

Topological Cones: Functional Analysis in a T_0 -Setting

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Dedicated to
Karl Heinrich Hofmann
at the occasion of his 75th birthday

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Abstract

Already in his PhD Thesis on compact Abelian semigroups under the direction of Karl Heinrich Hofmann the author was lead to investigate locally compact cones [18]. This happened in the setting of Hausdorff topologies. The theme of topological cones has been reappearing in the author's work in a non-Hausdorff setting motivated by the needs of mathematical models for a denotational semantics of languages combining probabilistic and nondeterministic choice. This is in the line of common work with Karl Heinrich Hofmann in *Continuous Lattices and Domains* [12].

Domain Theory is based on order theoretical notions from which intrinsic non-Hausdorff topologies are derived. Along these lines, domain theoretical variants of (sub-) probability measures have been introduced by Jones and Plotkin [15, 16]. Kirch [22] and Tix [36] have extended this theory to a domain theoretical version of measures and they have introduced and studied directed complete partially ordered cones as appropriate structures. Driven by the needs of a semantics for languages combining probability and nondeterminism, Tix [38, 37] and later on Plotkin and the author [39] developed basic functional analytic tools for these structures.

In this paper we extend this theory to topological cones the topologies of which are strongly non-Hausdorff. We carefully introduce these structures and their elementary properties. We prove Hahn-Banach type separation theorems under appropriate local convexity hypotheses. We finally construct a monad assigning to every topological cone C another topological cone $\mathcal{S}(C)$ the elements of which are nonempty compact convex subsets of C . For proving that this construction has good properties needed for the application in semantics we use the functional analytic tools developed before.

*Thanks to Gordon Plotkin for numerous discussions. Preliminary results have been announced at MFPS XXIII [20]. In his Master's thesis supervised by the author, B. Cohen [6] has worked out some of those results.

1 Introduction

Cones, more precisely *convex cones*, frequently occur in functional analysis, often as positive cones of ordered vector spaces (see for example [24, Luxemburg and Zaanen: *Riesz Spaces*]) or of locally convex ordered topological vector spaces (see for example [33, H.H. Schaefer: *Topological vector spaces*]). In the first case the interplay between the order and the convex structure is in the center of interest, in the second case a topology is added and one obtains a richer structure. Not only positive cones are of interest, but also ordered cones as independent objects, still embeddable in vector spaces, but not as positive cones. A class of lattice-ordered cones is investigated in detail in the book by Boboc, Bucur and Cornea entitled *Order and Convexity in Potential Theory: H-cones* [4]. Also largely motivated by potential theory, topological cones that are embeddable in locally convex topological vector spaces have been studied by G. Choquet and his school. An excellent recent source is the book by R. Becker entitled *Cônes Convexes en Analyse* [3]. On these cones a convex structure, an order and a Hausdorff topology are interacting.

A cone is asymmetric in the sense that it need not contain the negatives $-a$ of its elements a . We are going to make the topology asymmetric, too, in the sense that we throw away the Hausdorff separation axiom T_2 and only keep the axiom T_0 which asserts that for any two distinct points there is an open set containing one and only one of these points. This may seem too weak as an axiom. We claim that we also gain something. Indeed, every T_0 -topology encodes an order, the *specialisation order*: $x \leq y$ iff x is in the closure of the singleton y . And quite often it also encodes a Hausdorff topology. Let us illustrate this by an example: In the Banach space $C(X)$ of all real valued continuous functions on a compact Hausdorff space with the supremum norm, consider the collection \mathcal{U} of all open upper sets U . Being an *upper* set means that, with every $g \in U$, the set U contains all $f \geq g$. It is readily seen that \mathcal{U} is a topology which is T_0 , but far from being Hausdorff. It is asymmetric in the sense that the negative $-U$ of an open upper set is never a member of \mathcal{U} . The specialisation order with respect to this topology \mathcal{U} is just the usual pointwise ordering, and we can recover the original norm topology by symmetrisation. Thus the asymmetric topology \mathcal{U} encodes the order as well as the original topology.

Thus we are going to consider cones with a T_0 -topology such that addition and multiplication with nonnegative scalars are continuous. Our topological cones will be ordered cones with respect to their specialisation order. As we use the asymmetric topology of open upper sets on the nonnegative reals, our topological cones will have 0 as their smallest element. As we do not require the cancellation axiom for addition, our cones are not always embeddable in vector spaces. This allows us to include *infinite* elements in our cones, and we can consider cones of nonnegative extended real valued functions (admitting the value $+\infty$).

In [19], a different notion of a locally convex cone has been introduced and studied in detail. There, the uniform structure of a locally convex vector space has been axiomatised in an asymmetric way. Within that framework one can treat spaces of compact convex subsets and spaces of set-valued functions and derive, for example, Korovkin type approximation theorems which include the classical as well as the case of set-valued functions. The common feature of the two approaches is the essential use of T_0 -topologies that encode an order structure via the specialisation order.

In this paper we begin a detailed investigation of topological cones in our asymmetric sense. It is surprising that for most of our results addition need not be jointly but only separately continuous in

each of its two arguments. We deduce Hahn-Banach type separation theorems and the existence of continuous linear functionals under the appropriate local convexity assumptions.

In analogy to the Vietoris hyperspace we construct a powercone by considering the collection $\mathcal{S}(C)$ of all nonempty compact convex upper sets in a topological cone C . Under an appropriate topology, $\mathcal{S}(C)$ is again a topological cone with a continuous meet operation. This construction is free in the sense that \mathcal{S} is a monad over the category of topological cones and continuous linear maps. But the algebras of this monads are not known, so that it is not clear in general what $\mathcal{S}(C)$ is free for. By restricting to an appropriate subcategory, we can characterize the algebras of the monad, and for this we need the full power of our Hahn-Banach separation theorems.

This last example allows finally to explain the motivation behind all of this work: denotational semantics, more precisely, domain theoretical models for languages combining probabilistic and non-deterministic features (see [39]). A program P of such a language is interpreted as a map assigning to every element x of a state space X a certain set of subprobability distributions on X . In [39], the state spaces were supposed to be dcpo's (= directed complete partially ordered sets). Dcpo's carry an intrinsic topology, the Scott topology. The extended probability distributions form a dcpo cone $\mathcal{V}(X)$, and the appropriate sets of subprobability distributions are the nonempty compact convex saturated subsets, and they form again a cone $\mathcal{SV}(X)$. In order to be able to interpret composition of programs the construction \mathcal{SV} must be a free one in the sense that it constitutes a monad. Monads have algebras; but over dcpo's in general, the algebras of the monad \mathcal{SV} are not known. They are only known if one restricts to continuous dcpo's as state spaces.

There is another relevant class of state spaces not covered by [39], the locally stably compact spaces X , that is, locally compact sober T_0 -spaces in which the intersection of any two compact saturated sets is compact. In this case, the extended probabilistic powerdomain $\mathcal{V}(X)$ is a stably compact topological cone (see [17, 2]), but not a continuous dcpo. The algebras of the monad \mathcal{V} are not well-understood yet. The last section of this paper is devoted to the second step, namely the construction of the powercone $\mathcal{S}(C)$ of nonempty compact convex saturated subsets of a stably compact topological cone with appropriate local convexity properties. For this monad we are able to characterize the algebras.

Our developments are quite analogous to those for dcpo cones obtained by R. Tix [37, 38] and later by G.D. Plotkin and the author [39]. We take great care to observe which axioms are needed for our results. In particular, for most results we only need to require the separate continuity of addition. Thus the dcpo cones of Tix are contained as a special case in our developments; indeed, in these cones addition is only separately and not jointly continuous with respect to the Scott topology, in general.

2 Preliminaries

For subsets A of a partially ordered set P we use the following notations:

$$\downarrow A =_{def} \{x \in P \mid x \leq a \text{ for some } a \in A\}, \quad \uparrow A =_{def} \{x \in P \mid x \geq a \text{ for some } a \in A\}$$

and we say that A is a *lower* or *upper* set, if $\downarrow A = A$ or $\uparrow A = A$, respectively. Upper sets will also be called *saturated* and $\uparrow A$ will be called the *saturation* of A . For an element $a \in P$, we have the special

cases:

$$\downarrow a = \{x \in P \mid x \leq a\}, \quad \uparrow a = \{x \in P \mid x \geq a\}$$

For least upper and greatest lower bounds, we use the usual notations $\vee, \bigvee, \wedge, \bigwedge$. When we write $x = \bigvee \uparrow x_i$ then we mean that $(x_i)_i$ is a directed family and that x is its least upper bound.

Topological spaces will always be supposed to be T_0 -spaces. Every partially ordered set P can be endowed with two simple T_0 -topologies, the *upper topology* ν a subbasis for the closed sets of which is given by the sets $\downarrow x, x \in P$, and dually with the *lower topology* ω , a subbasis for the closed sets of which is given by the sets $\uparrow x, x \in P$. The common refinement of the upper and the lower topology is the *interval topology* denoted by λ .

We denote by \mathbb{R} the reals with the usual linear order, addition and multiplication, and by \mathbb{R}_+ the subset of nonnegative reals $r \geq 0$. The letters r, s, t, \dots will always denote nonnegative reals. Further

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, \quad \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$$

denote the reals and nonnegative reals extended by $+\infty$ (but not $-\infty$). Addition, multiplication and the order are extended to $+\infty$ in the usual way. In particular, $+\infty$ becomes the greatest element and we put $0 \cdot (+\infty) = 0$. On $\overline{\mathbb{R}}$ and its subsets, the upper topology has as open subsets simply the open upper intervals $]r, +\infty[= \{x \mid x > r\}$ and the lower topology has as open subsets simply the open lower intervals $\{x \mid x < r\}$. The interval topology is the usual topology.

For topological spaces X, Y, Z a function $f: X \times Y \rightarrow Z$ will be said to be *separately continuous* if, for each fixed $x \in X$, the function $y \mapsto f(x, y): Y \rightarrow Z$ is continuous and similarly for each fixed $y \in Y$. We say that f is *jointly continuous* if it is continuous for the product topology on $X \times Y$. We will use the following topological lemma (see [39, Lemma 1.2]):

Lemma 2.1. *Let $f: X \times Y \rightarrow Z$ be a separately continuous map of topological spaces. Then:*

$$\overline{f(\overline{A} \times \overline{B})} = \overline{f(\overline{A} \times B)} = \overline{f(A \times \overline{B})} = \overline{f(A \times B)}$$

for subsets $A \subseteq X$ and $B \subseteq Y$, where \overline{A} denotes the closure of A .

We refer to [12] for detailed background information on the order theoretical and topological notions used in this paper.

3 Cones and ordered cones

We want to consider structures that are close to vector spaces but less symmetric in the sense that elements do not have additive inverses. Accordingly, scalar multiplication is restricted to nonnegative real numbers.

Definition 3.1. *A cone is defined to be a commutative monoid C together with a scalar multiplication by nonnegative real numbers satisfying the same axioms as for vector spaces; that is, C is endowed*

with an addition $(x, y) \mapsto x + y: C \times C \rightarrow C$ which is associative, commutative and admits a neutral element 0, i.e., for all $x, y, z \in C$ it satisfies:

$$\begin{aligned}x + (y + z) &= (x + y) + z \\x + y &= y + x \\x + 0 &= x\end{aligned}$$

and with a scalar multiplication $(r, x) \mapsto r \cdot x: \mathbb{R}_+ \times C \rightarrow C$ satisfying for all $x, y \in C$ and all $r, s \in \mathbb{R}_+$:

$$\begin{aligned}r \cdot (x + y) &= r \cdot x + r \cdot y \\(r + s) \cdot x &= r \cdot x + s \cdot x \\(rs) \cdot x &= r \cdot (s \cdot x) \\1 \cdot x &= x \\0 \cdot x &= 0\end{aligned}$$

An ordered cone is a cone C endowed with a partial order \leq such that addition and multiplication by fixed scalars $r \in \mathbb{R}_+$ are order preserving, that is, for all $x, y, z \in C$ and all $r \in \mathbb{R}_+$:

$$x \leq y \implies x + z \leq y + z \text{ and } r \cdot x \leq r \cdot y$$

For a cone, scalar multiplication need only be defined for strictly positive reals and the above properties have to be satisfied for $r, s > 0$ only. Then one may extend scalar multiplication to $r = 0$ by defining $0 \cdot x = 0$ and the axioms are satisfied for all nonnegative reals.

