Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic*

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Abstract

In his foundation of probability theory, Bruno de Finetti devised a betting scheme where a bookmaker offers bets on the outcome of events ϕ occurring in the future. He introduced a criterion for coherent bookmaking, and showed that coherent betting odds are given by some probability distribution. While de Finetti dealt with yes-no events and boolean propositional logic, Mundici generalized the theory to the continuous spectrum events formalized within Łukasiewicz logic.

Both de Finetti and Mundici assume that the bookmaker/bettor roles can be interchanged. In this paper we deal with a more realistic situation, dropping the reversibility assumption. Working in the framework of Łukasiewicz logic, we introduce a coherence criterion for non-reversible bookmaking. Our main tool is given by 'imprecise probabilities', which are formulated in terms either of compact convex sets of probabilities or equivalently in terms of suitable sublinear functionals (see Section 5). Our main result is Theorem 8.3 which states that our coherence criterion arises from imprecise probabilities just as de Finetti's criterion arises from probabilities.

Throughout, we will work with MV-algebras. They play the same role for Łukasiewicz logic as Boolean algebras play for classical logic. Unital abelian lattice-ordered groups will provide an intermediate structure: while being categorically equivalent to MV-algebras, they are more akin to the Banach space $C(X)$. Functional analytic methods, developed in Section 6, are used for the proof of our main result.

1 Introduction

De Finetti [8], see also [7], suggested the following approach to probability: the probability of an event ϕ (whose outcome is unknown now and will be known later) is a number α between 0 and 1 that a rational and reversible bookmaker (called *Ada* in the sequel) would propose for the following bet: a bettor (called *Blaise* in the sequel) bets a real number λ and pays $\lambda\alpha$ to Ada now. If ϕ turns out to be true, then he will get back λ from Ada, and if ϕ turns out to be false, then he will get nothing. Denoting the possible truth values $v(\phi) = 1$ and 0 for 'true' and 'false', respectively, the return for Blaise at the end of the transaction will be $\lambda(v(\phi) - \alpha)$. Ada may accept bets on different events $\phi_1, \ldots, \phi_n$. In this case, she chooses a finite set $\Gamma = \{(\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)\}$, where, for $i = 1, \ldots, n$, ϕ_i is an event and α_i is Ada's *betting odd*, that is, the ratio that Ada chooses for a bet on ϕ_i of the form described above. Such a finite set Γ is called a *book*. We will write $(\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ instead of $\{(\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)\}$. If, for $i = 1, \ldots, n$, Blaise bets λ_i on ϕ_i, and the truth value of ϕ_i is $v(\phi_i)$, then the return for Blaise will sum up to $\sum_{i=1}^{n} \lambda_i(v(\phi_i) - \alpha_i)$. This game may be extended to infinite books: given an

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infinite set Δ of pairs (ϕ, α), where ϕ is an event and α is a real number between 0 and 1, Blaise chooses a finite subset $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ of Δ and then the game proceeds as above.

We point out that in this game Ada is assumed to be a *reversible* bookmaker, that is, if Blaise thinks that Ada's betting odd α for the event ϕ is too high, he can reverse the terms of the bet just betting a negative amount λ of money on ϕ. Since paying $\lambda < 0$ is the same as receiving $-\lambda$, a negative stake amounts to swapping the roles of bookmaker and bettor. Games where the bookmaker is reversible will be called *reversible betting games* in the sequel.

De Finetti's well known coherence criterion is as follows: a book Δ is said to be *rational* or *coherent* if there is no *winning strategy* or *Dutch Book* for Blaise, that is, if there is no finite subset $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ of Δ and no system of bets $\lambda_1, \ldots, \lambda_n$ on $\phi_1, \ldots, \phi_n$ such that Blaise's payoff $\sum_{i=1}^{n} \lambda_i(v(\phi_i) - \alpha_i)$ is positive regardless of the truth values $v(\phi_1), \ldots, v(\phi_n)$.

Stated informally, de Finetti [8] proved the following:

Proposition 1.1. *A book Δ is coherent if and only if there is a probability distribution P on events such that, if $(\phi, \alpha) \in \Delta$, then $P(\phi) = \alpha$.*

To be precise, events are supposed to be described by propositions, that is, by terms of propositional logic up to logical equivalence; they form a Boolean algebra, the Tarski-Lindenbaum algebra. More generally, events may be considered as elements of some Boolean algebra $\mathbf{B}$. The possible truth values are given by the homomorphisms v of the Boolean algebra $\mathbf{B}$ onto the two element Boolean algebra; we call them *valuations*. Equipped with the topology of pointwise convergence, they form a compact Hausdorff space, the Stone space $X_{\mathbf{B}}$ of $\mathbf{B}$. Probability distributions over $\mathbf{B}$ can be described in various equivalent ways:

(i) normalized additive maps from $\mathbf{B}$ to the unit interval;
(ii) regular Borel probability measures on the Stone space $X_{\mathbf{B}}$;
(iii) normalized positive linear functionals on the vector lattice $C(X_{\mathbf{B}})$ of all continuous real-valued functions on $X_{\mathbf{B}}$.

The goal of the present paper is to extend both de Finetti's criterion and the various characterizations of 'probability distributions' to non-reversible betting games and to many-valued events as considered in Łukasiewicz logic.

Our starting point will be Mundici's result [18], which generalizes de Finetti's considerations to many-valued events, but remaining in the setup of reversible games. In this context, events will be equivalence classes of formulas modulo equivalence in Łukasiewicz logic or in a consistent theory T of Łukasiewicz logic. Thus, events may be regarded as elements of an MV-algebra $\mathbf{A}$. A *valuation* will be a homomorphism from $\mathbf{A}$ into the unit interval considered as an MV-algebra $\mathbf{I}$ with the operations $\oplus$ and $\neg$ defined by $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$. Given a finite book $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$, the rules of the game are as before, with the only difference that $\phi_1, \ldots, \phi_n$ are now events represented by formulas of Łukasiewicz logic and can take values in the real interval $[0, 1]$. Once again, if, for $i = 1, \ldots, n$, Blaise bets λ_i on ϕ_i and v is any valuation, then the return for Blaise will be $\sum_{i=1}^{n} \lambda_i(v(\phi_i) - \alpha)$. As for classical events, the *coherence criterion* is the nonexistence of a *Dutch Book*, that is, of a system of bets $\lambda_1, \ldots, \lambda_n$ on $\phi_1, \ldots, \phi_n$ such that the return $\sum_{i=1}^{n} \lambda_i(v(\phi_i) - \alpha_i)$ for Blaise is strictly positive regardless of the valuation v. For his generalization of de Finetti's theorem Mundici [18] uses *states*, the latter being the natural generalization of additive normalized $[0, 1]$-valued maps on Boolean algebras. In [18] he proves:

Proposition 1.2. *Let $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ be a finite book over events represented by formulas of Łukasiewicz logic. The following are equivalent:*

(1) *There is a state μ on the Lindenbaum algebra of Łukasiewicz logic such that $\mu(\phi_i) = \alpha_i$ for $i = 1, \ldots, n$.*
(2) *Γ is coherent in the sense explained above, that is, there is no system of bets $\lambda_1, \ldots, \lambda_n$ on $\phi_1, \ldots, \phi_n$ such that Blaise's payoff $\sum_{i=1}^{n} \lambda_i(v(\phi_i) - \alpha_i)$ is strictly positive for every valuation v.*

The same result holds for infinite books, where an infinite book Δ is said to be coherent if every finite subset Γ of Δ is coherent.

As in the classical case of Boolean algebras, the states of an MV-algebra $\mathbf{A}$ have the following characterizations (see [14, 20, 15]). Here $X_{\mathbf{A}}$ denotes the space of all valuations, that is, the set of all MV-algebra homomorphisms v from $\mathbf{A}$ to $\mathbf{I}$ with the topology of pointwise convergence:

(i) normalized additive maps from $\mathbf{A}$ to the unit interval;
(ii) regular Borel probability measures on the space $X_{\mathbf{A}}$;
(iii) normalized positive linear functionals on the vector lattice $C(X_{\mathbf{A}})$.

In this paper, we extend these results to non-reversible betting games using imprecise probabilities.

There are reasons which suggest the use of non-reversible games, (see [12, 24, 25, 1]). Games in which only nonnegative bets λ are allowed will be called *non-reversible betting games.* The denomination 'non-reversible' is a bit misleading, as reversible games may be simulated by games with nonnegative bets λ. Indeed, if the book, besides a betting odd α for ϕ, contains a betting odd $1-\alpha$ for $\neg\phi$, then the game becomes equivalent to a reversible game: betting $\lambda < 0$ on ϕ in a reversible betting game is equivalent to betting $-\lambda$ on $\neg\phi$ in the sense that Blaise receives the same payoff in both cases.

Reversible betting games or, equivalently, non-reversible games with a book containing both (ϕ, α) and $(\neg\phi, 1-\alpha)$ are problematic for the following reason: In many situations the bookmaker may be uncertain about the probability α of an event ϕ. If she chooses α too high, the book will be unattractive to the bettor; if she fixes α too low, she risks to lose a lot of money. Thus, the bookmaker may want to choose an 'imprecise' probability for ϕ, modelled by an interval $[\beta, \alpha]$, $\beta < \alpha$, of probabilities. In her book she will offer (ϕ, α) and $(\neg\phi, 1-\beta)$ so that the book remains attractive for the bettor and the risk of losing big amounts is decreased. Such books are no longer covered by de Finetti's or Mundici's approach.

In the case of non-reversible games the nonexistence of a winning strategy for Blaise is a necessary but not sufficient condition for coherence. For instance, if both ϕ and ψ may be falsified by the same valuation, then Blaise has no winning strategy for the book $\Gamma = (\phi, \frac{1}{3}), (\neg\phi, 1), (\psi, \frac{1}{3}), (\neg\psi, 1), (\phi \vee \psi, 1), (\neg(\phi \vee \psi), 1)$, because if both ϕ and ψ are false, then Blaise cannot win anything. However, this book does not look rational: indeed, two bets on ϕ and on ψ are more advantageous for Blaise than a single bet on $\phi \vee \psi$, and Ada might make her book more attractive for Blaise by reducing her betting odd on $\phi \vee \psi$, e.g., to $\frac{2}{3}$, without any loss of money if Blaise plays his best strategy.

More precisely, a *bad bet* is one for which there is a system of bets which gives Blaise a strictly better payoff regardless of the truth values of the events involved. In our case, a bad bet is betting 1 on $\phi \vee \psi$ and a better system of bets, which gives Blaise a better payoff in any case, is betting 1 on both ϕ and ψ.

In this paper the nonexistence of bad bets will be chosen as the coherence criterion for non-reversible games. It turns out that coherent books can be modelled by 'imprecise probabilities".

Prima facie, imprecise probabilities may be modelled by sets of probabilities. Probabilities on Boolean algebras are replaced by states on MV-algebras in the many-valued setting. Thus, an imprecise state on an MV-algebra $\mathbf{A}$ can be viewed as a set K of states. Imprecise states in this sense allow to extend Mundici's result to non-reversible games (see Theorem 8.3):

For a book Δ on any MV-algebra, the following conditions are equivalent:
(1) *There is no bad bet based on Δ.*
(2) *There is a set K of states such that $\alpha = \max\{\mu(\phi) : \mu \in K\}$ for every $(\phi, \alpha) \in \Delta$.*

This investigation is preceded by a general discussion in which we compare the nonexistence of a bad bet with other rationality criteria, which are equivalent in the case of reversible betting games. We show that one of these criteria becomes too strong in the non-reversible case, in the sense that it is not reasonable to expect it to hold, and that another criterion becomes too weak, in the sense that it does not ensure the rationality of the book. This argument supports our choice of nonexistence of bad bets as our rationality criterion.

Changing the perspective, there is a dual coherence criterion that requires to avoid 'good bets'. Formally one just has to interchange the roles of $<$ and $>$ and simultaneously max and min.

Calling any set of states (probabilities) an 'imprecise state (probability)' may not be completely appropriate, since very different sets of states represent the same 'imprecise state': To any nonempty set K of states we assign an *upper state* $K^*\colon \mathbf{A} \to [0,1]$ and a lower state $K_*\colon \mathbf{A} \to [0,1]$ defined by $K^*(a) = \sup\{\mu(a) : \mu \in K\}$

and $K_*(a) = \inf\{\mu(a) : \mu \in K\}$ for all $a \in \mathbf{A}$. Lower and upper states are conjugates of one another, where the conjugate $u^\dagger$ of a map $u\colon \mathbf{A} \to [0,1]$ is defined by $u^\dagger(x) = 1 - u(\neg x)$.

Different sets of states may yield the same upper [lower] state. For every upper state u, the set u^* of all states $\mu \leq u$ is convex and compact in the topology of pointwise convergence. As a matter of fact, it is the largest set of states such that $K^* = u$. This correspondence establishes a bijection between convex compact sets of states and upper states, and dually for lower states. In the theorem above, the set K of states can be chosen convex and compact. Using the terminology of upper states, it can be rephrased as follows:

For a book Δ on an MV-algebra The following conditions are equivalent:
(1) *There is no bad bet based on Δ.*
(2) *There is an upper state u such that $u(\phi) = \alpha$ for all $(\phi, \alpha) \in \Delta$.*

There are three equivalent ways to describe imprecise probabilities on MV-algebras (see Section 7):

(i) upper (resp., lower) states, that is, pointwise joins (resp., pointwise meets) of sets of states;
(ii) compact convex sets of regular Borel probability measures on the space $X_{\mathbf{A}}$ of valuations;
(iii) strongly normalized positive sublinear (resp., superlinear) functionals on the vector lattice $C(X_{\mathbf{A}})$.

The description of upper states as pointwise suprema of states is not satisfactory. One would like to characterize them by a few natural axioms. We do not have such an axiomatic characterization except for 2-divisible MV-algebras, that is, MV-algebras in which we can multiply any element by $\frac{1}{2}$ (see Proposition 5.13).

A main tool used in the paper is the Yosida representation of an MV-algebra by continuous real-valued functions on the valuation space. As a surprise to the authors, the coherence criterion of nonexistence of bad bets, when transferred to the function space representing the MV-algebra, turned out to correspond to Walley's coherence criterion for upper previsions on a set of gambles (see [24, Section 2.5]). Thus, our approach yields a novel justification for Walley's choice for coherence. At the other hand we substantially make use of Walley's *natural extension* which guarantees that coherent upper previsions can be extended to sublinear functionals dominated by the sup-functional (see [24, Section 3.3] and 6.8).

