LECTURE 4

Tropical Polytopes

There is a natural notion of convexity in tropical geometry which was introduced by Develin and Sturmfels [**4**]. In some respect this concept resembles classical convexity but with a stronger combinatorial flavor.

1. Tropical Convexity

Given two points p, q in $\mathbb{R}^d$ the *tropical line segment* is the set

$$[p, q]_{\text{trop}} \ := \ \{(\lambda \odot p) \oplus (\mu \odot q) : \ \lambda, \mu \in \mathbb{R}\} \ .$$

For $\alpha, \beta \in \mathbb{R}$ it is immediate that

$$(\lambda \odot (p + \alpha\mathbb{1})) \oplus (\mu \odot (q + \beta\mathbb{1})) \ = \ ((\lambda + \alpha) \odot p) \oplus ((\mu + \beta) \odot q) \, ,$$

and hence the tropical line segment of two points p, q in the tropical torus $\mathbb{T}^{d-1} = \mathbb{R}^d/\mathbb{R}\mathbb{1}$ is well defined.

Proposition 4.1 ([**4**, Proposition 3]). *The tropical line segment $[p, q]_{\text{trop}}$ between two points p, q in $\mathbb{T}^{d-1}$ is the union of at most $d - 1$ ordinary line segments.*

Proof. Up to relabeling the coordinates we can assume that

$$q_1 - p_1 \ \le \ q_2 - p_2 \ \le \ \dots \ \le q_d - p_d \, . \tag{4.1}$$

In order to simplify the formulae below let $r_i = q_i - p_i$. By construction the following d points are contained in $[p, q]_{\text{trop}}$:

$$\begin{array}{rclcl} p & = & (r_1 \odot p) \oplus q & = & (q_1, r_1 + p_2, \dots, r_1 + p_d) \, , \\ & & (r_2 \odot p) \oplus q & = & (q_1, q_2, r_2 + p_3, \dots, r_2 + p_d) \, , \\ & & \vdots & \vdots & \vdots \\ & & (r_{d-1} \odot p) \oplus q & = & (q_1, q_2, \dots, q_{d-1}, r_{d-1} + p_d) \, , \\ q & = & (r_d \odot p) \oplus q & = & (q_1, q_2, q_3, \dots, q_d) \, . \end{array} \tag{4.2}$$

Observe that $(q_1, r_1 + p_2, \dots, r_1 + p_d) = r_1 \odot p$ to see that this really describes the point p in $\mathbb{T}^{d-1}$ as claimed above. The direction from p to $(q_1, q_2, r_2 + p_3, \dots, r_2 + p_d)$ is given by the difference

$$\begin{aligned} (q_1, & q_2, r_2 + p_3, \dots, r_2 + p_d) - (q_1, r_1 + p_2, \dots, r_1 + p_d) \\ & = \ (0, r_2 - r_1, r_2 - r_1, \dots, r_2 - r_1) \, . \end{aligned}$$

A direct computation shows that the tropical line segment between p and $(q_1, r_1 + p_2, \dots, r_1 + p_d)$ coincides with the ordinary line segment. Similarly for any two other consecutive points. We conclude that the ordering of the coordinate directions according to (4.1) gives a parameterization of $[p,q]_{\mathrm{trop}}$ into at most $d-1$ linear pieces, one for each strict inequality. □

EXAMPLE 4.2. The tropical line segment between $(0,1,0)$ and $(0,4,1)$ in $\mathbb{T}^2$ is the union of the ordinary line segments $[(0,1,0),(0,2,1)]$ and $[(0,2,1),(0,4,1)]$. The intermediate point $(0,2,1)$ equals $(1\odot(0,1,0))\oplus(0,4,1)$.

A set $S \subseteq \mathbb{T}^{d-1}$ is *tropically convex* if for any two points $p,q \in S$ the segment $[p,q]_{\mathrm{trop}}$ is contained in S. The *tropical convex hull* of an arbitrary set $S \subseteq \mathbb{T}^{d-1}$ is the smallest tropically convex set containing S; it is denoted by $\mathrm{tconv}(S)$. Clearly, $\mathrm{tconv}(\mathrm{tconv}(S)) = \mathrm{tconv}(S)$. Moreover, the intersection of tropically convex sets is again tropically convex.

