Weakly Compact Spaces of Homomorphisms in Asymmetric Topology

Klaus Keimel\textsuperscript{a,1}

\textsuperscript{a}Fachbereich Mathematik, Technische Universität, D–64289 Darmstadt, Germany.

Abstract

The classical Bourbaki-Alaoglu theorem asserts that the polar $U^\circ$ of a neighborhood of 0 in a locally convex topological vector space $V$ is a weak*compact subset of the dual space $V^*$. G. Plotkin has proved an analogous theorem in the framework of continuous d-cones, a kind of asymmetric version of topological vector spaces in Domain Theory.

In this paper we extend Plotkin’s results. We consider topological spaces $X$ with a finitary continuous algebraic structure in the sense of universal algebra instead of a cone structure. Linear functionals are replaced by continuous algebra homomorphisms into a test algebra $R$ replacing the reals. Our main result, Theorem 5.9, concerns the compactness of the space $C^*$ of continuous homomorphisms from $X$ to $R$ under appropriate hypotheses. We exhibit conditions under which $C^*$ is not only a space but also an algebra, as in the classical situation. This leads us to the notion of entropicity in the sense of universal algebra.

The background for our investigation is Domain Theory and its use in denotational semantics (see [7]). Thus our spaces are strongly non-Hausdorff. This paper can be seen as a contribution to asymmetric topology and analysis.

Keywords: Asymmetric topology, semantic domains, weak topologies, spaces of homomorphisms, entropic algebras

1. Introduction

The Bourbaki-Alaoglu theorem asserts that the polar $U^\circ$ of a neighborhood of 0 in a locally convex topological vector space $V$ is a weak*compact subset of the dual space $V^*$ (see, e.g., E. Rudin [18, Section3.15]). G. D. Plotkin [16] has proved an analogous theorem in the framework of continuous d-cones.

Here, a cone is meant to be an asymmetric version of a vector space, that is, a set $C$ with an addition $+$ (commutative and associative) with neutral element 0 and a scalar multiplication by nonnegative real numbers satisfying the same laws as in vector spaces, but asymmetric in the sense that one cannot subtract and that scalar multiplication is restricted to nonnegative reals. In the spirit of Domain theory, Plotkin considers cones that are ordered in such a way that directed subsets always have a supremum; addition and scalar multiplication are supposed to be continuous in the sense that they respect suprema of directed sets. A continuous d-cone is one in which every element can be approximated from below by its relatively finite parts. These notions are basic in Domain Theory (see, e.g., [7]) which has been developed for its use in semantics of programming languages. In particular d-cones occur when dealing with models for languages with probabilistic choice (see, e.g., [9, 12, 19]). The theory has been extended to models of languages combining ordinary non-determinism with probabilistic choice [20]. In establishing the equivalence between state transformer semantics and predicate transformer semantics for these models of mixed choice, quite a lot of functional analytic tools had to be developed in this domain theoretical setting, like Hahn-Banach type separation theorems, sublinear, superlinear and linear functionals, compactness criteria, etc. (see [11, 8]).

In this paper we extend Plotkin’s results to algebraic structures other than cones. We consider topological spaces $X$ with a finitary continuous algebraic structure in the sense of universal algebra instead of a cone structure. Instead of linear...
functionals we consider continuous algebra homomorphisms into a test algebra $R$ replacing the reals. Our main result, Theorem 5.9, concerns the compactness of the space $C^*$ of continuous homomorphisms from $X$ to $R$. It extends Plotkin’s Theorem 2 in [16]. We finally exhibit conditions under which $C^*$ is not only a space but also an algebra, as in the classical and in Plotkin’s situation. This leads us to the notion of entropicity in the sense of universal algebra. Our generalization should be of interest in Semantics when modelling algebraic effects other than probability and nondeterminism.

The background for our investigation is Domain Theory and its use in denotational semantics (see [7, sec. VII-2]). Thus our spaces are strongly non-Hausdorff. Continuity in our sense is closely related to lower semicontinuity in classical analysis. This paper can be seen as a contribution to asymmetric topology and analysis.

2. Preliminaries

2.1. Intrinsic topologies on dcpos

We consider directed complete partially ordered sets $P$, sets with a partial order $\leq$ in which every directed family $(x_i)$, has a least upper bound denoted by $\bigvee_i x_i$; we call them dcpos for short. We denote by $P^{\text{op}}$ the poset $P$ with the opposite order $\geq$.

On a dcpo we consider the following intrinsic topologies (for details we refer to [7]):

- The upper topology $\nu(P)$: The principal ideals $\downarrow x = \{y \in P \mid y \leq x\}, x \in P$, form a subbasis for the closed sets.
- The lower topology $\omega(P)$: The principal filters $\uparrow x = \{y \in P \mid y \geq x\}, x \in P$, form a subbasis for the closed sets.
- The interval topology $i(P) = \nu(P) \lor \omega(P)$: The topology generated by the upper and the lower topology; the closed intervals $[x,y] = \uparrow x \cap \downarrow y$ form a subbasis for the closed sets.
- The Scott topology $\sigma(P)$: The closed sets are those subsets that are lower sets closed for suprema of directed subsets, that is, subsets $A \subseteq P$ with the properties: (1) If $y \leq x$ and $x \in A$, then $y \in A$; (2) $\bigvee \uparrow D \in A$ for every directed subset $D$ of $A$.
- The Lawson topology $\lambda(P) = \omega(P) \lor \sigma(P)$: The topology generated by the Scott and the lower topology.

Clearly the Scott topology is finer than the upper topology and the Lawson topology is finer than the interval topology.

If $P$ is filtered complete, that is, directed complete for the opposite order, then we may consider the topology $\sigma^{\text{op}}(P)$, the Scott topology on the order-dual poset, and $\lambda^{\text{op}}(P) = \nu(P) \lor \sigma^{\text{op}}(P)$, the Lawson topology on the order-dual poset.

2.2. Completely distributive and linked bicontinuous lattices

On a dcpo $P$, the way-below relation is defined by $y \ll x$ if, for every directed subset $D \subseteq P$ such that $x \leq \bigvee \uparrow D$, there is a $d \in D$ such that $y \leq d$. We say that $P$ is a continuous dcpo if, for every $x \in P$, the set of all $y \ll x$ is directed and $x = \bigvee \uparrow \{y \in P \mid y \ll x\}$. Every complete lattice is a dcpo and, if it is continuous as a dcpo, it is called a continuous lattice. A complete lattice $L$ is said to be bicontinuous if it is continuous and if its order dual $L^{\text{op}}$ is a continuous lattice, too.

Our basic example $\overline{\mathbb{R}}_+$, the nonnegative real numbers completed by $+\infty$, is a bicontinuous; the Scott topology agrees with the upper topology, the dual Scott topology with the lower topology, and the Lawson topology with its dual and with the interval topology. The same holds for any completely distributive lattice (see [7, sec. VII-2]):

**Proposition 2.1.** Every completely distributive lattice $R$ is bicontinuous; the Scott topology agrees with the upper topology, the dual Scott topology with the lower topology, and the Lawson topology with its dual and with the interval topology:

$\nu(R) = \sigma(R), \omega(R) = \sigma^{\text{op}}(R), \lambda(R) = i(R) = \lambda^{\text{op}}(R)$.

---

$^2$Often the name weak upper topology is used for what we call simply upper topology, and similarly for the lower topology in the next item.
A completely distributive lattice is tacitly supposed to be a complete lattice. By the previous proposition, a completely distributive lattice is linked in the following sense:

**Definition 2.2.** A complete lattice $L$ is called **linked** if its Scott topology $\sigma$ agrees with the upper topology $\nu$ and its dual Scott topology $\sigma^{op}$ agrees with the lower topology $\omega$.

In a linked complete lattice the Lawson topology, the dual Lawson topology, and the interval topology all agree. Our definition seems to differ from the one adopted in [7, VII-2.5]; there, a complete lattice is called linked if its Lawson topology $\lambda$ agrees with the Lawson topology $\lambda^{op}$ of the order dual. The two definitions are equivalent for complete lattices:

**Lemma 2.3.** If in a complete lattice $L$ the Lawson topology $\lambda$ agrees with the dual Lawson topology $\lambda^{op}$, then $L$ is linked.

**Proof.** Since $L$ is a complete lattice, the closed sets for the lower topology $\omega$ agree with the upper sets $A$ closed for the Lawson topology by [7, III-3.18(ii)]. If the Lawson topology agrees with the dual Lawson topology, the closed upper sets for the Lawson topology agree with the closed upper sets for the dual Lawson topology. By [7, III-1.6] the closed upper sets for the dual Lawson topology agree with the closed sets for the dual Scott topology. Thus $\omega = \sigma^{op}$. By passing to the order dual, one obtains $\nu = \sigma$. 

**Examples 2.4.** Every order complete chain is a completely distributive lattice with the lattice operations $\max$ and $\min$.

Completely distributive lattices are linked bicontinuous. They can be represented as complete sublattices of powers of the unit interval with the usual order. A complete lattice is completely distributive if and only if it is distributive and linked bicontinuous (see [7, Proposition VII-2.10]).

Every finite lattice is linked bicontinuous, and every complete sublattice of a product of finite lattices is linked bicontinuous.

### 2.3. Substructures and quotients

We need several results concerning the behavior of our topologies when passing to specific subsets of complete lattices and images under specific maps.

Let $L$ be a complete lattice and $M$ a subset of $L$. Let us suppose that the least upper bound of every subset $A$ of $M$ belongs to $M$. Every element $g \in L$ has a greatest lower bound in $M$, namely $\hat{g} = \bigvee \{f \in M \mid f \leq g\}$. We call $\hat{g}$ the lower $M$-envelope of $g$.