Every real vector space is a cone, and every ordered vector space is an ordered cone. Cones may occur as subsets of real vector spaces: such subsets C are cones if $0 \in C$, if $a, b \in C \implies a + b \in C$ and if $a \in C, r \in \mathbb{R}_+ \implies ra \in C$.

Every direct product of (ordered) cones with pointwise addition and scalar multiplication (and order) is again a(n ordered) cone. But unlike for vector spaces, addition in cones need not satisfy the cancellation property, and cones need not be embeddable in vector spaces. For example $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ are ordered cones that are not embeddable in vector spaces. The same holds for $\overline{\mathbb{R}}^n$ and $\overline{\mathbb{R}}_+^n$ and for infinite powers $\overline{\mathbb{R}}^I$ and $\overline{\mathbb{R}}_+^I$.

A cone C is embeddable in a real vector space if and only if it satisfies cancellation:

$$(C) \quad x + y = x + z \implies y = z$$

and an ordered cone is embeddable in an ordered vector space if and only if it satisfies order cancellation:

$$(OC) \quad x + y \leq x + z \implies y \leq z$$

The embeddings are achieved by the standard method of formal differences $a - b$ with $a, b \in C$ where $a - b = a' - b'$ iff $a + b' = a' + b$ and $a - b \leq a' - b'$ iff $a + b' \leq a' + b$, respectively, in the ordered case.

As in real vector spaces, there is a notion of convexity in cones. Because of the possible existence of *infinite* elements in cones, convex sets may look unusual.

Definition 3.2. A subset A of a cone C is convex if, for all $a, b \in A$, the convex combination $ra + (1 - r)b$ belongs to A for every real number r with $0 \leq r \leq 1$.

This notion of convexity should not be mixed up with the notion of order-convexity: A subset A of an ordered set is *order-convex* if, for all $a, b \in A$ and all $c \in C$, the relation $a \leq c \leq b$ implies $c \in A$.

We will frequently use the simple observation that, in an ordered cone C , for every convex subset A , the lower set $\downarrow A$ and the upper set or saturation $\uparrow A$ are also convex.

For nonempty subsets A and B of a cone we may define the sum and multiplication by scalars $r \geq 0$ in the straightforward manner by

$$A + B = \{a + b \mid a \in A, b \in B\} \quad , \quad r \cdot A = \{ra \mid a \in A\}.$$

All the cone axioms are satisfied except for one of the distributivity laws: $rA + sA$ need not be equal to $(r + s)A$. This can be seen by taking $A = \{0, 1\} \subseteq \mathbb{R}_+$ and $r = s = 1$. For this reason we restrict ourselves to convex sets:

Lemma 3.3. For a nonempty convex set A of a cone we have $(r + s)A = rA + sA$ for all $r, s \in \mathbb{R}_+$.

Proof. One always has $(r + s)A \subseteq rA + sA$; for the converse inclusion, let $x \in rA + sA$, that is, $x = ra + sb$ for some $a, b \in A$; then $x = (r + s) \cdot (\frac{r}{r+s}a + \frac{s}{r+s}b) \in (r + s)A$ by the convexity of A (this proof only works for $r + s > 0$, but the case $r = s = 0$ is straightforward). \square

As a consequence we have:

Example 3.4. Ordered by inclusion, the nonempty convex subsets of a cone form an ordered cone; the singletons are the minimal elements.

Lemma 3.5. For an element a in an ordered cone C , one has $a \geq 0$ if and only if, for $r, s \in \mathbb{R}_+$:

$$(m) \quad r \leq s \Rightarrow r \cdot a \leq s \cdot a$$

i.e., if and only if $r \mapsto r \cdot a: \mathbb{R}_+ \rightarrow C$ is order preserving.

Proof. If $a \geq 0$ and $r \leq s$, then $s - r \geq 0$, whence $(s - r) \cdot a \geq (s - r) \cdot 0 = 0$. It follows that $s \cdot a = r \cdot a + (s - r) \cdot a \geq r \cdot a$. Conversely, if (m) holds, then $0 = 0 \cdot a \leq 1 \cdot a = a$, as $0 < 1$. \square

Definition 3.6. An ordered cone C with the property that $a \geq 0$ for all $a \in C$ will be called a pointed ordered cone.

In pointed ordered cones, scalar multiplication $(r, x) \mapsto rx: \mathbb{R}_+ \times C \rightarrow C$ is also order preserving in the first argument by the preceding lemma. The cone of all nonempty convex subsets of a cone ordered by inclusion is not pointed, because none of the non-zero singletons is above 0. But \mathbb{R}_+ , $\overline{\mathbb{R}}_+$ and arbitrary powers $\overline{\mathbb{R}}_+^I$ are pointed.

Definition 3.7. Let C and D be cones. A function $f: C \rightarrow D$ is called linear, if it is homogeneous:

$$f(r \cdot a) = r \cdot f(a) \text{ for all } a \in C \text{ and all } r \in \mathbb{R}_+$$

and additive:

$$f(a + b) = f(a) + f(b) \text{ for all } a, b \in C.$$

If D is an ordered cone, f is called superadditive if

$$f(a + b) \geq f(a) + f(b) \text{ for all } a, b \in C.$$

and subadditive if

$$f(a + b) \leq f(a) + f(b) \text{ for all } a, b \in C.$$

We say that f is sublinear (resp. superlinear), if f is homogeneous and subadditive (resp. superadditive).

Maps from a set X or from a cone C into $\overline{\mathbb{R}}$ are called *functionals*. Note that functionals are allowed to have the value $+\infty$ (but not $-\infty$). It is clear now what we mean by sublinear, superlinear and linear functionals.

A homogeneous function always maps 0 to 0. For pointed ordered cones C , this implies that order preserving homogeneous functionals have nonnegative values, i.e., they map C into $\overline{\mathbb{R}}_+$.

Example 3.8. (Order preserving functionals) For a pointed ordered cone C , we denote by:

C'_{sub}	the set of all order preserving sublinear functionals $f: C \rightarrow \overline{\mathbb{R}}_+$
C'_{sup}	the set of all order preserving superlinear functionals $f: C \rightarrow \overline{\mathbb{R}}_+$
$C' = C'_{sub} \cap C'_{sup}$	the set of all order preserving linear functionals $f: C \rightarrow \overline{\mathbb{R}}_+$

Under pointwise defined addition and multiplication by nonnegative scalars and pointwise order, the three sets just defined are pointed ordered cones, subcones of the product cone $\overline{\mathbb{R}}_+^C$. The constant zero functional is the smallest element and the functional having the value $+\infty$ everywhere except at 0 is the greatest element. The pointed ordered cone C' is called the *order dual* of C .

4 (Semi-) Topological cones

Recall that a real topological vector space is a vector space over the reals endowed with a Hausdorff topology in such a way that addition and scalar multiplication are jointly continuous. The scalars are endowed with the usual topology. For cones we continue our programme by using *asymmetric* topologies. On the extended reals $\overline{\mathbb{R}}$ and on its subsets \mathbb{R}_+ and $\overline{\mathbb{R}}_+$ we use the *upper topology* ν the only open sets for which are the open intervals $]r, +\infty] = \{s \mid s > r\}$. This upper topology is T_0 , but far from being Hausdorff.

Convention 4.1. If not specified otherwise, in the sequel we will use the upper topology on the extended reals $\overline{\mathbb{R}}$ and its subsets \mathbb{R}_+ and $\overline{\mathbb{R}}_+$.

For any topological space X , there are less functions $f: \overline{\mathbb{R}} \rightarrow X$ which are continuous with respect to the upper topology on $\overline{\mathbb{R}}$ than those which are continuous with respect to the usual topology. This fact will have striking consequences for topological cones in our sense. On the other hand, there are more functions $f: X \rightarrow \overline{\mathbb{R}}$ which are continuous with respect to the upper topology ν on $\overline{\mathbb{R}}$ than those which are continuous with respect to the usual topology λ . In agreement with classical analysis we shall adopt the terminology *lower semicontinuous* (*l.s.c.*, for short) for those functions $f: X \rightarrow \overline{\mathbb{R}}$ which are continuous with respect to the upper topology.¹

Any T_0 -space X comes with an intrinsic order, the *specialisation order* which is defined by $x \leq y$ if the element x is contained in the closure of the singleton $\{y\}$ or, equivalently, if every open set containing x also contains y . In this paper, an order on a topological space will always be the specialisation order. Open sets are upper (= saturated) sets and closed sets are lower sets with respect to the specialisation order. The saturation $\uparrow A$ of a subset A can also be characterised to be the intersection of the open sets containing A . In Hausdorff spaces, the specialisation order is trivial. But on $\overline{\mathbb{R}}$ with the upper topology, the specialisation order is just the usual linear order. Continuous maps between topological spaces preserve the respective specialisation orders. For more details, see e.g. [12], Section 0-5.

Definition 4.2. A semitopological cone is cone with a T_0 -topology such that addition and scalar multiplication are separately continuous, that is:

$$\text{TC1a} \quad a \mapsto ra: C \rightarrow C \text{ is continuous for every fixed } r > 0$$

$$\text{TC1b} \quad r \mapsto ra: \mathbb{R}_+ \rightarrow C \text{ is continuous for every fixed } a \in C$$

$$\text{TC2a} \quad b \mapsto a + b: C \rightarrow C \text{ is continuous for every fixed } a \in C$$

As for topological vector spaces, TC1a implies that multiplication $x \mapsto rx: C \rightarrow C$ with a fixed scalar $r > 0$ is linear and a homeomorphism, multiplication by r^{-1} giving the reciprocal map. Thus rU is open for every open set U , and rA is closed for every closed set A .

As we use the upper topology on \mathbb{R}_+ , axiom TC1b has the dramatic consequence that the topology on C cannot satisfy the Hausdorff separation property: As continuous maps preserve the respective specialisation orders, TC1b implies firstly that, for every $a \in C$, the map $r \mapsto ra: \mathbb{R}_+ \rightarrow C$ is order preserving, that is, the rays in the cone are nontrivially ordered (except for the singleton ray $\{0\}$), and secondly that the cone is pointed, as $0 = 0 \cdot a \leq 1 \cdot a = a$. In particular, the specialisation order on C is nontrivial and the topology is not Hausdorff. As continuous maps preserve the specialisation order, we conclude that *every semitopological cone is a pointed ordered cone*.

Definition 4.3. A topological cone is a cone C endowed with a T_0 -topology such that addition and scalar multiplication are jointly continuous, i.e.:

$$\text{TC1} \quad \text{Scalar multiplication } (r, a) \mapsto ra: \mathbb{R}_+ \times C \rightarrow C \text{ is jointly continuous}$$

$$\text{TC2} \quad \text{Addition } (a, b) \mapsto a + b: C \times C \rightarrow C \text{ is jointly continuous}$$

¹It is somewhat unfortunate that those functions are called *lower semicontinuous* which are continuous with respect to the *upper* topology. But we do not want to deviate from the terminology in classical analysis and the one adopted in [12].

A topological vector space satisfies TC1a and TC2, but not TC1b, as we use the upper topology on $\overline{\mathbb{R}}$. Thus, a topological vector space is not a (semi-)topological cone in our sense. At the other hand:

Example 4.4. (Topological cones from topological vector spaces) Let V be an ordered topological vector space with a closed positive cone V_+ and let C be a subcone of V_+ . The open upper subsets of C form a topology, and C is a topological cone in our sense with respect to this topology. One may ask the question to characterise those topological cones that arise from cones in topological vector spaces in this way.

There are (semi-)topological cones that are far from being embeddable in vector spaces:

Example 4.5. (Join-semilattices as cones) Every \vee -semilattice S with a smallest element 0 may be considered to be a pointed ordered cone: if we define multiplication with nonnegative reals on S by:

$$re = e \text{ for all } r > 0, \quad re = 0 \text{ for } r = 0$$

then the axioms for a pointed ordered cone are satisfied (with the join operation \vee as addition). Convex sets in S look quite particular; these are just the \vee -subsemilattices of S . It follows that all upper sets are convex. Homogeneous maps $f: S \rightarrow \overline{\mathbb{R}}_+$ admit only the values 0 and $+\infty$. Thus linear, sublinear and superlinear functionals on S are maps into $\{0, +\infty\}$. The linear maps between two \vee -semilattices with 0 are the semilattice homomorphisms preserving 0 ; order preserving sublinear maps are linear. All order preserving maps are superlinear.

