D-COMPLETIONS AND THE d-TOPOLOGY

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Abstract. In this article we study a type of directed completion for posets and T_0-spaces introduced by O. Wyler [Wy81] that we call the D-completion. The category D of monotone convergence spaces is a full reflective subcategory of the category of topological spaces and continuous maps, and the D-completion gives the reflection of a topological space into D, and as such exhibits a useful universal property that characterizes it. The direct definition and theory of the D-completion is based on a variant of the Scott topology, which we call the d-topology. For partially ordered sets the D-completion turns out to be a natural dcpo-completion that generalizes the rounded ideal completion.

1. Introduction

In the theory of lattices and partially ordered sets completions of various types play a basic role. For the theory of posets as they are used for modeling in theoretical computer science, the appropriate type of completeness is directed completeness, the existence of suprema for directed subsets. A topological analogue occurs in the theory of T_0-spaces, where one requires that directed sets in the order of specialization have suprema, and that a directed set converges to its supremum. Such spaces are called monotone convergence spaces in [GHKLMS] and they form an appropriate type of completion for T_0-spaces. As far as we know, O. Wyler [Wy81] was the first to introduce the notion of a monotone convergence space under the name of a d-space and to give a construction for the d-completion of a T_0 space. Wyler observed that this completion was a categorical reflection of the category of T_0-spaces onto the full subcategory of d-spaces. Hence this completion is the universal monotone convergence space completion.

Y. Ershov [Ers99] later investigated d-spaces in their own right. He has shown that, for any subspace X_0 of a d-space X, there is a smallest d-subspace containing X_0, namely the smallest dcpo containing X_0, and that the inclusion is d-completion, a result we rederive via our categorical machinery. This result implicitly gives the uniqueness of the completion.

Dcpos endowed with the Scott topology are basic examples of monotone convergence spaces, and is well-known they form a cartesian closed

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category. This is not true of the category of monotone convergence spaces. Hence in [BSS06] Battenfeld, Schröder, and Simpson considered monotone convergence spaces that are also compactly generated as a full subcategory of the cartesian closed category of compactly generated spaces. They showed that they form a reflective cartesian closed subcategory and that the monotone convergence space completion of a compactly generated space is again a compactly generated space.

Our results in this paper heavily overlap those of Wyler and Ershov just mentioned. However, our approach differs from theirs in at least two significant ways. First of all, we view the monotone convergence space completion as one manifestation of class of potential completions constructed from various full subcategories of the category of T_0-spaces. It is appropriate to label these completions by the subcategories defining them, and hence we denote the monontone convergence space completion as the D-completion, where D is the full subcategory of monotone convergence spaces. Secondly, we introduce a new topology, which we call the d-topology, which facilitates a number of the arguments by allowing standard arguments of a topological nature.

This theory of D-completions can be applied to obtain a universal dcpo-completion of a poset by passing to the Scott the universal monotone convergence space completion, and then taking its order of specialization. This dcpo-completion was also introduced and studied by D. Zhao and T. Fan [ZF], but from a more direct order-theoretic approach. Many of our results in the order theoretic setting, which we derive as corollaries of our topological theory, were previously obtained by them. We do show additionally that the D-completion generalizes the round ideal completion and connect it with the c-spaces of Erné [Ern81] and the α-spaces of Ershov [Ers97].

We close the paper with a more detailed historical appendix that supplements the preceding historical remarks.

We assume throughout a familiarity on the part of the reader with basic notions of domain theory and the related topological notions, as they may be found, for example, in [AJ95] or [GHKLMS].

Throughout the paper we assume that all topological spaces under consideration are T_0 and let TOP_0 denote the category of T_0-spaces and continuous maps. Besides facilitating ease of presentation, this assumption is equivalent to assuming that the order of specialization ($x \leq y$ iff x belongs to the closure of the singleton $\{y\}$) is a partial order, which is the case of interest to us. Since TOP_0 is a full reflective subcategory of the category TOP of all topological spaces and continuous maps, a number of the results of this paper extend to TOP by composing with the T_0-reflection functor, as we occasionally point out.

2. Reflectors and universal completions

We remind the reader that the notion of a full reflective subcategory offers a convenient setting to deal with all kinds of completions. A full subcategory K of a category C is called *reflective* if the inclusion functor has a left adjoint, which then is called a *reflector*. In many cases, reflectors are exhibited in the following way:

Definition 2.1. A morphism $\eta\colon C \to \widetilde{C}$ of an object C in C to an object $\widetilde{C}$ in K is called a *universal* K*-ification* if the following universal property is satisfied:

For every object K in K and every morphism $f\colon C \to K$, there is a unique morphism $\widetilde{f}\colon \widetilde{C} \to K$ such that $\widetilde{f} \circ \eta_C = f$:

$$\begin{array}{ccc} C & \xrightarrow{\ \eta\ } & \widetilde{C} \\ & \searrow^{f} & \downarrow{\scriptstyle \widetilde{f}} \\ & & K. \end{array}$$

Universal K-ifications, if they exist, are unique (as morphisms) up to isomorphism. More precisely, let $\eta\colon C \to \widetilde{C}$ be a universal K-ification. If $\eta'\colon C \to K$ is another universal K-ification, then there is a unique isomorphism $h\colon \widetilde{C} \to K$ such that $\eta' = h \circ \eta$. Conversely, if $h\colon \widetilde{C} \to K$ is an isomorphism, then $\eta' = h \circ \eta\colon C \to K$ is also a universal K-ification.

If every object C in C has a universal K-ification, a reflector $R\colon \mathsf{K} \to \mathsf{C}$ can be found by choosing a universal K-ification $\eta_C\colon C \to \widetilde{C}$ for every object C in C and by defining $R(C) = \widetilde{C}$ and $R(g) = \widetilde{\eta_D \circ g}$ for C-morphisms $g\colon C \to D$; that is, $R(g)$ is the unique morphism such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\ \eta_C\ } & R(C) = \widetilde{C} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle R(g) = \widetilde{\eta_D \circ g}} \\ D & \xrightarrow[\ \eta_D\ ]{} & R(D) = \widetilde{D}. \end{array}$$

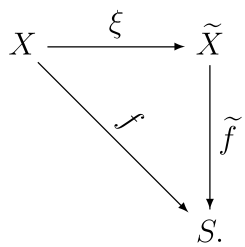
3. Sober spaces

In this section we recall some standard results about sober spaces that will be important for the developments of this paper. The sobrification of a space is a first example of a universal K-ification.

Recall that a subset C of a topological space is *irreducible* if it is nonempty and if $C \subseteq A \cup B$, where A and B are closed, implies that

$C \subseteq A$ or $C \subseteq B$. The closure of an irreducible set is irreducible. The closure $\overline{\{x\}}$ of a singleton is irreducible, and the point x is called a *generic point* of this irreducible set. A space X is *sober* if every closed irreducible subset C has a unique generic point (see, e.g., [GHKLMS, Section O-5] for basic definitions and results on T_0-spaces, sober spaces, and the order of specialization). Sober spaces are T_0-spaces and form the objects of a full subcategory SOB of TOP_0. Indeed SOB is a reflective subcategory of TOP_0, and more generally of TOP [GHKLMS, Proposition V-4.7(iii)]. For defining the reflector, one may proceed as indicated in the preceding section:

Definition 3.1. A *sobrification* of a space X consists of a sober space $\widetilde{X}$ and a continuous map $\xi\colon X \to \widetilde{X}$ which enjoys the following universal property: For every continuous map f from X to a sober space S, there is a unique continuous map $\widetilde{f}\colon \widetilde{X} \to S$ such that $\widetilde{f} \circ \xi = f$:

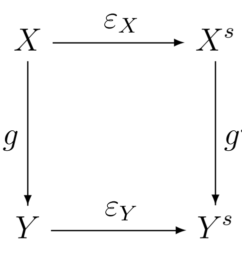


A standard construction for the sobrification of a space X is to set

$$X^s := \{A \subseteq X : A \text{ is closed and irreducible}\}$$

topologized by open sets $U^s := \{A \in X^s : A \cap U \neq \emptyset\}$ for each open subset U of X. If we define $\varepsilon_X\colon X \to X^s$ by $\varepsilon_X(x) = \overline{\{x\}}$, then we obtain a sobrification (see, e.g., [GHKLMS, Exercise V-4.9]), which we call the *standard sobrification.* For a continuous map f from X to a sober space S, the image $f(C)$ of an irreducible closed set $C \subseteq X$ is irreducible in S and the extension $\widetilde{f}\colon X^s \to S$ may be defined by $\widetilde{f}(C) = y$, where y is the unique generic point of the irreducible closed set $\overline{f(C)}$ in S. *We henceforth denote the standard sobrification by* $\varepsilon_X : X \to X^s$.