In general, the upper topology on a subset $M$ of a complete lattice $L$ is coarser than the topology induced by the upper topology on $L$. But:

**Lemma 2.5.** If $M$ is a subset of a complete lattice $L$ closed for arbitrary joins, the upper topology of $M$ agrees with the subspace topology induced by the upper topology on $L$: $\nu(M) = \nu(L)|_M$.

**Proof.** For $g \in L$ and $f \in M$, one has $f \leq g$ iff $f \leq \hat{g}$. Thus $\downarrow g \cap M = \{f \in M \mid f \leq \hat{g}\}$ with $\hat{g} \in M$, that is, intersecting a subbasic closed set of the upper topology on $L$ with $M$ yields a subbasic closed set of the upper topology on $M$. 

Recall that, on a continuous dcpo, the Lawson topology is Hausdorff and, on a continuous lattice, even compact (see, e.g.,[7, III-1.10, III-1.11]).

**Proposition 2.6.** Let $L$ be a continuous lattice and $M$ a subset that is closed in $L$ for arbitrary meets and directed joins. Then:

(a) $M$ is also a continuous lattice.

(b) $M$ is closed in $L$ for the Lawson topology.

(c) The Scott, lower and Lawson topologies of $M$ are the subspace topologies induced by the Scott, lower and Lawson topologies on $L$, respectively.
Proof. For (a) and (b) we refer to [7, Theorem I-2.1, Theorem III-1.12]. (c) The lower topology on \( M \) is induced by the lower topology on \( L \) by the order dual of Lemma 2.5. A Scott-closed subset \( A \) of \( M \) is closed for the Lawson topology \( \lambda(M) \), hence compact for the Lawson topology \( \lambda(L) \), since \( M \) is Lawson-closed in \( L \). It follows that \( \downarrow A \) is Lawson-closed, hence Scott-closed in \( L \). We infer that the Scott-closed subsets of \( M \) are the Scott-closed subsets of \( L \) intersected with \( M \).

From Lemma 2.5 and Proposition 2.6 we deduce:

**Corollary 2.7.** Let \( L \) be a linked bicontinuous (resp., completely distributive) lattice and \( M \) a subset closed in \( L \) for arbitrary meets and joins. Then:

(a) The lower, upper and interval topologies of \( M \) are the subspace topologies induces by the lower, upper and interval topologies on \( L \), respectively.

(b) \( M \) is closed in \( L \) for the interval topology.

(c) \( M \) is a linked bicontinuous (resp., completely distributive) lattice.

Let us summarize information from [7, Theorem I-2.11(iii), Proposition II-2.1, Lemma III-1.2]:

**Proposition 2.8.** Let \( L \) and \( M \) be complete lattices and \( f : L \rightarrow M \) a surjective map preserving arbitrary meets and joins. If \( L \) is a continuous lattice, then \( M \) is continuous, too, and the map \( f \) is continuous for the respective Scott, lower and Lawson topologies.

We deduce:

**Corollary 2.9.** Let \( L \) and \( M \) be complete lattices and \( f : L \rightarrow M \) a surjective map preserving arbitrary joins and meets. If \( L \) is linked bicontinuous (resp., completely distributive), then \( M \) is linked bicontinuous (resp., completely distributive), too; and \( f \) is continuous for the respective upper, lower and interval topologies.

**Proof.** For the completely distributive case this follows from the equational characterization of completely distributive lattices. Suppose that \( L \) is linked bicontinuous. By Proposition 2.8, \( M \) is continuous and dually continuous; moreover, the map \( f \) is continuous for the respective Lawson topologies, and similarly for dual Lawson topologies. Since these topologies are compact and Hausdorff, \( f \) is a quotient map for the respective Lawson topologies as well as for the dual Lawson topologies. Since on \( L \) the Lawson topology is supposed to agree with the dual Lawson topology, the same follows for the quotient topologies on \( M \). Using Lemma 2.3 we infer that \( M \) is linked, too.

2.4. Compact ordered and stably compact spaces

According to L. Nachbin [15], a compact ordered space is a compact topological space \( X \) with a partial order \( \leq \) the graph of which is closed in \( X \times X \).

There is an alternative approach to compact ordered spaces. Every topological T\(_0\)-space carries a natural order, the specialization order \( y \leq x \) if \( y \) belongs to the closure of the singleton \( \{x\} \). A subset is called saturated if it is an upper set for the specialization order.

A topological space \( X \) is stably compact [7, VI-6.7] if (a) \( X \) is locally compact and compact, (b) the intersection of finitely many compact saturated subsets is compact, and (c) \( X \) is sober.

Every stably compact space \( (X, \tau) \) becomes a compact ordered space if we endow it with its specialization order and the patch topology \( \tau_p \) generated by the given topology \( \tau \) and the co-compact topology \( \tau_c \), the closed sets of which are the compact saturated sets in \( X \). Conversely, every compact ordered space \( (X, \kappa, \leq) \) gives rise to a stably compact space \( (X, \kappa^*) \) by considering the topology \( \tau = \kappa^* \) of \( \kappa \)-open upper sets.

Every continuous lattice is a compact ordered space when endowed with its Lawson topology. The associated topology of open upper sets is the Scott topology, and the associated co-compact topology is the lower topology. In particular, a linked bicontinuous lattice is a compact ordered space for the interval topology; the associated topology of open upper sets is the upper topology, and the associated co-compact topology is the lower topology.

We will use the following lemma which can be applied to subsets of continuous lattices that are closed for the Lawson topology and, in particular, to subsets of linked bicontinuous lattices that are closed for the interval topology:
Lemma 2.10. Let \( D \) be a closed subspace of a compact ordered space \((X, \iota, \leq)\).

(a) \( D \) is a compact ordered space for the order and the topology \( \tau_D \) induced from \( X \).
(b) The topology \( \tau^* \) (resp., \( \kappa^* \)) on \( D \) formed by the \( \kappa \)-open upper (resp., lower) subsets of \( D \) is the subspace topology induced by the topology \( \iota^* \) (resp., \( \iota_* \)) of \( \iota \)-open upper (resp., lower) sets in \( X \).
(c) With the topology \( \kappa^* \), \( D \) becomes a stably compact space and \( \kappa_* \) is the associated co-compact topology.

Proof. (a) is straightforward. (b) Let \( A \) be a \( \iota \)-closed lower subset of \( D \). Since \( A = \downarrow A \cap D \), it suffices to show that the lower set \( \downarrow A \) generated by \( A \) in \( X \) is closed for the topology \( \iota \) on \( X \). As a closed subsets of the compact space \( X \), \( A \) is compact. We conclude that the lower set \( \downarrow A \) generated by \( A \) in \( X \) is \( \iota \)-closed. Indeed, Nachbin [15, Proposition 4, p. 44] has shown, that the lower set generated by a compact subset in any ordered topological space is closed. \( \square \)

3. Function spaces

From now on, let \( R \) denote a linked bicontinuous lattice. For any set \( X \), the set \( R^X \) of all functions \( f : X \to R \) with the pointwise defined order is a linked bicontinuous lattice, too. Its upper topology is the product topology on \( R \), i.e., \( \nu(R^X) = \nu(R)^X \); the analogous statement holds for the lower and for the interval topology.

For a subset \( D \) of \( R^X \), the topology induced by the upper topology on \( R^X \) is called the weak*upper topology \( w^*(D) \). A subbasis for its closed sets is given by

\[
A_{x,r} = \{ f \in D \mid f(x) \leq r \}, \quad x \in C, \ r \in R.
\]

It is the weakest topology on \( D \) for which all the point evaluations \( f \mapsto f(x) : D \to R, \ x \in X \), are continuous for the upper topology on \( R \).

In a similar way, the weak*lower topology \( w^*(D^{op}) \) on \( D \) has as subbasic closed sets the sets of the form

\[
B_{x,s} = \{ f \in D \mid f(x) \geq s \}, \quad x \in X, \ s \in R.
\]

The weak* topology is the common refinement of the two previous topologies. It is induced by the interval topology on \( R^X \).

The terms weak*upper topology, etc., have been chosen according to the terminology in functional analysis, where the topology of pointwise convergence on the dual of a topological vector space is called the weak* topology.

Our first example of a significant subset of \( R^X \) is the set \( M(X, R) \) of all order preserving functions \( f : X \to R \) in the case that \( X \) is a poset. Property (a) in the following Lemma is straightforward, the remaining claims follow from Corollary 2.7.

Lemma 3.1. (a) \( M(X, R) \) is closed in \( R^X \) for arbitrary pointwise suprema and infima.
(b) \( M(X, R) \) is a linked bicontinuous lattice; it is completely distributive if and only if \( R \) is.
(c) \( M(X, R) \) is closed in \( R^X \) for the interval topology. The intrinsic lower, upper and interval topologies on \( M(X, R) \) are those induced by the corresponding topologies on \( R^X \); in particular, the intrinsic lower, upper and interval topologies on \( M(X, R) \) are the weak*lower, weak*upper and weak*topologies, respectively.

We now turn to the topological situation. Let \( X \) be a topological space. With its specialization order \( X \) is a poset and, as before, we consider the lattice \( M(X, R) \) of all functions \( f : X \to R \) preserving the specialization order. We denote by \( L(X, R) \) the set of all lower semicontinuous functions \( f : X \to R \); by lower semicontinuity\(^3\) we mean that \( f \) is continuous for the upper topology on \( R \), that is, the set \( \{ x \in X \mid f(x) \leq r \} \) is closed in \( X \) for all \( r \in R \). As lower semicontinuous functions preserve the specialization order, \( L(X, R) \) is contained in \( M(X, R) \).