Let us consider now a (semi-)topological \vee -semilattice S with a smallest element 0 such that all open sets are upper sets. (Note that a (semi-)topological semilattice remains a (semi-)topological semilattice if we replace its topology by the topology of its open upper sets.) The scalar multiplication defined as above is continuous, and we may consider S to be a (semi-)topological cone. All open sets are convex. In particular, as a cone, S is locally convex (see 4.8 below). But the Separation and the Strict Separation Theorems (see 9.1, 10.5 below) are trivial in this case. Indeed, mapping a closed subsemilattice to 0 and its complement to $+\infty$ is a lower semicontinuous \vee -semilattice homomorphism separating A strictly from its complement.

In a topological vector space, addition considered as a map $(a, b) \mapsto a + b: C \times C \rightarrow C$ is an open map. This is no longer true in topological cones. Occasionally, we will need a slight weakening of the openness of addition:

Definition 4.6. We say that the addition is almost open in C if the following axiom is satisfied:

$$\text{TC3} \quad \uparrow(U + V) \text{ is open for all open sets } U, V \subseteq C,$$

As for topological vector spaces, we have a notion of local convexity. Actually, the notion of local convexity splits into several notions of various strength. The first notion is a kind of convex modification of the axiom T_0 :

Definition 4.7. A cone C with a topology is called convex- T_0 if, for any two distinct points, there is an open convex set containing one of these points but not the other.

The second notion is close to the classical one:

Definition 4.8. A cone C with a topology is called locally convex, if each point has a neighbourhood basis of open convex neighbourhoods.

For cones, there is another notion of local convexity which does not make sense in topological vector spaces as it implies local compactness:

Definition 4.9. A cone with a topology is called locally convex-compact if each of its points has a neighbourhood basis of compact convex sets.

Clearly, *locally convex-compact* implies *locally compact*, but not *locally convex*, in general. A sufficient condition for the latter implication to hold is axiom TC3 which implies that the interior of a convex saturated set is open (see 4.10(b) below).

As in vector spaces, the *convex hull* $\text{conv } A$ of a subset A of a cone, i.e., the smallest convex set containing A , is the set of all finite convex combinations $\sum r_i a_i$ of elements a_i of A with $r_i \geq 0$ and $\sum r_i = 1$.

Lemma 4.10. Let C be a cone endowed with a topology.

(a) If the axioms TC1a and TC2a are satisfied, the closure of a convex set is convex. Hence, the closure of the convex hull $\overline{\text{conv } A}$ of an arbitrary subset A is the smallest closed convex subset containing A .

(b) If the axioms TC1a and TC3 are satisfied, the interior of every convex saturated set is convex. If in addition the topology is locally convex, the saturation of the convex hull $\uparrow(\text{conv } U)$ of any open set U is open; thus, $\uparrow(\text{conv } U)$ is the smallest open convex set containing U .

(c) If C is a topological cone, the convex hull $\text{conv} \left(\bigcup_{i=1}^n K_i \right)$ of finitely many compact convex sets K_i ($i = 1, \dots, n$) and its saturation are compact. In particular, the convex hull of any finite subset and the saturation thereof are compact.

Proof. (a) Take a convex set A . By TC1a and TC2a, the map $(a, b) \mapsto ra + (1 - r)b$ is separately continuous for $0 < r < 1$. By Lemma 2.1, $r\overline{A} + (1 - r)\overline{A} \subseteq rA + (1 - r)A \subseteq \overline{A}$. Thus, \overline{A} is convex.

(b) Let A be any convex saturated set. Denote by $\text{int } A$ the interior of A . For $0 < r < 1$, the set $\uparrow(r \text{int } A + (1 - r) \text{int } A)$ is open by TC1a and TC3, and contained in A . Hence, $\text{int } A$ is convex.

Now let U be open and let $\sum_i r_i u_i$ be a finite convex combination of elements $u_i \in U$. Using local convexity, we may find an open convex neighbourhood U_i of u_i contained in U for every i , and we obtain $\sum_i r_i u_i \in \uparrow \sum_i r_i U_i \subseteq \uparrow \text{conv}(U)$. TC1a and TC3 together ensure that $\uparrow \sum_i r_i U_i$ is open, and the saturation of a convex set always is convex.

(c) Let Δ denote the standard n -simplex considered as a subspace of \mathbb{R}_+^n . If C is a topological cone, the map

$$\left((r_1, \dots, r_n), (x_1, \dots, x_n) \right) \mapsto \sum_{i=1}^n r_i x_i : \mathbb{R}_+^n \times C^n \rightarrow C^n$$

is continuous. The image of the compact set $\Delta \times K_1 \times \dots \times K_n$ under this continuous map is compact on the one hand, and is the convex hull of the union of the K_i on the other hand. \square

5 Functionals and weak* topologies

Arbitrary powers $\overline{\mathbb{R}}_+^I$ with the product topology ν^I of the upper topology ν on $\overline{\mathbb{R}}_+$ are topological cones.

Definition 5.1. *On every subcone \mathcal{D} of $\overline{\mathbb{R}}_+^I$ the product topology ν^I induces a topology which will be called the weak*upper topology on \mathcal{D} .*

A subbasis for the weak*upper opens is given by the sets:

$$W_{x,r}^* = \{f \in \mathcal{D} \mid f(x) > r\}, \quad x \in I, \quad 0 \leq r < +\infty$$

It is the weakest topology on \mathcal{D} for which the evaluation maps $f \mapsto f(x): \mathcal{D} \rightarrow \overline{\mathbb{R}}_+$ are lower semicontinuous for all $x \in I$. In a similar way, the *weak*lower topology* on \mathcal{D} , which comes from the product of the lower topology on $\overline{\mathbb{R}}_+$ has the subbasic open sets:

$$L_{x,s}^* = \{f \in \mathcal{D} \mid f(x) < s\}, \quad x \in I, \quad 0 < s \leq +\infty$$

The *weak* topology* is the common refinement of the two previous topologies and is induced from the product topology of the usual interval topology on $\overline{\mathbb{R}}_+$.

Every subcone \mathcal{D} of $\overline{\mathbb{R}}_+^I$ is a topological cone, when endowed with the weak*upper topology, which explains why this topology plays a more important role for us than the other two. As the subbasic opens $W_{x,r}^*$ are convex, \mathcal{D} is a locally convex topological cone.

The terms *weak*upper topology*, etc., have been chosen according to the terminology in functional analysis, where the topology of pointwise convergence on the dual of a topological vector space is called the *weak* topology*.

Example 5.2. For any pointed ordered cone C , the subcones C'_{sub} , C'_{sup} and C' in $\overline{\mathbb{R}}_+^C$ of order preserving sublinear, superlinear and linear functionals, respectively, are locally convex topological cones, when endowed with the weak*upper topology (see Example 3.8).

Example 5.3. (Lower semicontinuous functions) For any topological space X , we denote by $\mathcal{L}(X)$ the set of all lower semicontinuous functions $f: X \rightarrow \overline{\mathbb{R}}_+$. Clearly, $\mathcal{L}(X)$ is a subcone of $\overline{\mathbb{R}}_+^X$, hence a locally convex topological cone with respect to the weak*upper topology.

The pointwise infimum of finitely many and the pointwise supremum of any nonempty family of lower semicontinuous functions is lower semicontinuous. Because of the latter, there is a greatest lower semicontinuous function $\check{g}: X \rightarrow \overline{\mathbb{R}}_+$ below every function $g: X \rightarrow \overline{\mathbb{R}}_+$. The function \check{g} is called the *lower semicontinuous envelope* of g . It is given explicitly by:

$$\check{g}(x) = \bigvee_{U \in \mathcal{U}_x} \bigwedge_{u \in U} g(u) = \sup \{r \in \mathbb{R}_+ \mid \exists U \in \mathcal{U}_x . r < g(u) \text{ for all } u \in U\}$$

where \mathcal{U}_x is the filter of all neighbourhoods of $x \in X$.

Assigning its lower semicontinuous envelope \check{g} to every function $g \in \overline{\mathbb{R}}_+^X$ yields a projection $\Psi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+^X$, that is, an order preserving map satisfying $\Psi \circ \Psi = \Psi \leq \text{id}_{\overline{\mathbb{R}}_+^X}$, the image of which is

$\mathcal{L}(X)$. As for every projection, the corestriction $\Psi: \overline{\mathbb{R}}_+^X \rightarrow \mathcal{L}(X)$ preserves arbitrary meets. But notice that the meet of an infinite family in $\mathcal{L}(X)$ is not formed pointwise; it is the lower semicontinuous envelope of the pointwise meet. In general, Ψ does not preserve arbitrary joins. But we have:

Lemma 5.4. *The projection $\Psi: \overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+^X$, assigning to each function $g: X \rightarrow \overline{\mathbb{R}}_+$ its lower semicontinuous envelope \check{g} , is linear and preserves finite suprema.*

Proof. It is straightforward that $r \cdot \check{g} = (rg)^\checkmark$. Let $g, h \in \overline{\mathbb{R}}_+^X$. As $\check{g} \leq g$ and $\check{h} \leq h$, we have $\check{g} + \check{h} \leq g + h$, whence $\check{g} + \check{h} \leq (g + h)^\checkmark$. For the converse inequality, suppose that $r < (g + h)^\checkmark(x)$. Then there is an open neighbourhood U of x such that $r < \bigwedge_{u \in U} (g + h)(u) = \bigwedge_{u \in U} (g(u) + h(u)) \leq \bigwedge_{u \in U} g(u) + h(x) \leq \check{g}(x) + h(x)$. As this holds for all $x \in C$, we have $(g + h)^\checkmark \leq \check{g} + h$. We now get $(g + h)^\checkmark = (g + h)^\checkmark \leq (\check{g} + h)^\checkmark \leq \check{g} + \check{h}$, where the last inequality holds for the same reason as before.

It remains to show that $(f \vee g)^\checkmark = \check{f} \vee \check{g}$. It is straightforward that the left hand side is dominated by the right hand side. For the converse inequality, consider any $x \in X$ and let $\check{f}(x) \vee \check{g}(x) > r$. Then $\check{f}(x) > r$ or $\check{g}(x) > r$. In the first case (in the second case one argues in the same way), there is a open neighbourhood U of x such that $f(u) > r$ for every $u \in U$, and this implies that $(f \vee g)(u) = f(u) \vee g(u) > r$ for all $u \in U$, whence $(f \vee g)^\checkmark(x) > r$. \square

Example 5.5. (Lower semicontinuous functionals) For any cone C with a topology, we denote by:

C_{sub}^{t*}	the set of all lower semicontinuous sublinear functionals $f: C \rightarrow \overline{\mathbb{R}}_+$
C_{sup}^{t*}	the set of all lower semicontinuous superlinear functionals $f: C \rightarrow \overline{\mathbb{R}}_+$
$C^{t*} = C_{sub}^{t*} \cap C_{sup}^{t*}$	the set of all lower semicontinuous linear functionals $f: C \rightarrow \overline{\mathbb{R}}_+$

Under pointwise defined addition and multiplication by nonnegative scalars, C_{sub}^{t*} , C_{sup}^{t*} and C^{t*} are subcones of $\overline{\mathbb{R}}_+^C$. Thus they are locally convex topological cones when endowed with the weak*upper topology. As lower semicontinuous functionals preserve the specialisation order, the topological duals C_{sub}^{t*} , C_{sup}^{t*} and C^{t*} are subcones of their order theoretical analogues C'_{sub} , C'_{sup} and C' (see Example 3.8).