Another main tool is the embedding of MV-algebras into abelian lattice-ordered groups with a distinguished strong order unit, an embedding that can be extended to an equivalence between the category of MV-algebras and unital abelian lattice-ordered groups with their respective homomorphisms (see, for example, [6]). Despite the equivalence of categories, lattice-ordered groups show a better behaviour then MV-algebras. They allow a smooth characterization of upper and lower states without any divisibility hypothesis (see Definition 5.5 and Theorem 7.4).

Terminological Remark: We use the *functional* as a generic term denoting real-valued maps defined on lattice ordered groups. This usage has its origins in analysis, where it was first used for real-valued maps defined on function spaces, later on generalized to real-valued functions defined on vector spaces.

2 Preliminaries

For all concepts of universal algebra we refer to [4]. For concepts of many-valued logic, we refer to [11] and, for MV-algebras in particular, we also refer to [6]. For lattice-ordered groups we refer to [3]. For the basic functional analytical tools used in this paper see [22, 21].

The central algebraic structures used in this paper are MV-algebras. For *Lukasiewicz logic* (see [11] for an axiomatization) they play the same role as Boolean algebras play for classical propositional logic.

Definition 2.1. *An* MV-algebra *is an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is an involutive unary operation on A, $\mathbf{1} = \neg 0$ is an absorbing element, that is, $a \oplus \mathbf{1} = \mathbf{1}$, and the identities $\neg\neg a = a$ and $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$ hold.*

A homomorphism *$H : \mathbf{A} \to \mathbf{A}'$ of MV-algebras is a map satisfying $H(0) = 0$, $H(a \oplus b) = H(a) \oplus H(b)$ and $H(\neg a) = \neg H(a)$ for all $a, b \in \mathbf{A}$.*

In any MV-algebra $\mathbf{A}$, we further define

$$a \odot b = \neg(\neg a \oplus \neg b), \;\; a \vee b = a \oplus (b \odot \neg a), \;\; a \wedge b = a \odot (\neg a \oplus b).$$

An MV-algebra $\mathbf{A}$ is a distributive lattice with respect to $\vee$ and $\wedge$ with top element $\mathbf{1}$ and bottom element 0.

MV-algebras generalize Boolean algebras and are closely related to abelian lattice-ordered groups:

Definition 2.2. *An* abelian lattice-ordered group *(shortly, an* abelian ℓ-group*) is an algebra* $\mathbf{G} = (G, +, 0, \vee, \wedge)$ *such that* $(G, +, 0)$ *is a commutative group and* $(G, \vee, \wedge)$ *a lattice such that* $f + (g \vee h) = (f + g) \vee (f + h)$ *holds for all* $f, g, h \in G$*. The* positive cone *of an abelian* ℓ*-group is the set* $\mathbf{G}_+$ *of elements* $g \geq 0$ *in* $\mathbf{G}$*. We write* $n \cdot g$ *or simply* ng *for the* n*-fold sum* $g + \cdots + g$*. A* vector lattice *is an abelian* ℓ*-group which is also a vector space over the reals such that* $rg \geq 0$ *for all* $r \in \mathbb{R}_+$*,* $g \in \mathbf{G}_+$*.*

An element $\mathbf{1}$ *of an abelian* ℓ*-group* $\mathbf{G}$ *is a* strong order unit *if, for every* $g \in \mathbf{G}$*, there is a natural number* n *such that* $g \leq n \cdot \mathbf{1}$*. An* ℓ*-group* $\mathbf{G}$ *with a distinguished strong order unit* $\mathbf{1}$ *will be called a* unital ℓ*-group.*

Example 2.3. The reals $\mathbb{R}$ with the usual addition and lattice operations $r \vee s = \max(r, s)$ and $r \wedge s = \min(r, s)$ form an abelian ℓ-group with 1 as a strong order unit.

Example 2.4. For a compact Hausdorff space X, the continuous real-valued functions defined on X form an abelian ℓ-group, even a vector lattice, $C(X)$, with the constant function $\mathbf{1}$ as a strong order unit. The operations $f + g$, $f \vee g$, $f \wedge g$ are defined pointwise: $(f + g)(x) = f(x) + g(x)$, $(f \vee g)(x) = \max(f(x), g(x))$, $(f \wedge g)(x) = \min(f(x), g(x))$ for all $x \in X$. We will always consider $C(X)$ as a unital abelian ℓ-group with $\mathbf{1}$ as the distinguished strong order unit.

Definition 2.5. *Let* $\mathbf{G}$ *and* $\mathbf{G}'$ *be unital* ℓ*-groups. A map* $H \colon \mathbf{G} \to \mathbf{G}'$ *is said to be*
- *a* homomorphism *if* $H(g + h) = H(g) + H(h)$ *for all* $g, h \in \mathbf{G}$*,*
- *an* ℓ*-homomorphism if, in addition,* $H(g \vee h) = H(g) \vee H(h)$ *for all* $g, h \in \mathbf{G}$*,*
- unital *or* normalized *if* $H(\mathbf{1}) = \mathbf{1}$*,*
- positive *if* $H(g) \geq 0$ *for all* $g \geq 0$ *in* $\mathbf{G}$*.*

The connection between unital abelian ℓ-groups and MV-algebras can be described as follows according to [6, Chapter 2]:

Let $\mathbf{G}$ be a unital abelian ℓ-group. Consider the interval $[0, \mathbf{1}] = \{g \in \mathbf{G} : 0 \leq g \leq \mathbf{1}\}$ and, for $g, h \in [0, \mathbf{1}]$, define

$$\neg g = \mathbf{1} - g, \quad g \oplus h = (g + h) \wedge \mathbf{1}.$$

Then $\Gamma(\mathbf{G}) = ([0, \mathbf{1}], \oplus, \neg, 0)$ is an MV-algebra. The derived operations on $\Gamma(\mathbf{G})$ are the lattice operations $\vee$ and $\wedge$ on $\mathbf{G}$ restricted to $[0, \mathbf{1}]$ and $g \odot h = (g + h - \mathbf{1}) \vee 0$.

Any unital ℓ-homomorphism $H \colon \mathbf{G} \to \mathbf{G}'$ of unital abelian ℓ-groups maps the interval $[0, \mathbf{1}] \subseteq \mathbf{G}$ into the interval $[0, \mathbf{1}] \subseteq \mathbf{G}'$ and its restriction $\Gamma H = H|_{[0,\mathbf{1}]} \colon \Gamma(\mathbf{G}) \to \Gamma(\mathbf{G}')$ is a homomorphism of MV-algebras. Thus, Γ becomes a functor from the category of unital abelian ℓ-groups and unital ℓ-homomorphisms to the category of MV-algebras and MV-algebra homomorphisms. It is an important fact, that Γ establishes an equivalence between the two categories. In particular, we will rely on the following(see [6, Chapter 2], [17, 3.1(i)]):

Theorem 2.6. *For every MV-algebra* $\mathbf{A}$ *there is a unital abelian* ℓ*-group* $\mathbf{G}_\mathbf{A}$*, unique up to isomorphism, such that* $\mathbf{A}$ *is isomorphic to* $\Gamma(\mathbf{G}_\mathbf{A})$*. The elements of* $\mathbf{G}_\mathbf{A}$ *can be written in the form* $\sum_{i=k}^{n} \varepsilon_k a_k$ *with* $a_k \in \mathbf{A}$ *and* $\varepsilon_k = \pm 1$ *for* $k = 1, \ldots, n$*.*

For any two unital abelian ℓ*-groups* $\mathbf{G}$ *and* $\mathbf{G}'$*, restriction* $H \mapsto H|_{[0,\mathbf{1}]}$ *yields a one-to-one correspondence between the unital* ℓ*-homomorphisms* $H \colon \mathbf{G} \to \mathbf{G}'$ *and the MV-algebra homomorphisms* $\Gamma(\mathbf{G}) \to \Gamma(\mathbf{G}')$*.*

Henceforth, every MV-algebra $\mathbf{A}$ will be tacitly considered to be embedded into its *enveloping* unital abelian ℓ-group $\mathbf{G}$; in symbols, $\mathbf{A} = \Gamma(\mathbf{G})$ with $\mathbf{G} = \mathbf{G}_\mathbf{A}$ as given by Theorem 2.6. For elements $a, b \in \mathbf{A}$, we can form the sum $a + b$ in $\mathbf{G}$ and we have $a + b = a \oplus b$ provided that $a + b \in \mathbf{A}$. In this way, we will use $+$ as a partial operation on $\mathbf{A}$. One can check internally in the MV-algebra $\mathbf{A}$ whether $a + b$ is defined: $a + b$ is defined in $\mathbf{A}$ if and only if $a \odot b = 0$.

Example 2.7. Of particular interest is the MV-algebra $\Gamma(\mathbb{R})$ obtained from the ℓ-group $\mathbb{R}$ of the reals with 1 as a strong unit: it is the real unit interval $[0, 1]$ with the operations $r \oplus s = \min\{r + s, 1\}$ and $\neg r = 1 - r$ denoted shortly by

$$\mathbf{I} = ([0, 1], \oplus, \neg, 0).$$

Example 2.8. The MV-algebra obtained from the unital ℓ-group $C(X)$ of real-valued continuous functions on a compact Hausdorff space with the constant function $\mathbf{1}$ as a strong order unit is $C(X, \mathbf{I})$, the set of all continuous functions $g\colon X \to \mathbf{I}$ with the operations $g \oplus h = (g + h) \wedge \mathbf{1}$ and $\neg g = \mathbf{1} - g$.

By Chang's completeness theorem (see [5] or [6]), the class of MV-algebras coincides with the variety generated by the standard MV-algebra $\mathbf{I}$. Actually, MV-algebras are generated by $\mathbf{I}$ as a quasivariety. Thus, in order to check if an equation, or a quasi-equation, is valid in all MV-algebras, it is sufficient to check it in $\mathbf{I}$. We will tacitly use this fact in the sequel.

3 Valuations

In this section, let $\mathbf{A}$ be an MV-algebra and $\mathbf{G}$ the enveloping unital abelian ℓ-group, that is, $\mathbf{A} = \Gamma(\mathbf{G})$.

Definition 3.1. *A* valuation *on* $\mathbf{A}$ *is an MV-algebra homomorphism* $v\colon \mathbf{A} \to \mathbf{I}$*. A* valuation *on* $\mathbf{G}$ *is an* ℓ*-homomorphism* $v\colon \mathbf{G} \to \mathbb{R}$ *with* $v(\mathbf{1}) = 1$*.*

We denote by $X_\mathbf{A}$ *and* $X_\mathbf{G}$ *the sets of all valuations on the MV-algebra* $\mathbf{A}$ *and the unital* ℓ*-group* $\mathbf{G}$*, respectively.*

Notice that $X_\mathbf{A}$ is a subset of $\mathbf{I}^\mathbf{A}$, the set of all functions from $\mathbf{A}$ to $\mathbf{I}$. We endow the unit interval $\mathbf{I}$ with the usual compact Hausdorff topology and $X_\mathbf{A}$ with the topology of pointwise convergence, that is, the topology induced by the product topology on $\mathbf{I}^\mathbf{A}$.

Similarly, $X_\mathbf{G}$ as a subset of $\mathbb{R}^\mathbf{G}$, the set of all functions from $\mathbf{G}$ to $\mathbb{R}$, will also be equipped with the topology of pointwise convergence.

FACT 3.2. $X_\mathbf{A}$ *and* $X_\mathbf{G}$ *are compact Hausdorff spaces for the respective topologies of pointwise convergence and* $v \mapsto v|_\mathbf{A}\colon X_\mathbf{G} \to X_\mathbf{A}$ *is a homeomorphism.*

Proof. We first notice that the map $v \mapsto v|_{[0,\mathbf{1}]}\colon X_\mathbf{G} \to X_\mathbf{A}$ is a bijection by Theorem 2.6.

Recall that $\mathbf{I}^\mathbf{A}$ is compact Hausdorff with respect to the product topology by the Tychonoff Theorem. We claim that $X_\mathbf{A}$ is a closed subset of $\mathbf{I}^\mathbf{A}$, hence, a compact Hausdorff space with respect to the topology of pointwise convergence. Indeed, let $(v_i)_i$ be a generalized sequence (a net) of valuations on $\mathbf{A}$ converging pointwise to a function v. Then $v(\mathbf{1}) = \lim_i v_i(\mathbf{1}) = 1$ and, for every $x \in \mathbf{A}$, we have $v(\neg x) = \lim_i v_i(\neg x) = \lim_i(1 - v_i(x)) = 1 - \lim_i v_i(x) = 1 - v(x)$ and $v(x \oplus y) = \lim_i v_i(x \oplus y) = \lim_i(v_i(x) \oplus v_i(y)) = \lim_i v_i(x) \oplus \lim_i v_i(y) = v(x) \oplus v(y)$. Thus, v is a valuation too. We just have used that the operations $\oplus$ and $\neg$ are continuous on the unit interval.

In order to prove that $v \mapsto v|_\mathbf{A}\colon X_\mathbf{G} \to X_\mathbf{A}$ is a homeomorphism, we consider a generalized sequence of valuations (v_i) on $\mathbf{G}$. We claim: If $v_i(x)$ converges to $v(x)$ for every $x \in [0, \mathbf{1}]$, then $v_i(g)$ converges to some real number $v(g)$ for every $g \in \mathbf{G}$ and the pointwise limit v is a valuation on $\mathbf{G}$.

Indeed, let $g \in \mathbf{G}$. As stated in Theorem 2.6, there are elements $a_1, \dots, a_n \in [0, \mathbf{1}]$ such that $g = \sum_{k=1}^n \varepsilon_k a_k$, where $\varepsilon_k = \pm 1$. Then $v_i(g) = \sum_{k=1}^n \varepsilon_k v_i(a_k)$ for every i. Since all the summands converge, the sum also converges, that is, $v_i(g)$ converges to some real number $v(g)$. As in the preceding claim, one verifies that v is a valuation on $\mathbf{G}$. □

Remark 3.3. Let $H\colon \mathbf{A} \to \mathbf{A}'$ be an MV-algebra homomorphism. For every valuation v on $\mathbf{A}'$, the composed map $v \circ H$ is a valuation on $\mathbf{A}$ and we obtain an induced map $X_H\colon X_{\mathbf{A}'} \to X_\mathbf{A}$ defined by $X_H(v) = v \circ H$ which is easily seen to be continuous.

