism.)

Of course, d-homomorphisms are also relaxed d-morphisms. We record the following straightforward observation:

Lemma 7.1. *The composition of relaxed d-morphisms between d-algebras of d-signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$ yields relaxed d-morphisms.*

For a d-algebra R of d-signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$, we denote by $[R^X \circ_{\rightarrow_r} R]$ the set of all relaxed d-morphisms $\varphi: R^X \rightarrow R$. The pointwise supremum of a directed family of relaxed d-morphisms is again a relaxed d-morphism. Thus, these relaxed d-morphisms form a sub-dcpo of $[R^X \rightarrow R]$. As in the propositions 3.1 and 3.2 we have:

Proposition 7.1. *Let R be a d-algebra of d-signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$.*

(a) *For every state transformer $t: X \rightarrow [R^Y \rightarrow R]$ such that $t(x)$ is a relaxed d-morphism for each $x \in X$, the Kleisli lifting $t^\dagger: [R^X \rightarrow R] \rightarrow [R^Y \rightarrow R]$ maps relaxed d-morphisms to relaxed d-morphisms, so that our continuation monad $([R^- \rightarrow R], \delta, \dagger)$ restricts to a monad $([R^X \circ_{\rightarrow_r} R], \delta, \dagger)$.*

(b) *Under the bijective correspondences P and Q (see lemma 1.2, the predicate transformers corresponding to the state transformers $t: X \rightarrow [R^Y \circ_{\rightarrow_r} R]$ are the relaxed d-morphisms $s: R^Y \rightarrow R^X$, that is:*

$$[R^Y \rightarrow R]^X \cong [R^Y \circ_{\rightarrow_r} R^X]$$

The proofs are the same as for the corresponding claims in 2.1(c) and 3.2. We just have to replace the equality by the appropriate inequality (\leq in case $\omega \in \Omega^{\leq}$ and \geq in case $\omega \in \Omega^{\geq}$) every time that we have used the homomorphism property there.

We now turn to the question under what circumstances, the relaxed d-morphisms form a subalgebra of $[R^X \rightarrow R]$.

We attack this question more generally and consider a d-algebra B of d-signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$ and we ask the question, whether the set of relaxed d-morphisms $\varphi: B \rightarrow R$ is a subalgebra of $[B \rightarrow R]$.

In order to answer this question we consider an operation $\sigma \in \Omega$ of arity n and we have to show that $\sigma(\varphi_1, \dots, \varphi_n)$ is a relaxed morphism for all relaxed d-morphisms $\varphi_1, \dots, \varphi_n: B \rightarrow R$, that is, for all $\omega \in \Omega^{\leq}$ of arity m and all $f_1, \dots, f_m \in B$, we have:

$$\sigma(\varphi_1, \dots, \varphi_n)(\omega(f_1, \dots, f_m)) \leq \omega(\sigma(\varphi_1, \dots, \varphi_n)(f_1), \dots, \sigma(\varphi_1, \dots, \varphi_n)(f_m)), \quad (13)$$

and analogously, with the reverse inequality, for $\omega \in \Omega^{\geq}$.

Definition 7.3. We will say that an operation σ of arity n on a d-algebra A *subcommutes* with an operation ω of arity m (equivalently, ω *supercommutes* with σ) if, for all $x_{ij} \in A, i = 1, \dots, n, j = 1, \dots, m$:

$$\sigma(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})) \leq \omega(\sigma(x_{11}, \dots, x_{n1}), \dots, \sigma(x_{1m}, \dots, x_{nm})). \quad (14)$$

This is equivalent to the statement that σ is an ω -submorphism. Whenever this subcommutativity law holds in R , it also holds in R^X and R^{R^X} and in subalgebras thereof. As a consequence, inequation (13) holds if σ subcommutes with ω . Indeed,

$$\begin{aligned} \sigma(\varphi_1, \dots, \varphi_n)(\omega(f_1, \dots, f_m)) &= \sigma(\varphi_1(\omega(f_1, \dots, f_m)), \dots, \varphi_n(\omega(f_1, \dots, f_m))) \\ &\quad \text{(the operation } \sigma \text{ being defined pointwise)} \\ &\leq \sigma(\omega(\varphi_1(f_1), \dots, \varphi_1(f_m)), \dots, \omega(\varphi_n(f_1), \dots, \varphi_n(f_m))) \\ &\quad \text{(the } \varphi_i \text{ being } \omega\text{-submorphisms)} \\ &\leq \omega(\sigma(\varphi_1(f_1), \dots, \varphi_n(f_1)), \dots, \sigma(\varphi_1(f_m), \dots, \varphi_n(f_m))) \\ &\quad \text{(since } \sigma \text{ subcommutes with } \omega, \text{ equation (14))} \end{aligned}$$

This leads to the following definition:

Definition 7.4. We will say that the d-algebra R of d-signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$ is *relaxed entropic*, if every $\sigma \in \Omega$ is a relaxed morphism, that is, if σ subcommutes with every $\omega \in \Omega^{\leq}$ and supercommutes with every $\omega \in \Omega^{\geq}$.