Lemma 5.6. *The cones C_{sub}^{t*} , C_{sup}^{t*} and C^{t*} are sober for the weak*upper topology.*

Proof. The space of all lower semicontinuous functions $f: C \rightarrow \overline{\mathbb{R}}_+$ is sober in the weak*upper topology and C_{sub}^{t*} , C_{sup}^{t*} and C^{t*} are subspaces given by equations and inequations, hence by equalisers, which implies that they are sober, too (see [36, Chapter 5] and [14]). \square

The following lemma gives sufficient conditions for the projection Ψ to map C' , C'_{sub} and C'_{sup} onto C^{t*} , C_{sub}^{t*} and C_{sup}^{t*} , respectively. The first two claims occur implicitly in [37] for continuous d-cones. Plotkin has proved them in the framework of continuous d-cones [29]:

Lemma 5.7. *Let C be a cone with a topology and let $g: C \rightarrow \overline{\mathbb{R}}_+$ be any function. For its lower semicontinuous envelope \check{g} we have:*

- (a) *If g is homogeneous and if axiom TC1a is satisfied, then \check{g} is homogeneous, too.*
- (b) *If g is subadditive and axiom TC2a is satisfied, then \check{g} is subadditive, too.*

(c) If g is superadditive and preserves the specialisation order and if axiom TC3 is satisfied, then \check{g} is superadditive, too.

Proof. (a) Clearly, $\check{g}(0) = 0$ which implies that $g(sx) = sg(x)$ for $s = 0$. For $s > 0$, we note that, by (TC1a), U is an open neighborhood of x if and only if sU is an open neighborhood of sx . Thus, $r < \check{g}(x)$ iff $r < g(u)$ for all u in some open neighbourhood U of x iff $sr < sg(u) = g(su)$ for all u in some open neighbourhood U of x iff $sr < g(v)$ for all v in some open neighbourhood V of sx iff $sr < \check{g}(sx)$.

(b) Suppose that g is subadditive. In order to show that \check{g} is subadditive, too, we take any two elements $x, y \in C$ and we show that $r < \check{g}(x + y)$ implies $r \leq \check{g}(x) + \check{g}(y)$.

So let $r < \check{g}(x + y)$. By the definition of \check{g} there is an open neighbourhood W of $x + y$ such that $r \leq g(w)$ for all $w \in W$. As addition is supposed to be separately continuous, there is a neighbourhood U of x such that $u + y \in W$ for every $u \in U$. Again, for every $u \in U$, there is a neighborhood V_u of y such that $u + V_u \subseteq W$. For every $u \in U$ and all $v \in V_u$, we then have $r \leq g(u + v) \leq g(u) + g(v)$ from which we conclude in a first step that, for every $u \in U$, $r \leq g(u) + \check{g}(y)$, and in a second step that $r \leq \check{g}(x) + \check{g}(y)$.

(c) Take $r < \check{g}(x) + \check{g}(y)$. Then there are $a < \check{g}(x)$ and $b < \check{g}(y)$ such that $r < a + b$. So there are open neighbourhoods U and V of x and y , respectively, such that $g(u) > a$ for all $u \in U$ and $g(v) > b$ for all $v \in V$. By hypothesis, $\uparrow(U + V)$ is open, too, hence a neighbourhood of $x + y$. For all $w \in \uparrow(U + V)$ there are $u \in U, v \in V$ such that $w \geq u + v$, whence $g(w) \geq g(u + v) \geq g(u) + g(v) > a + b > r$ by the monotonicity and superadditivity of g . So we get that $\check{g}(x + y) > r$. \square

Definition 5.8. For a cone C with a topology, the cone C^* of all lower semicontinuous linear functionals endowed with the weak*upper topology is called the topological dual of C , or simply the dual cone.

Example 5.9. (Bidual) For any semitopological cone C , we may form its bidual C^{**} , that is, the cone of all linear functionals on the dual cone C^* that are lower semicontinuous with respect to the weak*upper topology on C^* . As for vector spaces, there is a canonical map $x \mapsto x^{**}: C \rightarrow C^{**}$ defined by $x^{**}(f) = f(x)$ for every $f \in C^*$. This map is linear and continuous, if we endow C^{**} with its weak*upper topology. This map is injective if and only if the lower semicontinuous linear functionals separate the points of C . In Theorem 9 we will see that this is guaranteed by the weak local convexity condition convex-T_0 as in topological vector spaces.

On C we may also consider the *weak topology* which is induced from the weak*topology on C^{**} , more precisely, it is the weakest topology on C for which the lower semicontinuous linear functionals $f \in C^*$ remain lower semicontinuous. A subbasis for the open sets of this weak topology is given by:

$$W_{f,r} =_{\text{def}} \{x \in C \mid f(x) > r\}, f \in C^*, 0 < r < +\infty$$

Definition 5.10. We will say that the semitopological cone C is reflexive, if the map $x \mapsto x^{**}: C \rightarrow C^{**}$ is bijective. (We do not require this map to be a homeomorphism.)

Example 5.11. (Reflexive cones) Some important instances of reflexive topological cones are known. Firstly, the cone $\mathcal{L}(X)$ of all lower semicontinuous functions defined on a locally compact space X with values in $\overline{\mathbb{R}}_+$ with the Scott topology, and its dual cone, the extended probabilistic powerdomain

$\mathcal{V}(X)$ (see Tix [36]) with the upper weak*topology. Secondly, the cone of all nonnegative hyperharmonic functions on an open subset of \mathbb{R}^n with the Scott topology and its dual (see [30, 4]). Thirdly, the round ideal completions of standard H-cones in the sense of [4] (see Rauch [30]).

In the classical setting, a locally convex topological vector space E is reflexive in the sense of the above definition, as E always is the dual of its dual E^* , if the latter is endowed with the weak*-topology.

Problem. Give necessary and/or sufficient conditions for a semitopological cone C to be reflexive. A contribution to the problem for the case of continuous d-cones (see the next section) would also be welcome.

6 s-Cones, d-cones and bd-cones, c-cones

In domain theory interesting classes of semitopological cones have been introduced by Kirch [22] and Tix [38, 37]. A more accessible source is [39]. We would like to tie up that domain theoretical approach with the present topological one. Before we enter the subject, let us recall some facts from domain theory (see [12]).

The basic notion is that of a Scott-continuous function: A function f from a poset P to a poset Q is called *Scott-continuous* if it is order preserving and if, for every directed subset D of P which has a least upper bound in P , the image $f(D)$ has a least upper bound in Q and $f(\bigvee^\uparrow D) = \bigvee^\uparrow f(D)$.

Every poset carries some intrinsic topologies. Besides the upper, lower and interval topologies addressed in the Preliminaries, two more topologies are of interest. One is the *Scott topology* σ , the closed sets of which are the lower sets that are closed with respect to suprema of directed subsets, as far as these suprema exist. The Scott topology is finer than the upper topology. A function between posets is continuous with respect to the respective Scott topologies if and only if it is Scott-continuous in the sense defined above. The common refinement of the Scott topology and the lower topology is the *Lawson topology* λ which is finer than the interval topology.

Definition 6.1. An s-cone is a cone with a partial order such that addition and scalar multiplication:

$$(a, b) \mapsto a + b: C \times C \rightarrow C, \quad (r, a) \mapsto ra: \mathbb{R}_+ \times C \rightarrow C$$

are Scott-continuous. An s-cone is called a [b]d-cone if its order is [bounded] directed complete, i.e., if each [upper bounded] directed subset has a least upper bound.

Example 6.2. For any pointed ordered cone C , the cones C'_{sup} , C'_{sub} and C' are d-cones and for any cone C with a topology, the cones C^*_{sup} , C^*_{sub} and C^* are d-cones. In general, their Scott topology is finer than the weak*upper topology.

Indeed, for families of functionals, the property of being order preserving is preserved under arbitrary pointwise suprema and infima, sublinearity is preserved under arbitrary pointwise suprema and directed pointwise infima, and superlinearity under arbitrary pointwise infima and pointwise directed suprema, lower semicontinuity is preserved under arbitrary pointwise suprema and binary pointwise infima.

An s-cone need not be a topological cone with respect to the Scott-topology. The point is a subtle one: The product of two Scott topologies need not be the Scott topology of the product; it may be coarser (see [12, Exercise II-4.26]). Thus, addition need not be jointly continuous. Of course it still is continuous in each argument separately. Thus:

Proposition 6.3. *Every s-cone, hence every [b]d-cone, is a semitopological cone with respect to its Scott topology.*

In the following, s-cones and [b]d-cones will always be considered as semitopological cones with their Scott topology, if not specified otherwise.

A d-cone always has a greatest element that we denote by ∞ . Indeed, as $a+b$ always is a common upper bound for a and b , a d-cone is a directed set, so $\bigvee^\uparrow C$ exists and is the greatest element of C . And this element satisfies $a + \infty = \infty$. Thus, a d-cone does not satisfy cancellation and cannot be embedded in an ordered vector space. On the other hand, the positive cone V_+ of an ordered vector space V is always an s-cone, and if V is bounded directed complete (such ordered vector spaces have been called *monotone complete* in the literature) then V_+ is a bd-cone yielding plenty of examples of bd-cones. The standard embedding of an ordered cone with order cancellation into an ordered vector space yields the following proposition:

Proposition 6.4. *A s-cone admits a Scott-continuous embedding into the positive cone of a bounded directed complete ordered vector space if and only if it satisfies order cancellation (OC).*

The annoying feature that s-cones and [b]d-cones need not be topological cones disappears, if we restrict our attention to the continuous case. Let us begin with some generalities which can be found in [12, Section I-1].

In any poset P , the *way-below relation* \ll is defined by: $x \ll y$ iff, for any directed subset $D \subseteq P$ for which $\sup D$ exists, the relation $y \leq \sup D$ implies the existence of a $d \in D$ with $x \leq d$. The poset P is called *continuous* if, for every element y in P , the set $\downarrow y =_{def} \{x \in P \mid x \ll y\}$ is directed and $y = \bigvee^\uparrow \downarrow y$.

Definition 6.5. *A continuous s-cone is an s-cone whose underlying poset is continuous. The same applies to continuous [b]d-cones.*

The Scott topology of a continuous poset has the remarkable property that every point y has a neighbourhood base consisting of principal filters $\uparrow x$, $x \ll y$. This leads to a slightly more general class of topological spaces:

Definition 6.6. *In an arbitrary T_0 -space X we define the (topological) way-below relation by $x \ll y$ if the principal filter $\uparrow x$ (with respect to the specialisation order) is a neighbourhood of y , and we say that X is a c-space, if every point $y \in X$ has a neighbourhood basis of sets of the form $\uparrow x$.*

The notion of a c-space has been studied by Ern  [7, pp. 75ff.], [8], and by Ershov [9, 10] under the name of an α -space. Ern  [8] has shown that c-spaces are equivalent to abstract bases in the sense of [1].

We have seen that every continuous poset with its Scott topology is a c-space. But not every c-space is a continuous poset with respect to its specialisation order. If we endow the reals with the

Alexandroff topology, i.e., all upper sets are declared open, then we obtain a c-space which is not a continuous poset for its specialisation order; indeed the intervals $[r, +\infty[$ are Alexandroff open; thus, $r \ll r$ for the topological way-below relation but not for the order theoretical way-below.

Proposition 6.7. *If in a c-space every directed subset which has a supremum in the specialisation order converges to this supremum, then the c-space is a continuous poset with respect to its specialisation order and the topological way-below relation coincides with the order theoretical one.*

The following lemma is due to Ershov [10, Proposition 2]:

Lemma 6.8. *For a c-space X and arbitrary topological spaces Y, Z , a function $f: X \times Y \rightarrow Z$ is jointly continuous iff it is separately continuous.*

As \mathbb{R}_+ with the upper topology is a c-space, this lemma implies:

Corollary 6.9. (a) *Separate continuity of scalar multiplication in a cone with a topology implies joint continuity, i.e., axiom TC1 is equivalent to the conjunction of the axioms TC1a and TC1b.*

(b) *If C is a cone with a c-space topology, separate continuity of the addition implies joint continuity, i.e., axiom TC2a implies axiom TC2.*

(c) *A semitopological cone with a c-space topology is a topological cone.*

Definition 6.10. *A c-cone will be a semitopological cone which topologically is a c-space.*

Example 6.11. Examples of c-cones are:

(a) \mathbb{R}_+^n with the product topology of the upper topology on \mathbb{R}_+ .

(b) Continuous s-cones and, in particular, continuous [b]d-cones.

(c) For a compact Hausdorff space X , the set $C_+(X)$ of all functions $f: X \rightarrow \mathbb{R}_+$ which are continuous with respect to the usual topology λ on \mathbb{R}_+ , where a neighborhood basis for a $g \in C_+(X)$ is given by the sets of the form: $\{f \in C_+(X) \mid f(x) \geq g(x) - \varepsilon \text{ for all } x \in X\}, \varepsilon > 0$.