Similarly, every unital ℓ-homomorphism $H\colon \mathbf{G} \to \mathbf{G}'$ of unital abelian ℓ-groups induces a continuous map $X_H\colon X_{\mathbf{G}'} \to X_\mathbf{G}$ on the respective valuation spaces.

In this way one obtains a contravariant functor X from the category of MV-algebras and MV-algebra homomorphisms to the category of compact Hausdorff spaces and continuous maps; and similarly for the category of unital abelian ℓ-groups and unital ℓ- homomorphisms. By the preceding FACT 3.2, the functor Γ establishes an isomorphism between the two valuation space functors: For a unital abelian ℓ-group $\mathbf{G}$ and $\mathbf{A} = \Gamma(\mathbf{G})$, the map $v \mapsto v|_{\mathbf{A}} \colon X_{\mathbf{G}} \to X_{\mathbf{A}}$ is a homeomorphism.

In the sequel we often will identify the valuations on the MV-algebra $\mathbf{A}$ with their extensions to the enveloping unital abelian ℓ-group $\mathbf{G}$. Thus, we will identify the compact Hausdorff space $X_{\mathbf{A}}$ of all valuations on $\mathbf{A}$ with the space $X_{\mathbf{G}}$ of all valuations on $\mathbf{G}$, and we denote this space simply by X, when no confusion may occur.

The two valuation space functors have adjoints: To every compact Hausdorff space X we assign the unital ℓ-group $C(X)$ of all continuous real-valued functions defined on X with the constant function $\mathbf{1}$ as unit and the MV-algebra $\Gamma(C(X)) = C(X, \mathbf{I})$. To every continuous function $f \colon X \to Y$ between compact Hausdorff spaces we assign the unital ℓ-homomorphism $C(f) \colon C(Y) \to C(X)$ defined by $C(f)(h) = h \circ f$ and in the same way an MV-algebra homomorphism $C(Y, \mathbf{I}) \to C(X, \mathbf{I})$.

The *Yosida representation* [26, 27] yields a natural transformation of the identity functor to the functor $C \circ X$. To every element g of a unital abelian ℓ-group $\mathbf{G}$ we assign its *Gelfand transform* $\widehat{g} \colon X_{\mathbf{G}} \to \mathbb{R}$ defined by

$$\widehat{g}(v) = v(g) \text{ for every valuation } v.$$

The function $\widehat{g} \colon X_{\mathbf{G}} \to \mathbb{R}$ is continuous. Indeed, for any generalized sequence of valuations $(v_i)_i$ converging pointwise to a valuation v we have $\widehat{g}(v) = v(g) = \lim_i v_i(g) = \lim_i \widehat{g}(v_i)$. In this way we obtain an ℓ-homomorphism from $\mathbf{G}$ onto the ℓ-subgroup

$$\widehat{\mathbf{G}} = \{\widehat{g} : g \in \mathbf{G}\} \subseteq C(X_{\mathbf{G}}).$$

The unit $\mathbf{1}$ of $\mathbf{G}$ is mapped to the constant function $\mathbf{1}$. The restriction of this representation to $\mathbf{A} = \Gamma(\mathbf{G})$ yields an MV-algebra homomorphism from $\mathbf{A}$ onto the MV-subalgebra

$$\widehat{\mathbf{A}} = \{\widehat{a} : a \in \mathbf{A}\} \subseteq C(X_{\mathbf{G}}, \mathbf{I}).$$

The ℓ-subgroup $\widehat{\mathbf{G}}$ of $C(X_{\mathbf{G}})$ and the MV-subalgebra $\widehat{\mathbf{A}}$ of $C(X_{\mathbf{G}}, \mathbf{I})$ both separate the points of $X_{\mathbf{G}}$. For this, it suffices to check that the functions $\widehat{a}$, $a \in \mathbf{A}$, separate the points of $X_{\mathbf{G}}$: If v and w are two different valuations, there is an $a \in \mathbf{A}$ such that $v(a) \neq w(a)$, whence $\widehat{a}(v) \neq \widehat{a}(w)$.

The kernel of the map $g \mapsto \widehat{g}$ is the *radical* of $\mathbf{G}$. It consists of the *infinitesimals*, the elements $g \in \mathbf{G}$ such that $-\mathbf{1} \leq ng \leq \mathbf{1}$ for all natural numbers n (see [6, 3.6.4]).

Definition 3.4. *A unital abelian ℓ-group [an MV-algebra] is called* semisimple *if it contains no non-zero infinitesimals, that is, if $g \mapsto \widehat{g}$ is one-to-one.*

Thus, a unital abelian ℓ-group [MV-algebra] is semisimple if and only if it is isomorphic to a separating ℓ-group (resp., MV-algebra) of continuous functions containing the constant function $\mathbf{1}$ defined on the valuation space X with values in $\mathbb{R}$ (resp., $\mathbf{I}$) (see [6, 3.6.8]).

In the sequel we could restrict ourselves to semisimple MV-algebras. This is justified in our context: Firstly, free MV-algebras and in particular the Lindenbaum algebra for Łukasiewicz logic are semisimple. Secondly, in a rational game one will always identify events that have a priori the same value under every possible valuation. Indeed, the MV-algebras $\mathbf{A}$ and $\widehat{\mathbf{A}}$ have the same valuations: Every valuation v on $\mathbf{A}$ defines a valuation $\widehat{a} \mapsto \widehat{a}(v) = v(a)$ on $\widehat{\mathbf{A}}$ and every valuation w on $\widehat{\mathbf{A}}$ defines a valuation $a \mapsto \widehat{a} \mapsto w(\widehat{a})$ on $\mathbf{A}$, and these two constructions are mutually inverse.

An element g of an abelian ℓ-group $\mathbf{G}$ is said to be *divisible by a natural number* $n > 1$, if there is an element $h \in \mathbf{G}$ such that such $g = nh$. The element h, if it exists, is uniquely determined by g and n and will be denoted by $h = \frac{1}{n}g$. An element a of an MV-algebra $\mathbf{A}$ will be said to be divisible by n if a is divisible by n in the enveloping unital ℓ-group $\mathbf{G}$; and if this is the case, then $\frac{1}{n}a$ also belongs to $\mathbf{A}$.

Definition 3.5. *We say that an abelian ℓ-group $\mathbf{G}$ [an MV-algebra $\mathbf{A}$] is* 2-divisible*, if each of its elements is divisible by* 2.

If an MV-algebra $\mathbf{A}$ is 2-divisible, the same holds for its enveloping unital abelian ℓ-group $\mathbf{G}$; indeed, since every element g of $\mathbf{G}$ is representable as a sum of elements of the form $\pm a_i$ with $a_i \in \mathbf{A}$, then the sum of the elements $\pm\frac{1}{2}a_i$ is $\frac{1}{2}g$.

In a 2-divisible MV-algebra $\mathbf{A}$ and similarly in a 2-divisible ℓ-group $\mathbf{G}$, every element is divisible by any power 2^n, $n \in \mathbb{N}$. We will write $\frac{m}{2^n}a$ for $m \cdot \frac{1}{2^n}a$ whenever m is a natural number; thus, we have a 'scalar multiplication' $q \cdot a$ of elements $a \in \mathbf{G}$ by nonnegative *dyadic* rational numbers $q = \frac{m}{2^n}$. For elements $a \in \mathbf{A}$ we restrict this scalar multiplication to dyadic rational numbers in the unit interval.

Every unital abelian ℓ-group and every MV-algebra is embeddable in a 2-divisible one. We do not need this fact in general but only for a special case: Let X be a compact Hausdorff space and $\widehat{\mathbf{G}}$ an ℓ-subgroup of $C(X)$ containing the constant function $\mathbf{1}$. It is straightforward that $\widehat{\mathbf{G}}_2 = \{\frac{1}{2^n}g : g \in \widehat{\mathbf{G}}, n \in \mathbb{N}\}$ is a 2-divisible ℓ-subgroup of $C(X)$ containing $\widehat{\mathbf{G}}$. For the MV-algebra $\widehat{\mathbf{A}} = \Gamma(\widehat{\mathbf{G}})$, the MV-subalgebra $\widehat{\mathbf{A}}_2 = \Gamma(\widehat{\mathbf{G}}_2)$ of $C(X, \mathbf{I})$ is 2-divisible and contains $\widehat{\mathbf{A}}$. One should notice that the 2-divisible hull $\widehat{\mathbf{A}}_2$ of the MV-algebra $\widehat{\mathbf{A}}$ contains not only the functions $\frac{1}{2^n}g, g \in \widehat{\mathbf{A}}, n \in \mathbb{N}$, but also finite sums of such. It suffices to consider the 2-divisible hull of the four element Boolean algebra. Since every semisimple unital abelian ℓ-group can be represented as a unital ℓ-subgroup of $C(X)$ and since every semisimple MV-algebra is a subalgebra of $C(X, \mathbf{I})$ for some compact Hausdorff space X, this special case is sufficiently general for our purposes.

Let X be a compact Hausdorff space. Every continuous real-valued function f defined on X has a maximal and a minimal value. We denote these by

$$\mathrm{SUP}(f) = \max_{x \in X} f(x) \quad \text{and} \quad \mathrm{INF}(f) = \min_{x \in X} f(x).$$

Using the Yosida representation we can use the same notation for elements g of a unital abelian ℓ-group $\mathbf{G}$ and the elements of an MV-algebra $\mathbf{A}$ by defining

$$\mathrm{SUP}(g) = \mathrm{SUP}(\widehat{g}) \quad \text{and} \quad \mathrm{INF}(g) = \mathrm{INF}(\widehat{g}).$$

On a unital abelian ℓ-group $\mathbf{G}$ the functionals SUP and INF can also be defined directly:

$$\mathrm{SUP}(g) = \inf\left\{\frac{n}{m} \mid mg \leq n\mathbf{1}, n \in \mathbb{Z}, 0 \neq m \in \mathbb{N}\right\}, \tag{1}$$

$$\mathrm{INF}(g) = \sup\left\{\frac{n}{m} \mid mg \geq n\mathbf{1}, n \in \mathbb{Z}, 0 \neq m \in \mathbb{N}\right\}. \tag{2}$$

The supremum norm on $C(X)$ is given by

$$||f|| = \sup_{x \in X} |f(x)| = \max(\mathrm{SUP}(f), -\mathrm{INF}(f)).$$

The topology associated with this norm will be referred to as the *topology of uniform convergence*. Since $C(X, \mathbf{I})$ is contained in $C(X)$, it can be equipped with the induced topology.

The classical Stone Weierstraß Theorem tells us: a subalgebra is dense in $C(X)$ for the topology of uniform convergence provided it contains the constant function $\mathbf{1}$ and separates the points of X. There is a similar Stone-Weierstraß Theorem for linear subspaces of $C(X)$ that are also sublattices (see, e.g., [22, Chapter V, 8.1]). Together with [2, Proposition 1.4] we have the following versions of the Stone-Weierstraß Theorem:

Theorem 3.6. *Let X be a compact Hausdorff space.*
(1) *Every 2-divisible unital ℓ-subgroup of $C(X)$ separating the points of X is dense in $C(X)$ for the topology of uniform convergence.*
(2) *Every 2-divisible MV-subalgebra of the MV-algebra $C(X, \mathbf{I})$ separating the points of X is dense in $C(X, \mathbf{I})$ for the topology of uniform convergence.*

Since the functions in the Yosida representation of a unital abelian ℓ-groups are separating, the Stone-Weierstraß Theorem 3.6 yields:

Corollary 3.7. (1) *For every unital abelian ℓ-group $\mathbf{G}$, the 2-divisible hull $\widehat{\mathbf{G}}_2$ of its Yosida representation $\widehat{\mathbf{G}}$ is dense in $C(X_{\mathbf{G}})$ for the topology of uniform convergence.*
(2) *For every MV-algebra $\mathbf{A}$, the 2-divisible hull $\widehat{\mathbf{A}}_2$ of its Yosida representation $\widehat{\mathbf{A}}$ is dense in $C(X_{\mathbf{A}}, \mathbf{I})$ for the topology of uniform convergence.*

We remark that the valuations of the MV-algebra $C(X_{\mathbf{A}}, \mathbf{I})$ are the point evaluations $\delta_v(f) = f(v)$, $v \in X_{\mathbf{A}}$. Thus, the space of valuations of $C(X_{\mathbf{A}}, \mathbf{I})$ coincides with the space $X_{\mathbf{A}}$ of valuations of the original MV-algebra $\mathbf{A}$. This simple but crucial observation will allow us to reduce the study of probabilities and imprecise probabilities over MV-algebras to the study of probabilities and imprecise probabilities over compact Hausdorff spaces.

4 Probabilities and states

Let $\mathbf{A}$ be an MV-algebra and $\mathbf{G}$ be a unital abelian ℓ-group such that $\mathbf{A} = \Gamma(\mathbf{G})$. We proceed to the first important notion in our context.

Definition 4.1. (See [18]) *A* state *on the unital abelian ℓ-group $\mathbf{G}$ is a functional $\mu\colon \mathbf{G} \to \mathbb{R}$ which is*
- positive*, that is, $\mu(g) \geq 0$ for all $g \geq 0$,*
- normalized*, that is, $\mu(\mathbf{1}) = 1$, and*
- additive*, that is, $\mu(g + h) = \mu(g) + \mu(h)$ for all $g, h \in \mathbf{G}$.*

A state *on the MV-algebra $\mathbf{A}$ is a map $\mu\colon \mathbf{A} \to \mathbf{I}$ which is*
- normalized*, that is, $\mu(\mathbf{1}) = 1$, and*
- additive*, that is, $\mu(a + b) = \mu(a) + \mu(b)$ for all $a, b \in \mathbf{A}$ such that $a + b$ is defined in $\mathbf{A}$.*

We denote by $\mathcal{P}(\mathbf{A})$ the set of all states on $\mathbf{A}$ and by $\mathcal{P}(\mathbf{G})$ the set of all states on $\mathbf{G}$.