In a relaxed entropic algebra, any two $\sigma, \omega \in \Omega_{\leq}$ commute and similarly for $\sigma, \omega \in \Omega_{\geq}$. By the arguments preceding the definition we have:

Proposition 7.2. *If R is a relaxed entropic d-algebra, the relaxed d-morphisms from any d-algebra B to R form a subalgebra of $[B \rightarrow R]$.*

Specializing to $B = R^X$ for a dcpo X gives us:

Corollary 7.1. *If R is a relaxed entropic d-algebra, the collection $[R^X \circ_{\rightarrow_r} R]$ of all relaxed d-morphisms $\varphi: R^X \rightarrow R$ is a d-subalgebra of $[R^X \rightarrow R]$.*

We now turn to the question whether $[R^X \circ_{\rightarrow_r} R]$ equals the d-algebra $\mathcal{F}_R X$ generated by the projections. Again we do not have a general answer, but at least a containment relation:

Proposition 7.3. *If R is a relaxed entropic d-algebra, then the d-subalgebra $\mathcal{F}_R X$ of $[R^X \rightarrow R]$ generated by the projections $\delta_X(x), x \in X$, is a d-subalgebra of $[R^X \circ_{\rightarrow_r} R]$.*

Proof. The projections $\delta_X(x)$ are d-homomorphisms, hence also relaxed d-morphisms and, in particular, contained in $[R^X \circ_{\rightarrow_r} R]$. If R is relaxed entropic, then $[R^X \circ_{\rightarrow_r} R]$ is a d-subalgebra of $[R^X \rightarrow R]$ by Proposition 13. Thus, $[R^X \circ_{\rightarrow_r} R]$ contains the d-subalgebra $F_R X$ which is generated by the projections. \square

Whether we have equality $F_R X = [R^X \circ_{\rightarrow_r} R]$, has to be decided in each special case separately.

8 Example: Combining nondeterminism and extended probability

We now combine extended probability and nondeterminism that had been considered separately in Section 6. For this purpose our domain R of observations will be $\overline{\mathbb{R}}_+ = \{r \in \mathbb{R} \mid r \geq 0\} \cup \{+\infty\}$ endowed both with its cone structure and a semilattice operation, \max or \min , which we denote by \vee and \wedge , respectively.

8.1 The angelic case

We first look at $\overline{\mathbb{R}}_+$ as a d-cone (see Definition 6.1) with \vee as an additional binary operation. Considered separately as a \vee -semilattice and as a cone, $\overline{\mathbb{R}}_+$ is entropic. Also multiplication by scalars commutes with the binary operation \vee ; indeed,

$$r(x \vee y) = rx \vee ry$$

holds for all $r \in \overline{\mathbb{R}}_+$ and all $x, y \in \overline{\mathbb{R}}_+$. But the two binary operations $+$ and \vee do not commute so that $(\overline{\mathbb{R}}_+, +, \vee, r \cdot -, 0)$ is not entropic. But the inequational law :

$$(x_1 + y) \vee (x_2 + z) \leq (x_1 \vee x_2) + (y \vee z), \quad (15)$$

holds for all $x_1, x_2, y, z \in \overline{\mathbb{R}}_+$, that is, \vee subcommutes with $+$ (and $+$ supercommutes with \vee). Thus, we put \vee into Ω^{\leq} and $+$ in Ω^{\geq} ; the constant 0 and the unary operations $x \rightarrow rx$ will be both in Ω^{\leq} and Ω^{\geq} , and we have:

Lemma 8.1. *$\overline{\mathbb{R}}_+$ with the binary operations $+$, \vee , the unary operations $x \mapsto rx$, $r \in \overline{\mathbb{R}}_+$, and the constant 0 is a relaxed entropic d-algebra.*