It may be surprising that a semitopological cone C in our sense is always compact; indeed, principal filters $\uparrow x$ are always compact and $C = \uparrow 0$. But in the T_0 -setting compactness is a weak property. It does not imply local compactness. Therefore, the following, which applies in particular to continuous s-cones and continuous [b]d-cones, is noteworthy:

Lemma 6.12. *Every c-cone C is a locally convex and locally convex-compact topological cone, hence, also locally compact.*

Proof. Local convex-compactness is a consequence of the facts that principal filters $\uparrow x$ are compact and convex and that every point has a neighbourhood basis of principal filters in a c-space. For local convexity, take an arbitrary neighbourhood U of an element $y \in X$. By the c-space property we can find a sequence of elements $x_n \in U$ such that $\dots \ll x_{n+1} \ll x_n \ll \dots \ll x_0 \ll y$. Then the set $V = \bigcup_n \uparrow x_n$ is contained in U , it is open by the construction of the x_n , and it is convex as the sets $\uparrow x_n$ form an increasing sequence of convex sets. \square

In a c-cone, the property of addition being almost open can be identified with a property of the way-below relation:

Definition 6.13. In a c -cone, and similarly in a continuous s -cone, the relation \ll is said to be additive, if:

$$x' \ll x \text{ and } y' \ll y \text{ imply } x' + y' \ll x + y.$$

Lemma 6.14. In a c -cone and similarly in a continuous s -cone C , addition is almost open (i.e., axiom TC3 is satisfied) if and only if its way-below relation is additive.

Proof. Suppose first that the way-below relation is additive. Let U and V be open sets in C . For $z \in \uparrow(U + V)$ there are $u \in U$ and $v \in V$ such that $u + v \leq z$. As we are in a c -cone, there are elements $u' \in U$ and $v' \in V$ such that $u' \ll u$ and $v' \ll v$. By additivity we have $u' + v' \ll u + v$, whence $u' + v' \ll z$. Thus $\uparrow(u' + v')$ is a neighbourhood of z contained in $\uparrow(U + V)$ which proves that $\uparrow(U + V)$ is open.

Conversely, let $x' \ll x$ and $y' \ll y$. Then $x' + y' \in \uparrow(\uparrow x + \uparrow y)$. If we suppose addition to be almost open, the set $\uparrow(\uparrow x + \uparrow y)$ is open. Thus there is an element z in it with $z \ll x' + y'$. As then $x + y \leq z$, we conclude that $x + y \ll x' + y'$. \square

7 Minkowski functionals

Let C be a semitopological cone. (The attentive reader will observe that, in this section, we only use the axiom TC1a,b). The order on C is always meant to be the specialisation order. Recall that 0 is the smallest element of C .

Definition 7.1. For a subset A of C we define its lower Minkowski functional $F_A: C \rightarrow \overline{\mathbb{R}}_+$ by:

$$F_A(x) =_{def} \inf\{r \in \mathbb{R}_+ \mid x \in rA\}.$$

It is understood that $F_A(x) = +\infty$, if $x \notin rA$ for all $r \in \mathbb{R}_+$. Conversely, for every functional $F: C \rightarrow \overline{\mathbb{R}}_+$ we consider the following subset of C :

$$A_F =_{def} \{x \in C \mid F(x) \leq 1\}$$

We first notice:

$$A \subseteq B \implies F_A \geq F_B, \quad F \leq G \implies A_F \supseteq A_G$$

i.e., we have order reversing maps between the set of subsets $A \subseteq C$ and the functionals $F: C \rightarrow \overline{\mathbb{R}}_+$.

We want to characterise properties of subsets $A \subseteq C$ by properties of their lower Minkowski functionals. Let us begin with a nonempty closed set $A \subseteq C$. By TC1b, the map $r \mapsto rx$ is continuous. Thus, $\{s \in \mathbb{R}_+ \mid sx \in A\}$ is a nonempty closed subset of \mathbb{R}_+ for the upper topology, i.e., either $sx \in A$ for all $s \in \mathbb{R}_+$ or $\{s \in \mathbb{R}_+ \mid sx \in A\} = [0, s_0]$ for some $s_0 \in \overline{\mathbb{R}}_+$. It follows that either $x \notin rA$ for all $r \in \mathbb{R}_+$, hence $F_A(x) = +\infty$, or $\{r \in \mathbb{R}_+ \mid x \in rA\} = [r_0, +\infty[$ for some $r_0 \in \mathbb{R}_+$, hence $r_0 = F_A(x)$. So either $x \notin rA$ for all $r \in \mathbb{R}_+$. Using the convention $\infty \cdot A = C$, we have:

$$(0) \quad F_A(x) = \min\{r \in \overline{\mathbb{R}}_+ \mid x \in rA\} = \sup\{r \in \mathbb{R}_+ \mid x \notin rA\}$$

and:

$$(1) \quad r \geq F_A(x) \iff x \in rA$$

In passing let us note the following:

Remark 7.2. Let (r_i) be any family of real numbers with $\inf_i r_i > 0$. For every closed subset A of C , one has $(\inf_i r_i) \cdot A = \bigcap_i r_i A$ and, for every element x , one has $\inf_i (r_i \cdot x)$ exists and $(\inf_i r_i) \cdot x = \inf_i (r_i \cdot x)$.

Proof. Let $r = \inf_i r_i > 0$. Clearly $rA \subseteq \bigcap_i r_i A$. For the converse inclusion, let $x \notin rA$. By (1), $r < F_A(x)$. Thus, there is an i such that $r_i < F_A(x)$, that is, $x \notin r_i A \supseteq \bigcap_i r_i A$. Applying the above result to the case $A = \downarrow x$ yields the second claim. \square

The following proposition is crucial:

Proposition 7.3. a) The lower Minkowski functional F_A of a nonempty closed subset A of C is homogeneous and lower semicontinuous and $A = \{x \in C \mid F_A(x) \leq 1\}$.

b) For every homogeneous lower semicontinuous functional F , the set A_F is nonempty and closed in C , and F is the lower Minkowski functional of A_F .

c) The map $A \mapsto F_A$ establishes an order anti-isomorphism between the collection of all nonempty closed subsets of C and the set of all homogeneous lower semicontinuous functionals $F: C \rightarrow \overline{\mathbb{R}}_+$, the reciprocal map being $F \mapsto A_F$.

Proof. a) We first show that F_A is homogeneous. As $0 \in rA$ for all $r > 0$, we have $F_A(0) = 0$. For $s > 0$, we have by (1): $r \geq F_A(sx) \iff sx \in rA \iff x \in \frac{r}{s}A \iff \frac{r}{s} \geq F_A(x) \iff r \geq sF_A(x)$, whence $F_A(sx) = sF_A(x)$.

By (1) above we have $\{x \in C \mid F_A(x) \leq r\} = rA$. As rA is closed by TC1a, this shows the continuity of F_A . For $r = 1$, we obtain the last claim in (a).

(b) Let F be a lower semicontinuous functional. Clearly, the set A_F of all $x \in C$ with $F(x) \leq 1$ is closed. If F is homogeneous, then $F(0) = 0$ and, hence, A_F contains 0. Further, A_F is the lower Minkowski functional of F . Indeed, $F(x) \leq r \iff F(\frac{1}{r}x) = \frac{1}{r}F(x) \leq 1 \iff \frac{x}{r} \in A_F \iff x \in rA_F$. Thus, $F(x) = \inf\{r \in \mathbb{R}_+ \mid x \in rA_F\}$, i.e. F is the lower Minkowski functional of A_F .

(c) is a direct consequence of (a) and (b). \square

Turning to proper open subsets U of C we define their *upper Minkowski functional* F^U to be the lower Minkowski functional of their complement $C \setminus U$ which is nonempty and closed, i.e.:

$$(0') \quad F^U(x) = \min\{r \in \overline{\mathbb{R}}_+ \mid x \notin rU\} = \sup\{r \in \overline{\mathbb{R}}_+ \mid x \in rU\}$$

and:

$$(1') \quad r \geq F^U(x) \iff x \notin rU$$

From the previous proposition 7.3 we obtain:

Proposition 7.4. *Associating to every proper open subset U of C its upper Minkowski functional F^U establishes an order isomorphism between the set of all homogeneous continuous functionals on C and the set of all proper open subsets of C ordered by inclusion.*

We now turn to linearity properties of the lower Minkowski functional.

Lemma 7.5. *For a nonempty closed subset A of C , the lower Minkowski functional F_A is sublinear if and only if A is convex. It is superlinear if and only if the open complement $U = C \setminus A$ is convex.*

Proof. If F_A is superlinear, then $U = \{x \in C \mid F_A(x) > 1\} = C \setminus A$ is convex. Conversely, let $U = C \setminus A$ be convex. For any $0 < r < F_A(x)$ and any $0 < s < F_A(y)$ we have $x \in rU$ and $y \in sU$, whence $x + y \in rU + sU = (r + s)U$ by Lemma 3.3, as U is convex. Hence, $r + s \leq F_U(x + y)$, whence $F_U(x) + F_U(y) \leq F_U(x + y)$.

If F_A is sublinear, then $A = \{x \in C \mid F(x) \leq 1\}$ is convex. Conversely, let A be convex. Let $r = F_A(x)$ and $s = F_A(y)$. Then $x \in rA$, $y \in sA$ and, consequently, $x + y \in rA + sA = (r + s)A$ by Lemma 3.3, as A is convex. Hence, $r + s \geq F_A(x + y)$. \square

From the preceding lemma 7.5 and the propositions 7.3, 7.4 we conclude:

Proposition 7.6. (a) *Assigning its lower Minkowski functional F_A to every nonempty closed convex subset A of C induces an order anti-isomorphism between the complete lattices $\mathcal{H}(C)$ of nonempty closed convex subsets and the ordered cone C_{sub}^* of lower semicontinuous sublinear functionals on C .*

(b) *Assigning the upper Minkowski functional F^U to every proper open convex subset U of C induces an order isomorphism between the complete lattices $\mathcal{O}_c(C)$ of proper open convex subsets and the ordered cone C_{sup}^* of lower semicontinuous superlinear functionals on C .*

8 A Sandwich Theorem

It is our aim now to prove a number of Hahn-Banach type theorems for semitopological cones. We will use a sandwich theorem due to W. Roth (see [31], Theorem 2.6) for ordered cones:

Theorem 8.1. *Let C be an ordered cone. Let $p: C \rightarrow \overline{\mathbb{R}}$ be sublinear and $q: C \rightarrow \overline{\mathbb{R}}$ superlinear functionals, respectively, such that $a \leq b \Rightarrow q(a) \leq p(b)$. (The latter is satisfied if $q \leq p$ and one of p, q is order preserving.) Then there are order preserving sublinear functionals $\Lambda: C \rightarrow \overline{\mathbb{R}}$ that are minimal among those that satisfy $q \leq \Lambda \leq p$, and all of those minimal Λ are linear.*

The following topological version of Roth's Sandwich Theorem has been proved by Tix [38, 37] for continuous d-cones (see also [39]).