The pointwise supremum of any family of lower semicontinuous functions is lower semicontinuous. Hence, there is a greatest lower semicontinuous function \( \hat{g} : X \to R \) below every function \( g : X \to R \). The function \( \hat{g} \) is called the lower semicontinuous envelope of \( g \).

\(^3\)It may look strange that lower semicontinuity refers to the upper topology. For the sake of consistency with \cite{11} we stick to this terminology.
Associating its lower semicontinuous envelope $\hat{g}$ to every function $g: X \to R$ yields a projection, i.e., an order preserving map $\Psi: R^X \to R^X$ satisfying $\Psi \circ \Psi = \Psi \leq \text{id}_{R^X}$, the image of which is $\mathcal{L}(X, R)$.

The natural embedding $e: \mathcal{L}(X, R) \to \mathcal{M}(X, R)$ preserves arbitrary suprema. Its adjoint, the corestriction $\Psi: R^X \to \mathcal{L}(X, R)$ of the projection, preserves arbitrary infima. $\mathcal{L}(X, R)$ is a complete lattice. Suprema of families of lower semicontinuous functions are formed pointwise as in $R^X$. Of course, infinite infima in $\mathcal{L}(X, R)$ are not pointwise as in $R^X$: the infimum is the lower semicontinuous envelope of the the pointwise infimum.

We will only be interested in a very special class of topological spaces, the so-called c-spaces (see, e.g., Ernè [4]; Ershov [6] has introduced them independently under the name of $\alpha$-spaces). In a topological space $X$ we use the topological way-below relation $y \ll x$ when $\uparrow y$ is a neighborhood of $x$. We say that $X$ a c-space if for every point $x$ in an open set $U$ there is a $y \in U$ such that $y \ll x$, that is, if every point $x$ has a basis of neighborhoods of the form $\uparrow y$.

C-spaces are closely related to continuous dcpo's. Indeed, every continuous dcpo is a c-space with respect to its Scott topology and the topological way-below relation agrees with the order theoretical way-below relation. Conversely, the sobrification of every c-space is a continuous dcpo with respect to its specialization order (see [14]).

When $X$ is a c-space, the lower semicontinuous envelope of an order preserving function $g: X \to R$ is given by the simple formula

$$\hat{g}(x) = \bigvee_{y \ll x} g(y)$$

for every $x \in X$.

**Proposition 3.2.** Let $X$ be a c-space and $R$ a linked bicontinuous lattice.

(a) The collection $\mathcal{L}(X, R)$ of all lower semicontinuous maps $f: X \to R$ is closed in $\mathcal{M}(X, R)$ for arbitrary (pointwise) joins and finite (pointwise) meets.

(b) $\mathcal{L}(X, R)$ is a complete lattice, the meet of an arbitrary family in $\mathcal{L}(X, R)$ being the lower semicontinuous envelope of the pointwise meet.

(c) The projection map $\Psi: \mathcal{M}(X, R) \to \mathcal{L}(X, R)$ assigning to every order preserving function $g: X \to R$ its lower semicontinuous envelope $\hat{g}$ preserves arbitrary joins and arbitrary meets.

(d) $\mathcal{L}(X, R)$ is a linked bicontinuous lattice; it is completely distributive if and only if $R$ is.

(e) The projection $\Psi = (g \mapsto \hat{g}): \mathcal{M}(X, R) \to \mathcal{L}(X, R)$ is continuous for the respective upper (= Scott), lower (= dual Scott) and interval (= Lawson) topologies.

(f) The upper (= Scott) topology of $\mathcal{L}(X, R)$ is induced by the upper topology on $\mathcal{M}(X, R)$. It agrees with the weak-upper topology. A subbasis for the closed sets is given by the sets:

$$A_{x,r} = \{ g \in \mathcal{L}(X, R) \mid g(x) \leq r \} \ , x \in X, r \in R .$$

(g) The lower (= dual Scott) topology of $\mathcal{L}(X, R)$ has a subbasis for the closed sets of the form:

$$B_{U,r} = \{ g \in \mathcal{L}(X, R) \mid g(x) \geq r \text{ for all } x \in U \}$$

where $U$ ranges over the open subsets of $X$ and $r$ over $R$.

**Proof.** (a) and (b) are clear from the preceding remarks. (c) The canonical embedding $\mathcal{L}(X, R) \to \mathcal{M}(X, R)$ preserves arbitrary joins. Thus, its adjoint $\Psi$ preserves arbitrary meets. For showing that $\Psi$ preserves arbitrary joins, let $(g_i)$ be a family of functions in $\mathcal{M}(X, R)$ and $g = \sup_i g_i$. Then $\hat{g}(x) = \bigvee_{y \ll x} g(y) = \bigvee_{y \ll x} \bigvee_i g_i(y) = \bigvee_i \hat{g}_i(x)$ for every $x \in X$.

(d) and (e) are consequences of (c) in the light of Corollary 2.9. (f) results from Lemma 2.5.

(g) By definition, the sets $B_f = \{ g \in \mathcal{L}(X, R) \mid f \leq g \} , f \in \mathcal{L}(X, R)$, form a subbasis for the closed sets of the lower topology on $\mathcal{L}(X, R)$. For an open subset $U$ of $X$ and an element $r \in R$, the function $U \searrow r$ with value $r$ on $U$ and $\bot$ elsewhere is lower semicontinuous, and every lower semicontinuous function $f: X \to R$ is the join of such one-step functions. Thus $B_f$ is an intersection sets of the form $B_{U \searrow r} = \{ g \in \mathcal{L}(X, R) \mid U \searrow r \leq g \} = \{ g \in \mathcal{L}(X, R) \mid g(x) \geq r \text{ for all } x \in U \}$ and the latter sets also form a subbasis for the closed sets of the lower topology.  \[\square\]
The lower topology on \( L(X, R) \) may be strictly coarser than the topology induced by the lower topology on \( \mathcal{M}(X, R) \).

The topology on a subset \( D \subseteq L(C, R) \) with a subbasis for the closed sets of the form

\[ B_{U,r} = \{ g \in D \mid g(x) \geq r \text{ for all } x \in U \} \]

where \( U \) ranges over the open subsets of \( X \) and \( r \) over \( R \), has been called the lower-open topology by Plotkin [16].

Property (c) of the previous proposition is noteworthy, since the projection map \( \Psi : R^X \to L(X, R) \) does not preserve finite joins. Indeed, if we consider maps from the unit interval to itself, we may take the function \( f \) with value 1 on the rationals and zero else and \( g = 1 - f \). Then \( \hat{f} = \hat{g} = 0 \) but \( \hat{f} \vee \hat{g} = 1 \).

**Remark 3.3.** Since \( \Psi : \mathcal{M}(X, R) \to \mathcal{M}(X, R) \) preserves arbitrary suprema (under the hypotheses of the previous lemma), \( \Psi \) also has an upper adjoint \( \Phi \): Indeed, \( \Phi(g) \) is the greatest order preserving function such that \( \Psi(\Phi(g)) = \Psi(g) \). It is concretely given by \( \Phi(g)(x) = \bigwedge_{x \leq y} g(y) \) for all \( x \in X \).

### 4. Adding algebraic structure

We recall a few concepts from universal algebra. Every algebra has a well defined signature \( \Omega \) which consists of operation symbols \( \omega \). Each operation symbol \( \omega \) has a preassigned arity which will be supposed to be a natural number in this paper.

In most examples there are no operation symbols of arity \( n > 2 \). Thus we can restrict our attention to nullary, unary and binary operation symbols, and in many cases there are only finitely many operation symbols altogether. The operation symbols of arity 0 are also called constants.

For simplicity we will often use the binary operation symbol \( + \) in proofs (without supposing commutativity or any other special property to hold). By using this familiar symbol, proofs become easy to read without loosing generality.

**Definition 4.1.** An algebra of signature \( \Omega \) consists of a set \( A \) together with operations

\[ \omega^A : A^n \to A, \]

one for each \( \omega \in \Omega \) of arity \( n \). A map \( h : A \to B \) from one algebra \( A \) of signature \( \Omega \) to another one, \( B \), is a homomorphism if

\[ h(\omega^A(a_1, \ldots, a_n)) = \omega^B(h(a_1), \ldots, h(a_n)) \]

for every \( \omega \in \Omega \) of arity \( n \) and all \( a_1, \ldots, a_n \in A \).

We fix a signature \( \Omega \) for the rest of this paper and all algebras are understood to be of this signature. For simplicity, we will omit the superscript \( A \) in \( \omega^A \) for denoting the interpretation of an operation symbol in a particular algebra.

We need an order theoretical and a topological version of algebras:

**Definition 4.2.** A partially ordered algebra (a po-algebra, for short) is an algebra \( A \) endowed with a partial order \( \leq \) in such a way that each operation \( \omega \in \Omega \) is order preserving as a map from \( A^n \) to \( A \) where \( n \) is the arity of \( \omega \). A topological algebra is an algebra \( A \) endowed with a topology in such a way each \( \omega \in \Omega \) is continuous as a map from \( A^n \) to \( A \).