According to the general procedure in the previous section, let X be any dcpo and consider the relaxed d-morphisms $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$. In view of our signature $\Omega = \Omega^{\leq} \cup \Omega^{\geq}$, such a relaxed d-morphism is characterized by the following equalities and inequalities:

$$\varphi(0) = 0, \quad (16)$$

$$\varphi(rf) = r\varphi(f), \quad (17)$$

$$\varphi(f + g) \leq \varphi(f) + \varphi(g) \quad (18)$$

$$\varphi(f \vee g) \geq \varphi(f) \vee \varphi(g) \quad (19)$$

for all $f, g \in \overline{\mathbb{R}}_+^X$ and all $r \in \overline{\mathbb{R}}_+$. Since the first equality is a consequence of the second for $r = 0$, and since the last inequality is always satisfied for order preserving maps, we can omit these two and we see that the relaxed morphisms are nothing but the *sublinear functionals*. We denote by $\mathcal{H}\mathcal{V}X$ the set of all these Scott-continuous sublinear functionals φ .

Since linear functionals are sublinear, $\mathcal{H}\mathcal{V}X$ contains $\mathcal{V}X$. Moreover, $\mathcal{H}X$ is a retract of $\mathcal{H}\mathcal{V}X$: just use the retraction of $\overline{\mathbb{R}}_+$ onto $\{0, +\infty\}$ mapping all $r > 0$ onto $+\infty$ and notice that $+$ and \vee agree on $\{0, +\infty\}$.

As a special case of Corollary 7.1 and Proposition 7.1 we obtain:

Proposition 8.1. (a) *For every dcpo X , the set $\mathcal{H}\mathcal{V}X$ of Scott-continuous sublinear functionals $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ is a d-subcone and a d- \vee -subsemilattice of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$; in fact, $(\mathcal{H}\mathcal{V}X, \delta, \dagger)$ is a monad over the category DCPO subordinate to the continuation monad.*

(b) *There is a canonical one-to-one correspondence between state transformers $t: X \rightarrow \mathcal{H}\mathcal{V}Y$ and sublinear predicate transformers $s: \overline{\mathbb{R}}_+^Y \rightarrow \overline{\mathbb{R}}_+^X$.*

By Proposition 8.1(a), $\mathcal{H}\mathcal{V}X$ is a subalgebra of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$ and by Proposition 7.3 it contains the d-cone \vee -subsemilattice $\mathcal{F}_{\overline{\mathbb{R}}_+} X$ generated by the projections $\widehat{x}, x \in X$. For dcpos X in general, equality need not hold. For this we have to require continuity::

Proposition 8.2. *For every continuous dcpo X , the subalgebra $\mathcal{F}_{\overline{\mathbb{R}}_+} X$ of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$ generated by the projections agrees with the d-cone \vee -semilattice $\mathcal{H}\mathcal{V}X$ of all Scott-continuous sublinear functionals $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$.*

Proof. (Sketch) This proposition has been proved in [13, 25]. The proof uses the following ingredients: Let X be a continuous dcpo. Then $\overline{\mathbb{R}}_+^X$ is a continuous d-cone. We firstly use a Hahn-Banach type theorem [13, 5.9(1)] that tells us that every Scott-continuous sublinear functional $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ is pointwise the sup of a family of Scott-continuous linear functionals $\mu_i: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$. In Lemma 6.1 we have seen that every μ_i is pointwise the supremum of a directed family of finite linear combinations $\sigma_{ij} = \sum_{k=1}^n r_{kj} \delta_X(x_{kj})$ of projections (Dirac measures). Thus, φ is the supremum of the σ_{ij} . Taking finite suprema of the σ_{ij} one obtains a directed family $\bigvee_{j=1}^m \sum_{k=1}^n r_{kj} \delta_X(x_{kj})$ of finite suprema of simple valuations. This shows that φ belongs to the d-cone \vee -subsemilattice generated by the projections. \square

In [13, 25] one finds another representation of $\mathcal{H}\mathcal{V}X$, namely as the collection of all nonempty Scott-closed convex subsets of the d-cone $\mathcal{V}X$ ordered by inclusion. The equivalence of the two presentation is given as follows: To every sublinear functional $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ we assign the set $C(\varphi)$ of all linear functionals $\mu \in \mathcal{V}X$ such that $\mu \leq \varphi$. Then $C(\varphi)$ is a nonempty closed convex subset of $\mathcal{V}X$. In [13, Corollary 6.3] it is shown that the assignment $\varphi \mapsto C(\varphi)$ is an order isomorphism from $\mathcal{H}\mathcal{V}X$ onto the collection of nonempty Scott-closed convex subsets of $\mathcal{V}X$ ordered by inclusion.