This standard sobrification yields a reflector form the category TOP_0 to the full subcategory SOB of sober spaces: For a continuous map $g\colon X \to Y$, we define $g^s = (\varepsilon_Y \circ g)^{\sim}\colon X^s \to Y^s$, which is the unique continuous map making the following diagram commutative:



g^s maps an irreducible closed set $C \in X^s$ to the irreducible closed set $\overline{f(C)} \in Y^s$.

It is useful to have conditions to determine when a map is a sobrification map. We recall that $f\colon X \to Y$ is a *quasihomeomorphism* if the correspondence $U \mapsto f^{-1}(U)$ is a bijection from the lattice of open sets $\mathcal{O}(Y)$ to $\mathcal{O}(X)$. We also recall that the *strong topology* of X has as a subbasis of open sets $\mathcal{O}(X) \cup \mathcal{L}$, where $\mathcal{L}$ is the collection of all lower sets

$$A = \downarrow A := \{y : y \leq x \text{ for some } x \in A\}$$

in the order of specialization. The closure of a subset A with respect to the strong topology will be denoted by $cl_s(A)$.

Proposition 3.2. *For a continuous $f : X \to Y$, where Y is sober, the following are equivalent.*

(1) *f is a sobrification map.*
(2) *f is an embedding and $f(X)$ is dense in the strong topology.*
(3) *f is a quasihomeomorphism.*

Proof. For the equivalence of (1), (2), and (3), see Exercises V-5.32 and V-5.34 of [GHKLMS] and the accompanying hints. □

Remark 3.3. Any order preserving map between posets is continuous with respect to the lower set topologies $\mathcal{L}$ (the dual Alexandroff discrete topologies), and hence any continuous $f : X \to Y$ is also continuous with respect to the strong topologies.

Proposition 3.4. *For a subset X of a sober space S, the closure $cl_s(X)$ of X with respect to the strong topology is sober with the relative topology from S, and $cl_s(X)$ is a sobrification of X.*

Proof. Let A be an irreducible subset of X. Its closure $\overline{A}$ in S has a generic point a. We claim that a is in the strong closure of A: Indeed, a basic neigbourhood of a for the strong topology is of the form $U \cap L$, where U is an open set and L a lower set in S, both containing a. Then $A \subseteq \downarrow a \subseteq L$, and U meets A, as a is in the closure of A. Thus, $U \cap L$ meets A.

As the strong topology on a subspace is induced from the strong topology on the whole space, X is strongly dense in $cl_s(X)$. So, $cl_s(X)$ is a sobrification of X by Proposition 3.2. □

The preceding proposition has two immediate consequences:

Corollary 3.5. *The sober subspaces of a sober space are the same as the closed sets for the strong topology.*

Corollary 3.6. *Let $j\colon X \to S$ be an embedding of a space X into a sober space S. Then the corestriction of j to the strong closure $cl_s j(X)$ of $j(X)$ in S is a sobrification.*

4. Completions subordinate to sobrification

We consider a full subcategory K of the category TOP_0 of T_0-spaces. The objects of K will be called K-spaces. We require the following properties:

(1) Homeomorphic copies of K-spaces are K-spaces.
(2) All sober spaces are K-spaces.
(3) In a sober space S, the intersection of any family of K-subspaces is a K-space.

The specialists for the empty set will notice that property (2) is redundant, as it is a consequence of (3).

As every K-space can be embedded into a sober space, for example its standard sobrification, property (3) holds for any K-space K instead of S. Thus, the K-subspaces of any K-space form a closure system, though not necessarily a topology of closed sets. As with every closure system, we have an associated closure operator: For every subset Z, the intersection of all K-subspaces containing Z is again a K-subspace, hence the smallest K-subspace containing Z; it is called the K-closure of Z and denoted by $cl_{\mathsf{K}}(Z)$. Note that if we again assume that K is embedded in its sobrification, then the strong closure of $Z \subseteq K$ is its sobrification (Corollary 3.6), which by (2) is a K-space, and hence contains $cl_{\mathsf{K}}(Z)$. It follows that $cl_{\mathsf{K}}(Z)$ is contained in the strong closure in K, since the relative strong topology agrees with the strong topology of K.

A subspace of a K-space K is said to be K-*dense* if its K-closure is all of K. Note from the preceding paragraph that a K-dense subspace of K is also dense for the strong topology. We require one more property for our category K:

(4) Continuous maps $f\colon S \to T$ between sober spaces S and T are K-continuous, that is, for every K-subspace K of T, the inverse image $f^{-1}(K)$ is a K-subspace of S or, equivalently, $f\big(cl_{\mathsf{K}}(Z)\big) \subseteq cl_{\mathsf{K}} f(Z)$ for every subset $Z \subseteq S$.

Using the sobrification functor one sees that property (4) holds for arbitrary K-spaces, i.e., continuous functions between K-spaces are K-continuous.

Remark 4.1. We note that the category SOB of sober spaces has all the properties (1) through (4) and is in fact the smallest such category. By Corollary 3.5, K-continuity is continuity with respect to the strong topology in this case.

In the remainder of this section, K will be a category satisfying properties (1) through (4).

Definition 4.2. For an arbitrary space X, a map $\eta\colon X \to K$ into a K-space K is called a K-*completion* if η is an embedding onto a K-dense subspace of K.

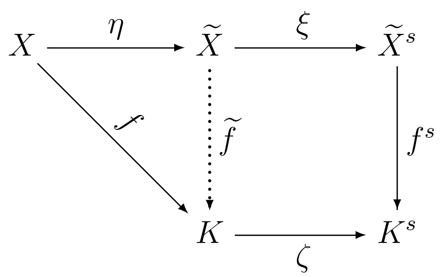
The existence of K completions follows from sobrification:

Definition 4.3. For a topological space X, we define X^k to be the subspace of the standard sobrification X^s obtained by taking the K-closure of $\varepsilon_X(X)$ in X^s, equipped with the subspace topology from X^s. The corestriction of ε_X from X into X^k, denoted $\eta_X \colon X \to X^k$, is a K-completion of X, which we call the *standard* K-*completion*.

Our main result is the following:

Theorem 4.4. *Let $j \colon X \to Y$ be a topological embedding of a space X into a K-space Y. Let $\widetilde{X} = cl_{\mathsf{K}}(j(X))$ be the K-closure of $j(X)$ in Y, equipped with the relative topology from Y. Then the corestriction $\eta \colon X \to \widetilde{X}$ of j is not only a K-completion but even a universal K-ification of X.*

Proof. Let $f \colon X \to K$ be any continuous map into a K-space K. Let $\xi \colon \widetilde{X} \to \widetilde{X}^s$ and $\zeta \colon K \to K^s$ be sobrifications of $\widetilde{X}$ and K, respectively. As $\eta(X)$ is also dense in $\widetilde{X}$ for the strong topology and as $\xi(\widetilde{X})$ is dense in $\widetilde{X}^s$ for the strong topology, $\xi(\eta(X))$ is also dense in $\widetilde{X}^s$ for the strong topology. By Proposition 3.2, the composition $\xi \circ \eta \colon X \to \widetilde{X}^s$ is a sobrification. The universal property of the sobrification of X yields a unique continuous map $f^s \colon \widetilde{X}^s \to K^s$ such that the following outer diagram commutes:



By Proposition 3.2(2), ζ is an embedding, hence $\zeta(K)$ is a K-space, and thus K-closed in K^s. By commutativity, $f^s(\xi(\eta(X))) = \zeta(f(X)) \subseteq \zeta(K)$. By property (4) and the comment following it, the continuous map $f^s \circ \xi$ is K-continuous, and hence the K-closure $\widetilde{X}$ of $\eta(X)$ is carried into $cl_{\mathsf{K}}(\zeta(K)) = \zeta(K)$, i.e., $f^s(\xi(\widetilde{X})) \subseteq \zeta(K)$. As ζ is an embedding, we thus can define $\widetilde{f} \colon \widetilde{X} \to K$ to be $\zeta^{-1} \circ f^s \circ \xi$, and this is a continuous map such that $\widetilde{f} \circ \eta = \zeta^{-1} \circ f^s \circ \xi \circ \eta = \zeta^{-1} \circ \zeta \circ f = f$.