We will deal with topological algebras \( A \), where the underlying topological space is a c-space. Such topological algebras will be called c-space algebras. A second class of topological algebras are those where the underlying space is a linked bicontinuous lattice with its upper topology, that is, the algebraic operations are continuous with respect to the upper topology. We will speak of linked bicontinuous algebras.\(^4\)

\(^4\)Let us stress that in a linked bicontinuous algebra the algebraic operations are continuous with respect to the upper topology only; bi-continuity only refers to the underlying space, not to a property of the algebraic operations.
Since continuous maps preserve the specialization order, each topological algebra is a partially ordered algebra with respect to its specialization order.

For partially ordered algebras, we will want to consider a relaxed notion of morphisms in the following sense:

**Definition 4.3.** Let \( A \) and \( B \) be partially ordered algebras. A map \( h: A \to B \) is said to be a supermorphism if for every \( \omega \in \Omega \) of arity \( n \) and all \( a_1, \ldots, a_n \in A \):

\[
h(\omega(a_1, \ldots, a_n)) \geq \omega(h(a_1), \ldots, h(a_n)).
\]

If we replace \( \geq \) by \( \leq \) we obtain the notion of a submorphism.

We also use the notions of sub- and supermorphisms for maps between topological algebras \( A \) and \( B \), where the order is understood to be the specialization order.

The terminology *supermorphism* and *submorphism* has been borrowed from functional analysis were maps are called superadditive if \( g(a + b) \geq g(a) + g(b) \) and subadditive if \( g(a + b) \leq g(a) + g(b) \) for all \( a, b \).

We fix again a signature \( \Omega \) and a linked bicontinuous algebra \( R \) of signature \( \Omega \). For any set \( X \), the product \( R^X \) with pointwise defined operations

\[
\omega(f_1, \ldots, f_n)(x) = \omega(f_1(x), \ldots, f_n(x))
\]

is also a linked bicontinuous algebra of signature \( \Omega \).

If \( X \) is a poset, the order preserving maps \( f: X \to R \) form not only a complete sublattice but also a subalgebra \( \mathcal{M}(X, R) \) of \( R^X \). Similarly, if \( X \) is a topological space, the lower semicontinuous functions \( f: X \to R \) form a subalgebra \( \mathcal{L}(X, R) \) contained in \( \mathcal{M}(X, R) \).

**Lemma 4.4.** Let \( R \) be a linked bicontinuous algebra and \( X \) a topological space. The projection \( \Psi: \mathcal{M}(X, R) \to \mathcal{L}(X, R) \) assigning to each order preserving function \( g: X \to R \) its lower semicontinuous envelope \( \hat{g} \) is a supermorphism. If \( X \) is a c-space, \( \Psi \) is a homomorphism.

**Proof.** We prove the claim for a binary operation denoted additively. Let \( g, h \in \mathcal{M}(X, R) \). As \( \hat{g} \leq g \) and \( \hat{h} \leq h \), we have \( \hat{g} + \hat{h} \leq g + h \), whence \( \hat{g} + \hat{h} \leq \hat{g + h} \). Thus, \( \Psi \) is a supermorphism.

Now let \( X \) be a c-space. Then \( \hat{g} + \hat{h}(x) = \bigvee_{y \in X} (g + h)(y) = \bigvee_{y \in X} g(y) + h(y) = \bigvee_{y \in X} g(y) + \bigvee_{y \in X} h(y) = \hat{g}(x) + \hat{h}(x) = (\hat{g} + \hat{h})(x) \) for all \( x \in X \). This shows that \( \Psi \) is a homomorphism. \( \square \)

5. A general Bourbaki-Alaoglu Theorem

We fix again a signature \( \Omega \) and a linked bicontinuous algebra \( R \).

For a po-algebra \( C \) we consider the product algebra \( R^C \) and the subset \( \mathcal{M}(C, R) \) of all order preserving \( g: C \to R \).

Therein, we consider the subsets

\[
C_{\text{sup}}' \quad \text{of all order preserving supermorphisms;}
\]

\[
C_{\text{sub}}' \quad \text{of all order preserving submorphisms;}
\]

\[
C' \quad \text{of all order preserving homomorphisms.}
\]

Note that \( C_{\text{sup}}' \) has a greatest element, the constant function with value \( \top \). Similarly, \( C_{\text{sub}}' \) has a least element, the constant function with value \( \bot \). But there need not exist any homomorphism from \( C \) to \( R \) so that \( C' \) may be empty.

The following lemma is straightforward:

**Lemma 5.1.** (a) The set \( C_{\text{sup}}' \) of all order preserving supermorphisms \( f: C \to R \) is closed in \( \mathcal{M}(C, R) \) with respect to arbitrary pointwise infima and directed pointwise suprema.

(b) The set \( C_{\text{sub}}' \) of all order preserving submorphisms \( f: C \to R \) is closed in \( \mathcal{M}(C, R) \) with respect to arbitrary pointwise suprema and filtered pointwise infima.

(c) The set \( C' \) of all order preserving homomorphisms \( f: C \to R \) is closed in \( \mathcal{M}(C, R) \) with respect to directed pointwise suprema and filtered pointwise infima.
Lemma 5.2. (a) \( C_{\text{sup}}' \) is a continuous lattice.
(b) \( C_{\text{sup}}' \) is closed in \( \mathcal{M}(C, R) \) for the Lawson (= interval = weak*) topology.
(c) The Lawson topology on \( C_{\text{sup}}' \) is induced by the Lawson (= interval = weak*) topology on \( \mathcal{M}(C, R) \); hence, the Lawson topology on \( C_{\text{sup}}' \) is the weak*topology.
(d) The Scott topology on \( C_{\text{sup}}' \) is induced by the Scott (= upper = weak*upper) topology on \( \mathcal{M}(C, R) \); hence, the Scott topology on \( C_{\text{sup}}' \) is the weak*upper topology.
(e) The lower topology on \( C_{\text{sup}}' \) is induced by the lower (= weak*lower) topology on \( \mathcal{M}(C, R) \); hence, the lower topology on \( C_{\text{sup}}' \) is the weak*lower topology.

Proof. By Lemma 3.1(b), \( \mathcal{M}(C, R) \) is a continuous lattice. Using Lemma 5.1(a), the claims (a) and (b) follow from Proposition 2.6(a), (b). The claims (c), (d) and (e) now follow from Proposition 2.6(c) using Lemma 3.1(c). \( \square \)

Using 2.10 for the collection \( C' = C_{\text{sub}}' \cap C_{\text{sup}}' \) of all order preserving algebra homomorphisms from \( C \) to \( R \), we obtain the following conclusions:

Corollary 5.3. (a) \( C' \) is closed in \( \mathcal{M}(C, R) \) for the interval (= weak*) topology, hence, a compact ordered space for the weak*topology.
(b) \( C' \) is a stably compact space for the weak*upper topology. The co-compact topology associated with the weak*upper topology is the weak*lower topology, and the patch topology is the weak*topology.

We now introduce the following continuous analogues to the spaces of morphisms above. For a a c-space algebra \( C \), denote by

\[
\begin{align*}
C_{\text{sup}}^* & \quad \text{the set of all lower semicontinuous supermorphisms } g: C \to R, \\
C_{\text{sub}}^* & \quad \text{the set of all lower semicontinuous submorphisms } g: C \to R, \\
C^* & \quad \text{the set of all lower semicontinuous homomorphisms } g: C \to R.
\end{align*}
\]

Clearly \( C_{\text{sup}}^* = C_{\text{sup}}' \cap \mathcal{L}(C, R) \) (and similarly for \( C_{\text{sub}}^* \) and \( C^* \)). Thus, \( C_{\text{sup}}^* \) is closed for directed pointwise suprema and \( C_{\text{sub}}^* \) is closed for arbitrary pointwise suprema. The pointwise infimum \( g \) of a family \( (g_i) \) of continuous supermorphisms is a supermorphism but need not be continuous. In order to form the infimum of the \( g_i \) in \( \mathcal{L}(C, R) \) we have to pass to the lower semicontinuous envelope \( \hat{g} \) of the pointwise infimum \( g \). But is \( \hat{g} \) still a supermorphism? In order to answer this question, we need an additional property:

Definition 5.4. An \( n \)-ary operation \( \omega \) on a c-space \( C \) is said to preserve the way-below relation, if

\[
x_1 \ll y_1, \ldots, x_n \ll y_n \implies \omega(x_1, \ldots, x_n) \ll \omega(y_1, \ldots, y_n)
\]

A binary operation denoted additively preserves the way-below relation if \( x_1 \ll y_1, x_2 \ll y_2 \implies x_1 + x_2 \ll y_1 + y_2 \); in this case, the way-below relation has been called additive in [16].

The following lemma gives sufficient conditions for the projection \( \Psi: \mathcal{M}(C, R) \to \mathcal{L}(C, R) \) to map \( C_{\text{sup}}' \) onto \( C_{\text{sup}}^* \). Plotkin has proved this property in the framework of continuous d-cones [16, Proposition 1].

Lemma 5.5. Let \( C \) be a c-space algebra, \( R \) a linked bicontinuous algebra and \( g: C \to R \) an order preserving map.

(a) If \( g \) is a submorphism, then its lower semicontinuous envelope \( \hat{g} \) is a submorphism, too; hence, \( \Psi \) maps \( C_{\text{sub}}' \) onto \( C_{\text{sub}}^* \).

Suppose in addition that all operations \( \omega \in \Omega \) preserve the way-below relation on \( C \).

(b) If \( g \) is a supermorphism, then \( \hat{g} \) is a supermorphism, too; hence, \( \Psi \) maps \( C_{\text{sup}}' \) onto \( C_{\text{sup}}^* \).