Restricting a state μ on $\mathbf{G}$ to the MV-algebra $\mathbf{A}$ yields a state on $\mathbf{A}$. In [18, Theorem 2.4], Mundici showed:

Theorem 4.2. *For every state μ on the MV-algebra $\mathbf{A}$, there is a unique state on the enveloping unital abelian ℓ-group $\mathbf{G}$ extending μ. Thus, $\mu \mapsto \mu|_{\mathbf{A}}$ is a bijection between the set $\mathcal{P}(\mathbf{G})$ of states on $(\mathbf{G}, e)$ and the set $\mathcal{P}(\mathbf{A})$ of states on $\mathbf{A}$.*

By definition, $\mathcal{P}(\mathbf{G})$ and $\mathcal{P}(\mathbf{A})$ are subsets of $\mathbb{R}^{\mathbf{G}}$ and $\mathbf{I}^{\mathbf{A}}$, respectively, and we endow them with the respective topologies of pointwise convergence. Moreover, $\mathcal{P}(\mathbf{G})$ and $\mathcal{P}(\mathbf{A})$ are convex subsets of $\mathbb{R}^{\mathbf{G}}$ and $\mathbf{I}^{\mathbf{A}}$, respectively, in the sense that the convex combination $p\mu + (1-p)\nu$ of any two states μ and ν is again a state for all $p \in [0, 1]$.

FACT 4.3. *The states of a unital abelian ℓ-group $\mathbf{G}$ form a compact convex subset $\mathcal{P}(\mathbf{G})$ of the vector space $\mathbb{R}^{\mathbf{G}}$. Similarly, $\mathcal{P}(\mathbf{A})$ is a compact convex subset of $\mathbf{I}^{\mathbf{A}}$. The map $\mu \mapsto \mu|_{\mathbf{A}}\colon \mathcal{P}(\mathbf{G}) \to \mathcal{P}(\mathbf{A})$ is an affine homeomorphism. The valuations form closed subspaces $X_{\mathbf{G}}$ and $X_{\mathbf{A}}$ of $\mathcal{P}(\mathbf{G})$ and $\mathcal{P}(\mathbf{A})$, respectively.*

Proof. We first recall that the map $\mu \mapsto \mu|_{\mathbf{A}}\colon \mathcal{P}(\mathbf{G}) \to \mathcal{P}(\mathbf{A})$ is bijective by Theorem 4.2. It is easily seen to be affine.

$\mathcal{P}(\mathbf{A})$ is a closed subspace of $\mathbf{I}^{\mathbf{A}}$, hence compact. Indeed, let $(\mu_i)_i$ be a generalized sequence of states on $\mathbf{A}$ converging pointwise to some function μ. Then $\mu(\mathbf{1}) = \lim_i \mu_i(\mathbf{1}) = 1$ and, for any two elements $a, b \in \mathbf{A}$ such that $a + b$ is defined in $\mathbf{A}$, we have $\mu(a + b) = \lim_i \mu_i(a + b) = \lim_i(\mu_i(a) + \mu_i(b)) = \lim_i \mu_i(a) + \lim_i \mu_i(b) = \mu(a) + \mu(b)$. This shows that μ is a state.

The claim that $\mu \mapsto \mu|_{\mathbf{A}}$ is a homeomorphism is proved in exactly the same way as the corresponding claim for valuations in FACT 3.2. □

Remark 4.4. Every MV-algebra homomorphism $H\colon \mathbf{A} \to \mathbf{A}'$ induces a continuous map $\mathcal{P}(H)\colon \mathcal{P}(\mathbf{A}') \to \mathcal{P}(\mathbf{A})$ defined by $\mathcal{P}(H)(\mu) = \mu \circ H$, and similarly for unital ℓ-group homomorphisms. Moreover, these maps are affine, that is, they preserve convex combinations.

In this way, $\mathcal{P}$ becomes a contravariant functor from the category of unital abelian ℓ-groups and unital ℓ-homomorphisms to the category of compact convex sets and continuous affine maps; and similarly for the category of MV-algebras and MV-algebra homomorphisms. The functor Γ establishes a natural isomorphism between the two state space functors.

On Boolean algebras, the states are the additive normalized $[0, 1]$-valued maps. These maps on Boolean algebras have often been called probability measures (see, for example Sikorski [23]). It has been argued in detail in [18] why states play the same role for MV-algebras as probability measures for Boolean algebras. Independently, Kroupa and Panti [14, 20] gave a precise formulation for this claim:

Theorem 4.5. *Let $\mathbf{A}$ be an MV-algebra and $\mathbf{G}$ a unital abelian ℓ-group such that $\mathbf{A} = \Gamma(\mathbf{G})$. There are canonical bijective correspondences between the set $\mathcal{P}(\mathbf{A})$ of states on the MV-algebra $\mathbf{A}$, the set $\mathcal{P}(\mathbf{G})$ of states on the unital abelian ℓ-group $\mathbf{G}$ and the set $\mathcal{P}(X)$ of regular Borel probability measures on the compact Hausdorff space $X = X_{\mathbf{A}} \cong X_{\mathbf{G}}$ of valuations.*

A proof of Theorem 4.5 along the line of Panti's proof will also result from our more general considerations on imprecise probabilities.

5 Imprecise probabilities, upper and lower states

This section is devoted to giving a precise definition of 'imprecise probabilities'.

We will treat the case of an MV-algebra $\mathbf{A}$ in parallel with the case of a unital ℓ-group $\mathbf{G}$, where $\mathbf{A} = \Gamma(\mathbf{G})$. We denote by X the compact space of valuations according to FACT 3.2 and by $\mathcal{P}(\mathbf{G})$ the compact convex set of all states on $\mathbf{G}$ identified with the space $\mathcal{P}(\mathbf{A})$ of states on $\mathbf{A}$ according to FACT 4.3. Recall the correspondence between states and regular Borel probability measures on the space X (Theorem 4.5).

As a FIRST APPROACH we say that an 'imprecise probability' is represented by a set of regular Borel probability measures, equivalently, by a set S of states.

Every set S of states gives rise to two functionals $S^*\colon \mathbf{G} \to \mathbb{R}$ and $S_*\colon \mathbf{G} \to \mathbb{R}$ defined by

$$S^*(g) = \sup_{\mu \in S} \mu(g) \quad \text{and} \quad S_*(g) = \inf_{\mu \in S} \mu(g) \qquad \text{for all } g \in \mathbf{G}.$$

Restricting S^* and S_* to $\mathbf{A}$, yields functions with values in $\mathbf{I}$. If S is a closed subset of $\mathcal{P}(\mathbf{G})$, then S is compact and, since the function $\mu \mapsto \mu(g)$ is continuous for every fixed g by the definition of the topology on $\mathcal{P}(\mathbf{G})$, the supremum and the infimum are attained in some $\mu \in S$, that is,

$$S^*(g) = \max_{\mu \in S} \mu(g) \quad \text{and} \quad S_*(g) = \min_{\mu \in S} \mu(g) \qquad \text{for all } g \in \mathbf{G}.$$

The two functionals S^* and S_* represent two opposite views of imprecise probabilities: Given a set S of probabilities, the optimistic observer is interested in the best case, the highest possible probability $S^*(g)$ of an event g to occur, whilst the pessimistic observer will consider the worst case, the lowest possible probability $S_*(g)$ of an event g. The interval $[S_*(g), S^*(g)]$ represents the imprecise probability of the event g.

Conversely, for every functional $u\colon \mathbf{G} \to \mathbb{R}$, we may consider the following sets of states:

$$u^* = \{\mu \in \mathcal{P}(\mathbf{G}) : \mu(g) \leq u(g) \text{ for all } g \in \mathbf{G}\}, \qquad u_* = \{\mu \in \mathcal{P}(\mathbf{G}) : \mu(g) \geq u(g) \text{ for all } g \in \mathbf{G}\}.$$

In the same way we define u^* and u_* for functions $u\colon \mathbf{A} \to \mathbf{I}$.

Lemma 5.1. *u^* and u_* are closed and convex subsets of the space $\mathcal{P}(\mathbf{G})$ (resp., $\mathcal{P}(\mathbf{A})$) of all states.*

Proof. For every fixed $g \in \mathbf{G}$, the set of all states μ satisfying the inequality $\mu(g) \leq u(g)$ is convex and closed for the topology of pointwise convergence. Since u^* is the intersection of all of these sets, it is also convex and closed. The proof for u_* is similar. □

We immediately obtain the following two Galois correspondences (see [10, Section 3.1]) between functionals and sets of states:

Lemma 5.2.

(0)	$u_1 \leq u_2 \implies u_1^* \subseteq u_2^*$ *and* $u_{1*} \supseteq u_{2*}$,	$S_1 \subseteq S_2 \implies S_1^* \leq S_2^*$ *and* $S_{1*} \geq S_{2*}$,
(1)	$u^{**} \leq u \leq u_{**}$,	$S \subseteq S^{**}$ *and* $S \subseteq S_{**}$,
(2)	$u^{***} = u^*$, $u_{***} = u_*$,	$S^{***} = S^*$ *and* $S_{***} = S_*$.

Let us denote by $\overline{\mathcal{U}}$ the collection of all those functionals u that are pointwise suprema of states, that is, functionals of the form $u = S^*$ for some set S of states, and similarly by $\underline{\mathcal{U}}$ the collection of all those functionals u that are pointwise infima of states, that is, of the form $u = S_*$ for some set S of states. We denote by $\mathcal{K}^*$ the collection of those sets of states that are of the form u^* and by $\mathcal{K}_*$ the collection of those sets of states that are of the form u_* for some functional u. Note that, by Lemma 5.1, the members of $\mathcal{K}^*$ and of $\mathcal{K}_*$ are closed convex subsets of $\mathcal{P}(\mathbf{G})$.

Since we have a Galois correspondence, the maps $u \mapsto u^*$ and $S \mapsto S^*$ establish a one-to-one correspondence between $\overline{\mathcal{U}}$ and $\mathcal{K}^*$. Similarly, the maps $u \mapsto u_*$ and $S \mapsto S_*$ establish a one-to-one correspondence between $\underline{\mathcal{U}}$ and $\mathcal{K}_*$. We will see that the sets $\mathcal{K}^*$ and $\mathcal{K}_*$ are identical. For this we introduce a duality operator on functionals. We have to distinguish the case of ℓ-groups from that of MV-algebras. For any functional $u\colon \mathbf{G} \to \mathbb{R}$ the *conjugate* functional $u^\# : \mathbf{G} \to \mathbb{R}$ is defined by

$$u^\#(g) = -u(-g) \quad \text{for all } g \in \mathbf{G}$$

and for a function $u\colon \mathbf{A} \to \mathbf{I}$ we define the *conjugate* $u^\dagger : \mathbf{A} \to \mathbf{I}$ by

$$u^\dagger(x) = 1 - u(\neg x) \quad \text{for all } x \in \mathbf{A}.$$

The following properties are straightforward:

Lemma 5.3. *For functionals $u, w\colon \mathbf{G} \to \mathbb{R}$ and any set S of states we have:*
(1) $u^{\#\#} = u$;
(2) $u \leq w$ *if and only if* $u^\# \geq w^\#$;
(3) $\mu^\# = \mu$ *for every state* μ *on* $\mathbf{G}$;
(4) $u^* = (u^\#)_*$ *and* $u_* = (u^\#)^*$;
(5) $S^* = (S_*)^\#$ *and* $S_* = (S^*)^\#$.
For functions $u\colon \mathbf{A} \to \mathbf{I}$ the same properties hold with $u^\dagger$ in place of $u^\#$.

By Lemma 5.3(4), every member of $\mathcal{K}^*$ is also a member of $\mathcal{K}_*$ and conversely. Thus, the collections $\mathcal{K}^*$ and $\mathcal{K}_*$ of compact convex sets of states coincide. We will simply denote them by $\mathcal{K}$. By Lemma 5.3(5), the map $u \mapsto u^\#$ induces a bijection between $\overline{\mathcal{U}}$ and $\underline{\mathcal{U}}$.

For the optimistic as well as for the pessimistic observer, two sets S and T of states represent the same 'imprecise' probability, if they have the same pointwise supremum $S^* = T^*$ or, equivalently, if they have the same pointwise infimum $S_* = T_*$; indeed, since $S_* = (S^*)^\#$ by Lemma 5.2(4), $S^* = T^*$ implies $S_* = T_*$. Thus, a SECOND APPROACH would be to say that 'imprecise probabilities' are represented by those functionals which are either suprema (upper states) or, dually, infima (lower states) of sets of states, that is, the functionals belonging to $\overline{\mathcal{U}}$ and $\underline{\mathcal{U}}$, respectively.

Every imprecise probability S is represented by a largest set of states which is compact and convex, namely $S^{**} = S_{**}$. This is not completely satisfactory:

Problem 5.4. *1. Characterize those compact convex subsets of $\mathcal{P}(\mathbf{G})$ which belong to $\mathcal{K}$, that is, which arise as sets of the form u^* or u_*.*

2. Characterize those functionals which are suprema (or infima) of nonempty sets of states, that is, the functionals belonging to $\overline{\mathcal{U}}$ and $\underline{\mathcal{U}}$, respectively, without referring to states.

The answer to Problem 1 will be that $\mathcal{K}$ is the collection of all nonempty closed convex subsets of the state space $\mathcal{P}(\mathbf{G})$ (see Theorem 7.4). For Problem 2 we also have a solution. For MV-algebras this solution is not quite satisfactory in general, as it heavily refers to the enveloping unital abelian ℓ-group. Only under a divisibility hypothesis we have an intrinsic solution for MV-algebras.