For the uniqueness, suppose that a continuous map $g \colon \widetilde{X} \to K$ satisfies $g \circ \eta = f$. As $\xi \colon \widetilde{X} \to \widetilde{X}^s$ is a sobrification, there is an extension $g^s \colon \widetilde{X}^s \to K^s$ such that $\zeta \circ g = g^s \circ \xi$. But then $g^s \circ \xi \circ \eta = \zeta \circ g \circ \eta = \zeta \circ f$, which implies that $g^s = f^s$ by the uniqueness of the extension f^s. Thus, $g = \widetilde{f}$. $\square$

The next corollary is an immediate consequence of the Theorem 4.4.

Corollary 4.5. *Every* K*-completion of a space* X *is a universal* K*-ification. Thus, any two* K*-completions of* X *are homeomorphic.*

Let $\xi : X \to X^*$ be a K-completion, and let $A \subseteq X$. Let A^* be the K-closure of $\xi(A)$ in X^*. Then the restriction and corestriction $\xi|_A : A \to A^*$ is a K-completion of A. By the previous corollary, it is the universal K-ification of A.

Remark 4.6. From the preceding theorem it follows that the K-spaces form a full reflective subcategory of TOP_0. A reflector is given by assigning to every space X its standard K-completion X^k and, for a continuous function $f\colon X \to Y$ of topological spaces, we just define $f^k = (\eta_Y \circ f)^\sim\colon X^k \to Y^k$. This is the unique continuous map from X^k to Y^k that makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X^k \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^k} \\ Y & \xrightarrow{\eta_Y} & Y^k. \end{array}$$

If this reflector is composed with the reflector functor from TOP to TOP_0, then clearly the composition is a reflector from TOP to K.

Remark 4.7. As K is a full reflective subcategory of TOP_0, the limit of any diagram of K-spaces, which exists in the category TOP_0, is again a K-space, i.e., the category K is complete. In particular, the product of any family of K-spaces is a K-space and the same holds for equalizers.

In general, the reflection from TOP_0 to K does not preserve limits; in particular equalizers need not be preserved. But sometimes the reflection preserves products:

Remark 4.8. Let $X_i, i \in I$, be a family of spaces. For every $i \in I$, let $\eta_i\colon X_i \to X_i^k$ be the standard K-completion. There is an induced embedding $\eta\colon \prod_i X_i \to \prod_i X_i^k$. By the preceding remark, $\prod_i X_i^k$ is a K-space. If the condition

$$cl_{\mathsf{K}}\big(\prod_i \eta_i(X_i)\big) = \prod_i X_i^k$$

is satisfied, then $\eta\colon \prod_i X_i \to \prod_i X_i^k$ is a K-completion, i.e.,

$$\big(\prod_i X_i\big)^k \cong \prod_i X_i^k.$$

Since for topological spaces the closure of the product is the product of the closures, this condition will be satisfied if the K-closed subsets form a topology of closed sets in any K-space.

As a special case we obtain a result of R.-E. Hoffmann [Hoff79, 1.4] for the sobrification: The sobrification of a product space is the product of the sobrifications of the factors, since the sober subsets are the closed subsets for the strong topology by Corollary 3.5.

For a subspace X of a K-space K we may consider the subspaces $X \cap L$ of X, where L ranges over the K-subspaces of K. These subspaces form a closure system on X. We obtain the same closure system on X, if L ranges over the K-subspaces of $cl_{\mathsf{K}}(X)$ only. More generally:

Definition 4.9. Let X be a space and $j\colon X \to K$ an embedding into a K-space. A subspace Y of X is called a *relative* K*-subspace*, if there is a K-subspace L of K such that $Y = j^{-1}(L \cap j(X))$.

We need to show that the notion of a relative K-subspace does not depend on the choice of the embedding j: First, we notice that we may assume that the image $j(X)$ is K-dense in K, i.e., that j is a K-completion. As any two K-completions are homeomorphic, we are done.

The relative K-subspaces of a space X form a a closure system. It follows from considering X as a subspace of a sober space, from our assumption (2), and from Corollary 3.5 that all strongly closed sets are relative K-subspaces. The following lemma is then immediate.

Lemma 4.10. *For a space X with the specialization order and the closure system of relative* K*-subspaces, each principal lower set $\downarrow x$ and its complement is strongly closed, hence a relative* K*-subspace. In particular, the strong topology and any refinement thereof is Hausdorff and totally order disconnected.*

We denote again by $cl_{\mathsf{K}}(Z)$ the smallest relative K-subspace containing a subset Z, and we say that Z is K*-dense in* X if $X = cl_{\mathsf{K}}(Z)$.

Continuous maps are always strongly continuous, and from (4) we derive that continuous maps are also K-continuous in the sense that $f(cl_{\mathsf{K}}(Z)) \subseteq cl_{\mathsf{K}}(f(Z))$. Since a K-dense set is strongly dense (since strongly closed sets are relative K-subspaces), we conclude from the previous lemma by a standard topological argument for Hausdorff spaces:

Corollary 4.11. *If two continuous maps f, g from a space X to another space Y agree on a* K*-dense subset, then they agree everywhere.*

5. The d-topology

In this section we take a brief detour to develop some background that will be helpful for considering particular examples of completions subordinate to sobrification, a topic that we pick up in the next section.

Recall that for a partially ordered set, or simply poset, P, a subset A is defined to be *Scott-closed* if it is a lower set and for every directed subset $D \subseteq A$ that possesses a supremum $\bigvee D$, it is the case that $\bigvee D \in A$. We drop the lower set condition and define a subset A of P to be *d-closed* if for every directed subset $D \subseteq A$ that possesses a supremum $\bigvee D$, it is the case that $\bigvee D \in A$. It is immediate that an arbitrary intersection of d-closed sets is again d-closed and almost immediate that the same is true for finite unions. (Indeed, let A and B be d-sets and D a directed subset of $A \cup B$ which has a supremum $\bigvee D$ in P. If $A \cap D$ is cofinal in D, then $\bigvee D = \bigvee (A \cap D) \in A$. If not, then $D \setminus \downarrow(A \cap D)$ is a cofinal subset of D which is contained in B; and then $\bigvee D = \bigvee (D \setminus \downarrow(A \cap D)) \in B$.) Hence the d-closed sets form the closed sets for a topology, called the *d-topology.* We define the *d-topology of a* T_0*-space* to be the d-topology of the associated order of specialization.

There are some elementary facts that one can quickly derive about this topology.

Lemma 5.1. (1) *A lower set is d-closed if and only if it is Scott-closed.*
(2) *Any upper set is d-closed, and hence any lower set is d-open.*
(3) *If* $\bigvee D$ *exists for a directed set* D*, then the directed set converges to* $\bigvee D$ *in the d-topology.*

Proof. (1) and (2) are immediate. For (3) let U be a d-open set containing $\bigvee D$. If $\bigvee D \in D$, then trivially the directed set is eventually in U. Suppose that D is not eventually contained in U. Then there exists a cofinal subset E of D contained in the complement $X \setminus U$. But then $\bigvee D = \bigvee E \in X \setminus U$ by closedness, a contradiction. □

A poset is a *dcpo* if every directed subset has a supremum. A *sub-dcpo* of a poset P is a subset A with the property that every directed subset of A has a supremum in P which belongs to A. The proof of the following lemma is almost immediate, hence omitted, and the results will be used freely in the following.