8.2 The demonic case

We now look at $\overline{\mathbb{R}}_+$ as a d-cone with the additional binary operation \wedge ($x \wedge y = \min(x, y)$). As in the angelic case this algebra is not entropic, since addition and meet do not commute. But we have the inequation

$$(x_1 + y) \wedge (x_2 + z) \geq (x_1 \wedge x_2) + (y \wedge z), \quad (20)$$

that is, \wedge supercommutes with $+$ (and $+$ subcommutes with \wedge). Thus, we put $+$ into Ω^{\leq} and \wedge in Ω_{\geq} ; the constant 0 and unary operations $x \rightarrow rx$ will be both in Ω^{\leq} and Ω_{\geq} , and we have:

Lemma 8.2. *$\overline{\mathbb{R}}_+$ with the binary operations $+$, \wedge , the unary operations $x \mapsto rx$, $r \in \overline{\mathbb{R}}_+$, and the constant 0 is a relaxed entropic d-algebra.*

Similarly as in the angelic case, the relaxed morphisms $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ are characterized by

$$\varphi(rf) = r\varphi(f), \quad (21)$$

$$\varphi(f + g) \geq \varphi(f) + \varphi(g), \quad (22)$$

$$(23)$$

for all $f, g \in \overline{\mathbb{R}}_+^X$ and all $r \in \overline{\mathbb{R}}_+$. We see that the relaxed d-morphisms are nothing but the Scott-continuous superlinear functionals. We denote by $\mathcal{S}\mathcal{V}X$ the set of all these superlinear functionals φ . Since linear functionals are superlinear, $\mathcal{S}\mathcal{V}X$ contains $\mathcal{V}(X)$.

As a special case of Corollary 7.1 and Proposition 7.1 we obtain:

Proposition 8.3. (a) *For every dcpo X , the set $\mathcal{S}\mathcal{V}X$ of Scott-continuous superlinear functionals $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ is a d-subcone and a d- \wedge -subsemilattice of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$; in fact, $(\mathcal{S}\mathcal{V}X, \delta, \dagger)$ is a monad over the category \mathbf{DCPO} subordinate to the continuation monad.*

(b) *There is a canonical one-to-one correspondence between state transformers $t: X \rightarrow \mathcal{S}\mathcal{V}Y$ and superlinear predicate transformers $s: \overline{\mathbb{R}}_+^Y \rightarrow \overline{\mathbb{R}}_+^X$.*

By Proposition 8.3(a), $\mathcal{S}\mathcal{V}X$ is a d-cone \wedge -subsemilattice of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$ and by Proposition 7.3 it contains the d-cone \wedge -subsemilattice $\mathcal{F}_{\overline{\mathbb{R}}_+} X$ generated by the projections \hat{x} , $x \in X$. For dcpos X in general, equality need not hold. For this we have to require even more than continuity:

Proposition 8.4. *For every continuous coherent dcpo X , the d-cone \wedge -subsemilattice $\mathcal{F}_{\overline{\mathbb{R}}_+} X$ of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$ generated by the projections agrees with the d-cone \wedge -semilattice $\mathcal{S}\mathcal{V}X$ of all Scott-continuous superlinear functionals $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$.*

The coherence property required in the previous proposition means that the intersection of any two Scott-compact saturated subsets of X is Scott-compact. This proposition follows from results in [13, 25]. One uses that, for a continuous coherent dcpo X , the function space $\overline{\mathbb{R}}_+^X$ is a coherent continuous d-cone. A Hahn-Banach type theorem [13, 5.9(2)] tells us that every Scott-continuous superlinear functional $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$ is pointwise the infimum of a family of Scott-continuous linear functionals $\mu: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$. But now the arguments become more sophisticated than in the angelic case. One has to show that φ is pointwise the supremum of a directed family of finite infima of finite linear combinations of projections.

In [13, 25] one finds another representation of $\mathcal{S}\mathcal{V}X$, namely as the collection of all nonempty Scott-compact saturated convex subsets of the d-cone $\mathcal{V}X$ ordered by inclusion. The equivalence of the two presentation is given as follows: To

every Scott-continuous superlinear functional $\varphi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+$, we assign the set $K(\varphi)$ of all linear functionals $\mu \in \mathcal{V}X$ such that $\mu \geq \varphi$. Then $K(\varphi)$ is a nonempty Scott-compact saturated convex subset of $\mathcal{V}X$. In [13, Corollary 6.6] it is shown that the assignment $\varphi \mapsto K(\varphi)$ is an order isomorphism from $\mathcal{S}\mathcal{V}X$ onto the collection of nonempty Scott-compact saturated convex subsets of $\mathcal{V}X$ ordered by reverse inclusion.

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