LEMMA 4.3. *For $S \subseteq \mathbb{T}^{d-1}$ we have*

$$\mathrm{tconv}(S) \;=\; \{(\lambda_1\odot p_1)\oplus(\lambda_2\odot p_2)\oplus\cdots\oplus(\lambda_n\odot p_n):\ \lambda_i\in\mathbb{R},\ p_i\in S\}\ .$$

A *tropical polytope* is the tropical convex hull of finitely many points.

EXAMPLE 4.4. The tropical convex hull of the four points $v_1=(0,1,0)$, $v_2=(0,4,1)$, $v_3=(0,3,3)$, and $v_4=(0,0,2)$ in $\mathbb{T}^2$ is shown in Figure 4.1. The line segment discussed in Example 4.2 is $[v_1,v_2]_{\mathrm{trop}}$. The intermediate point $(1\odot v_1)\oplus v_2$ is labeled w_2.

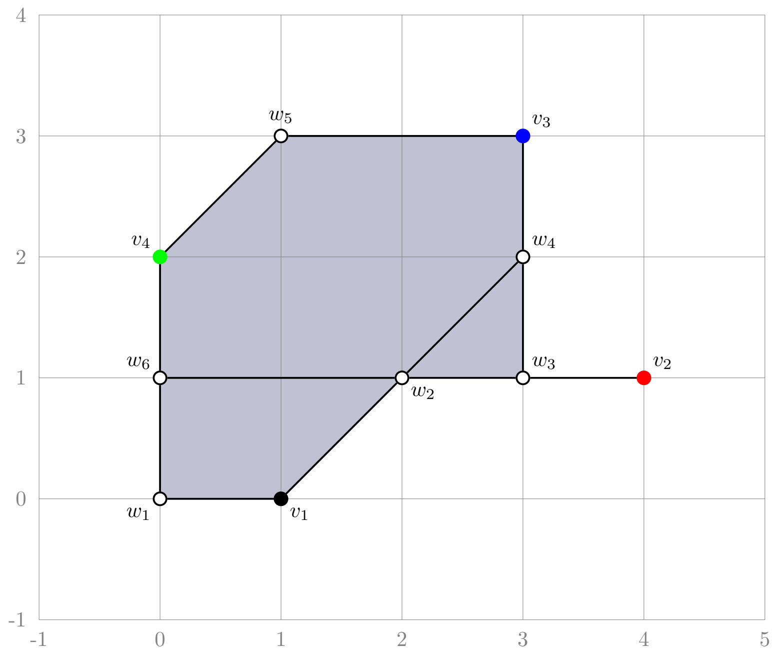

FIGURE 4.1. Tropical convex hull of four points in $\mathbb{T}^2$

2. Arrangements of Tropical Hyperplanes

The tropical hyperplane in $\mathbb{T}^{d-1}$ with apex $-a$ is the tropical hypersurface of the tropical polynomial

$$\langle a, x\rangle_{\text{trop}} \;:=\; (a_1 \odot x_1) \oplus (a_2 \odot x_2) \oplus \cdots \oplus (a_d \odot x_d)$$

in d indeterminates. All tropical hyperplanes in $\mathbb{T}^2$ look like the one in Figure 1.2 (left). A max-tropical hyperplane $H^{\max}(-a)$ is the image of the reflection at the apex $-a$ of the min-tropical hyperplane $H(-a) = H^{\min}(-a)$; the max-tropical hyperplane is shown in Figure 1.2 (right). From Theorem 1.8 it follows that, as a polytopal complex, it is isomorphic to the codimension-2-skeleton of the positive orthant in $\mathbb{R}^d$. The apex plays the role of the origin. In particular, the complement of a tropical hyperplane consists of precisely $d+1$ connected components, its *open sectors*. A *closed sector* is the topological closure of an open sector. The closed sectors of $H(0)$ are the sets $\mathrm{S}_1, \mathrm{S}_2, \dots, \mathrm{S}_d$, where

$$\mathrm{S}_k \;:=\; \left\{p \in \mathbb{T}^{d-1} :\; k \in \operatorname{argmin}\{p_1, p_2, \dots, p_d\}\right\} .$$

Analogously $-a + \mathrm{S}_k$ is the k-th closed sector of $H(-a)$).