(c) If \( g \) is a homomorphism, its lower semicontinuous envelope \( \hat{g} \) is a homomorphism, too; hence, \( \Psi \) maps \( C' \) onto \( C^* \).
Proof. For the proof we suppose that $\omega$ is a binary operation $\oplus$.

(a) We suppose that $g: C \to R$ is order preserving and subadditive. We use the continuity of $\oplus$ on $C$: For every $z \ll x + y$ there are $z_1 \ll x$ and $z_2 \ll y$ such that $z \ll z_1 + z_2$. Thus:

$$\hat{g}(x + y) = \bigvee_{z \ll x + y} g(z) \leq \bigvee_{z_1 \ll x, z_2 \ll y} g(z_1 + z_2) \leq \bigvee_{z_1 \ll x, z_2 \ll y} g(z_1) + g(z_2) \quad \text{since } g \text{ is subadditive}$$

$$\leq \bigvee_{z_1 \ll x} g(z_1) + \bigvee_{z_2 \ll y} g(z_2) \quad \text{by continuity of } \oplus \text{ on } R$$

$$= \hat{g}(x) + \hat{g}(y).$$

(b) We now suppose $\oplus$ to preserve $\ll$ on $C$ and $g$ to be order preserving and superadditive. Then:

$$\hat{g}(x) + \hat{g}(y) = \bigvee_{u \ll x, v \ll y} g(u) + \bigvee_{v \ll y} g(v)$$

$$= \bigvee_{u \ll x, v \ll y} g(u) + g(v) \quad \text{by continuity of } \oplus \text{ on } R$$

$$\leq \bigvee_{u \ll x, v \ll y} g(u + v) \quad \text{since } g \text{ is superadditive}$$

$$\leq \bigvee_{u \ll x + y} g(w) \quad \text{since } + \text{ preserves } \ll \text{ on } C$$

$$= \hat{g}(x + y).$$

(c) is a direct consequence of (a) and (b) using of course that $C' = C'_{\text{sub}} \cap C'_{\text{sup}}.$ \qed

Lemma 5.6. Let $R$ be a linked bicontinuous algebra and $C$ a $c$-space algebra. Then:

(a) $C'_{\text{sub}}$ is closed in $\mathcal{L}(C, R)$ with respect to arbitrary suprema and filtered infima.

Suppose in addition that all basic operations $\omega \in \Omega$ preserve the way-below relation on $C$.

(b) $C'_{\text{sup}}$ is closed in $\mathcal{L}(C, R)$ with respect to directed suprema and arbitrary infima.

(c) $C^*$ is closed in $\mathcal{L}(C, R)$ for directed joins and filtered meets.

Proof. By Lemma 5.1 and Proposition 3.2(a), $C'_{\text{sup}}$ is closed in $\mathcal{L}(C, R)$ with respect to directed and $C'_{\text{sub}}$ for arbitrary joins.

(a) Now consider a filtered family of lower semicontinuous submorphisms $g_i: C \to R$. The pointwise meet $\hat{g}$ of this family is still a submorphism, but not necessarily lower semicontinuous. The lower semicontinuous envelope $\hat{g}$ is the meet of the $g_i$ in the lattice $\mathcal{L}(C, R)$. By Lemma 5.5, $\hat{g}$ is a submorphism, too.

(b) Now consider a family of lower semicontinuous supermorphisms $g_i: C \to R$. The pointwise meet $\hat{g}$ of this family is still a supermorphism, but not necessarily lower semicontinuous. The lower semicontinuous envelope $\hat{g}$ is the meet of the $g_i$ in the lattice $\mathcal{L}(C, R)$. Under the hypothesis that the algebraic operations on $C$ preserve the way-below relation, $\hat{g}$ is a supermorphism, too, by Lemma 5.5.

(c) follows immediately from (a) and (b). \qed

We now proceed to the main results in this paper:

Proposition 5.7. Let $R$ be a linked bicontinuous algebra and $C$ a $c$-space algebra in which all basic operations $\omega \in \Omega$ preserve the way-below relation. Then:

(a) $C'_{\text{sup}}$ is a continuous lattice.

(b) $C'_{\text{sup}}$ is closed in $\mathcal{L}(C, R)$ for the Lawson (= interval) topology.

(c) The Lawson (resp., Scott, lower) topology on $C^*$ is induced by the Lawson (resp., Scott lower) topology on $\mathcal{L}(C, R)$. 

10
(d) The Scott topology on $C^*_{sup}$ agrees with the weak*upper topology. Thus, it has a subbasis for the Scott-closed sets of the form

$$A_{x,r} = \{ g \in C^*_{sup} \mid g(x) \leq r \}, x \in C, r \in R.$$  

(e) The lower topology on $C^*_{sup}$ has a subbasis for the closed sets of the form

$$B_{U,r} = \{ g \in C^*_{sup} \mid g(x) \geq r \text{ for all } x \in U \}$$

where $U$ ranges over the open subsets of $C$ and $r \in R$.

(f) The map $\Psi$ from $C^*_{sup}$ onto $C^*_{sup}$ is surjective and preserves directed joins and arbitrary meets; hence, it is continuous for the Scott, lower and Lawson topologies, respectively.

**Proof.** (a), (b) and (c) follow from Lemma 5.6(b) together with 2.6. (d) and (e) follow from (c) using Proposition 3.2(f),(g). (f) The projection $\Psi : M(C, R) \to L(C, R)$ preserves all meets and joins by Proposition 3.2(b). By Lemma 5.5(b), $\Psi$ maps $C^*_{sup}$ onto $C^*_{sup}$. Since $C^*_{sup}$ and $C^*_{sup}$ are closed in $M(C, R)$ and $L(C, R)$, respectively, for arbitrary meets and directed joins, the claim follows. \qed

Dually, for the case of submorphisms we do not need the hypothesis that the operations on $C$ preserve the way-below relation. Otherwise the proof is analogous to the proof of the previous proposition:

**Proposition 5.8.** Let $R$ be a linked bicontinuous algebra and $C$ a c-space algebra.

(a) $C^*_{sub}$ is a dually continuous lattice.

(b) $C^*_{sub}$ is closed in $L(C, R)$ for the dual Lawson (= interval) topology.

(c) The dual Lawson (resp., dual Scott, upper) topology on $C^*_{sub}$ is induced by the dual Lawson (resp., dual Scott, upper) topology on $L(C, R)$.

(d) The dual Scott topology on $C^*_{sub}$ is induced by the lower topology on $L(C, R)$. Thus, the dual Scott topology on $C^*_{sub}$ has a subbasis for the closed sets of the form

$$B_{U,r} = \{ h \in C^*_{sub} \mid h(x) \geq r \text{ for all } x \in U \},$$

where $U$ ranges over the open subsets of $C$ and $r \in R$.

(e) The upper topology on $C^*_{sub}$ is induced by the upper topology on $L(C, R)$, hence, agrees with the weak*upper topology.

(f) The map $\Psi : C^*_{sub} \to C^*_{sub}$ is surjective and preserves arbitrary joins and directed meets, hence, is continuous for the dual Lawson, dual Scott and upper topologies, respectively.

We now put the information of the two previous propositions together and we deduce our main result:

**Theorem 5.9.** Let $R$ be a linked bicontinuous algebra and $C$ a c-space algebra in which all operations $\omega \in \Omega$ preserve the way-below relation. Let $\kappa$ denote the restriction of the interval topology $\iota$ on $L(C, R)$ to $C^*$. Then:

(a) $C^*$ is closed for the interval topology $\iota$ in the linked bicontinuous lattice $L(C, R)$ of all lower semicontinuous functions $f : C \to R$, hence, a compact ordered space for the pointwise order and the subspace topology $\kappa = \iota|_{C^*}$.

(b) With respect to its weak*upper topology, $C^*$ is a stably compact space. A subbasis for the closed lower sets is given by the sets:

$$A_{x,r} = \{ h \in C^* \mid h(x) \leq r \}, x \in C, r \in R.$$  

The weak*upper topology on $C^*$ agrees with the topology of $\kappa$-open upper sets.

(c) The topology formed by the $\kappa$-open lower sets of $C^*$ is the co-compact topology associated with the weak*upper topology. It is induced by the lower (= dual Scott) topology on $L(C, R)$. A subbasis for the closed upper sets is given by:

$$B_{U,r} = \{ h \in C^* \mid h(x) \geq r \text{ for all } x \in U \}$$

where $U$ ranges over the open subsets of $C$ and $r \in R$.

(d) The map $\Psi : C^* \to C^*$ assigning to every order preserving homomorphism $f : C \to R$ its lower semicontinuous envelope preserves directed joins and filtered meets. It is continuous for the respective compact Hausdorff topologies.
Proof. (a) By the items (b) in the propositions 5.7 and 5.8, both \( C^{\ast}_{\text{sup}} \) and \( C^{\ast}_{\text{sub}} \) are closed in \( \mathcal{L}(C, R) \) for the latter’s interval topology. Thus \( C^{\ast} = C^{\ast}_{\text{sup}} \cap C^{\ast}_{\text{sub}} \) is also closed in \( \mathcal{L}(C, R) \) for the interval topology. Since a linked bicontinuous lattice is a compact Hausdorff topological lattice for its interval topology, every closed subset is a compact ordered space.

(b) The topology of \( \iota\text{-open upper sets in } \mathcal{L}(C, R) \) is the upper (= weak upper) topology. It follows that the topology of \( \kappa = \iota|_C \)-open upper sets of \( C^{\ast} \) is the weak upper topology on \( C^{\ast} \) by Lemma 2.10. Thus, \( C^{\ast} \) is a stably compact space with respect to its weak upper topology.