Lemma 5.2. *Let* $(P, \leq)$ *be a poset or a space equipped with its order of specialization.*

(1) *A sub-dcpo of* P *is d-closed, and the converse holds if* P *is a dcpo.*
(2) *For any nonempty subset* X *of a dcpo* P*, the closure* $cl_d(X)$ *in the d-topology is the intersection of all sub-dcpos containing* X*.*
(3) *If* Q *is a d-closed subset, then the d-topology of* Q *agrees with the relative d-topology.*

A function $f : P \to Q$ between two posets is *Scott-continuous* if it preserves the order and all existing directed sups.

Lemma 5.3. *Let* $f : P \to Q$ *be a function between posets. The following are equivalent:*

(1) f *is Scott-continuous;*
(2) f *is continuous with respect to the Scott topology;*
(3) f *is order preserving and d-continuous.*

Proof. For the equivalence of (1) and (2), see [GHKLMS, Chapter II-2] (there it is proved for dcpos, but the same proof works in general).

(3)⇒(2): If f is d-continuous and order preserving, then the inverse of a Scott-closed set will be a lower set that is d-closed, hence Scott-closed by Lemma 5.1(1).

(1)⇒(3): If f is Scott-continuous, then it is order-preserving. Let B be d-closed in B, and let $A = f^{-1}(B)$. If D is a directed set in A with supremum $\bigvee D$, then $f(\bigvee D) = \bigvee(f(D))$, since f preserves all existing directed sups. Since B is d-closed, $\bigvee(f(D)) \in B$, and thus $\bigvee D \in A$. □

Note that, in contrast to Scott-continuous maps, it need not be the case that a d-continuous map is order-preserving. Indeed a bijection from a chain of length 2 to an antichain of length 2 provides a simple counterexample.

On the product $P \times Q$ of two posets P and Q we have the intrinsic d-topology and the product of the d-topologies on P and Q. In general, the former is finer than the latter. It may be strictly finer: There are dcpos P and Q with the property that the Scott topology on $P \times Q$ is strictly finer than the product of the two Scott topologies (see [GHKLMS, p.197]). By Lemma 5.1(1) the corresponding property holds for the d-topology. Thus the closure operators are different for the two topologies on $P \times Q$. It is noteworthy, that the two closure operators agree for subsets of the form $A \times B$:

Lemma 5.4. *For subsets A and B of posets P and Q, respectively, we have $cl_d(A \times B) = cl_d(A) \times cl_d(B)$.*

Proof. As the projections from $P \times Q$ onto its factors are d-continuous, $cl_d(A \times B) \subseteq cl_d(A) \times cl_d(B)$. Conversely, for every $a \in A$ set B_a equal to all $y \in cl_d(B)$ such that $(a, y) \in cl_d(A \times B)$. Then B_a contains B and is closed under existing directed sups (since $cl_d(A \times B)$ is), hence is equal to $cl_d(B)$. Thus, we obtain $A \times cl_d(B) \subseteq cl_d(A \times B)$. Repeating the argument yields $cl_d(A) \times cl_d(B) \subseteq cl_d(A \times B)$. □

6. The monotone convergence space completion

In this and the following section we will look at particular instances of completions subordinate to sobrifications. Our first example will be the subcategory D of monotone convergence spaces and continuous maps.

We recall that a T_0-space is a *monotone convergence space* (terminology from [GHKLMS]) if every subset D that is directed in the order of specialization has a supremum to which it converges. These were previously studied first by O. Wyler [Wy81] and later by Y. Ershov [Ers99] under the name of d-spaces. Useful standard facts (from [GHKLMS, Section II-3]) include the observations that a monotone convergence space is a dcpo in the order of specialization and that each of its open sets is Scott-open.

Remark 6.1. Since the closed subsets of a monotone convergence space X are Scott-closed, by Lemma 5.1 we have that the closed sets and the upper sets are closed in the d-topology. Since these sets form a subbasis for the closed subsets of the strong topology, the d-topology refines the strong topology of X. It follows that the d-topology is Hausdorff and the point-closures $\downarrow x$ are clopen (so that the d-topology is totally order disconnected).

We record a basic lemma.

Lemma 6.2. *Let $f\colon X \to Y$ be a continuous map of topological spaces. If D is a subset of X which is directed and has a supremum in the specialization order to which it converges, then f(D) is directed and has a supremum in Y to which it converges, and $f(\bigvee D) = \bigvee f(D)$.*

Proof. (i) Since f is continuous it is order preserving (see [GHKLMS, Section O-5]) for the order of specialization. Let D be a directed set in X with supremum d. By order preservation $f(D)$ is a directed set and $f(c) \leq f(d)$ for all $c \in D$, i.e., $f(d)$ is an upper bound for $f(D)$. Since D converges to d, by continuity $f(D)$ converges to $f(d)$. If b is also an upper bound of $f(D)$, then $\downarrow b = \overline{\{b\}}$ is a closed set containing $f(D)$, and hence contains the limit point $f(d)$, i.e., $f(d) \leq b$. Thus $f(d)$ is the supremum of $f(D)$. □

We say that a subspace A of a T_0-space X is a *monotone convergence subspace* if it is a monotone convergence space in the subspace topology. The following lemma gives alternative characterizations of the monotone convergence subspaces; indeed, in a monotone convergence space they are precisely the d-closed subsets.

Lemma 6.3. *Consider the following properties for a subset A of a T_0-space X:*

(1) *A is a monotone convergence subspace.*
(2) *A is a sub-dcpo.*
(3) *A is d-closed.*

Then (1) *implies* (2) *implies* (3)*, and all three are equivalent if X is a monotone convergence space.*

Proof. (1)⇒(2): Apply Lemma 6.2 to the inclusion map.

(2)⇒(3): Lemma 5.2(1).

(3)⇒(1): Assume X is a monotone convergence space, and let D be a directed set in A. Then D has a supremum d to which it converges. Since A is d-closed, $d \in A$. Thus D converges to d in the subspace topology. □

Lemma 6.4. *The category* D *of monotone convergence spaces satisfies the properties (1) through (4) of Section 4. Furthermore, for any monotone convergence space, the* D*-closed subspaces are precisely those closed in the d-topology, hence form a topology of closed sets.*

Proof. (1) Straightforward. (2) Every sober space is a monotone convergence space [GHKLMS, Section II-3]. (3) The monotone convergence subspaces of a sober space, or more generally of a montone convergence space, are the d-closed ones by 6.3, which are closed under intersections. (4) Let $f: S \to T$ be a continuous map of sober spaces or, more generally, monotone convergence spaces and M a monotone convergence subspace of T. Then M is a sub-dcpo. As f preserves directed suprema by Lemma 6.2, the preimage $f^{-1}(M)$ is a also sub-dcpo, hence a monotone convergence subspace by Lemma 6.3. Thus f is D-continuous.

The preceding Lemma tells us that, in a monotone convergence space, the monotone convergence subspaces form a closure system and, by Lemma 6.3, the D-closed subsets are precisely the d-closed subsets. In particular, the D-closed sets form a topology which agrees with the d-topology. □

It follows immediately from the preceding that the closure operator cl_{D} coincides with the topological closure operator cl_d and the D-dense subsets agree with the d-dense ones.

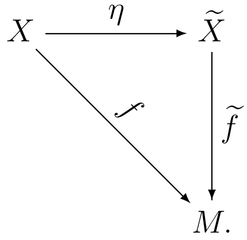
We now may proceed consistent with our general categorical framework defining the notion of a D-completion and the standard D-completion:

Definition 6.5. An embedding $\eta: X \to \widetilde{X}$ of a space X onto a d-dense subset of a monotone convergence space $\widetilde{X}$ is called a D-*completion*.

Definition 6.6. For a topological space X, we define X^d to be the subspace of the standard sobrification X^s obtained by taking the d-closure of $\varepsilon_X(X)$ in X^s, that is, X^d as a topological space is equal to $(cl_d(\varepsilon_X(X)), \tau)$, where τ is the subspace topology from X^s. The corestriction of ε_X from X into X^d, denoted $\eta_X : X \to X^d$, is a D-completion of X, which we call the *standard* D-*completion*.