The notions of upper and lower states defined below constitute an attempt to attack problem 2. We first restrict our attention to ℓ-groups:

Definition 5.5. *A functional $u\colon \mathbf{G} \to \mathbb{R}$ on a unital abelian ℓ-group $\mathbf{G}$ is said to be an* upper state*, if it is*
- order preserving*, that is, $u(g) \leq u(h)$ for all $g \leq h$ in $\mathbf{G}$,*
- strongly normalized *if $u(\mathbf{1}) = 1$ and $u(-\mathbf{1}) = -1$,*
- $\mathbb{N}$-homogeneous*, that is, $u(ng) = nu(g)$ for all $g \in \mathbf{G}$ and all $n \in \mathbb{N}$, and*

- subadditive*, that is,* $u(g+h) \le u(g) + u(h)$ *for all* $g, h \in \mathbf{G}$.
Similarly, u *is a* lower state*, if* u *is*
- *order preserving, strongly normalized,* $\mathbb{N}$*-homogeneous and*
- superadditive*, that is,* $u(g+h) \ge u(g) + u(h)$ *for all* $g, h \in \mathbf{G}$.
We denote by $\overline{\mathcal{P}}(\mathbf{G})$ *and* $\underline{\mathcal{P}}(\mathbf{G})$ *the sets of upper and lower states on* $\mathbf{G}$*, respectively.*

Clearly every state is an upper and a lower state, more precisely, the states are those functionals which are simultaneously upper and lower states.

For later use, let us note the following property of upper and lower states u on $\mathbf{G}$:

(SN) $$u(g + m\mathbf{1}) = u(g) + m \text{ for all } g \in \mathbf{G} \text{ and all } m \in \mathbb{Z}.$$

Indeed, an upper state u is subadditive, $\mathbb{N}$-homogeneous and strongly normalized so that $u(g + m\mathbf{1}) \le u(g) + u(m\mathbf{1}) = u(g) + |m|u(\mathrm{sgn}(m)\mathbf{1}) = u(g) + m$ and, conversely, $u(g) + m = u(g + m\mathbf{1} - m\mathbf{1}) + m \le u(g + m\mathbf{1}) + u(-m\mathbf{1}) + m = u(g + m\mathbf{1}) + |-m|u(\mathrm{sgn}(-m)\mathbf{1}) + m = u(g + m\mathbf{1}) - m + m = u(g + m\mathbf{1})$, where $\mathrm{sgn}(m)$ denotes the sign of $m \in \mathbb{Z}$.

The defining properties of upper and lower states are closely related to Walley's coherence conditions for upper and lower previsions (see [24, Definition 2.3.3] on linear spaces of bounded real valued functions. Let us state this explicitly for the upper case:

Proposition 5.6. *A functional* u *on a unital abelian* ℓ*-group* $\mathbf{G}$ *is an upper state if and only if it is*
- $\mathbb{N}$*-homogeneous,*
- *subadditive, and*
- upper bounded*, that is,* $u(g) \le \mathrm{SUP}(g)$ *for every* $g \in \mathbf{G}$.

Proof. Let u be subadditive and $\mathbb{N}$-homogeneous.

(a) Suppose that u is upper bounded. We first show that u is order preserving: Let $g \le h$. By subadditivity, $u(g) = u(h+(g-h)) \le u(h)+u(g-h)$. Since u is upper bounded and $g-h \le 0$, we have $u(g-h) \le \mathrm{SUP}(g-h) \le 0$ and, consequently, $u(g) \le u(h)$. We now show that u is strongly normalized: Clearly, by boundedness, we have $u(m\mathbf{1}) \le \mathrm{SUP}(m\mathbf{1}) = m$. The reverse inequality, $m \le u(m\mathbf{1})$, follows from a similar argument and subadditivity: $0 = u(0) = u(m\mathbf{1}-m\mathbf{1}) \le u(m\mathbf{1})+u(-m\mathbf{1}) \le u(m\mathbf{1})+\mathrm{SUP}(-m\mathbf{1}) = u(m\mathbf{1})-m$. Thus, u is strongly normalized.

(b) Suppose conversely that u is order preserving and strongly normalized. If $ng \le m\mathbf{1}$ for integers $n > 0$ and m, then $nu(g) = u(ng) \le u(m\mathbf{1}) = m$, whence $u(g) \le \frac{m}{n}$. Thus, $u(g) \le \inf\{\frac{m}{n} \mid ng \le m\mathbf{1}\} = \mathrm{SUP}(g)$, that is, u is upper bounded. □

Remark 5.7. As for states, the upper states of a unital abelian ℓ-group $\mathbf{G}$ form a convex subset $\overline{\mathcal{P}}(\mathbf{G})$ of $\mathbb{R}^{\mathbf{G}}$, which is closed for the topology of pointwise convergence. It is even compact.

Indeed, for every $g \in \mathbf{G}$, there is a natural number n_g such that $-n_g\mathbf{1} \le g \le n_g\mathbf{1}$. For every upper state u we then have $-n_g \le u(g) \le n_g$, which implies that the set of upper states is contained in the product $\prod_{g\in\mathbf{G}}[-n_g, n_g] \subseteq \mathbb{R}^{\mathbf{G}}$ of compact intervals.

Let $H\colon \mathbf{G} \to \mathbf{G}'$ be a unital ℓ-homomorphism of unital abelian ℓ-groups. For every upper state u on $\mathbf{G}'$ the composed map $u \circ H$ is an upper state on $\mathbf{G}$. Thus H induces a map $\overline{\mathcal{P}}(H)\colon \overline{\mathcal{P}}(\mathbf{G}') \to \overline{\mathcal{P}}(\mathbf{G})$ defined by $\overline{\mathcal{P}}(H)(u) = u \circ H$. This map is continuous and affine, as one easily verifies, and $\overline{\mathcal{P}}$ becomes a contravariant functor from the category of unital abelian ℓ-groups and unital ℓ-homomorphisms to the category of compact convex sets and continuous affine maps.

The compact convex set $\overline{\mathcal{P}}(\mathbf{G})$ also carries a pointwise defined partial order. Since the defining properties of upper states are all preserved under taking pointwise suprema of functionals, the pointwise supremum of any nonempty set of upper states is an upper state. It follows that there is a greatest upper state, the pointwise supremum of all upper states. This greatest upper state is the functional SUP:

$$\mathrm{SUP}(g) = \inf\{\frac{m}{n} \mid ng \le m\mathbf{1}\} \text{ for } g \in \mathbf{G}.$$

In particular, $\overline{\mathcal{P}}(\mathbf{G})$ is a $\vee$-semilattice. The $\vee$-semilattice operation is continuous and satisfies the distributive law $u+(v\vee w)=(u+v)\vee(u+w)$. The map $\overline{\mathcal{P}}(H)$ induced by a unital ℓ-homomorphism is not only continuous and affine but also a $\vee$-semilattice homomorphism.[1]

We now turn to the case of MV-algebras:

Definition 5.8. *For an MV-algebra* $\mathbf{A}$*, an* upper *(resp.,* lower*)* state *is a function* w *on* $\mathbf{A}$ *which is the restriction* $w=u|_{\mathbf{A}}$ *of an upper (resp., lower) state* u *on its enveloping unital abelian* ℓ*-group* $\mathbf{G}$*. We denote by* $\overline{\mathcal{P}}(\mathbf{A})$ *and* $\underline{\mathcal{P}}(\mathbf{A})$ *the sets of upper and lower states on* $\mathbf{A}$*, respectively.*

Upper and lower states on MV-algebras have similar properties as on unital abelian ℓ-groups.

Remark 5.9. The upper states on $\mathbf{A}$ form a compact convex subset of $\mathbf{I}^{\mathbf{A}}$ and the map assigning to every upper state u on $\mathbf{G}$ its restriction $w=u|_{\mathbf{A}}$ to $\mathbf{A}$ is continuous and affine. This map is no longer a bijection. Indeed, Walley [24, p. 82] exhibited an example of a Boolean algebra and two different lower states on the enveloping unital abelian ℓ-group which agree on the Boolean algebra.

Every MV-algebra homomorphism $H\colon \mathbf{A}\to\mathbf{A}'$ induces an affine continuous map $\overline{\mathcal{P}}(\mathbf{A}')\to\overline{\mathcal{P}}(\mathbf{A})$ on the respective spaces of upper states in such a way that we obtain a contravariant functor from the category of MV-algebras and MV-algebra homomorphisms to the category of compact convex sets and affine continuous maps. Moreover, for an MV-algebra homomorphism $H\colon \mathbf{A}\to\mathbf{A}'$ and its unique extension to a unital ℓ-homomorphism $\widetilde{H}\colon \mathbf{G}\to\mathbf{G}'$ between the enveloping unital abelian ℓ-groups, the following diagram commutes:

$$\begin{array}{ccc}
\overline{\mathcal{P}}(\mathbf{G}) & \xleftarrow{\ \overline{\mathcal{P}}(\widetilde{H})\ } & \overline{\mathcal{P}}(\mathbf{G}') \\
{\scriptstyle u\mapsto u|_{\mathbf{A}}}\downarrow & & \downarrow{\scriptstyle u\mapsto u|_{\mathbf{A}'}} \\
\overline{\mathcal{P}}(\mathbf{A}) & \xleftarrow{\ \overline{\mathcal{P}}(H)\ } & \overline{\mathcal{P}}(\mathbf{A}')
\end{array}$$

We want to relate upper and lower states to sets of states. The next results hold for unital abelian ℓ-groups $\mathbf{G}$ as well as for MV-algebras $\mathbf{A}$. Recall that the states are those functionals which are simultaneously upper and lower states, that is, $\mathcal{P}(\mathbf{G})=\overline{\mathcal{P}}(\mathbf{G})\cap\underline{\mathcal{P}}(\mathbf{G})$.

As the pointwise supremum of any nonempty set of upper states is an upper state we have for unital abelian ℓ-groups as well as for MV-algebras:

Lemma 5.10. *The pointwise supremum* S^* *of any nonempty set* S *of states is an upper state and the pointwise infimum* S_* *is a lower state.* SUP *is the greatest upper and* INF *the least lower state.*

Proposition 5.11. *A functional* $u\colon \mathbf{G}\to\mathbb{R}$ *on a unital abelian* ℓ*-group is an upper state if and only if its conjugate* $u^{\#}$ *is a lower state. Moreover, every upper state* u *dominates its conjugate lower state* $u^{\#}$*.*

A map $w\colon \mathbf{A}\to\mathbf{I}$ *on an MV-algebra* $\mathbf{A}$ *is an upper state if and only if its conjugate* $w^{\dagger}$ *is a lower state. Every upper state* w *dominates its conjugate lower state* $w^{\dagger}$*.*

Proof. For a functional u on a unital abelian ℓ-group, it is easy to verify that its conjugate $u^{\#}$ is order preserving, strongly normalized, $\mathbb{N}$-homogeneous, if and only if u is order preserving, strongly normalized, $\mathbb{N}$-homogeneous, respectively, whereas $u^{\#}$ is superadditive if and only if u is subadditive.

For an upper state u, one has $0=u(g-g)\le u(g)+u(-g)$ by subadditivity, whence $u^{\#}(g)=-u(-g)\le u(g)$ for every $g\in\mathbf{G}$.

Let w be an upper state on an MV-algebra $\mathbf{A}$. By definition, $w=u|_{\mathbf{A}}$ for some upper state u on the enveloping unital abelian ℓ-group $\mathbf{G}$. Using property (SN) for u, we observe that $w^{\dagger}(a)=1-w(\neg a)=1-u(\mathbf{1}-a)=1-(1+u(-a))=-u(-a)=u^{\#}(a)$ for every $a\in\mathbf{A}$, that is, $w^{\dagger}=u^{\#}|_{\mathbf{A}}$. As $u^{\#}$ is a lower state on $\mathbf{G}$ by the first claim, its restriction $w^{\dagger}$ to $\mathbf{A}$ is a lower state on $\mathbf{A}$. □

[1] One can show that the state space $\mathcal{P}(\mathbf{G})$ is the 'free' compact convex set over the valuation space $X_{\mathbf{G}}$ and that $\overline{\mathcal{P}}(\mathbf{G})$ is the 'free' compact convex $\vee$-semilattice over the valuation space $X_{\mathbf{G}}$. But these latter statement are out of the range of this paper.

The above definition of upper and lower states for MV-algebras looks artificial, as it refers heavily to the enveloping unital abelian ℓ-group. One would prefer a definition by conditions that one can check directly in the MV-algebra $\mathbf{A}$. Of course, an upper state on $\mathbf{A}$ inherits the following properties from the defining properties of upper states on its enveloping unital abelian ℓ-group: It is

- order preserving,
- strongly normalized in the sense that $u(g+\frac{1}{n}\mathbf{1}) = u(g)+\frac{1}{n}$, whenever $\mathbf{1}$ is divisible by n and $g+\frac{1}{n}\mathbf{1}$ is defined in $\mathbf{A}$.
- $\mathbb{N}$-*homogeneous* in the sense that $u(na) = nu(a)$ for all $a \in \mathbf{A}$ and all $n \in \mathbb{N}$ such that the n-fold sum $na = a+a+\cdots+a$ is defined in $\mathbf{A}$, and
- *subadditive* in the sense that $u(a+b) \leq u(a)+u(b)$ for all $a, b \in \mathbf{A}$ such that $a+b$ is defined in $\mathbf{A}$.

These properties are not sufficient for a function u on an arbitrary MV-algebra to be the restriction of an upper state on the enveloping unital abelian ℓ-group.

Example 5.12. For a simple counterexample one may consider the Boolean algebra of all subsets of the three element set $\{1, 2, 3\}$. One defines $u(\emptyset) = 0$, $u(\{1, 2, 3\}) = 1$ and $u(Y) = \frac{1}{2}$ for the proper nonempty subsets Y. The above properties are all satisfied. But u is not a restriction of an upper state on the enveloping unital abelian ℓ-group. One can check this claim directly. Alternatively, one can show that u does not satisfy Walley's upper coherence condition in Theorem 6.8(1) which characterizes the extendibility to an upper state: We replace the subsets by their characteristic functions; in particular, let f_1, f_2, f_3 be the characteristic functions of $\{2, 3\}, \{1, 3\}, \{1, 2, \}$, respectively, and $f_0 = 0$; then $\mathrm{INF}(f_1+f_2+f_3) = 2 > \frac{3}{2} = u(f_1)+u(f_2)+u(f_3)$.