(c) The topology of \( \iota\text{-open lower sets in } \mathcal{L}(C, R) \) is the lower topology. It follows that the topology of \( \kappa \)-open lower sets of \( C^{\ast} \) is induced by the lower topology on \( \mathcal{L}(C, R) \) again by Lemma 2.10. Thus, a subbasis for the closed upper sets of \( C^{\ast} \) is given by the sets of the form as indicated in the claim.

(d) follows directly from items (f) in propositions 5.7 and 5.8. \( \square \)

Plotkin [16] has a point of view slightly different from ours. Let us reformulate our result in his terms:

**Corollary 5.10.** Let \( R \) be a linked bicontinuous algebra and \( C \) a c-space algebra in which all operations \( \omega \in \Omega \) preserve the way-below relation. Then the set \( C^{\ast} \) of all lower semicontinuous homomorphisms \( h: C \to R \) endowed with the weak upper topology is a stably compact space. The co-compact topology associated with the weak upper topology is the 'lower-open topology' a subbasis for the closed sets of which is given by the sets of the form

\[
B_{U, r} = \{ h \in C^{\ast} \mid h(x) \geq r \text{ for all } x \in U \}
\]

where \( U \) ranges over the open subsets of \( C \) and \( r \) over \( R \).

We may consider \( C^{\ast} \) to be a kind of dual of the c-space algebra \( C \) if we compare this situation with the classical setting of a topological vector space \( C \) and the dual space \( C^{\ast} \) of continuous linear functionals. Our result is a generalization of a result of G. D. Plotkin [16, Theorem 2]: There Plotkin considers continuous d-cones with an additive way-below relation. In his setting the dual \( C^{\ast} \) of a d-cone \( C \) is also a cone, in the same way as the dual of topological vector space also becomes a vector space in an obvious way. In our general setting, \( C^{\ast} \) does not carry any algebraic structure. We therefore proceed to discuss conditions under which the dual \( C^{\ast} \) becomes an algebra of the same type as \( C \).

### 6. The dual algebra \( C^{\ast} \)

In the previous section we were only interested in the topological properties of the space \( C^{\ast} \) of all lower semicontinuous homomorphisms from a c-space algebra \( C \) to a fixed linked bicontinuous algebra \( R \). The question arises naturally under which circumstances \( C^{\ast} \) can be considered not only as a topological space but also as an algebra dual to the algebra \( C \).

The pattern behind these questions is, of course, that the continuous linear functionals on a topological vector space \( V \) also form a vector space \( V^{\ast} \) that can be endowed with the weak*topology. The classical Alaoglu-Bourbaki Theorem asserts that the polar \( U^{\circ} = \{ h \in V^{\ast} \mid h(u) \leq 1 \text{ for all } u \in U \} \) is weak*compact for every neighborhood \( U \) of 0 in \( V \) (see, e.g., [18, Section 3.15]).

Clearly the function space \( \mathcal{L}(C, R) \) is not only a linked bicontinuous lattice but also a linked bicontinuous algebra in a natural way. It suffices to define the algebraic operations \( \omega \in \Omega \) pointwise. If \( \omega \) is of arity \( n \), we define \( \omega(f_1, \ldots, f_n) \) for \( f_1, \ldots, f_n \in \mathcal{L}(C, R) \) by

\[
\omega(f_1, \ldots, f_n)(x) = \omega(f_1(x), \ldots, f_n(x)) \text{ for all } x \in C .
\]

The operation \( \omega \) is indeed lower semicontinuous as a map \( \mathcal{L}(C, R)^n \to \mathcal{L}(C, R) \). We naturally are led to the question, under which hypotheses \( C^{\ast} \) is a subalgebra of the algebra \( \mathcal{L}(C, R) \) of all lower semicontinuous functions from \( C \) to \( R \).

We borrow our answer to this question from Universal Algebra; see, e.g., the monograph [17] by Romanowska and Smith.

**Definition 6.1.** We will say that an operation \( \sigma \) of arity \( n \) and an operation \( \omega \) of arity \( m \) on an algebra \( R \) commute if, for all \( x_{ij} \in R, \ i = 1, \ldots, n, \ j = 1, \ldots, m: \)

\[
\sigma(\omega(x_{11}, \ldots, x_{1m}), \ldots, \omega(x_{n1}, \ldots, x_{nm})) = \omega(\sigma(x_{11}, \ldots, x_{n1}), \ldots, \sigma(x_{1m}, \ldots, x_{nm})).
\]

12
Such an equational law is also called an entropic law. It can also be expressed by the commutativity of the following diagram:

\[
\begin{array}{ccc}
(R^n)^n & \xrightarrow{\sigma^n} & R^n \\
\omega^n & \downarrow & \downarrow \omega \\
R^n & \xrightarrow{\sigma} & R
\end{array}
\]

**Definition 6.2.** An algebra \( R \) of signature \( \Omega \) is called entropic if any two operations \( \sigma, \omega \in \Omega \) commute.

Let us specify what it means to commute for constants \( c, c' \), unary operations \( \lambda, \mu \) and binary operations \( * \) and \( + \):

\[
c = c' \text{ (commuting constants are equal)}, \quad \lambda(c) = c, \quad c * c = c * c = c,
\]

\[
\lambda(\mu(x)) = \mu(\lambda(x)), \quad \lambda(x + y) = \lambda(x) + \lambda(y),
\]

\[
(x_1 * x_2) + (x_3 * x_4) = (x_1 + x_2) * (x_3 + x_4).
\]

A binary operation \( * \) commutes with itself if \((x_1 * x_2) * (x_3 * x_4) = (x_1 * x_3) * (x_2 * x_4)\). An entropic algebra has at most one constant. A commutative, associative binary operation commutes with itself. Thus, commutative semigroups, commutative monoids, commutative groups and semilattices are entropic. Vector spaces are entropic when considered as algebras with a constant 0, a binary operation + and unary operations \( \lambda \), multiplication by the scalars \( \lambda \) in the base field. Similarly, modules over commutative rings are entropic. On the other hand, semirings and rings are not entropic, since addition and multiplication do not commute. Similarly lattices, even distributive lattices are not entropic.

We use the central property of entropic algebras in the following form:

**Lemma 6.3.** (Compare, e.g., [17, Proposition 5.1]) If the linked bicontinuous algebra \( R \) is entropic and \( C \) any c-space algebra, the lower semicontinuous supermorphisms, submorphisms, and homomorphisms \( h: C \to R \), respectively, form subalgebras \( C^\ast_{\text{sub}}, C^\ast_{\text{sup}} \) and \( C^\ast \) of the algebra \( \mathcal{L}(C, R) \), respectively.

**Proof.** We just consider the case of submorphism. Consider any operation \( \omega \in \Omega \), say of arity \( n \). For any \( h_1, \ldots, h_n \in C^\ast_{\text{sub}} \), we have to show that \( \omega(h_1, \ldots, h_n) \) is a submorphism, too. For this we have to consider any \( \sigma \in \Omega \), say of arity \( m \), and we have to show that, for all \( x_1, \ldots, x_m \in C \), we have

\[
\omega(h_1, \ldots, h_n)(\sigma(x_1, \ldots, x_m)) \leq \sigma(\omega(h_1(x_1), \ldots, h_n(x_1)), \ldots, \omega(h_1(x_m), \ldots, h_n(x_m))).
\]

For better readability, we use a binary operation \( + \) as a typical example for \( \sigma \); we then have indeed:

\[
\omega(h_1, \ldots, h_n)(x_1 + x_2) = \omega(h_1(x_1 + x_2), \ldots, h_n(x_1 + x_2))
\]

(the operation \( \omega \) being defined pointwise)

\[
\leq \omega(h_1(x_1) + h_1(x_2), \ldots, h_n(x_1) + h_n(x_2))
\]

(the \( h_i \) being submorphisms)

\[
= \omega(h_1(x_1), \ldots, h_n(x_1)) + \omega(h_1(x_2), \ldots, h_n(x_2))
\]

(since \( \omega \) commutes with \( + \) in \( R \))

\[
= \omega(h_1, \ldots, h_n)(x_1) + \omega(h_1, \ldots, h_n)(x_2)
\]

We obtain:

**Proposition 6.4.** Suppose that \( R \) is an entropic linked bicontinuous algebra and \( C \) a c-space algebra in which the operations \( \omega \in \Omega \) preserve the way below relation. Then the lower semicontinuous homomorphisms, submorphisms, and supermorphisms \( h: C \to R \), form topological algebras \( C^\ast, C^\ast_{\text{sub}}, \text{and } C^\ast_{\text{sup}} \) with respect to their weak*upper topologies.
Thus, in the entropic case, $C^*$ may be considered to be the algebra dual to $C$. Every $x \in C$ defines a map $\tilde{x} = (h \mapsto h(x)) : C^* \to R$ which clearly is a lower semicontinuous algebra homomorphism. Thus we have a continuous map $x \mapsto \tilde{x} : C \to C^{**} \subseteq L(C^*, R)$ which clearly is an algebra homomorphism. This raises the

**Question 6.5.** Under which hypotheses is this algebra homomorphism an isomorphism?

### 7. Examples

Let us consider some special cases of our results.

#### 7.1. Semilattices

We consider the three-element chain $R = \{ \bot < u < \top \}$ with semilattice operation $x \sqcup y = u$ if $x \neq y$ (and $x \sqcup x = x$, of course). This semilattice operation is order preserving, but it is not the meet or join operation associated with the order $\preceq$ of $R$. (One can think of $R$ to consist of the nonempty subsets of the two-element set $\{0, 1\}$ with the order $\{0\} < \{0, 1\} < \{1\}$ and union as semilattice operation; one may interpret $\bot$ as 'impossible', $u$ as 'may' and $\top$ as 'must'.)