Our Main Theorem 4.4 tells us that D-completions are universal:

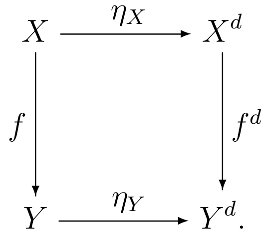
Theorem 6.7. *Let $j: X \to Y$ be a topological embedding of a space X into a monotone convergence space Y. Let $\widetilde{X} = cl_d(j(X))$ be the d-closure of $j(X)$ in Y, equipped with the relative topology from Y. Then the corestriction $\eta: X \to \widetilde{X}$ is a universal D-completion, that is, for every continuous map f from X to a monotone convergence space M, there is a unique continuous map $\widetilde{f}: \widetilde{X} \to M$ such that $\widetilde{f} \circ \eta = f$:*



Since universal objects are unique up to isomorphism, we have the

Corollary 6.8. *Any two* D*-completions of* X *are homeomorphic.*

Remark 6.9. From the preceding theorem it follows that the monotone convergence spaces form a full reflective subcategory D of TOP_0. A reflector is given by assigning to every space X its standard D-completion X^d and, for a continuous function $f\colon X \to Y$ of topological spaces, we define $f^d = (\eta_Y \circ f)^{\sim}\colon X^d \to Y^d$. This is the unique continuous map from X^d to Y^d such that the following diagram commutes:



If this reflector is composed with the reflector from TOP to TOP_0, then clearly the composition is a reflector from TOP to D.

Corollary 6.10. *For a continuous* $\eta\colon X \to \widetilde{X}$ *the following are equivalent.*

1. (1) $\eta\colon X \to \widetilde{X}$ *is a* D*-completion.*
2. (2) $\eta\colon X \to \widetilde{X}$ *satisfies the universal property of Theorem 6.7.*
3. (3) η *is an embedding,* $\widetilde{X}$ *is a monotone convergence space, and no proper subdcpo of* $\widetilde{X}$ *contains* $\eta(X)$*.*
4. (4) η *is an embedding with the property that if* $i : X \to Y$ *is continuous,* $j : Y \to \widetilde{X}$ *is continuous and injective, and* $ji = \eta$*, and if* Y *is a monotone convergence space, then* j *is a homeomorphism.*

Proof. That (1) implies (2) follows from Theorem 6.7, and conversely by the same theorem any D-completion is universal, hence homeomorphic to any other universal completion. Thus all universal completions are D-completions. The equivalence of (1) and (3) follows from Lemma 6.3.

We show that (1) implies (4). First note that i is a topological embedding since the restriction of $\eta^{-1} \circ j$ to $i(X)$ gives the continuous inverse. Thus the corestriction $i : X \to Y_1$, where Y_1 is the d-closure of $i(X)$ in Y, is a D-completion. Then j restricted to Y_1 is a morphism between D-completions of X, and hence by Corollary 6.8 a homeomorphism from Y_1 onto $\widetilde{X}$. Since j is injective on Y, it follows that $Y_1 = Y$.

Finally assume (4). Let Y be the d-closure of $\eta(X)$ in $\widetilde{X}$, let $i : X \to Y$ be the corestriction of η, and let $j : Y \to \widetilde{X}$ be the inclusion. The $i : X \to Y$ is a D-completion, $ji = \eta$, and by hypothesis j is a homeomorphism. Hence $\eta : X \to \widetilde{X}$ is a D-completion, and (1) is satisfied. □

Condition (4) can be useful in applications. We make a brief detour for one application and include another in the next section. Since the category of monotone convergence spaces is not cartesian closed, in [BSS06] Battenfeld, Schröder, and Simpson considered monotone convergence spaces that are also compactly generated as a full subcategory of the cartesian closed category of compactly generated spaces. They showed that they form a reflective cartesian closed subcategory. We rederive an important result of theirs using the preceding machinery. Recall that a space is *compactly generated* if it is a direct limit of compact Hausdorff spaces.

Corollary 6.11. *The* D*-completion of a compactly generated space is again compactly generated.*

Proof. Let X be compactly generated and let $\eta : X \to \widetilde{X}$ be a D-completion. Let Y be $\widetilde{X}$ equipped with its compactly generated topology, let $i : X \to Y$ be defined by η, and let $j : Y \to X$ be the identity map (as a set map). Since $\widetilde{X}$ is a monotone convergence space, its Scott topology refines the given topology. By [ELS04, Theorem 4.7] the dcpo $\widetilde{X}$ endowed with the Scott topology is compactly generated, and hence refines the topology of Y. Thus Y is a monotone convergence space. Applying Corollary 6.10(4), we conclude that j is a homeomorphism. □

As the monotone convergence spaces form a full reflective subcategory of TOP_0, the limit of every diagram of monotone convergences spaces is again a monotone convergence space. By Lemma 5.4, the d-closure of $X_1 \times \cdots \times X_n$ in $X_1^k \times \cdots \times X_n^k$ is all of $X_1^k \times \cdots \times X_n^k$. From Remark 4.8 we conclude:

Proposition 6.12. *For finitely many spaces $X_1, \ldots, X_n$, the product $X_1^k \times \cdots \times X_n^k$ of the* D*-completions of the individual spaces X_i is a* D*-completion of the product space $X_1 \times \cdots \times X_n$.*

The next corollary is an immediate consequence of Theorem 6.7.

Corollary 6.13. *Let $\xi : X \to \widetilde{X}$ be a* D*-completion, and let $A \subseteq X$. Let $\widetilde{A}$ be the d-closure of $\xi(A)$ in the d-topology of X^m. Then the restriction and corestriction $\xi|_A : A \to \widetilde{A}$ is a* D*-completion of A.*

Remark 6.14. Wyler [Wy81] defines a space to be a *d-space* if the closure of every (for the specialization order) directed subset D has a generic point x. It then follows, that $x = \bigvee D$ and that D converges to x, that is, a d-space is a monotone convergence space. Conversely, when D is a directed subset of a monotone convergence space, then D converges to $x = \bigvee D$, whence x is in the closure of D and then $\downarrow x$ is the closure of D. Thus, the notions *d-space* and *monotone convergence space* agree.

7. Directed completions of posets

In this section we apply the preceding construction and results to posets and dcpos by endowing them with the Scott topology. As far as the authors know, D. Zhou and T. Fan were the first to introduce and study this useful completion in [ZF], and most of the following results may be found there, where they are derived in a more order-theoretic rather than categorical or topological manner. For instance taking the d-closure of a set is replaced by taking the smallest sub-dcpo containing the set.

To apply the results of the preceding section, we embed the partially ordered sets as objects in the category TOP_0 by equipping each partially ordered set P with the Scott topology, denoted $\sigma(P)$. If we take for morphisms not all monotone maps, but only those between posets that are Scott-continuous, then we obtain a full subcategory of the category TOP_0. In this section we henceforth tacitly consider posets as topological spaces in this fashion.

The following straightforward observation is crucial:

Lemma 7.1. *If D is a directed subset of a poset that has a least upper bound $\bigvee D$, then D converges to $\bigvee D$ with respect to the Scott topology.*

The preceding lemma implies that a dcpo is a monotone convergence space with respect to the Scott topology. The next lemma gives conditions for a D-completion to carry the Scott topology.

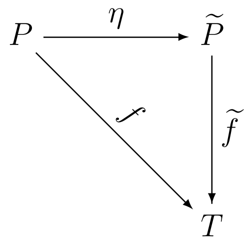
Lemma 7.2. *Let $\xi : X \to \widetilde{X}$ be a D-completion for a space X, let σ denote the Scott topology on $\widetilde{X}$ with respect to the order of specialization, and suppose that $\xi : X \to (\widetilde{X}, \sigma)$ is continuous. Then the original topology on $\widetilde{X}$ is the Scott topology.*

Proof. Let Y denote $\widetilde{X}$ endowed with the Scott topology, which refines the given topology since $\widetilde{X}$ is a monotone convergence space. Let $i : X \to Y$ be defined by η. Since with respect to the order of specialization $\widetilde{X}$ is a dcpo, Y is a monotone convergence space. Applying Corollary 6.10(4) to the identity $j : Y \to \widetilde{X}$, we obtain the desired conclusion. □

Definition 7.3. A *dcpo-completion* of a poset P is a dcpo $\widetilde{P}$ together with a map $\eta\colon P \to \widetilde{P}$ which is an embedding for the Scott topologies and has the property that the image $\eta(P)$ is dense in $\widetilde{P}$ for the d-topology.