Proposition 4.5. *A tropical hyperplane and each open or closed sector of a tropical hyperplane is tropically convex.*

Proof. Let H be the tropical hyperplane in $\mathbb{T}^{d-1}$ with apex $-a$. If p and q are points on H then the minima $\langle a, p\rangle_{\text{trop}}$ and $\langle a, q\rangle_{\text{trop}}$ both are attained at least twice. For $\lambda, \mu \in \mathbb{R}$ we obtain

$$\langle a, (\lambda \odot p) \oplus (\mu \odot q)\rangle_{\text{trop}} \;=\; (\lambda \odot \langle a, p\rangle_{\text{trop}}) \oplus (\mu \odot \langle a, q\rangle_{\text{trop}}) \tag{4.3}$$

by applying the distributive laws. On the right hand side we see that this minimum is attained at least twice in the first term or at least twice in the second term. This implies that $(\lambda \odot p) \oplus (\mu \odot q)$ is contained in H. This proves that tropical hyperplanes are tropically convex.

Now consider two points p, q in the closed sector $-a + \mathrm{S}_k$ of H. The minima $\langle a, p\rangle_{\text{trop}}$ and $\langle a, q\rangle_{\text{trop}}$ are both attained in the k-th coordinate. Scrutinizing (4.3) again shows that $(\lambda \odot p) \oplus (\mu \odot q)$ is also contained in the k-th closed sector of H, for all $\lambda, \mu \in \mathbb{R}$. We conclude that $-a + \mathrm{S}_k$ is tropically convex, and a similar argument gives the same result for the open sectors. ☐

As we have the choice between min and max for the tropical addition, we also have two notions of tropical convexity, one with respect to min and the other with respect to max. Notice that the relation $-\min(x, y) = \max(-x, -y)$ implies that

$$-\operatorname{tconv}^{\min}(S) \;=\; \operatorname{tconv}^{\max}(-S)$$

holds for all subsets S of $\mathbb{T}^{d-1}$. Consider a point $q \in \mathbb{T}^{d-1}$ with $0 \leq q_1 \leq q_2 \leq \cdots \leq q_d$. By Proposition 4.1 the max-tropical line segment from 0 to q is the union of the $d-1$ ordinary line segments between any two consecutive of the d points

$$\begin{aligned} 0 &= (q_d, q_d, \ldots, q_d) \\ & \quad (q_{d-1}, q_{d-1}, \ldots, q_{d-1}, q_d) \\ & \quad \vdots \\ & \quad (q_2, q_2, q_3, \ldots, q_d) \\ q &= (q_1, q_2, q_3, \ldots, q_d) \end{aligned}$$

Their min-tropical convex hull is the Minkowski sum

$$\begin{aligned} &(q_2 - q_1) \cdot [0, (0,1,1,\ldots,1)] \; + \; (q_3 - q_2) \cdot [0, (0,0,1,\ldots,1)] \\ &\quad + \; \ldots \; + \; (q_d - q_{d-1}) \cdot [0, (0,0,\ldots,0,1)] \,. \end{aligned}$$

Here $[a, b]$ denotes the ordinary line segment in $\mathbb{R}^d$. Since the vectors $(0,1,1,\ldots,1)$, $(0,0,1,\ldots,1)$, …, $(0,0,\ldots,0,1)$ in $\mathbb{R}^d$ are linearly independent this zonotope is a parallelotope. The same computation shows that for arbitrary points $p, q \in \mathbb{T}^{d-1}$ the set

$$c(p,q) \;:=\; \mathrm{tconv}^{\min}(\mathrm{tconv}^{\max}(p,q))$$

is a parallelotope.

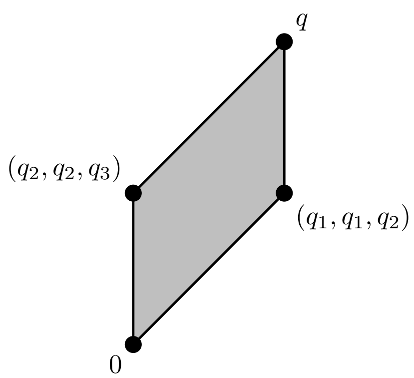

FIGURE 4.2. Bi-tropical convex hull of two points in $\mathbb{T}^2$

As a parallelotope is convex in the ordinary sense we obtain an interesting relationship between tropical and ordinary convexity. A subset of $S \subseteq \mathbb{T}^{d-1}$ is *bi-tropically convex* if its is both min- and max-tropically convex.