Sandwich Theorem 8.2. *Let C be a semitopological cone. Let $q: C \rightarrow \overline{\mathbb{R}}_+$ be superlinear and lower semicontinuous and let $p: C \rightarrow \overline{\mathbb{R}}_+$ be sublinear. If $q \leq p$, then there is a lower semicontinuous linear functional $\Lambda: C \rightarrow \overline{\mathbb{R}}_+$ such that $q \leq \Lambda \leq p$.*

Proof. We can apply Roth's sandwich theorem 8.1 to our situation with the specialisation order on C . As q is lower semicontinuous, it preserves the specialisation order, and as $q \leq p$, the hypotheses of Roth's sandwich theorem are satisfied. Thus, there is an order preserving linear functional Λ such that $q \leq \Lambda \leq p$ and such that there is no order preserving sublinear functional strictly between q and Λ . We now show that Λ is lower semicontinuous. As $q \leq \Lambda$ and as q is lower semicontinuous, Λ has indeed a lower semicontinuous envelope $\check{\Lambda}$ with $q \leq \check{\Lambda} \leq \Lambda$. Lemma 5.7(a),(b) implies that $\check{\Lambda}$ is sublinear, too. The minimality property of Λ now implies $\Lambda = \check{\Lambda}$, that is, Λ is lower semicontinuous. \square

Corollary 8.3. *Every lower semicontinuous superlinear functional q on a semitopological cone C is the pointwise infimum of lower semicontinuous linear functionals; more precisely,*

$$q(a) = \min\{\Lambda(a) \mid q \leq \Lambda \in C^*\} \text{ for every } a \in C.$$

Proof. Consider any element $a \in C$ and define a functional p by $p(x) = q(x)$ if $x = sa$ for some $s \in \mathbb{R}_+$ and $p(x) = +\infty$ else. Then p is sublinear and $q \leq p$. By the Sandwich Theorem 8.2 there is a lower semicontinuous linear functional Λ such that $q \leq \Lambda \leq p$; in particular, $q(a) \leq \Lambda(a) \leq p(a) = q(a)$, whence $\Lambda(a) = q(a)$. \square

Note. The attentive reader will notice that axiom T1b has not been used in the proof. Thus, our results apply to more general situations than semitopological cones. In particular, in the Sandwich Theorem one may replace $\overline{\mathbb{R}}_+$ by $\overline{\mathbb{R}}$.

9 A Separation Theorem

We now turn to a geometric version of the Hahn-Banach theorem for cones. For the case of d-cones, the results in this section are due to [38, 37, 39]

Separation Theorem 9.1. *In a semitopological cone C consider a nonempty convex subset A and an open convex subset U . If A and U are disjoint, then there exists a lower semicontinuous linear functional $\Lambda: C \rightarrow \overline{\mathbb{R}}_+$ such that $\Lambda(a) \leq 1 < \Lambda(u)$ for all $a \in A$ and all $u \in U$.*

Proof. By Lemma 4.10(a), the closure of A is also convex, and still disjoint from the open set U . Thus, we may suppose A to be closed. In order to apply the Sandwich Theorem 8.2, let q be the upper Minkowski functional of U and p the lower Minkowski functional of A . As $A \cap U = \emptyset$, we have $q \leq p$. By Lemma 7.5, q is superlinear, p is sublinear and both are lower semicontinuous. Now, we apply the Sandwich Theorem 8.2 to get a lower semicontinuous linear functional Λ with $q \leq \Lambda \leq p$. Since $a \in A$ implies $p(a) \leq 1$ and as $u \in U$ implies $q(u) > 1$, we have $\Lambda(a) \leq p(a) \leq 1 < q(u) \leq \Lambda(u)$ for all $a \in A$ and all $u \in U$. \square

Corollary 9.2. *For every open convex set U in a semitopological cone C and every element $a \in C$ not contained in U , there is a lower semicontinuous linear functional Λ such that $\Lambda(a) \leq 1$ but $\Lambda(u) > 1$ for all $u \in U$*

The previous corollary implies that the lower semicontinuous linear functionals separate the points of a locally convex topological cone. The result holds for the more general class of convex- T_0 cones which are characterised by the property that, for any two elements a and b with $a \not\geq b$, there is an open convex set containing b but not a :

Corollary 9.3. *For elements a, b in a convex- T_0 semitopological cone C with $a \not\geq b$, there is a linear lower semicontinuous functional $\Lambda: C \rightarrow \overline{\mathbb{R}}_+$ such that $\Lambda(a) < \Lambda(b)$.*

Corollary 9.4. *Let C be a locally convex semitopological cone. For every nonempty closed convex subset A and every $b \in C$ not contained in A , there is a lower semicontinuous linear functional Λ such that $\Lambda(a) \leq 1$ for every $a \in A$ but $\Lambda(b) > 1$.*

Proof. As the complement of A is a neighbourhood of b , there is a convex open neighbourhood U of b disjoint from A and we may apply the Separation Theorem 9.1 in order to find the desired lower semicontinuous linear functional Λ . \square

Recall that the weak upper topology is defined to be the coarsest topology such that all lower semicontinuous linear functionals on C remain lower semicontinuous. Clearly, the *weak upper topology* is locally convex. The previous corollary tells us that, for a locally convex cone, the closed convex sets are the same with respect to the original and the weak upper topology.

Corollary 9.5. *Every lower semicontinuous sublinear functional p on a locally convex semitopological cone is pointwise the supremum of lower semicontinuous linear functionals.*

Proof. Let A be the closed convex set which has p as its Minkowski functional. By corollary 9.4, A is the intersection of closed convex sets A_i for which the lower Minkowski functional Λ_i is linear and lower semicontinuous. From the order anti-isomorphism between the closed convex sets and lower semicontinuous linear functionals (see 7.6) the claim follows. \square

10 A Strict Separation Theorem

We now present another Hahn-Banach type separation theorem of a geometric flavour and some of its consequences. For continuous d-cones, this theorem is due to G.D. Plotkin (see [39]) with slightly different proofs. We use the basic idea of his proof in a more general setting. We begin with a quite special situation.

Lemma 10.1. *Let K be a compact convex subset of the cone $\overline{\mathbb{R}}_+^n$. Then every neighbourhood of K contains a convex open neighbourhood of K .*

Proof. By a classical theorem due to Caratheodory (see e.g. [5]) each point in the convex hull of a set V in \mathbb{R}^n is a convex combination of $n + 1$ points of V . The same holds for the cone $\overline{\mathbb{R}}_+^n$. Let us reformulate this theorem:

Let Δ denote the standard n -simplex considered as a subspace of $\overline{\mathbb{R}}_+^{n+1}$. The map

$$\left((r_0, r_1, \dots, r_n), (x_0, x_1, \dots, x_n) \right) \mapsto \sum_{i=0}^n r_i x_i: \Delta \times (\overline{\mathbb{R}}_+^n)^{n+1} \rightarrow \overline{\mathbb{R}}_+^n$$

is continuous. The image of $\Delta \times V^{n+1}$ is the convex hull of V .

Thus $\Delta \times K^{n+1}$ is mapped onto K . If U is any neighborhood of K , then there is an open neighborhood V of K such that $\Delta \times V^{n+1}$ is mapped into U . This follows from the continuity of the map and the compactness of Δ and of K . Hence, the convex hull of V as well as its saturation $\uparrow(\text{conv } V)$ are contained in U . As the way-below relation is additive on $\overline{\mathbb{R}}_+^n$, the interior of the convex set $\uparrow(\text{conv } V)$ is convex, too, by 4.10. \square

Proposition 10.2. *Let C be a locally convex semitopological cone. Suppose that K is a compact convex set and that A is a nonempty closed convex set disjoint from K . Then there is a convex open set U including K and disjoint from A .*

Proof. Consider an element v of K . As v is not in A , by Corollary 9.4 of the Separation Theorem, there is a lower semicontinuous linear functional g such that $g(v) > 1$ and $g(y) \leq 1$ for all y in A . So

$$U_g := \{x \mid g(x) > 1\}$$

is an open set containing v . As K is compact we can cover it by a finite collection U_{g_1}, \dots, U_{g_n} of such open sets. The map $\bar{g}: C \rightarrow \overline{\mathbb{R}}_+^n$ defined by:

$$\bar{g}(x) = (g_1(x), \dots, g_n(x))$$

is linear and continuous. So we have that $\bar{g}(A) \subset \downarrow \mathbf{1}$, where $\mathbf{1}$ is the point in $\overline{\mathbb{R}}_+^n$ with all coordinates 1, and $\bar{g}(K)$ is compact, convex, and disjoint from $\downarrow \mathbf{1}$ (any x in K is in some U_{g_i} , so $g_i(x) > 1$, and we have that $\bar{g}(x) \not\leq \mathbf{1}$).

Lemma 10.1 now yields an open convex set V containing $\bar{g}(K)$ disjoint from $\downarrow \mathbf{1}$ and hence from $\bar{g}(A)$. The preimage of V under \bar{g} is an open convex subset U of C containing K and disjoint from A . \square

Remark 10.3. In the preceding proposition as well as the Strict Separation Theorem the hypothesis can be weakened in the sense that K need only be compact in the weak upper topology on C . In the same vein, the conclusion of Proposition 10.2 can be strengthened, as the open convex neighborhood produced is even open in the upper weak topology.

Corollary 10.4. *In a locally convex semitopological cone, every compact convex saturated set is the intersection of its open convex neighborhoods.*

Strict Separation Theorem 10.5. *Let C be a locally convex semitopological cone. Suppose that K is a compact convex set and that A is a nonempty closed convex set disjoint from K . Then there is a lower semicontinuous linear functional f and an r such that $f(b) \geq r > 1 \geq f(a)$ for all b in K and all a in A .*

Proof. By Proposition 10.2 there is an open convex neighborhood U of K disjoint from A . By the Separation Theorem 9.1, we find a lower semicontinuous linear functional f such that $f(a) \leq 1$ for all $a \in A$ and $f(u) > 1$ for all $u \in U$, whence, $f(b) > 1$ for all $b \in K$. Every lower semicontinuous functional has a minimum value on the compact set K . Thus, there is an $r > 1$ such that $f(b) \geq r$ for all $b \in K$. \square

We now have the following strong local convexity properties. The idea for the second part of the proof is due to A. Jung:

Proposition 10.6. *Let C be a locally convex, locally convex-compact topological cone the topology of which is sober. Then every nonempty compact convex saturated subset Q has a neighbourhood basis of compact saturated convex neighbourhoods and a neighbourhood basis of open convex neighbourhoods.*

Proof. Let U be any convex neighborhood of Q . By local convex-compactness and Lemma 4.10(c), we may find a compact convex neighbourhood K of Q contained in U . By Corollary 10.4, Q is the intersection of the convex open sets containing it. Hence, Q is the intersection of its compact convex neighbourhoods. The family of compact convex neighbourhoods of Q is filtered: If K_1 and K_2 are two compact convex neighbourhoods of Q , their intersection $K_1 \cap K_2$ is also a convex neighborhood of Q which, by the above, contains a compact convex neighborhood of Q .

Let U be any open set containing Q . The statement at the end of the previous paragraph and the Hofmann-Mislove theorem (see [12], p. 147) imply that Q has a compact convex saturated neighbourhood K_1 contained in U . (This proves the first claim of our Proposition.) For the same reason, K_1 has a compact convex saturated neighbourhood K_2 contained in U . By induction we obtain an increasing sequence of compact convex saturated sets K_n contained in U such that K_n is in the interior of K_{n+1} . It follows that $V = \bigcup_n K_n$ is an open convex neighbourhood of Q contained in U which establishes the second claim. \square

As a continuous d-cone satisfies the hypotheses of the previous proposition, we obtain [39, Corollary 3.13]:

Corollary 10.7. *Every compact saturated convex set Q in a continuous d-cone C has a neighbourhood basis of compact saturated convex neighbourhoods and a neighbourhood basis of open convex neighbourhoods.*

In this corollary, the compact convex neighborhoods can be chosen to be of the form $\uparrow \text{conv } F$ for a finite set F .

11 The upper or Smyth powercone

As an application of our theory of semitopological cones we present a power space construction and its properties. It is related to the classical Vietoris hyperspace construction. In [39] this construction was carried through for continuous d-cones. There, the motivation came from the semantics of programming languages combining probabilistic and nondeterministic choice. Our motivation for generalising is also linked to the desire to close the gap between classical hyperspace constructions and the domain theoretical ones.