It seems that this notion of a dcpo-completion cannot be readily phrased in order theoretical terms. The following theorem shows that every *dcpo*-completion is universal:

Theorem 7.4. *Let $j\colon P \to Y$ be an embedding of a poset P with the Scott topology into a monotone convergence space Y. Let $\widetilde{P} = cl_d(j(X))$ be the d-closure of $j(X)$ in Y. Then the relative topology on $\widetilde{P}$ induced by the topology of Y agrees with the Scott topology associated with the specialization order. The corestriction $\eta\colon P \to \widetilde{P}$ of j is a universal dcpo-completion, that is, for every dcpo T and every Scott-continuous map $f\colon P \to T$, there is a unique Scott-continuous map $\widetilde{f}\colon \widetilde{P} \to T$ such that $f = f^d \circ \eta$:*



Proof. From Theorem 6.7 we know that $\widetilde{P}$ with the relative topology is a monotone convergence space with the universal property that, for any continuous map f from P to a monotone convergence space M, there is a unique continuous map $\widetilde{f}\colon \widetilde{P} \to M$ such that $f = \widetilde{f} \circ \eta$. This holds in particular for the special case that $M = T$ is a dcpo with its Scott topology. We deduce from Lemma 6.2 that j is Scott continuous, and then from Lemma 7.2 that the topology of $\widetilde{P}$ agrees with the Scott topology. □

Definition 7.5. We equip P with the Scott topology $\sigma(P)$ and define the *standard dcpo-completion* to be the standard D-completion of $(P, \sigma(P))$ equipped with its order of specialization. More specifically, we consider the sobrification of $(P, \sigma(P))$, $\varepsilon_P : P \to P^s$, let P^d be the closure of the image of P in the d-topology of P^s equipped with the relative specialization order from P^s, and call the corestriction $\eta_P : P \to P^d$ the *standard dcpo-completion.*

Note that η_P is a topological embedding for the respective Scott topologies since ε_P is a topological embedding and since the topology induced from P^s coincides with the Scott topology on P^d by the preceding Theorem.

Remark 7.6. The preceding theorem may be stated in categorical language as the existence of a reflection, but a bit of care is needed. The reflector given by the theorem is the left adjoint to the inclusion functor from the category of dcpos and Scott-continuous maps into the category of posets and Scott-continuous maps, not posets and order preserving maps.

By passing to Scott topologies, we obtain the following as a special case of Proposition 6.12 and Corollary 6.13.

Proposition 7.7. *The product $P_1^d \times \cdots \times P_n^d$ of the dcpo-completions of the posets $P_i, i = 1, \ldots, n$ is a dcpo-completion of the product $P_1 \times \cdots \times P_n$, provided the Scott topology of the product $P_1 \times \cdots \times P_n$ is the product of the Scott topologies of the factors.*

Corollary 7.8. *Let P be a poset, let $\xi : P \to \widetilde{P}$ be a dcpo-completion, and let $A \subseteq P$ such that its Scott topology agrees with the relative Scott topology. Let $\widetilde{A}$ be the d-closure of $\xi(A)$ in the d-topology of $\widetilde{P}$. Then the restriction and corestriction $\xi|_A : A \to \widetilde{A}$ is a dcpo-completion of A.*

8. Bounded completions

As a variant to monotone convergence spaces and D-completions, we consider *bounded monotone convergence spaces*, T_0-spaces in which every subset D that is directed in the order of specialization and is bounded above has a supremum to which it converges. A *bounded directed complete partially ordered set*, *bdcpo* for short, is a poset in which each principal ideal $\downarrow x$ is a dcpo.

Remark 8.1. In analogy to monotone convergence spaces, the bounded monotone convergence spaces form a full reflective subcategory BD of TOP_0. Indeed any monotone convergence space, in particular any sober space, is a bounded monotone convergence space. Collections of bounded monotone convergence subspaces are easily seen to be closed under intersection (although not necessarily under finite unions), since monotone convergence subspaces are. With these observations the proof of Lemma 6.4 readily adapts to conclude that the category BD of bounded monotone convergence spaces satisfies the properties (1) through (4) of Section 3, so that the results and categorical constructions of that section apply to this setting.

Definition 8.2. A *bounded* D*-completion*, BD*-completion* for short, of a space X consists of a bounded monotone convergence space $\widetilde{X}$ and a topological embedding $\zeta \colon X \to \widetilde{X}$ which maps X onto a d-dense subset of $\widetilde{X}$.

In light of the preceding remarks we have the following special case of Theorem 4.4 and the following Corollary 4.5:

Theorem 8.3. *Let $j : X \to Y$ be a topological embedding into a bounded monotone convergence space Y. Let $\widetilde{X}$ be the smallest bounded monotone convergence subspace of Y containing $j(X)$. Then the corestriction $\eta : X \to \widetilde{X}$ is a universal* BD*-completion and any two* BD*-completions are homeomorphic.*

The bounded monotone convergence space completion may be obtained in a direct fashion from the D-completion (monotone convergence space completion). We begin with a property reminiscent of the sobrification and strong topology.

Lemma 8.4. *If $\xi : X \to X^d$ is a D-completion, then $\xi(X) \cap \downarrow y$ is d-dense in $\downarrow y$ for all $y \in X^d$.*

Proof. Since X^d is the d-closure of $\xi(X)$, it follows that $\xi(X) \cap \downarrow y$ is d-dense in the d-open set $\downarrow y$ (Lemma 5.1). □

Lemma 8.5. *Let $\xi : X \to X^d$ be a D-completion. Then $X^b = \downarrow \xi(X)$ is the smallest bdcpo containing $\xi(X)$, and hence the corestriction $\zeta : X \to X^b = \downarrow \xi(X)$ is a BD-completion.*

Proof. Each principal ideal $\downarrow x$ of X^d is a monotone convergence subspace, and hence $\downarrow \xi(X)$ is a bounded monotone convergence subspace containing $\xi(X)$. We need to show it is the smallest one. Let $\widehat{X}$ be another bounded monotone convergence subspace containing $\xi(X)$, and let $x \in X$. Then $\widehat{X} \cap \downarrow \xi(x)$, the lower set of $\xi(x)$ in $\widehat{X}$, must be a monotone convergence subspace, hence d-closed. By the preceding lemma $\xi(X) \cap \downarrow \xi(x)$ is d-dense in $\downarrow \xi(x)$, and hence its d-closure $\downarrow \xi(x)$ is contained in the d-closed set $\widehat{X} \cap \downarrow \xi(x)$. This shows that $\downarrow \xi(X) \subseteq \widehat{X}$. □

The following corollary is essentially a reformulation of Lemma 8.5.

Corollary 8.6. *Let $j : X \to Y$ be a topological embedding, where Y is a monotone convergence space. Then the corestriction of j to the intersection of $\downarrow j(X)$ and the d-closure of $j(X)$ in Y is a bounded D-completion of X.*

As we used the sobrification functor to define the standard D-completion, we use the latter to define the standard bounded D-completion.

Definition 8.7. For a topological space X, let $\eta_X : X \to X^d$ be its standard D-completion. We define $X^b := \downarrow \eta_X(X)$ and call the corestriction of η_X, denoted $\zeta_X : X \to X^b$, the *standard bounded D-completion* of X.

Remark 8.8. By considering the Scott topology on posets and dcpos, the preceding results may be applied to construct *bounded* (also called *local*) *dcpo-completions* that give the reflections to the inclusion of the full subcategory of local dcpos into the category of posets with Scott-continuous maps. We may find a bounded *dcpo*-completion P^b of a poset P by forming first a *dcpo*-completion $j\colon P \to P^d$ and taking the corestriction of this map to $P^b = \downarrow j(P)$. Every bounded *dcpo*-completion is universal and, hence, any two bounded *dcpo*-completions are isomorphic.

These local completions have been considered by Zhao and Fan [ZF]. They point out that an earlier attempt by Mislove [Mi99] to construct such local completions is flawed because inadequate attention is paid to the fact that the Scott topology of a dcpo need not be sober.

9. Round ideal completions and ideal completions

In this section we show how round ideal completions can be realized in the framework of d-completions. We have to provide some topological prerequisites first.