We now look at c-space semilattices, that is, c-spaces with a continuous semilattice operation $\sqcup$ which need not be the join or meet for the specialization order. If we require that the semilattice operation preserves the way-below relation on $C$, that is, if $x \ll x', y \ll y' \Rightarrow x \sqcup y \ll x' \sqcup y'$, we may apply Theorem 5.9 and Proposition 6.4 and we obtain:

The set $C^*$ of lower semicontinuous semilattice homomorphisms from $C$ to $R$ is a stably compact topological semilattice with respect to the weak$^*$upper topology.

#### 7.2. Borel measures on compact Hausdorff spaces

Let us show how the classical case of the weak$^*$topology on the set of regular Borel measures on a compact Hausdorff space fits into our general framework. We identify regular Borel measures through the Riesz representation theorems.

For $X$ be a compact Hausdorff space and $C = C_*(X)$ the positive cone of all functions $f : X \to \mathbb{R}_+$ that are continuous with respect to the usual Hausdorff topology on $\mathbb{R}_+$. We topologize $C_+(X)$ so that it becomes a c-space by taking as neighborhood basis for $f \in C_+(X)$ the sets $U_{f, \varepsilon} = \{ g \in C_+(X) \mid f - \varepsilon \leq g \}$ for $\varepsilon > 0$. The topological way-below relation is $g \ll f$ iff $g \leq f - \varepsilon$ for some $\varepsilon > 0$. The algebraic structure of $C_+(X)$ is that of a cone: we have the binary operation $+$, the constant 0 and the unary operations of multiplication with scalars $r \geq 0$. The algebraic operations are continuous for the c-space topology just defined and one verifies that the algebraic operations preserve the topological way-below relation.

For $R$ we choose $\mathbb{R}_+$, the positive reals extended by $+\infty$ so that we have a complete chain. $\mathbb{R}_+$ is also a cone if addition and scalar multiplication are extended to $+\infty$ in the obvious way. We recall that cones are entropic algebras.

The homomorphisms from $C_+(X)$ to $\mathbb{R}_+$ are the maps $h$ satisfying $h(f + g) = h(f) + h(g)$, $h_rf = rh(f)$ for $r > 0$ and $h(0) = 0$; we call these maps linear functionals. In this case all linear functionals are order preserving. Thus $C^+_+(X)'$ is the cone of all linear functionals $h : C^+_+(X) \to \mathbb{R}_+$. Replacing the linearity by sublinearity and superlinearity, respectively, we obtain the cones $C^+_+(X)'_{\text{sub}}$ and $C^+_+(X)'_{\text{sup}}$ of order preserving sublinear and superlinear functionals. Topologically the three cones have the properties indicated in 5.2, 5.3.

Restricting our attention to lower semicontinuous functionals we obtain the cones $C^+_+(X)'_{\text{sub}}$, $C^+_+(X)'_{\text{sub}}$ and $C^+_+(X)'_{\text{sup}}$ of lower semicontinuous linear, sublinear and superlinear functionals, respectively. Topologically they have the properties indicated in Theorem 5.9 and propositions 5.7, 5.8. In particular, $C^+_+(X)'$ is stably compact for the weak$^*$upper topology $\tau$. The lower open topology is the co-compact topology associated with $\tau$. And $C^+_+(X)'$ is a compact ordered space for the patch topology $\tau_p$, the coarsest topology generated by the weak$^*$upper and the lower-open topology. We have shown:

The lower semicontinuous linear functionals $h : C^+_+(X) \to \mathbb{R}_+$ form a cone which is stably compact for the weak$^*$upper topology. The associated compact Hausdorff patch topology extends the classical weak$^*$topology on the subcone of regular Borel measures on $X$.  

14
Compared with the classical situation we have admitted the value $+\infty$ for the linear functionals. For this reason, $C_+(X)^*$ is a compactification of the space $M_+(X)$ of regular Borel measures. If we restrict to linear functionals $h: C_+(X) \to \mathbb{R}_+$ with finite values, these are automatically lower semicontinuous. The restriction of the patch topology $\tau_p$ on $C_+(X)^*$ to the cone $M_+(X)$ of linear functionals $h: C_+(X) \to \mathbb{R}_+$ yields the usual weak* topology. The set of probability measures can be identified with the set of those linear functionals with $h(1) = 1$ which clearly is a patch-closed hence compact subset of $C_+(X)^*$; similarly, the subprobability measures, those with $h(1) \leq 1$, form a patch-closed hence compact subset of $C_+(X)^*$.

We see that classical results on the weak* compactness of the space of probability and subprobability measures on compact Hausdorff spaces are covered by our general approach.

7.3. Borel measures on compact ordered spaces

Consider now a compact ordered space $(X, \tau, \leq)$. D. A. Edwards [3] has introduced the stochastic order on the set $M_+(X)$ of regular Borel measures on $X$ by $\mu \leq \nu$ if $\mu(U) \leq \nu(U)$ for every $\tau$-open upper set $U \subseteq X$, equivalently, $\int f d\mu \leq \int f d\nu$ for every order preserving continuous function $f: X \to \mathbb{R}_+$. In this way the Dirac measures $\delta_x, x \in X$ are ordered in the same way as the elements $x$. This example fits our general framework in the following way.

Let $CM_+(X)$ denote the cone of all continuous order preserving functions $f: X \to \mathbb{R}_+$. As a subcone of $C_+(X)$ it inherits a c-space structure. As in the previous example, the cone $CM_+(X)^*$ of all lower semicontinuous linear functionals $h: CM_+(X) \to \mathbb{R}_+$ is a cone that is stably compact for the weak* upper topology. The regular Borel measures on $X$ constitute the subcone $M_+(X)$ of all order preserving linear functionals $h: CM_+(X) \to \mathbb{R}_+$. The specialization order associated with the weak* upper topology induces the stochastic order and the patch topology induces the weak* topology on the regular Borel measures.

7.4. Cones

The two previous examples are special cases of the following situation: Consider a c-space $C$ with a continuous cone structure. Since cones are entropic, the developments of the previous section apply. The sets $C^*_{\text{sup}}, C^*_{\text{sub}}$ and $C^*$ of lower semicontinuous superlinear, sublinear and linear functionals $h: C \to \mathbb{R}_+$, respectively, are cones and they have the topological properties indicated in 5.7, 5.8 and 5.9. Note that we have to suppose that the way-below relation on $C$ is additive for the first and the third case. (Scalar multiplication always preserves the way-below relation.) It is natural to call $C^*$ the dual of the c-space cone $C$. In particular, $C^*$ is stably compact for the weak* upper topology.

Since continuous dcpos are c-spaces with respect to their Scott topology, these results can be applied to continuous d-cones and we have derived Plotkin’s main result, Theorem 2 in [16]. Our proof technique in Section 5 is a generalization of Plotkin’s use of the projection from the set $C^*$ of order preserving linear functionals onto the set $C^*$ of lower semicontinuous linear functionals. Plotkin has borrowed this idea from A. Jung [2], who had for the first time used this idea for an elegant proof of the fact that the probabilistic power domain over a stably compact space is stably compact for the weak* upper topology. Earlier proofs of this fact were quite technical and involved (see e.g. [10]). The probabilistic powerdomain over a stably compact space is equivalent to the dual cone $C^*$ treated in the previous subsection 7.3.

8. A relaxed setting

In order to include a larger class of examples we need to extend the quite narrow entropic setting. We will have to pay a price. We have to merge the three cases $C^*, C^*_{\text{sub}}$ and $C^*_{\text{sup}}$ into a single one.

We suppose the signature $\Omega$ to be the union of two subsets $\Omega^\leq$ and $\Omega^\geq$ which need not be disjoint. The subsets $\Omega^\leq$ and $\Omega^\geq$ will be kept fixed. We define a notion that merges the notions of sub- and supermorphisms:

**Definition 8.1.** Let $A$ and $A'$ be partially ordered algebras of signature $\Omega = \Omega^\leq \cup \Omega^\geq$. A map $h: A \to A'$ is called a relaxed morphism if, for all $\omega \in \Omega^\leq$ of arity $n$, we have

$$h(\omega(x_1, \ldots, x_n)) \leq \omega(h(x_1), \ldots, h(x_n))$$
and for all $\omega \in \Omega^\leq$ of arity $n$, we have

$$h(\omega(x_1, \ldots, x_n)) \geq \omega(h(x_1), \ldots, h(x_n))$$

This means that a relaxed morphism is a submorphism for the subsignature $\Omega^\leq$ and a supermorphism for the subsignature $\Omega^\geq$; it is a homomorphism for the subsignature $\Omega^\geq \cap \Omega^\leq$.

We now fix a c-space algebra $C$ and a relaxed entropic, linked bicontinuous algebra $R$ both of signature $\Omega = \Omega^\leq \cup \Omega^\geq$. We denote by $C_{lax}'$ and $C_{lax}^*$ the sets of order preserving and lower semicontinuous relaxed morphisms $h: C \to R$, respectively.