Under divisibility hypotheses, the upper states on MV-algebras can be characterized internally by the properties above:

Proposition 5.13. *Let $\mathbf{A}$ be a 2-divisible MV-algebra and $\mathbf{G}$ a unital abelian ℓ-group such that $\mathbf{A} = \Gamma(\mathbf{G})$. A function $w\colon \mathbf{A} \to \mathbf{I}$ is an upper state if and only if it is order preserving, strongly normalized, $\mathbb{N}$-homogeneous and subadditive. And if this is the case, then w has a unique extension to an upper state u on the enveloping unital abelian ℓ-group $\mathbf{G}$.*

Proof. Consider a 2-divisible MV-algebra $\mathbf{A}$ and an order preserving, $\mathbb{N}$-homogeneous, strongly normalized, subadditive function $w\colon \mathbf{A} \to \mathbf{I}$. The enveloping unital abelian ℓ-group is also 2-divisible.

For $g \in \mathbf{A}$ and $n \in \mathbb{N}$, by $\mathbb{N}$-homogeneity we have $2^n w(\frac{1}{2^n}g) = w(2^n \cdot \frac{1}{2^n}g) = w(g)$, whence $w(\frac{1}{2^n}g) = \frac{1}{2^n}w(g)$. Using $\mathbb{N}$-homogeneity again, we derive that $w(\frac{m}{2^n}g) = mw(\frac{1}{2^n}g) = \frac{m}{2^n}w(g)$ for all $m \leq 2^n$.

We now extend w to the positive cone $\mathbf{G}_+$ of $\mathbf{G}$: For every $g \in \mathbf{G}_+$, there is an n such that $g \leq 2^n\mathbf{1}$, whence $\frac{1}{2^n}g \in \mathbf{A}$, and we define $u(g) = 2^n w(\frac{1}{2^n}g)$. One first shows that u is well defined, that is, one shows that the value $u(g)$ does not depend on the particular choice of n; for this, one uses the homogeneity property exhibited in the previous paragraph. It is straightforward that u is $\mathbb{N}$-homogeneous on $\mathbf{G}_+$ and that it is the unique $\mathbb{N}$-homogeneous extension of w to $\mathbf{G}_+$. One easily shows that u is order preserving, strongly normalized and subadditive on $\mathbf{G}_+$.

In a second step we extend u to a functional on $\mathbf{G}$: For every $h \in \mathbf{G}$, there is a natural number m such that $-m\mathbf{1} \leq h$, that is, $0 \leq h+m\mathbf{1}$. We define $\widetilde{u}(h) = u(h+m\mathbf{1})-m$. The value of $\widetilde{u}(h)$ does not depend on the particular choice of m. The extension $\widetilde{u}$ is strongly normalized on $\mathbf{G}$ and clearly the only strongly normalized extension of u. Straightforward calculations show that $\widetilde{u}$ is an upper state on $\mathbf{G}$. □

We conclude:

FACT 5.14. *Let $\mathbf{A}$ be a 2-divisible MV-algebra and $\mathbf{G}$ be a unital abelian ℓ-group such that $\mathbf{A} = \Gamma(\mathbf{G})$. Then $u \mapsto u|_{\mathbf{A}}$ is an affine homeomorphism between the compact convex set of upper states on $\mathbf{G}$ and the compact convex set of upper states on $\mathbf{A}$, and similarly for lower states.*

Now let $\mathbf{A}$ be any MV-algebra and $\mathbf{G}$ a unital abelian ℓ-group such that $\mathbf{A} = \Gamma(\mathbf{G})$. We denote by X the compact Hausdorff space of all valuations on $\mathbf{A}$, identified with the space of valuations on $\mathbf{G}$. Recall that $\mathbf{G}$ has a (generally not one-to-one) representation $\widehat{\mathbf{G}}$ as a separating unital ℓ-subgroup of $C(X)$. Similarly, $\mathbf{A}$ is represented as a separating MV-subalgebra $\widehat{\mathbf{A}}$ of $C(X, \mathbf{I})$.

We are going to show that states, upper states and lower states on unital abelian ℓ-groups and on MV-algebras are in a one-to-one correspondence with states, upper states and lower states, respectively, on the functional representation.

FACT 5.15. *Let* $\mathbf{G}$ *be a unital abelian* ℓ*-group. Consider its Yosida representation* $g \mapsto \widehat{g}\colon \mathbf{G} \to \widehat{\mathbf{G}}$ *by continuous functions on the valuation space* X *of* $\mathbf{G}$*. Then the induced continuous affine maps* $\mathcal{P}(\widehat{\mathbf{G}}) \to \mathcal{P}(\mathbf{G})$ *and* $\overline{\mathcal{P}}(\widehat{\mathbf{G}}) \to \overline{\mathcal{P}}(\mathbf{G})$ *between the spaces of states and upper states, respectively, are homeomorphisms. The corresponding statement also holds for lower states on* $\mathbf{G}$ *and, similarly for states, upper and lower states on MV-algebras.*

Proof. It suffices to prove the claim for upper states on ℓ-groups. For every upper state $\widehat{u}$ on $\widehat{\mathbf{G}}$, the composition $g \mapsto \widehat{g} \mapsto \widehat{u}(\widehat{g})$ is an upper state on $\mathbf{G}$. Conversely, every upper state u on $\mathbf{G}$ factors through the map $g \mapsto \widehat{g}\colon \mathbf{G} \to \widehat{\mathbf{G}}$. Indeed, if g and h are elements such that $\widehat{g} = \widehat{h}$, there is an infinitesimal z such that $h = g+z$. Since $-\mathbf{1} \leq nz \leq \mathbf{1}$ for all natural numbers n and since u is order preserving, we have $-1 = u(-\mathbf{1}) \leq u(nz) = nu(z) \leq u(\mathbf{1}) = 1$ for all natural numbers n, whence $u(z) = 0$. By subadditivity, $u(h) = u(g+z) \leq u(g) + u(z) = u(g)$ and similarly $u(g) = u(h-z) \leq u(h) + u(-z) = u(h)$. The upper state $\widehat{u}$ on $\widehat{\mathbf{G}}$ corresponding to the upper state u on $\mathbf{G}$ is defined by $\widehat{u}(\widehat{g}) = u(g)$. This shows that the induced map is bijective on upper states on $\mathbf{G}$. The same then holds for lower states and for states. □

We now are in a position to apply classical tools of functional analysis to deal with states, upper and lower states through their counterparts in $C(X)$.

6 Tools from functional analysis

Throughout this section, X will be a compact Hausdorff space and $C(X)$ the unital vector lattice of all continuous functions $f\colon X \to \mathbb{R}$ with the supremum norm, the constant function $\mathbf{1}$ being the unit. We specialize the considerations of the previous section to the unital ℓ-group $C(X)$ and to the MV-algebra $\mathbf{A} = C(X, \mathbf{I})$.

Upper and lower states u of $C(X)$ are *homogeneous* in the sense that, for $f \in C(X)$, we have $u(rf) = ru(f)$ not only for natural numbers but for every nonnegative real number r. In order to verify this claim, we first notice that, for any positive natural number n, we have $nu(\frac{1}{n}f) = u(f)$, hence $u(\frac{1}{n}f) = \frac{1}{n}u(f)$, by $\mathbb{N}$-homogeneity. Again by $\mathbb{N}$-homogeneity, $u(\frac{m}{n}f) = mu(\frac{1}{n}f) = \frac{m}{n}u(f)$ for every natural number m. Thus $u(rf) = ru(f)$ for all nonnegative rational numbers r. As u is order preserving, this equality also holds for all nonnegative reals r.

Thus, upper and lower states on $C(X)$ are *sublinear* and *superlinear*, respectively. The states on $C(X)$ are the positive normalized linear functionals. Using Proposition 5.6, a functional u on $C(X)$ is seen to be an upper (resp., lower) state if and only if it is a *coherent upper* (resp., *lower*) *prevision* in Walley's terminology (see [24, Definition 2.3.3]).

We denote by $\mathcal{M}(X)$ the dual vector space of all linear functionals $\mu\colon C(X) \to \mathbb{R}$ which are continuous with respect to the supremum norm. We endow $\mathcal{M}(X)$ with the *weak* topology*, which is the weakest topology such that the evaluation maps $\mu \mapsto \mu(f)\colon \mathcal{M}(X) \to \mathbb{R}$ are continuous for all $f \in C(X)$. This topology can also be seen to be the topology of pointwise convergence: (μ_i converges to μ iff $\mu_i(f)$ converges to $\mu(f)$ for all $f \in C(X)$) induced by the product topology on $\mathbb{R}^{C(X)}$.

The positive linear functionals on $C(X)$ are automatically norm continuous. They form the positive cone $\mathcal{M}_+(X)$ of $\mathcal{M}(X)$. The normalized positive linear functionals on $C(X)$ are precisely the states. They form a convex subset $\mathcal{P}(X)$ of $\mathcal{M}_+(X)$ which is compact in the weak* topology. The weak* topology on $\mathcal{P}(X)$ agrees with the topology of pointwise convergence on state spaces as introduced in Section 4. Through the Riesz representation theorem (see [21]) the states, that is, the normalized positive linear functionals on $C(X)$ can be identified with the regular Borel probability measures on X.

We want to establish a one-to-one correspondence between nonempty closed convex subsets K of the compact convex state space $\mathcal{P}(X)$ and the upper states on $C(X)$. For this we will use tools from functional analysis. The first tool is a standard Hahn-Banach type theorem (see [21, Theorem 3.2]):

Lemma 6.1. *Let W be a linear subspace of a real vector space V. Let $v\colon W \to \mathbb{R}$ be a linear and $u\colon V \to \mathbb{R}$ a sublinear functional such that $v(f) \leq u(f)$ for all $f \in W$. Then there is a linear functional $\mu\colon V \to \mathbb{R}$ extending v such that $\mu(f) \leq u(f)$ for all $f \in V$.*

We apply this Hahn-Banach Theorem in the case $V = C(X)$:

Proposition 6.2. *Let $u\colon C(X) \to \mathbb{R}$ be an upper state. For any f_0 in the positive cone $C_+(X)$ there is a $\mu \in \mathcal{P}(X)$ such that $\mu(f) \leq u(f)$ for all $f \in C(X)$ and $\mu(f_0) = u(f_0)$.*

Proof. Fix $f_0 \in C_+(X)$ and consider the set $C = \{r \cdot \mathbf{1} + tf_0 : r \in \mathbb{R}, t \in \mathbb{R}_+\}$. Since u is supposed to be strongly normalized, the restriction of u is linear on C. Let W be the linear subspace generated by C. We define $v\colon W \to \mathbb{R}$ to be the unique linear extension of $u|_C$. Note that for $f \in W \setminus C$, we have $-f \in C$, hence $-v(f) = v(-f) = u(-f)$. Since $0 = u(0) \leq u(f) + u(-f)$ by the subadditivity of u, we deduce that $v(f) = -u(-f) \leq u(f)$. Thus v and u satisfy the hypotheses of Lemma 6.1 and, consequently, there is a linear functional μ on $C(X)$ which extends v and is dominated by u everywhere. Since $f_0 \in C$ and $\mathbf{1} \in C$, we have $\mu(f_0) = u(f_0)$ as desired and $\mu(\mathbf{1}) = u(\mathbf{1}) = 1$, that is, μ is normalized. Moreover μ is a positive linear functional. Indeed, if $f \geq 0$, then $-f \leq 0$ and consequently $-\mu(f) = \mu(-f) \leq u(-f) \leq u(0) = 0$, since u is order preserving, whence $\mu(f) \geq 0$. □

As a second tool we use the following lemma (see [22, Chapter IV, 1.2]):

Lemma 6.3. *For every weak*-continuous linear functional $F\colon \mathcal{M}(X) \to \mathbb{R}$ there is a uniquely determined $f \in C(X)$ such that $F(\mu) = \mu(f)$ for all $\mu \in \mathcal{M}(X)$.*

Our third tool is a Separation Theorem [22, Chapter II, 9.2]:

Lemma 6.4. *Let C be a closed convex subset in a locally convex topological vector space W and x an element of W not contained in C. Then there is a continuous linear functional F on W and a real number r such that $F(y) \leq r < F(x)$ for all $y \in C$.*

Proposition 6.5. *For every nonempty closed convex set K of states and every state $\mu_0 \notin K$, there is an $f \in C_+(X)$ and a real number r such that $\mu(f) \leq r < \mu_0(f)$ for all $\mu \in K$.*

Proof. Let C be the set of all $\nu \in \mathcal{M}(X)$ such that $\nu \leq \mu$ for some $\mu \in K$. Then C is a weak*-closed convex set not containing μ_0. Indeed, let $(\nu_i)_i$ be a generalized sequence (a net) in C which weak*-converges to some $\nu \in \mathcal{M}(X)$. For each i there is a $\mu_i \in K$ such that $\nu_i \leq \mu_i$. In the weak*-compact set K the generalized sequence $(\mu_i)_i$ has a generalized subsequence $(\mu_{i_j})_j$ which weak*-converges to some $\mu \in K$. Since $\nu_{i_j} \leq \mu_{i_j}$ we conclude that $\nu = \lim_j \nu_{i_j} \leq \lim_j \mu_{i_j} = \mu$, that is, $\nu \in C$. Since distinct states are incomparable, μ_0 is not contained in C. Since $\mathcal{M}(X)$ with its weak* topology is a locally convex topological vector space, by the Separation Theorem (Lemma 6.4), there is a continuous linear functional F on $\mathcal{M}(X)$ such that, for some real number r, $F(\mu) \leq r < F(\mu_0)$ for all $\mu \in K$. By Lemma 6.3, there is an $f \in C(X)$ such that $F(\nu) = \nu(f)$ for all $\nu \in \mathcal{M}(X)$. Thus, $\mu(f) \leq r < \mu_0(f)$ for all $\mu \in C$. It remains to show that $f \in C_+(X)$. Every $\nu \leq 0$ is contained in C, since $0 \leq \mu$ for all $\mu \in K$ which is nonempty by hypothesis. For every $t \in \mathbb{R}_+$ we also have $t\nu \leq 0$, whence $t\nu(f) \leq r$. We conclude that $\nu(f) \leq 0$ for every $\nu \leq 0$ and consequently $\nu(f) \geq 0$ for all $\nu \geq 0$. Hence, $f \geq 0$. □

For every set S of states the functional $S^*\colon C(X) \to \mathbb{R}$ defined by

$$S^*(f) = \sup_{\mu \in S} \mu(f)$$

is an upper state by Lemma 5.10. If S is weak*-closed, the supremum is attained, that is, for every $f \in C(X)$ there is a $\mu_0 \in S$ such that $S^*(f) = \mu_0(f)$.