Recall that a T_0-space X is called a *c-space* (according to Erné [Ern81, 2.11], [Ern91]) or an *α-space* (in the terminology of Ershov [Ers93, Ers97]) if each of its points has a neighborhood basis of principal filters $\uparrow x = \{y \in X \mid x \leq y\}$ with respect to the specialization order.

In a c-space X we may define $x \prec y$ if $\uparrow x$ is a neigborhood of y. This relation $\prec$ is *fully transitive* which means that it is transitive and satisfies the *strong interpolation property*:

$$\forall |F| < \infty,\ F \prec z \Rightarrow \exists y \prec x \text{ such that } F \prec y,$$

where we write $F \prec y$ if $x \prec y$ for all $x \in F$. For the case $F = \emptyset$, we interpret the condition to mean that for all $z \in P$, there exists $y \in P$ such that $y \prec z$. Thus a c-space equipped with $\prec$ is an *abstract basis*, by which we mean a pair $(X, \prec)$, where X is a set equipped with a fully transitive relation $\prec$ (see [AJ95]).

Conversely, for $(X, \prec)$ an abstract basis and $b \in X$, define

$$\Uparrow b := \{x \in X : b \prec x\}.$$

It follows from the strong interpolation property that the sets $\Uparrow b$ form a basis for a topology on X, which we call the *pseudoScott* topology. The resulting topological space is a c-space. In fact, the notion of a c-space is equivalent to that of an abstract basis, as Erné [Ern91] has shown, and the constructions of $\prec$ for a c-space and the pseudoScott topology for an abstract basis are inverse constructions giving the equivalence.

Given an abstract basis $(X, \prec)$, we define a *round ideal* to be a nonempty subset I of X with the properties (i) if $y \in I$ and $x \prec y$, then $x \in I$, and (ii) if $x, y \in I$, then there exists $z \in I$ such that $x \prec z$ and $y \prec z$. The *round ideal completion* is the partially ordered set $(RI(X), \sqsubseteq)$ consisting of all round ideals ordered by set inclusion. It is a basic result that the round ideal completion is a continuous domain and that the map $j : X \to RI(X)$ defined by $j(y) = \{x \in X : x \prec y\}$ sends X onto a basis of $RI(X)$. It was shown in [La97, Theorem 3.3] that, for an abstract basis X, the map $j : X \to RI(X)$ is a sobrification of X, if X is equipped with the pseudoScott topology and $RI(X)$ is equipped with the Scott topology. Since any c-space X is identical to the corresponding abstract basis equipped with the pseudoScott topology, $RI(X)$ equipped with the Scott topology is its sobrification. Since $j(X)$ is a basis, the image is d-dense, and thus $RI(X)$ is also the D-completion. We summarize:

Proposition 9.1. *For a c-space X, the map $j : X \to RI(X)$ is the* D*-completion of X, if the rounded ideal completion $RI(X)$ is equipped with the Scott topology.*

Remark 9.2. The preceding proposition makes possible an extension of the equivalence of Erné. Not only are c-spaces and abstract bases equivalent, but by taking the D-completion of a c-space and the rounded ideal completion of the corresponding abstract basis, one again obtains a c-space (the D-completion) and an abstract basis (the way-below relation of the rounded ideal completion) that are equivalent.

A special important case is that of a continuous poset P. In this case P is a c-space for the Scott topology, and the embedding $j : P \to RI(P)$, both equipped with the Scott topology, is the D-completion. See [La97, Example 3.5].

Proposition 9.3. *If P is a continuous poset, then P is a c-space, and the map $j : P \to RI(P)$ into the round ideal completion is a* D*-completion, provided both P and $RI(P)$ are equipped with the Scott topology. Thus the map j gives the universal dcpo-completion.*

Another special case is the ideal completion of a poset P. In this case the equivalence between c-spaces and abstract bases restricts to an equivalence between Alexandroff discrete spaces (spaces in which every upper set in the order of specialization is open) and abstract bases that are partially ordered sets with the partial order $\leq$ taken for the relation $\prec$, as can be easily verified. The rounded ideals in this case are simply the *ideals*, that is, the directed lower sets, and thus the rounded ideal completion coincides with the ideal completion. Proposition 9.1 and the following remark specialize to the following:

Proposition 9.4. *Let P be a poset, let $I(P)$ be its ideal completion, and let $\eta : P \to I(P)$ be the embedding $x \mapsto \downarrow x$, which identifies P with the compact elements of $I(P)$. If P is endowed with the Alexandroff discrete topology and $I(P)$ with the Scott topology, then $\eta : P \to I(P)$ is a* D*-completion. Hence the order of specialization of a* D*-completion of a poset endowed with the A-topology yields (up to isomorphism) the ideal completion.*

The fact that the ideal completion with the Scott topology is the sobrification of the Alexandroff discrete topology of a poset was observed early on by R.-E. Hoffman [Hoff79b].

10. The space of ideal closures

Let X be a space and $\varepsilon_X : X \to X^s$ be the standard sobrification of all closed, irreducible subsets. A subset I of X is an *ideal* if it is both directed and a lower set in the order of specialization. The lower set of any directed set is an ideal. A directed set D, in particular an ideal, and hence its closure is an irreducible subset of X, and thus $\overline{D} \in X^s$. A special case are the point closures $\varepsilon_X(x) = {\downarrow} x = \overline{\{x\}}$, the principal ideals. Since for any directed set D, $\overline{D}$ is the supremum in X^s of the directed set $\{{\downarrow} x : x \in D\}$ in the inclusion order, which is the order of specialization, it follows that $\overline{D}$ belongs to the standard D-completion $\widetilde{X}$, the closure of $\varepsilon_X(X)$ in the D-topology.

It follows from the preceding considerations that the space of closed subsets that are closures of directed sets, equivalently ideals (since a directed set and its lower set have the same closure), is contained in the D-completion. There are then several natural questions that arise. For example, in what cases is every member of $\widetilde{X}$ the closure of a directed set? Under what conditions is $\widetilde{X} = X^s$? In the following we give a class of spaces where both questions have positive answers generalizing Proposition 9.1.

Definition 10.1. A T_0 space X is called a quasi-c-space, if each of its points has a neighborhood basis of finitely generated upper sets ${\uparrow} F = \{y \in X \mid x \leq y \text{ for some } x \in F\}$ with respect to the specialization order, where F is finite.

In the same way as c-spaces are generalizations of continuous dcpos, quasi-c-spaces are generalization of quasicontinuous dcpos (see [GHKLMS] for more details), when the latter are equipped with the Scott topology.

Proposition 10.2. *Let X be a quasi-c-space. Then every closed irreducible subset is the closure of a directed set, and hence $X^s = \widetilde{X}$, that is, the D-completion coincides with the sobrification. Furthermore, with respect to its order of specialization, $\widetilde{X}$ is a quasicontinuous dcpo equipped with the Scott topology.*

Proof. Let A be a closed irreducible subset of X. Then $A = {\downarrow} A$, and hence for any finite set F, ${\uparrow} F \cap A = {\uparrow}(F \cap A) \cap A$. From this observation one argues that A in its relative topology satisfies the hypothesis of the theorem. Hence we may assume without loss of generality that $A = X$, and show that X is the closure of a directed set.

Define $\mathcal{F} = \{F \subseteq X : |F| < \infty, {\uparrow} F \text{ has nonempty interior}\}$. For $F_1, F_2 \in \mathcal{F}$, it follows from the irreducibility of X that $\mathrm{int}(F_1) \cap \mathrm{int}(F_2) \neq \emptyset$. Let z be in the intersection U of the interiors. By hypothesis there exists a finite set $F \subseteq U$ such that $x \in \mathrm{int}({\uparrow} F)$. Hence $F \in \mathcal{F}$, and $F \subseteq {\uparrow} F_1 \cap {\uparrow} F_2$. By Rudin's Lemma ([GHKLMS, Lemma III-3.3]) there exists a directed set D such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

We claim that D is dense in X. If not there exists U open with some $x \in U$ such that $U \cap D = \emptyset$. By hypothesis $x \in \mathrm{int}({\uparrow} F)$ for some finite set $F \subseteq U$. But then $F \in \mathcal{F}$ and hence meets D, a contradiction.