For a topological cone C , we consider the set $\mathcal{S}(C)$ of all nonempty compact convex saturated subsets $Q \subseteq C$. We endow $\mathcal{S}(C)$ firstly with the order of reverse inclusion

$$P \sqsubseteq Q \iff P \supseteq Q$$

Secondly we equip it with an addition, a scalar multiplication and an operation \sqcap defined as follows:

$$\begin{aligned} P \underset{s}{+} Q &= \uparrow(P + Q) \\ r \underset{s}{\cdot} P &= \uparrow(r \cdot P) \\ P \sqcap Q &= \uparrow \operatorname{conv}(P \cup Q) \end{aligned}$$

Thirdly, we define a topology Σ with the following basic open sets:

$$\mathcal{N}_U = \{Q \in \mathcal{S}(C) \mid Q \subseteq U\}, \quad \text{for } U \text{ open in } C.$$

The sets \mathcal{N}_U do indeed form a basis of a topology on $\mathcal{S}(C)$, since $\mathcal{N}_U \cap \mathcal{N}_V = \mathcal{N}_{U \cap V}$. Let us note also that the operations above are well-defined. Indeed, $P + Q$ is compact and convex as it is the image of the compact convex set $P \times Q$ under addition which is continuous and linear as a map from $C \times C$ to C . The saturation $\uparrow(P + Q)$ is compact and convex, too. Concerning scalar multiplication, for $r = 0$, we have $r \underset{s}{\cdot} Q = \uparrow 0$ and $\uparrow 0 = C$ is compact and convex; for $r > 0$, $r \underset{s}{\cdot} P = rP$ is compact convex and saturated, as $x \mapsto rx$ is a cone isomorphism. Finally, $\operatorname{conv}(P \cup Q) = \bigcup_{0 \leq r \leq 1} rP + (1-r)Q$ is the image of the compact convex set $\Delta_1 \times P \times Q$ under the continuous linear map $(r, s, x, y) \mapsto rx + sy: \mathbb{R}_+^2 \times C \times C \rightarrow C$, where $\Delta_1 = \{(r, s) \in \mathbb{R}_+^2 \mid r + s = 1\}$. The saturation of $\operatorname{conv}(P \cup Q)$ is compact and convex, too. Note that we also can write $P \sqcap Q = \bigcup_{0 \leq r \leq 1} (r \underset{s}{\cdot} P \underset{s}{+} (1-r) \underset{s}{\cdot} Q)$. As $r \underset{s}{\cdot} P = rP$ except for $r = 0$, we often will write simply rP instead of $r \underset{s}{\cdot} P$.

Theorem 11.1. *The collection $\mathcal{S}(C)$ with the operations $\underset{s}{+}$ and $\underset{s}{\cdot}$ and the topology Σ is a topological cone with a continuous meet-semilattice operation sqcap satisfying*

$$r \underset{s}{\cdot} (Q \sqcap R) = r \underset{s}{\cdot} Q \sqcap r \underset{s}{\cdot} R, \quad P \underset{s}{+} (Q \sqcap R) = (P \underset{s}{+} Q) \sqcap (P \underset{s}{+} R)$$

Its specialization order coincides with \sqsubseteq .

Proof. We begin with the last assertion. If $P \sqsubseteq Q$, then $P \supseteq Q$ and so every open set containing P also contains Q . Conversely, if $P \not\sqsubseteq Q$, then $P \not\supseteq Q$; thus, there is an element $x \in Q \setminus P$. As $\downarrow x$ is closed and disjoint of P , its complement is an open set containing P but not Q . We have shown that $P \sqsubseteq Q$ iff every open set containing P also contains Q .

For the proof of the continuity of the operations we use the following lemma from general topology (see e.g. [21, Chapter 5, Theorem 12]): If P, Q are compact subsets of spaces X, Y , respectively, and if U' is an open neighbourhood of $P \times Q$ in the product space $X \times Y$, then there are open sets W, V in X, Y , respectively, such that $P \times Q \subseteq W \times V \subseteq U'$.

For the continuity of scalar multiplication on $\mathcal{S}(C)$, we choose a $P \in \mathcal{S}(C)$, an $r \in \mathbb{R}_+$ and an open neighborhood U of rP in C . By the continuity of the scalar multiplication in C , there is an open neighborhood U' of $\{r\} \times P$ such that $sx \in U$ for all $(s, x) \in U'$. Using the topological lemma just indicated, there is an open neighborhood W of r in \mathbb{R}_+ and an open neighborhood V of P in C such that $tx \in U$ for all $s \in W$ and all $x \in V$.

For the continuity of addition, let $P, Q \in \mathcal{S}(C)$. Let $\mathcal{N}_U \subseteq \mathcal{S}(C)$ be basic open with $P \underset{s}{+} Q \in \mathcal{N}_U$. We have $P \underset{s}{+} Q \in \mathcal{N}_U \Leftrightarrow P \underset{s}{+} Q \subseteq U \Leftrightarrow \uparrow(P + Q) \subseteq U \Leftrightarrow P + Q \subseteq U$ since U is open, hence an upper set. By the continuity of addition in C , the set U' of all $(x, y) \in C \times C$ with $x + y \in U$ is open and it contains $P \times Q$. So we can again apply the topological lemma above in order to find

open neighborhoods W and V of P and Q , respectively, such that $W \times V \subseteq U'$. So we get open neighbourhoods \mathcal{N}_W and \mathcal{N}_V of P and Q in $\mathfrak{S}(C)$, respectively, such that, for all $P' \in \mathcal{N}_W$ and all $Q' \in \mathcal{N}_V$, we have $P' +_s Q' \in \mathcal{N}_U$. Hence, addition $+_s$ on $\mathfrak{S}(C)$ is continuous. The continuity of the operation \sqcap is proved along the same lines.

The verification of the cone axioms for the addition and the scalar multiplication on $\mathfrak{S}(C)$ is straightforward. The proof of the distributivity law, $(r + s) \cdot_s P = (r \cdot_s P) +_s (s \cdot_s P)$ uses Lemma 3.3.

Clearly, $P \sqcap Q = \uparrow \text{conv}(P \cup Q)$ is the smallest compact convex saturated set containing P and Q . Thus \sqcap is a meet-semilattice operation for the reverse inclusion order. It remains to show that scalar multiplication and addition distribute over \sqcap . As $P \mapsto r \cdot_s P$ is an order isomorphism for $r > 0$, it preserves the meet operation and we have $r \cdot_s (P \sqcap Q) = r \cdot_s P \sqcap r \cdot_s Q$ as desired. For addition we want to prove the identity $P +_s (Q \sqcap R) = (P +_s Q) \sqcap (P +_s R)$. We do this using that $P \sqcap Q = \bigcup_{0 \leq r \leq 1} (rP +_s (1-r)Q)$ and the cone structure of $\mathfrak{S}(C)$:

$$\begin{aligned} P +_s (Q \sqcap R) &= P +_s \bigcup_{0 \leq r \leq 1} (rQ +_s (1-r)R) \\ &= \bigcup_{0 \leq r \leq 1} (P +_s (rQ +_s (1-r)R)) \\ &= \bigcup_{0 \leq r \leq 1} (rP +_s (1-r)P +_s rQ +_s (1-r)R) \\ &= \bigcup_{0 \leq r \leq 1} (r(P +_s Q) +_s (1-r)(P +_s R)) \\ &= (P +_s Q) \sqcap (P +_s R) \end{aligned}$$

□

Definition 11.2. For any topological cone C , the topological cone $\mathfrak{S}(C)$ of all nonempty compact convex saturated subsets will be called the upper powercone over C .

We have shown that the upper powercone $\mathfrak{S}(C)$ over a topological cone C is a topological meet-semilattice cone in the following sense:

Definition 11.3. A topological cone D will be called a meet-semilattice cone, if any two elements a, b of D have a greatest lower bound $a \wedge b$ with respect to the specialisation order and if the following distributivity laws hold for all $a, b, c \in D$ and all $r \in \mathbb{R}_+$:

$$r(a \wedge b) = ra \wedge rb, \quad c + (a \wedge b) = (c + a) \wedge (c + b)$$

12 The upper powercone monad \mathfrak{S} and its algebras

We want to show now that the assignment $C \mapsto \mathfrak{S}(C)$ extends to an endofunctor of the category TopCone of topological cones and continuous linear maps which is in fact a monad.

We first observe that there is an obvious map

$$\eta_C = (x \mapsto \uparrow x): C \rightarrow \mathfrak{S}(C)$$

which is easily seen to be linear and topologically an embedding. For arbitrary topological cones C and D and every continuous linear map $u: C \rightarrow \mathfrak{S}(D)$ we want to define a *lifted* continuous linear \sqcap -preserving map

$$u^\dagger: \mathfrak{S}(C) \rightarrow \mathfrak{S}(D)$$

in such a way that

$$(K) \quad \eta_C^\dagger = \text{id}_{\mathcal{S}(C)} \quad u^\dagger \circ \eta = u \quad (u^\dagger \circ v)^\dagger = u^\dagger \circ v^\dagger$$

Then, for every continuous linear map $f: C \rightarrow D$ of topological cones, we obtain a continuous linear \sqcap -preserving map $\mathcal{S}(f) = (\eta_D \circ f)^\dagger: \mathcal{S}(C) \rightarrow \mathcal{S}(D)$, thus extending the assignment $C \mapsto \mathcal{S}(C)$ to a functor \mathcal{S} which is a monad over TopCone with unit η_C as above and multiplication $\mu_C = (\text{id}_{\mathcal{S}(C)})^\dagger$ (see e.g. [25]).

Thus, let C and D be topological cones and $u: C \rightarrow \mathcal{S}(D)$ a continuous linear map. For defining u^\dagger we need a lemma:

Lemma 12.1. *For every compact convex subset Q of C ,*

$$u^\dagger(Q) =_{\text{def}} \bigcup_{x \in Q} u(x)$$

is compact, convex and saturated in D .

Proof. As the sets $u(x) \in \mathcal{S}(D)$ are saturated, their union is saturated, too. For the compactness claim, suppose that $u^\dagger(Q) = \bigcup_{x \in Q} u(x)$ is covered by a family of basic open sets $\mathcal{N}(U_i)$, where the U_i are open sets in D . For every $x \in Q$, $u(x)$ belongs to some $\mathcal{N}(U_i)$. As u is continuous, there is a neighborhood V_i of x , such that $u(y) \in \mathcal{N}(U_i)$ for all $y \in V_i$. As Q is compact, it is covered by finitely many of the V_i s, say by V_{i_1}, \dots, V_{i_n} . Then, for every $x \in Q$, $u(x)$ is contained in one of $\mathcal{N}(U_{i_1}), \dots, \mathcal{N}(U_{i_n})$, that is, $u^\dagger(Q)$ is covered by this finite subfamily of the $\mathcal{N}(U_i)$ s.

For the convexity claim, let $y_1, y_2 \in u^\dagger(Q) = \bigcup_{x \in Q} u(x)$. Then $y_i \in u(x_i)$ for some $x_i \in Q$, for $i = 1, 2$. For $0 \leq p \leq 1$, we have $py_1 + (1-p)y_2 \in pu(x_1) + (1-p)u(x_2) = u(px_1 + (1-p)x_2)$ by the linearity of u . As Q is supposed to be convex, $px_1 + (1-p)x_2 \in Q$, whence $py_1 + (1-p)y_2 \in u^\dagger(Q)$. \square

Thus, we may define

$$u^\dagger: \mathcal{S}(C) \rightarrow \mathcal{S}(D)$$

by

$$u^\dagger(Q) = \bigcup_{x \in Q} u(x) \quad \text{for all } Q \in \mathcal{S}(C)$$

Proposition 12.2. *The lifted map u^\dagger is continuous, linear and \sqcap -preserving, and the lifting $u \mapsto u^\dagger$ satisfies the properties (K).*

Proof. u^\dagger is continuous: We take a basic open set \mathcal{N}_V of $\mathcal{S}(D)$, where V is an open set in D . As u is continuous, the preimage $U = u^{-1}(\mathcal{N}_V)$ is open in C . We show that the basic open set \mathcal{N}_U is the preimage of \mathcal{N}_V under the map u^\dagger . Indeed, suppose that $u^\dagger(Q) \in \mathcal{N}_V$, which is the same as $u^\dagger(Q) \subseteq V$. By definition this is equivalent to saying that, for all $x \in Q$, $u(x) \subseteq V$, i.e., $u(x) \in \mathcal{N}_V$, or else $x \in u^{-1}(\mathcal{N}_V) = U$. As this holds for all $x \in Q$, we have $Q \subseteq U$, i.e., $Q \in \mathcal{N}_U$.