Conversely, for every functional $u\colon C(X) \to \mathbb{R}$,

$$u^* = \{\mu \in \mathcal{P}(X) : \mu \leq u\}$$

is a weak*-closed convex subset of $\mathcal{P}(X)$ by Lemma 5.1.

Theorem 6.6. *The maps $K \mapsto K^*$ and $u \mapsto u^*$ are mutually inverse bijections between the set $\mathcal{K}(\mathcal{P}(X))$ of all nonempty closed convex subsets of $\mathcal{P}(X)$ and the set $\overline{\mathcal{P}}(X)$ of all upper states on $C(X)$.*

Proof. (a) We know that K^* is an upper state on $C(X)$. We now show that $K^{**} = K$, provided that K is convex and weak*-closed. Clearly, if $\mu \in K$, then $\mu \leq K^*$ by definition. Thus, $K \subseteq K^{**}$. For the reverse inclusion we use Proposition 6.5, which tells us that for every probability measure $\mu_0 \notin K$, there is an $f \in C_+(X)$ such that $\mu_0(f) > \sup_{\mu \in K} \mu(f) = K^*(f)$, that is, $\mu_0 \not\leq K^*$.

(b) We know that u^* is a closed convex subset of $\mathcal{P}(X)$. Clearly $u^{**} \leq u$. The reverse inequality follows directly from Proposition 6.2. □

Dually, for every lower state $u \colon C(X) \to \mathbb{R}$,

$$u_* = \{\mu \in \mathcal{P}(X) : \mu(f) \geq u(f) \text{ for all } f \in C(X)\}$$

is a nonempty weak* closed convex subset of $\mathcal{P}(X)$ and for every weak*-closed subset K of $\mathcal{P}(X)$ the functional defined by

$$K_*(f) = \min_{\mu \in K} \mu(f)$$

is a lower state and, by duality (see Proposition 5.11), we have:

Theorem 6.7. *The maps $K \mapsto K_*$ and $u \mapsto u_*$ are mutually inverse bijections between the set $\mathcal{K}(\mathcal{P}(X))$ of all nonempty closed convex subsets of $\mathcal{P}(X)$ and the set $\underline{\mathcal{P}}(X)$ of all lower states on $C(X)$.*

We finally want to characterize those real-valued functions u defined on some subset $D \subseteq C(X)$ that can be extended to upper or lower states on all of $C(X)$. For $f \in C(X)$, recall the functionals

$$\mathrm{SUP}(f) = \sup_{x \in X} f(x) = \max_{x \in X} f(x),$$

$$\mathrm{INF}(f) = \inf_{x \in X} f(x) = \min_{x \in X} f(x).$$

Note that $\mathrm{SUP}, \mathrm{INF} \colon C(X) \to \mathbb{R}$ are upper and lower states on $C(X)$, respectively, which are conjugate in the sense that $\mathrm{INF}(f) = -\mathrm{SUP}(-f)$. Moreover, SUP is the greatest upper and INF is the least lower state on $C(X)$.

Following Walley [24, p. 122 f., p.137], we consider the *natural upper extension* $\overline{u} \colon C(X) \to \mathbb{R}$ of an arbitrary function $u \colon D \to \mathbb{R}$ defined on some subset $D \subseteq C(X)$ given by

$$\overline{u}(g) = \inf\{s \in \mathbb{R} : g - s \leq \sum_{i=1}^{n} r_i(f_i - u(f_i)) \text{ for some } f_i \in D \text{ and } r_i \geq 0\}.$$

The *natural lower extension* $\underline{u} \colon C(X) \to \mathbb{R}$ is defined similarly:

$$\underline{u}(g) = \sup\{t \in \mathbb{R} : g - t \geq \sum_{j=1}^{n} r_i(f_i - u(f_i)) \text{ for some } f_i \in D \text{ and } r_i \geq 0\}.$$

We summarize those of Walley's results that will be used in the sequel:

Theorem 6.8. *Let X be a compact Hausdorff space. Consider a nonempty subset D of $C(X)$ and a function $u \colon D \to \mathbb{R}$.*

(1) *There is an upper state $\overline{u} \colon C(X) \to \mathbb{R}$ extending u if and only if u is* upper coherent *in the following sense:*

$(\overline{U})$ *For all $f_0, f_1, \ldots, f_n \in D$ and all $m \in \mathbb{N}$,* $\mathrm{INF}\big(\sum_{i=1}^{n} f_i - mf_0\big) \leq \sum_{i=1}^{n} u(f_i) - mu(f_0).$

And if this is the case, the natural upper extension $\overline{u}$ as defined above is the greatest upper state on $C(X)$ extending u.

(2) *There is a lower state $\underline{u}\colon C(X) \to \mathbb{R}$ extending u if and only if u is* lower coherent *in the following sense:*

$$(\underline{U}) \qquad \text{For all } f_0, f_1, \ldots, f_n \in D \text{ and all } m \in \mathbb{N}, \quad \sum_{i=1}^{n} u(f_i) - mu(f_0) \leq \mathrm{SUP}\big(\sum_{i=1}^{n} f_i - mf_0\big).$$

And if this is the case, the natural lower extension $\underline{u}$ as defined above is the least lower state extending u.

(3) *There is a state $\widetilde{u}\colon C(X) \to \mathbb{R}$ extending u if and only if the following condition (L) is satisfied:*

$$(L) \qquad \text{For all } f_1, \ldots, f_n, g_1, \ldots, g_m \in D, \quad \mathrm{SUP}\big(\sum_{i=1}^{n} f_i - \sum_{j=1}^{m} g_j\big) \geq \sum_{i=1}^{n} u(f_i) - \sum_{j=1}^{m} u(g_j).$$

In fact, Walley‘s natural extension is defined for all bounded functions $g\colon X \to \mathbb{R}$. Since continuous real-valued functions on compact Hausdorff spaces are bounded and have maxima and minima, we can restrict Walley's extensions to $C(X)$ and replace inf and sup by min and max, respectively.

7 Back to ℓ-groups and MV-algebras

We want to use the results of the previous section for proving results about imprecise probabilities on MV-algebras in general and for 2-divisible ones in particular.

As a first application of the results in the previous section we consider a compact Hausdorff space X and an ℓ-subgroup $\widehat{\mathbf{G}}$ of $C(X)$ containing the constant function $\mathbf{1}$. The canonical embedding of $\widehat{\mathbf{G}}$ into $C(X)$ induces a continuous affine map from the space of $\overline{\mathcal{P}}(X)$ of upper states on $C(X)$ to the space $\overline{\mathcal{P}}(\widehat{\mathbf{G}})$ of upper states on $\widehat{\mathbf{G}}$, namely $u \mapsto u|_{\widehat{\mathbf{G}}}$. We claim that this map is always surjective and even bijective, if the elements $g \in \widehat{\mathbf{G}}$ separate the points of X:

FACT 7.1. *Let $\widehat{\mathbf{G}}$ be an ℓ-subgroup containing the constant function $\mathbf{1}$ of the vector lattice $C(X)$ over a compact Hausdorff space X. Then every upper state u on $\widehat{\mathbf{G}}$ can be extended to an upper state $\overline{u}$ on $C(X)$. If the functions $g \in \widehat{\mathbf{G}}$ separate the points of X, the extension $\overline{u}$ is unique. The same applies to lower states and to states.*

Proof. Let u be an upper state on $\widehat{\mathbf{G}}$. We verify that u satisfies condition $(\overline{U})$ of Theorem 6.8(1) (compare [24, 2.5.5]): Let $g_0, g_1, \ldots, g_n \in \widehat{\mathbf{G}}$ and $m \in \mathbb{N}$ and set $g = \sum_{i=1}^{n} g_i$. Using $\mathbb{N}$-homogeneity, subadditivity and upper boundedness of u on $\mathbf{G}$, we obtain $mu(g_0) = u(mg_0) = u(g + (mg_0 - g)) \leq u(g) + u(mg_0 - g) \leq u(g) + \mathrm{SUP}(mg_0 - g)$. We conclude that $\mathrm{INF}(g - mg_0) = -\mathrm{SUP}(mg_0 - g) \leq u(g) - mu(g_0)$. Since $u(g) \leq \sum_{i=1}^{n} u(g_i)$ again by subadditivity, we have the desired inequality. From Theorem 6.8(1) we derive that u can be extended to an upper state $\overline{u}$ on $C(X)$.

For the claim of uniqueness, we need some preparations. Firstly we consider the 2-divisible hull $\widehat{\mathbf{G}}_2$ of $\widehat{\mathbf{G}}$ which consists of the functions representable in the form $f = \frac{1}{2^n} g$ for some $g \in \widehat{\mathbf{G}}$ and some $n \in \mathbb{N}$. Because of $\mathbb{N}$-homogeneity, $u(g) = \overline{u}(g) = \overline{u}(2^n f) = 2^n \overline{u}(f)$, whence $\overline{u}(f) = \frac{1}{2^n}\overline{u}(g)$. Thus, the values of $\overline{u}$ on $\widehat{\mathbf{G}}_2$ are uniquely determined by the values of u on $\widehat{\mathbf{G}}$.

Next we show that an upper state w on any ℓ-subgroup $\mathbf{H} \subseteq C(X)$ containing $\mathbf{1}$ is uniformly continuous for the supremum norm. Consider, indeed, any two elements $g, g' \in \mathbf{H}$. The subadditivity of w implies that $w(g) = w(g' + (g - g')) \leq w(g') + w(g - g')$, whence $w(g) - w(g') \leq w(g - g') \leq \mathrm{SUP}(g - g')$, since w is upper bounded. Similarly, $w(g') - w(g) \leq \mathrm{SUP}(g' - g)$. We conclude that $|w(g) - w(g')| \leq \max(\mathrm{SUP}(g - g'), \mathrm{SUP}(g' - g)) = ||g - g'||$.

In particular, the extension $\overline{u}$ of u is uniformly continuous on $C(X)$. If $\widehat{\mathbf{G}}$ separates the points, then its 2-divisible hull $\widehat{\mathbf{G}}_2$ also separates the points of X. By the Stone-Weierstraß Theorem 3.7, $\widehat{\mathbf{G}}_2$ is uniformly dense in $C(X)$. Thus, $\overline{u}$ is uniquely determined by its values on $\widehat{\mathbf{G}}_2$. □

Let $\mathbf{G}$ be a unital abelian ℓ-group and X the compact Hausdorff space of all valuations on $\mathbf{G}$. Collecting the information contained in the FACTS

Theorem 7.2. *The Yosida representation $\mathbf{G} \to C(X)$ induces an affine homeomorphism between the space $\overline{\mathcal{P}}(\mathbf{G})$ of upper states on $\mathbf{G}$ and the space $\overline{\mathcal{P}}(X)$ of upper states on $C(X)$, and similarly for lower states and for states.*

Theorem 4.5 due to Panti and Kroupa is a consequence of the preceding considerations: Consider an MV-algebra $\mathbf{A}$ and its valuation space X. Using also FACT 4.3, we have: The Yosida representation of $\mathbf{A}$ induces an affine homeomorphism between the space $\mathcal{P}(\mathbf{A})$ of states on $\mathbf{A}$ and the space $\mathcal{P}(X)$ of states on $C(X)$.

By Walley's example [24, p. 82] mentioned already in Remark 5.9, FACT 4.3 does not generalize from states to upper and lower states. But for 2-divisible MV-algebras FACT 5.14 furnishes the desired unique extension result and we conclude:

Corollary 7.3. *The Yosida representation $g \mapsto \widehat{g}\colon \mathbf{A} \to C(X, \mathbf{I})$ of a 2-divisible MV-algebra $\mathbf{A}$ induces an affine homeomorphism between the compact convex space $\overline{\mathcal{P}}(\mathbf{A})$ of upper states on $\mathbf{A}$ and the compact convex space $\overline{\mathcal{P}}(X)$ of upper states on $C(X, \mathbf{I})$. The corresponding statement holds for lower states.*

The following theorem contains the solution to our first Problem: Two sets S and T of states on a unital abelian ℓ-group $\mathbf{G}$ represent the same imprecise probability if and only if they have the same closed convex hull. More precisely:

Theorem 7.4. *For a unital abelian ℓ-group $\mathbf{G}$, $K \mapsto K^*$, where $K^*(g) = \max_{\mu\in K} \mu(g)$, is a bijection between the set of all nonempty closed convex subsets K of the state space $\mathcal{P}(\mathbf{G})$ and the set $\overline{\mathcal{P}}(\mathbf{G})$ of upper states.*

Proof. Let X be the valuation space of $\mathbf{G}$. The Yosida representation of $\mathbf{G}$ induces an affine homeomorphism between the state spaces $\mathcal{P}(\mathbf{G})$ of $\mathbf{G}$ and $\mathcal{P}(X)$ of $C(X)$, hence a bijection between the set $\mathcal{K}(\mathcal{P}(\mathbf{G}))$ of compact convex subsets of $\mathcal{P}(\mathbf{G})$ and the set $\mathcal{K}(\mathcal{P}(X))$ of compact convex subsets of $\mathcal{P}(X)$. By Theorem 6.6 we have a bijection between the set $\mathcal{K}(\mathcal{P}(X)$ and the set $\overline{\mathcal{P}}(X)$ of upper states on $C(X)$ given by $K \mapsto K^*$ where $K^*(f) = \max_{\mu\in K} \mu(f)$ for all $f \in C(X)$. Since the Yosida representation also induces a bijection between the upper states on $C(X)$ and the upper states on $\mathbf{G}$, we have proved our claim for ℓ-groups. □

For arbitrary MV-algebras we only know that two sets of states determine the same upper state if their closed convex hulls coincide. But different closed convex sets might determine the same upper state. We still have:

Theorem 7.5. *Let $\mathbf{A}$ be an MV-algebra and $\mathcal{P}(\mathbf{A})$ the compact convex state space of $\mathbf{A}$. Let K be a set of states and u an upper state on $\mathbf{A}$. With the notation of Lemma 5.2 we have:*