If we restrict to the subsignature $\Omega^\leq$, we may apply Proposition 5.8 to the set $C_{lax}^*_{\Omega^\leq}$ of lower semicontinuous maps $h: C \to R$ which are submorphisms for the subsignature $\Omega^\leq$. Similarly we may apply Proposition 5.7 the set $C_{lax}^*_{\Omega^\geq}$ of lower semicontinuous maps $h: C \to R$ which are supermorphisms for the subsignature $\Omega^\geq$. Since $C_{lax}^* = C_{lax}^*_{\Omega^\geq} \cap C_{lax}^*_{\Omega^\leq}$, we can deduce the following result for $C_{lax}^*$ in exactly the same way as the results for $C^*$ in Theorem 5.9:

**Proposition 8.2.** Let $R$ be a linked bicontinuous algebra and $C$ a c-space algebra in which all operations $\omega \in \Omega = \Omega^\leq \cup \Omega^\geq$ preserve the way-below relation. Let $\kappa$ denote the restriction of the interval topology $\iota$ on $L(C, R)$ to $C_{lax}^*$. Then:

(a) $C_{lax}^*$ is closed for the interval topology $\kappa$ in the linked bicontinuous lattice $L(C, R)$ of all lower semicontinuous functions $f: C \to R$, hence, a compact ordered space for the pointwise order and the subspace topology $\kappa = \iota|_{C_{lax}^*}$.

(b) With respect to its weak*upper topology, $C_{lax}^*$ is a stably compact space. A subbasis for the closed lower sets is given by the sets:

$$A_{x, r} = \{ h \in C_{lax}^* \mid h(x) \leq r \} , x \in C, r \in R .$$

The weak*upper topology on $C_{lax}^*$ agrees with the topology of $\kappa$-open upper sets.

(c) The topology formed by the $\kappa$-open lower sets of $C_{lax}^*$ is the co-compact topology associated with the weak*upper topology. It is induced by the lower (= dual Scott) topology on $L(C, R)$. A subbasis for the closed upper sets is given by:

$$B_{U, r} = \{ h \in C_{lax}^* \mid h(x) \geq r \text{ for all } x \in U \}$$

where $U$ ranges over the open subsets of $C$ and $r$ over $R$.

(d) The map $\Psi: C_{lax} \to C_{lax}$ assigning to every order preserving homomorphism $f: C \to R$ its lower semicontinuous envelope preserves directed joins and filtered meets. It is continuous for the respective compact Hausdorff topologies.

For an entropic algebra $A$ we required each $n$-ary operation $\omega \in \Omega$ to be a homomorphism from the algebra $A^n$ to $A$. We now consider a partially ordered algebra $A$ of signature $\Omega = \Omega^\leq \cup \Omega^\geq$.

**Definition 8.3.** The partially ordered algebra $A$ of signature $\Omega = \Omega^\leq \cup \Omega^\geq$ is said to be relaxed entropic if every operation $\omega \in \Omega$ of arity $n$ is a relaxed morphism from $A^n$ to $A$.

This means explicitly that for any operation $\sigma \in \Omega^\leq$, say of arity $m$, we have for all $x_{ij} \in A$, $i = 1, \ldots, n$, $j = 1, \ldots, m$:

$$\omega(\sigma(x_{11}, \ldots, x_{1m}), \ldots, \sigma(x_{n1}, \ldots, x_{nm})) \leq \sigma(\omega(x_{11}, \ldots, x_{n1}), \ldots, \omega(x_{1m}, \ldots, x_{nm})). \quad (2)$$

and for any operation $\sigma \in \Omega^\geq$, say of arity $m$, we have for all $x_{ij} \in A$, $i = 1, \ldots, n$, $j = 1, \ldots, m$:

$$\omega(\sigma(x_{11}, \ldots, x_{1m}), \ldots, \sigma(x_{n1}, \ldots, x_{nm})) \geq \sigma(\omega(x_{11}, \ldots, x_{n1}), \ldots, \omega(x_{1m}, \ldots, x_{nm})). \quad (3)$$

We note that any two operation in $\Omega^\leq$ have to commute so that a relaxed entropic po-algebra is entropic for the subsignature $\Omega^\leq$, and similarly for $\Omega^\geq$.

We now fix a c-space algebra $C$ and a relaxed entropic, linked bicontinuous algebra $R$ both of signature $\Omega = \Omega^\leq \cup \Omega^\geq$. We denote by $C_{lax}'$ and $C_{lax}^*$ the sets of order preserving and lower semicontinuous relaxed morphisms $h: C \to R$, respectively. Similarly as in Lemma 6.3 one shows:
Lemma 8.4. \( C'_{lax} \) and \( C^*_{lax} \) are subalgebras of \( M(C, R) \) and \( L(C, R) \), respectively.

We now can announce a relaxed version of Proposition 6.4:

Proposition 8.5. Suppose that \( R \) is a relaxed entropic, linked bicontinuous algebra and \( C \) a c-space algebra in which all operations \( \omega \in \Omega \) preserve the way-below relation. Then the lower semicontinuous relaxed morphisms \( h : C \to R \) form a stably compact topological algebra \( C^*_{lax} \), with respect to the weak*upper topology.

9. Examples

9.1. Lattices

Lattices are not entropic. For the two lattice operations \( \lor \) and \( \land \) one has the inequality

\[
(x \land y) \lor (z \lor u) \leq (x \lor z) \land (y \lor u),
\]

but never equality except for the one-element lattice. But this inequality is exactly what allows the claim that every lattice is relaxed entropic; we just let \( \lor \) and \( \land \) be the order preserving maps.

9.2. Max-plus algebras

Let \( C \) be a c-space with the structure of a cone, that is, a continuous commutative monoid structure \( + \) with neutral element 0 and a continuous multiplication by scalars \( r \geq 0 \) satisfying the same laws as in vector spaces. Suppose further that \( C \) is a \( \lor \)-semilattice with respect to its specialization order and that the following distributivity laws hold:

\[
x + (y \lor z) = (x + y) \lor (x + z) \quad \text{and} \quad r(x \lor y) = rx \lor ry \quad \text{for all scalars} \quad r \geq 0.
\]

The semilattice operation destroys the entropicity of the cone structure. But one has the inequality

\[
(x + y) \lor (z + u) \leq (x \lor z) + (y \lor u).
\]

Thus, if we put \( + \) in \( \Omega^\leq \) and \( \lor \) in \( \Omega^\geq \) and the constant 0 and the scalars \( r \geq 0 \) considered as unary operation in both \( \Omega^\leq \) and \( \Omega^\geq \), we have a relaxed entropic algebra. A typical example is \( \mathbb{R}_+ \) with its upper topology, the the usual addition, multiplication with scalars \( r \geq 0 \) and the semilattice operation \( \max \).

The relaxed homomorphisms between two such algebras have to satisfy \( h(rx) = rh(x) \), \( h(x + y) \leq h(x) + h(y) \) and \( h(x \lor y) \geq h(x) \lor h(y) \). Thus, the lower semicontinuous relaxed morphisms agree with the sublinear maps.

For a c-space algebra \( (C, +, \lor, r(r \geq 0), 0) \) the lower semicontinuous sublinear maps \( h : C \to \mathbb{R}_+ \) form an algebra \( C^*_{sub} \) satisfying the topological properties of Proposition 5.8.

9.3. Min-plus algebras

Let \( C \) be a c-space with a continuous cone structure as in the previous example. Suppose further that \( C \) is a \( \land \)-semilattice with respect to its specialization order and that the following distributivity laws hold:

\[
x + (y \land z) = (x + y) \land (x + z) \quad \text{and} \quad r(x \land y) = rx \land ry \quad \text{for all scalars} \quad r \geq 0.
\]

The semilattice operation destroys the entropicity of the cone structure. But one has the inequality

\[
(x + y) \land (z + u) \geq (x \land z) + (y \land u).
\]

Thus, if we put \( + \) in \( \Omega^\geq \) and \( \lor \) in \( \Omega^\leq \) and the constant 0 and the scalars \( r \geq 0 \) considered as unary operation in both \( \Omega^\leq \) and \( \Omega^\geq \), we have a relaxed entropic algebra. A typical example is \( \mathbb{R}_+ \) with its upper topology, the the usual addition, multiplication with scalars \( r \geq 0 \) and the semilattice operation \( \min \).
The relaxed homomorphisms between two such algebras have to satisfy \( h(rx) = rh(x), h(x+y) \geq h(x)+h(y) \) and \( h(x \wedge y) \geq h(x) \wedge h(y) \). Thus, the lower semicontinuous relaxed morphisms agree with the superlinear maps.

For a c-space algebra \((C, +, \wedge, r(\geq 0), 0)\) in which the way-below relation is additive, the lower semicontinuous superlinear maps \( h: C \to \mathbb{R}_+^\ast \) form an algebra \( C^{\ast \ast} \) satisfying the topological properties of Proposition 5.7.

9.4. A non-example

There are simple situations that do not fit under our general framework. It would be desirable to extend our framework in such a sense that, for example, the following situation would be covered too:

Let \( B \) be a Boolean algebra. Denote by \( 2 \) the two element Boolean algebra with the discrete topology. The set \( B^* \) of all homomorphisms \( h: B \to 2 \) as a subspace of the product space \( 2^B \) is a compact Hausdorff space (and totally disconnected), the famous Stone space corresponding to the Boolean algebra \( B \).

What is the relation of this example to our setting? Firstly, we may endow the Boolean algebra \( B \) with the discrete topology. It then is (trivially) a c-space and all the Boolean operations are continuous. So it is a c-space algebra. The specialization order is equality. The two element algebra \( 2 \) with the discrete topology is compact Hausdorff. It can be considered to be linked bicontinuous bi-dcpo (directed and filtered complete). The result mentioned in the previous paragraph would fit in our framework, if \( 2 \) were a lattice, not the two element antichain.

References