u^\dagger is linear: Firstly, $u^\dagger(rQ) = \bigcup_{x \in rQ} u(rx) = \bigcup_{x \in Q} ru(x) = r \cdot \bigcup_{x \in Q} u(x) = r \cdot u^\dagger(Q)$, and secondly, $u^\dagger(P \uparrow_s Q) = u^\dagger(\uparrow(P+Q)) = \bigcup_{x \in \uparrow(P+Q)} u(x) = \bigcup_{x \in P+Q} u(x) = \bigcup_{y \in P, z \in Q} u(y+z) =$

$$\bigcup_{y \in P, z \in Q} u(y) + u(z) = \bigcup_{y \in P} u(y) + \bigcup_{z \in Q} u(z) = u^\dagger(P) + u^\dagger(Q).$$

$$u^\dagger \text{ preserves } \sqcap: \text{ Indeed, } u^\dagger(P \sqcap Q) = u^\dagger\left(\bigcup_{0 \leq p \leq 1} (pP +_s (1-p)Q)\right) = \bigcup_{0 \leq p \leq 1} u^\dagger(pP +_s (1-p)Q) = \bigcup_{0 \leq p \leq 1} pu^\dagger(P) +_s (1-p)u^\dagger(Q) = u^\dagger(P) \sqcap u^\dagger(Q).$$

Let us verify the properties (K): Firstly, $\eta_C^\dagger(Q) = \bigcup_{x \in Q} \eta_C(x) = \bigcup_{x \in Q} \uparrow x = Q$, i.e., $\eta_C^\dagger(Q) = \text{id}_{\mathcal{S}(C)}$. Secondly, $u^\dagger(\eta_C(x)) = \bigcup_{y \in \uparrow x} u(y) = u(x)$, i.e., $u^\dagger \circ \eta_C = u$. Finally, $(u^\dagger \circ v)^\dagger = u^\dagger \circ v^\dagger$, as for every Q we have

$$\begin{aligned} u^\dagger(v^\dagger(Q)) &= \bigcup \{u(y) \mid y \in v^\dagger(Q)\} &= \bigcup \{u(y) \mid y \in \bigcup_{x \in Q} v(x)\} \\ &= \bigcup_{x \in Q} \bigcup \{u(y) \mid y \in v(x)\} &= \bigcup_{x \in Q} u^\dagger(v(x)) \\ &= (u^\dagger \circ v)^\dagger(Q) \end{aligned}$$

□

Remark 12.3. According to the general scheme, for a continuous linear map $f: C \rightarrow D$, the map $\mathcal{S}(f): \mathcal{S}(C) \rightarrow \mathcal{S}(D)$ is explicitly given by

$$\mathcal{S}(f)(Q) = \bigcup_{x \in Q} \uparrow f(x) = \uparrow f(Q)$$

and the monad multiplication $\mu_C: \mathcal{S}\mathcal{S}(C) \rightarrow \mathcal{S}(C)$ by

$$\mu_C(\mathcal{Q}) = \bigcup \{P \mid P \in \mathcal{Q}\}$$

where \mathcal{Q} is a compact convex saturated subset of $\mathcal{S}\mathcal{S}(C)$, that is, a collection \mathcal{Q} of compact convex saturated subsets P of C which itself is compact, convex, and saturated in $\mathcal{S}(C)$.

Let us turn to local convexity properties of the upper powercone.

Proposition 12.4. *Let C be a locally convex topological cone. Then the upper powercone $\mathcal{S}(C)$ is convex- T_0 .*

Proof. In a locally convex topological cone C , every compact convex saturated set Q is the intersection of its open convex neighborhoods by Corollary 10.4. Thus, if P is a compact convex saturated set not contained in Q , there is an open convex set U containing Q , but not P . In $\mathcal{S}(C)$, the basic open set \mathcal{N}_U is convex, it contains Q as an element, but not P . □

Lemma 12.5. *The upper powercone $\mathcal{S}(C)$ over a sober topological cone C is a d -cone with respect to the order \sqsubseteq of reverse inclusion. Its Scott topology is finer than the topology Σ .*

Proof. In a sober space, the intersection of any down-directed family of nonempty compact saturated sets Q_i is again nonempty compact by the Hofmann-Mislove Theorem [12, II-1.22] and (of course) saturated. If all the Q_i are convex, the same holds for their intersection. Thus, $\mathcal{S}(C)$ is directed complete with $\bigsqcup_i Q_i = \bigcap_i Q_i$ for every \sqsubseteq -directed family in $\mathcal{S}(C)$.

Let us show now that the basic open sets \mathcal{N}_U are Scott-open. Indeed, if $\bigcap_i Q_i \in \mathcal{N}_U$ for a down-directed family in $\mathcal{S}(C)$, then $\bigcap_i Q_i \subseteq U$ which implies that $Q_i \subseteq U$ for some i , as every sober space is well-filtered (see [12, II-1.21]).

The Scott continuity of scalar multiplication, addition and the operation \sqcap in $\mathcal{S}(C)$ follows from the continuity of these operations for the topology Σ and from the fact that every \sqsubseteq -directed family $(Q_i)_i$ converges to $\bigsqcup_i Q_i = \bigcap_i Q_i$ for the topology Σ by the Hofmann-Mislove theorem. \square

For the following results we need the full strength of our Separation Theorems:

Theorem 12.6. *Let C be a locally convex, locally convex-compact topological cone the topology of which is sober. Then the upper powercone $\mathcal{S}(C)$ is a continuous \sqcap -semilattice d-cone with respect to the reverse inclusion order \sqsupseteq . For P and Q in $\mathcal{S}(C)$, one has $P \ll Q$ iff P contains Q in its interior. The Scott topology coincides with the powercone topology Σ .*

Proof. In Lemma 12.5 we have seen that $\mathcal{S}(C)$ is a d-cone with respect to \sqsupseteq because of soberness.

If we suppose that C is locally convex and locally convex-compact then Proposition 10.6 tells us that, for every open set U containing a $Q \in \mathcal{S}(C)$, there is a compact convex saturated set $P \subseteq U$ with the property that Q is contained in the interior of P . Thus, in the space $\mathcal{S}(C)$, Q has a neighborhood basis of sets of the form $\uparrow_{\mathcal{S}(C)} P$. This shows that $(\mathcal{S}(C), \Sigma)$ is a c-space. But every c-space which is directed complete with respect to the specialisation order is a continuous domain; its topology coincides with the Scott topology and the topological way-below relation coincides with the order theoretic one (see [9, Theorem 4]). Thus, $P \ll Q$ iff $\uparrow_{\mathcal{S}(C)} P$ is a neighbourhood of Q , i.e., iff Q is contained in the interior of P . \square

As every continuous d-cone is locally convex and locally convex-compact, the theorem yields:

Corollary 12.7. *For every locally convex, locally convex-compact sober topological cone C , the upper powercone $\mathcal{S}(C)$ is locally convex and locally convex-compact.*

It is an open problem to characterise the algebras of the upper powercone monad \mathcal{S} over the category TopCone of topological cones and continuous linear maps. We can give an answer to the problem by restricting to the full subcategory LCCone of locally convex, locally convex-compact sober topological cones. The previous theorem and its corollary show that \mathcal{S} restricts to an endofunctor, hence a monad, on LCCone . By the following universal property the algebras of the monad \mathcal{S} over the category LCCone are identified to be the continuous \wedge -semilattice d-cones:

Theorem 12.8. *Let C be a locally convex, locally convex-compact sober topological cone. Further, let D be any continuous \wedge -semilattice d-cone. Then, for every continuous linear map $u: C \rightarrow D$ there is a unique continuous linear map $u^\dagger: \mathcal{S}(C) \rightarrow D$ preserving binary meets such that $u = u^\dagger \circ \eta$; the map u^\dagger is defined by $u^\dagger(Q) = \bigwedge_{x \in Q} u(x)$ for every $Q \in \mathcal{S}(C)$.*

$$\begin{array}{ccc}
 C & \xrightarrow{\eta_C} & \mathcal{S}(C) \\
 & \searrow u & \downarrow u^\dagger \\
 & & D
 \end{array}$$

Proof. We use the following Lemma [39, 4.21]: In a continuous \wedge -semilattice every compact subset has a meet, and a Scott-continuous map of continuous \wedge -semilattices preserving binary meets also preserves meets of compact sets.

By the lemma, each compact subset of D has a meet. Thus, a map u^\dagger is well-defined by $u^\dagger(Q) = \bigwedge_{x \in Q} u(x)$ for every $Q \in \mathcal{S}(C)$. Moreover, $u^\dagger(\eta_C(x)) = u^\dagger(\uparrow x) = \bigwedge_{y \in \uparrow x} u(y) = u(x)$, whence $u = u^\dagger \circ \eta$. Let us show that u^\dagger is continuous. Indeed, let $Q \in \mathcal{S}(C)$ and $y = u^\dagger(Q)$. For every $z \ll y$, the set $\uparrow z$ is an open neighborhood of y . By the continuity of u , its inverse image $U = u^{-1}(\uparrow z)$ is open and contains Q . Thus \mathcal{N}_U is a neighborhood of Q in $\mathcal{S}(C)$. And for every $P \in \mathcal{N}_U$ we have $P \subseteq U$, whence $u(P) \subseteq \uparrow z$ which implies $u^\dagger(P) = \bigwedge_{x \in P} u(x) \geq z$.

By definition, the maps $x \mapsto a + x$ and $x \mapsto rx$ from D into itself preserve binary meets. From the Lemma just cited we conclude that these maps preserve meets of compact sets. This implies the linearity of the map u^\dagger . This map also preserves binary meets. Indeed, let $P, Q \in \mathcal{S}(C)$. Then $P \sqcap Q = \uparrow \text{conv}(P \cup Q)$. Thus $u^\dagger(P \sqcap Q) = \bigwedge u(\uparrow \text{conv}(P \cup Q)) = \bigwedge u(\text{conv}(P \cup Q)) = \bigwedge u(P \cup Q)$, where the last equality holds because upper sets of points are convex.

It only remains to justify the uniqueness of u^\dagger . So, let $v: \mathcal{S}(C) \rightarrow D$ be any Scott-continuous map preserving finite meets such that $v \circ \eta_C = u$. For $x \in C$ we then have $v(\uparrow x) = v(\eta_C(x)) = u(x)$. Take any $Q \in \mathcal{S}(C)$. Then the set $\mathcal{Q} = \{\uparrow x \mid x \in Q\}$ is a compact subset of $\mathcal{S}(C)$ and $Q = \bigcap_{x \in \mathcal{Q}} \uparrow x$. By the lemma cited above, v preserved infima of compact sets. Hence $v(Q) = v(\bigcap_{x \in \mathcal{Q}} \uparrow x) = \bigwedge_{x \in \mathcal{Q}} v(\uparrow x) = \bigwedge_{x \in Q} u(x) = u^\dagger(Q)$. \square

Our category LCCone contains the category of continuous d-cones. Thus, we have generalised the results in [39, Section 4.2] from the category of continuous d-cones to the category LCCone . The algebras of the monad \mathcal{S} turn out to be same in both cases. The generalisation is relevant as we catch an important case not covered by [39]:

Recall that a topological space is *stably locally compact* if it is locally compact and sober and if the intersection of any two compact saturated subsets is sober. For such a space X , the extended probabilistic powerdomain $\mathcal{V}(X)$ of continuous valuations with the weak*upper topology is a locally convex, locally convex-compact, stably compact cone (see [2]). In particular, the cone $\mathcal{V}(X)$ belongs to our category LCCone . But $\mathcal{V}(X)$ is not a continuous d-cone, if X is not a continuous domain.

Let us note that, for a locally convex, locally convex-compact stably compact d-cone C , the upper cone $\mathcal{S}(C)$ is a lattice, hence a continuous lattice d-cone by 12.6. This applies in particular to the the extended probabilistic powerdomain $\mathcal{V}(X)$ over a stably locally compact space.

Combining the monad \mathcal{V} with the monad \mathcal{S} , we obtain a monad $\mathcal{S}\mathcal{V}$ over the category of stably locally compact spaces. This combined monad can serve for a semantics of languages combining probability and nondeterminism over stably compact state spaces (compare [26, 39]). One would like to characterise the algebras of the combined monad $\mathcal{S}\mathcal{V}$ over the category of stably locally compact spaces. By our results in this section we have solved this problem modulo a characterisation of the algebras of the monad \mathcal{V} over stably locally compact spaces which is still an open problem. A solution of this latter problem would include two nontrivial results: firstly, the characterisation of the algebras of the monad of probability measures on compact Hausdorff spaces as compact convex sets in locally convex topological vector spaces (see [35]); secondly the characterisation of the algebras of the extended probabilistic powerdomain monad \mathcal{V} over stably compact continuous domains as stably compact continuous d-cones (see [39]).

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