The last assertion now follows from Theorem 2 of [La85]. □

Remark 10.3. As every c-space is a quasi-c-space, Proposition 10.2 and Proposition 9.1 both apply to c-spaces. It follows that the sobrification, which equals the D-completion, can be obtained both as the rounded ideal completion and as the space of closures of (round) ideals.

In general, the closure of a directed set need not be directed. in particular, the closure of an ideal need not be an ideal, even in a continuous poset.

Example 10.4. Let $I = [0,1]$, the unit interval with its usual order, and let $P := I \times I \setminus \{(1,1)\}$ with the coordinatewise order and Scott topology. Then $I = [0,1) \times [0,1)$ is an ideal with closure P, which is not directed.

If it is directed, the closure of a directed set is a closed ideal, as closed sets are always lower sets. The closed ideals in a space X always form a subspace $clI(X)$ of the the D-completion containing the principal ideals $\downarrow x, x \in X$, called the *closed ideal completion.*

Proposition 10.5. *If X is a space with the property*

(H) *the closure of every directed set is directed*

then the members of the standard D*-completion are the closed ideals, i.e., the closed ideal completion is the standard* D*-completion .*

Proof. Since closed ideals always belong to the standard D-completion, we must establish the converse. For this it suffices to see that the subspace of closed ideals is d-closed, for then it will contain $\widetilde{X}$. Let $\mathcal{I}$ be a directed family of closed ideals. Since the order of specialization is the inclusion order, it follows that $\bigcup \mathcal{I}$ is again an ideal, and hence by our hypothesis so is its closure. Since the closure is the directed supremum of $\mathcal{I}$ in the standard sobrification X^s, we conclude that the subspace of closed ideals is d-closed. □

We include an important class of spaces which satisfy the hypotheses of the previous proposition.

Proposition 10.6. *Suppose that X is a join semilattice in its order of specialization, and that the join operation is continuous in each variable separately. Then the closure of every directed subset of X is directed.*

Proof. Let D be directed. By separate continuity $x \vee \overline{D} \subseteq \overline{x \vee D} \subseteq \overline{D}$ for each $x \in D$, i.e., $D \vee \overline{D} \subseteq \overline{D}$. From this fact and the same argument, $\overline{D} \vee y \subseteq \overline{D}$ for each $y \in \overline{D}$. Thus $\overline{D} \vee \overline{D} \subseteq \overline{D}$, so $\overline{D}$ is directed. □

11. Future directions

The authors view this paper as a first step toward the investigation of non-Hausdorff topological cones and their completions. Let us illustrate with some examples of the type that motivated this study. Although non-Hausdorff spaces occur rarely in classical mathematics, they can be found there, too.

Consider, for example, the space $\mathrm{C}(X)$ of all continuous real valued functions defined on a compact Hausdorff space X with the topology of uniform convergence and the usual pointwise order. Considering the collection τ of all open upper sets, we obtain a topology which is a c-space topology on $\mathrm{C}(X)$. Indeed, for $f \in \mathrm{C}(X)$, the sets $\uparrow(f-\varepsilon) = \{g \in \mathrm{C}(X) \mid f-\varepsilon \leq g\}$ form a basis for the τ-neigborhoods. The sobrification, which is equal to the monotone convergence space-completion (see Section 9), can be identified with the space of lower semicontinuous functions $h\colon X \to \mathbb{R} \cup \{+\infty\}$ with the Scott topology. The bounded completion yields the space of bounded lower semicontinuous functions $h\colon X \to \mathbb{R}$.

A variant of this example is obtained by looking at the set $\mathrm{A}(X)$ of real-valued continuous affine functions defined on a convex compact subset of a locally convex topological vector space. Again we obtain a c-space by looking at the topology τ of open upper sets for the topology of uniform convergence and pointwise order. The sobrification can be identified with the space of lower semicontinuous affine function $a\colon X \to \mathbb{R} \cup \{+\infty\}$ equipped with the Scott topology.

Generalizing these examples, we may look at ordered topological vector spaces. By this we mean topological vector spaces V together with a closed pointed convex cone P, i.e., with a closed subset P satisfying $P + P \subseteq P$, $rP \subseteq P$ for every real $r \geq 0$ and $P \cap -P = \{0\}$. Defining $x \leq y$ iff $y - x \in P$ we obtain a closed partial order on V such that addition and multiplication by nonnegative reals are order preserving. We may consider the topology τ on V consisting of the open upper sets of V. Clearly, the vector space ordering $\leq$ is the specialization order for the new topology τ.

One may study the spaces (V, τ) and their various completions obtained via the sobrification, the D-completion, and the bounded D-completion. These completions should also have algebraic operations extending the vector space operations and they should have concrete representations in special cases. The present paper can be seen as laying the groundwork towards the investigation of the structure of such completions.

12. A historical appendix

One findes a different categorical approach to completions that yields results overlapping ours in Section 4 in the work of L. Skula [Sk69], in which he seeks to characterize reflective subcategories of TOP. Skula considers reflectors $F : \mathsf{TOP} \to \mathsf{K}$, where K is a full subcategory of TOP_o, that are *strongly embedding*, that is, the image of X under the embedding $\varepsilon_X : X \to R(X)$ is strongly dense in $R(X)$ for every X.

This implies that $R(X)$ is contained in the sobrification of X. Skula's Theorem 3.4 asserts that a full subcategory K of TOP_0 is reflective with a strongly embedding reflector if and only if K is closed for arbitrary products and strongly closed subspaces. Note that our conditions (1)-(4) of Section 4 give alternative sufficient (and we suspect necessary) conditions for the existence of R. Skula goes on to show (Theorem 3.5) that if K is a full reflective subcategory of TOP_0 that contains a space that is not T_1, then the reflector is strongly embedding. A major insight missing at the time of Skula's work is the fact that the reflections considered are contained in the sobrification.

The earliest explicit appearance of the notion of a monotone convergence space, a.k.a. d-space, known to us is an SCS memo of O. Wyler from 1977, which was expanded into [Wy81]. He considers both d-sets (short for (upper) Dedekind complete), which are dcpos, and d-spaces. He points out that these spaces form a reflective subcategory, and in the latter paper he gives a construction of the d-completion (our D-completion) along the lines of what we call the standard D-completion. They appeared again in *A Compendium of Continuous Lattices* in 1980 under the name "monotone convergence spaces."

In [Ers99] Ershov has considered d-spaces and d-completions, our D-completions. His Proposition 8 is essentially the same as our Theorem 6.7 and his Corollary 9 is our Corollary 6.8; his proof uses transfinite induction instead of our topological approach.

The c-spaces of Erné [Ern81], [Ern91] (short for "core," since the cores $\uparrow x$ form a basis for the topology) were considered earlier (unnamed) by Banaschewski [Ba77] and by Hoffmann [Hoff81], who showed that posets with an auxiliary relation satisfying the properties of an abstract basis are equivalent to c-spaces, They were also considered by Ershov [Ers97] under the name of α-spaces, who showed that the round ideal completion is a continuous dcpo with the appropriate universal property.

From the direction of theoretical computer science, d-closed sets in dcpos were considered at an early stage by Scott and DeBakker (unpublished) and received the name 'admissible predicates.' They can be identified with the functions into Boole with the property that for every directed set D, if $f(d) =$ true for all $d \in D$, then $f(\sup D) =$ true. The admissible predicates became standard in semantics, as they allow fixed point induction. So one finds, for example, in the book of Loeckx and Sieber [LS84] the following:

> Theorem 4.29 (Induction Priciple of Scott, also called Fixed Point Induction Principle) Let P be an admissible predicate on a dcpo with a bottom element $\bot$ and $f\colon D \to D$ a Scott-continuous function. Then
>
> $$P(\bot) \text{ and } \forall d\,\big(P(d) \Rightarrow P(f(d))\big) \implies P(\mathrm{Fix} f)$$
>
> where $\mathrm{Fix} f = \sup_n f^n(\bot)$ is the least fixed point of f.

The closedness of admissible predicates under Boolean operations was an issue. The closedness of admissible predicates under intersections was used. It was known that admissible predicates are closed under finite unions, but the proof for this fact is nonconstructive and for this reason this fact is not useful in semantics. But the fact that upper sets and, in particular, that Scott-closed sets and their complements are admissible has been useful.

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