(1) *$u^* = (u^\dagger)_*$ is a closed convex subset of $\mathcal{P}(\mathbf{A})$.*
(2) *K^* is an upper state and $K_* = (K^*)^\dagger$ is its conjugate lower state.*
(3) *$u(a) = \max_{\mu\in u^*} \mu(a)$ for all $a \in \mathbf{A}$.*

Proof. (1) follows from Lemma 5.1 and Lemma 5.3(4). Claim (2) is a consequence of Lemma 5.3(5) together with Lemma 5.10 and Proposition 5.11. Claim (3): By definition, the upper state u on $\mathbf{A}$ can be extended to an upper state $\overline{u}$ on the enveloping unital abelian ℓ-group $\mathbf{G}$. By Theorem 7.4 we have $\overline{u}(g) = \max_{\overline{\mu}\in\overline{K}} \overline{\mu}(g)$ for all $g \in \mathbf{G}$, where $\overline{K}$ is the set of all states $\overline{\mu}$ on $\mathbf{G}$ such that $\overline{\mu} \leq \overline{u}$. Restricting all the functionals to $\mathbf{A}$ yields the desired result. □

For 2-divisible MV-algebras the same conclusions as for unital abelian ℓ-groups hold:

Theorem 7.6. *Let $\mathbf{A}$ be a 2-divisible MV-algebra and $\mathcal{P}(\mathbf{A})$ the compact convex set of all states on $\mathbf{A}$. The maps $u \mapsto u^*$ and $K \mapsto K^*$ are mutually inverse bijections between the set $\mathcal{K}(\mathcal{P}(\mathbf{A}))$ of nonempty closed convex subsets of $\mathcal{P}(\mathbf{A})$ and the set $\overline{\mathcal{P}}(\mathbf{A})$ of upper states on $\mathbf{A}$. The corresponding claim holds for lower states on $\mathbf{A}$.*

Proof. For a 2-divisible MV-algebra $\mathbf{A}$ we use Theorem 7.4 for the enveloping unital abelian ℓ-group $\mathbf{G}$. Then we apply FACT 4.3 that $\mu \mapsto \mu|_{\mathbf{A}}$ is an affine homeomorphism from the state space of $\mathbf{G}$ to the state space of $\mathbf{A}$ and that similarly, by FACT 5.14, $u \mapsto u|_{\mathbf{A}}$ is a bijection from the set of upper states on $\mathbf{G}$ to the set of upper states on $\mathbf{A}$. □

8 Upper and lower states and non-reversible bets

In this section we connect the results of the previous sections to the interpretation of imprecise probabilities in terms of bets.

As already outlined in the introduction, we imagine the following game: a bookmaker *Ada* proposes a book Δ consisting of pairs (ϕ, α) where ϕ is a proposition describing an event and α some number in the unit interval. We suppose the underlying logic to be Łukasiewicz logic and we identify propositions if they are equivalent in this logic so that the propositions ϕ are supposed to range over the Lindenbaum algebra of Łukasiewicz propositional logic, that is, the free countably generated MV-algebra. Our statements remain true if ϕ ranges over elements of an arbitrary MV-algebra $\mathbf{A}$.

A bettor *Blaise* chooses a finite subset $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ of Δ and bets λ_i on ϕ_i, $i = 1, \ldots, n$. The possible values that events may take are described by the valuations v of the MV-algebra $\mathbf{A}$. Depending on which of these possibilities will occur, the payoff from Ada to Blaise will be

$$P = \sum_{i=1}^{n} \lambda_i (v(\phi_i) - \alpha_i).$$

The game is *reversible* if the bets λ_i may be chosen among all real numbers and *non-reversible* if the λ_i are bound to be nonnegative.

To begin with, we introduce two types of rationality criteria for betting games.

Definition 8.1. *A* winning strategy *for Blaise based on a book Δ consists of a finite subset $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ of Δ and of a system of bets $\lambda_1, \ldots, \lambda_n$ on $\phi_1, \ldots, \phi_n$, respectively, such that for every valuation v the corresponding payoff from Ada to Blaise is strictly positive. Replacing 'strictly positive' by 'strictly negative' yields the notion of a* losing strategy *for Blaise.*

A bad bet *for Blaise consists of an element (ϕ, α) of Δ and of a bet δ on ϕ such that there is a finite subset $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ of Δ and a system $\lambda_1, \ldots, \lambda_n$ of bets on $\phi_1, \ldots, \phi_n$ which ensures to Blaise a better payoff independently of the valuation; in other terms,*

$$\sum_{i=1}^{n} \lambda_i (v(\phi_i) - \alpha_i) > \delta(v(\phi) - \alpha) \quad \textit{for every valuation } v,$$

equivalently,

$$\sum_{i=1}^{n} \lambda_i v(\phi_i) - \delta v(\phi) > \sum_{i=1}^{n} \lambda_i \alpha_i - \delta\alpha \quad \textit{for every valuation } v.$$

Replacing $>$ by $<$ in the above inequalities yields the notion of a good bet.

The following result implies that, in a reversible game, rationality prevents not only Ada, but also Blaise from sure loss.

Theorem 8.2. *Let Δ be a book in a reversible game. The following are equivalent:*
(1) *There is no winning strategy for Blaise.*
(2) *There is no losing strategy for Blaise.*
(3) *There is no bad bet for Blaise.*
(4) *There is no good bet for Blaise.*

Proof. (1) $\Leftrightarrow$ (2). If $\Gamma = (\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n)$ and $\lambda_1, \ldots, \lambda_n$ constitute a winning strategy for Blaise, then $-\lambda_1, \ldots, -\lambda_n$ constitute a losing strategy for him, and vice-versa.

(2) $\Leftrightarrow$ (3). If betting λ_i on ϕ_i, for $i = 1, \ldots, n$, is a losing strategy for Blaise, then betting λ_1 on ϕ_1 is a bad bet for him: a better alternative is betting $-\lambda_i$ on ϕ_i, for $i = 2, \ldots, n$. Conversely, if betting λ on ϕ is a bad bet and betting λ_i on ϕ_i, for $i = 1, \ldots, n$ constitutes a better alternative, then betting λ on ϕ and $-\lambda_i$ on ϕ_i, $i = 1, \ldots, n$, is a losing strategy.

(3) $\Leftrightarrow$ (4). Betting λ on ϕ is a bad bet iff betting $-\lambda$ on ϕ is a good bet. $\square$

The situation changes when dealing with non-reversible betting games. For instance, if the book consists of $(p, \alpha), (\neg p, 1-\alpha)$, where p is a propositional variable, then Blaise has no winning strategy, but he has both a losing strategy (betting 1 both on p and on $\neg p$) and a good bet (betting nothing). Thus, conditions (1), (2), (3), (4) are no longer equivalent. Moreover, the above book seems to be rational for a non reversible game. Hence, the presence of either a good bet or of a losing strategy for Blaise does not mean that the book is not coherent. Also, in the introduction we have argued that a book which does not admit a winning strategy for Blaise need not be rational. In the next theorem, the main result in this paper, we examine the remaining rationality criterion, the nonexistence of bad bets, and we relate it to imprecise probabilities in the form of closed convex sets of states and to upper states. In the same way, we relate the nonexistence of a good bet to lower states, and the nonexistence of either a good bet or a bad bet to states. One should notice the resemblance to Mundici's theorem 1.2.

Theorem 8.3. *Let Δ be a book in a non-reversible game over an MV-algebra $\mathbf{A}$ of events.*

(1) *The following are equivalent:*
  (1a) *There is no bad bet for Blaise based on Δ.*
  (1b) *There is an upper state u over the MV-algebra $\mathbf{A}$ of events such that, if $(\phi, \alpha) \in \Delta$, then $u(\phi) = \alpha$.*
  (1c) *There is a closed and convex set K of states on $\mathbf{A}$ such that, for all $(\phi, \alpha) \in \Delta$, $\alpha = \max_{\mu \in K} \mu(\phi)$.*

(2) *The following are equivalent:*
  (2a) *There is no good bet for Blaise based on Δ.*
  (2b) *There is a lower state u over the MV-algebra $\mathbf{A}$ of events such that, if $(\phi, \alpha) \in \Delta$, then $u(\phi) = \alpha$.*
  (2c) *There is a closed and convex set K of states on $\mathbf{A}$ such that, for all $(\phi, \alpha) \in \Delta$, $\alpha = \min_{\mu \in K} \mu(\phi)$.*

(3) *The following are equivalent:*
  (3a) *There is neither a good bet nor a bad bet for Blaise based on Δ.*
  (3b) *There is a state μ on $\mathbf{A}$ such that, for all $(\phi, \alpha) \in \Delta$, $\alpha = \mu(\phi)$.*

Proof. We only prove (1) using statement (1) of Theorem 6.8 in a crucial way. The proofs of (2) and (3) are similar using the statements (2) and (3).

We will use the Yosida representation $\phi \mapsto \widehat{\phi}$ of the MV-algebra $\mathbf{A}$ by a MV-subalgebra $\widehat{\mathbf{A}}$ of $C(X, \mathbf{I})$, the MV-algebra of all continuous functions $f : X \to \mathbf{I}$, where X is the compact Hausdorff space of all valuations on $\mathbf{A}$ and $\widehat{\phi}(v) = v(\phi)$ for every valuation v (see Section 3).

Let Δ be a book. Let us consider the situation where there is a bet based on Δ. By definition this means that there are $(\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n), (\psi, \alpha) \in \Delta$ and $\lambda_1, \ldots, \lambda_n, \delta \geq 0$ such that

$$\sum_{i=1}^{n} \lambda_i v(\phi_i) - \delta v(\psi) > \sum_{i=1}^{n} \lambda_i \alpha_i - \delta\alpha \quad \text{for every valuation } v \text{ of } \mathbf{A}.$$

Translated into the functional setting the last inequality reads

$$\sum_{i=1}^{n} \lambda_i \widehat{\phi}_i(v) - \delta\widehat{\psi}(v) > \sum_{i=1}^{n} \lambda_i \alpha_i - \delta\alpha \quad \text{for every } v \in X,$$

or equivalently,

$$\mathrm{INF}\big(\sum_{i=1}^{n} \lambda_i \widehat{\phi}_i - \delta\widehat{\psi}\big) > \sum_{i=1}^{n} \lambda_i \alpha_i - \delta\alpha,$$

where $\mathrm{INF}(f) = \min_{v \in X} f(v)$ for $f \in C(X)$. Note that the last inequality remains strict, since every continuous real-valued function on a compact Hausdorff space assumes its infimum in some point. Thus, there is no bad bet for the book Δ if, for all $(\phi_1, \alpha_1), \ldots, (\phi_n, \alpha_n), (\psi, \alpha) \in \Delta$ and $\lambda_1, \ldots, \lambda_n, \delta \geq 0$,

$$\mathrm{INF}\big(\sum_{i=1}^{n} \lambda_i \widehat{\phi}_i - \delta\widehat{\psi}\big) \leq \sum_{i=1}^{n} \lambda_i \alpha_i - \delta\alpha.$$

We can suppose that an event ϕ at most occurs once in a book Δ; indeed, a rational bookmaker will not propose $(\phi,\alpha),(\phi,\beta)$ with $\alpha\neq\beta$ in the same book. This implies that we can represent the book Δ by the function u defined on the set D all ϕ occurring in some $(\phi,\alpha)\in\Delta$ with value $u(\phi)=\alpha$. The previous inequality now reads:

$$(\overline{\mathrm{U}}') \qquad \mathrm{INF}\Big(\sum_{i=1}^{n}\lambda_i\widehat{\phi}_i-\delta\widehat{\psi}\Big)\leq\sum_{i=1}^{n}\lambda_i u(\widehat{\phi}_i)-\delta u(\widehat{\psi}),$$

and this inequality holds for all $\psi,\phi_1,\ldots,\phi_n\in D$ and all $\delta,\lambda_1,\ldots,\lambda_n\in\mathbb{R}_+$. According to Walley [24, Lemma 2.5.4] this condition $(\overline{U}')$ is equivalent to his upper coherence condition $(\overline{U})$ in Theorem 6.8(1) for the function u. Thus, the nonexistence of a bad bet is equivalent to the upper coherence of u. The upper coherence of u in turn is equivalent to the existence of an upper state $\overline{u}$ on $C(X)$ such that $\overline{u}(\widehat{\phi})=u(\phi)$ for all $\phi\in D$ by Theorem 6.8(1). Through the Yosida representation the upper states on $C(X)$ correspond bijectively to the upper states on the enveloping unital abelian ℓ-group $\mathbf{G}$ of $\mathbf{A}$ (see Theorem 7.2). And the upper states on $\mathbf{A}$ are by definition the restrictions of the upper states on $\mathbf{G}$. Thus we have proved that (1a) and (1b) are equivalent.

We now prove the equivalence of (1b) and (1c). By Theorem 7.5(3) we know that for every upper state u on $\mathbf{A}$ there is a compact convex set K of states such that $u(a)=\max_{\mu\in K}\mu(a)$ for all $a\in\mathbf{A}$. It suffices to take K to be the set of states μ on $\mathbf{A}$ such that $\mu\leq u$. Conversely, if K is a set of states on $\mathbf{A}$ such that $u(a)=\max_{\mu\in K}\mu(a)$ for all $a\in\mathbf{A}$, then u is an upper state by Lemma 5.10. □

In the above theorem one can replace the MV-algebra $\mathbf{A}$ by any unital abelian ℓ-group. Adapting Walley's terminology, one could say that one considers a unital abelian ℓ-group $\mathbf{G}$ of 'gambles' $g\in\mathbf{G}$. A book Δ would be a set of pairs (g,α) with $g\in\mathbf{G}$ and $\alpha\in\mathbb{R}$. A bet in a non-reversible game would consist in the choice of a finite subset $(g_1,\alpha_1),\ldots,(g_n,\alpha_n)$ of Δ and bets $\lambda_1,\ldots,\lambda_n\in\mathbb{R}_+$. Everything else remains as before. In particular, Theorem 8.3 remains valid for this extension. The advantage of ℓ-groups is that we have clearcut notions of 'imprecise probabilities': they correspond in a one-to-one way to the upper states and, equivalently, to the nonempty compact convex sets of states ('probabilities'). This remains true for MV-algebras that are 2-divisible. For MV-algebras in general we do not have a simple characterization of the upper states and different compact convex sets of states may correspond to the same upper state. The clarification of this situation is open to further research.

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