PROPOSITION 4.6. *A subset of $S \subseteq \mathbb{T}^{d-1}$ is bi-tropically convex if and only if for any two points $p, q \in S$ the parallelotope $c(p,q)$ is contained in S. In particular, such a set is convex in the ordinary sense.*

Key examples which combine min- and max-tropical convexity are the sectors of a tropical hyperplane.

COROLLARY 4.7. *Any open or closed sector of a* min*-tropical hyperplane is also* max*-tropically convex and conversely. In particular, open and closed sectors are convex in the ordinary sense.*

For the remainder of this section we fix an ordered sequence $V = (v_1, v_2, \ldots, v_n)$ of pairwise distinct points in $\mathbb{T}^{d-1}$. Each of these points is the apex of a unique max-tropical hyperplane. The sequence

$$\mathfrak{A}_V \;=\; (H^{\max}(v_1), H^{\max}(v_2), \ldots, H^{\max}(v_n))$$

is the *arrangement* of max-tropical hyperplanes induced by V. For each point $p \in \mathbb{T}^{d-1}$ we define the *type* of p with respect to V to be

$$\operatorname{type}_V(p) \;:=\; (T_1, T_2, \ldots, T_d)\,,$$

where $T_k = \{i \in [n] \colon p \in v_i + \mathrm{S}_k^{\max}\}$. Notice that between min and max we have the duality relation

$$p \in q + \mathrm{S}_k^{\max} \quad \text{if and only if} \quad q \in p + \mathrm{S}_k^{\min}\,.$$

Each point in $\mathbb{T}^{d-1}$ has a type, and therefore via its type function respect the sequence V induces a decomposition of $\mathbb{T}^{d-1}$ into finitely many cells. The *cell* of type T is

$$X_T^\circ \;:=\; \left\{p \in \mathbb{T}^{d-1} :\; \operatorname{type}_V(p) = T\right\}\,.$$

The cell X_T° is relatively open, that is, it is an open subset of its affine hull.

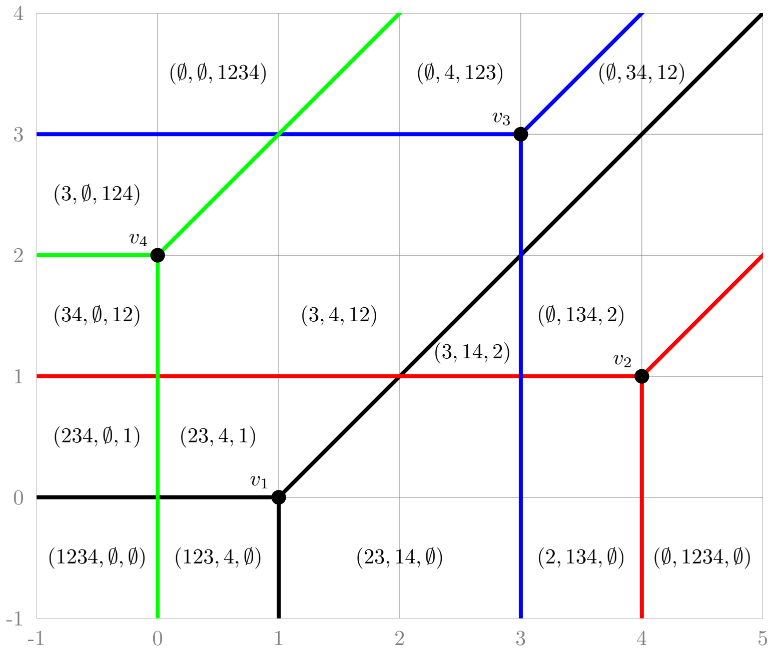

FIGURE 4.3. Arrangement of four max-tropical lines in $\mathbb{T}^2$

LEMMA 4.8 ([**4**, Corollary 12]). *A cell X_T° of type $T = (T_1, T_2, \ldots, T_d)$ is bounded if and only if none of the sets T_k is empty.*

PROOF. Let $p \in \mathbb{T}^{d-1}$ with $\operatorname{type}_V(p) = T$. Then T_k is empty if and only if the closed sector $p + \mathrm{S}_k^{\min}$ does not contain any point from the sequence V. If this is the case then the point $p+\lambda e_k$ has the same type T for arbitrary $\lambda \geq 0$. This implies that the cell X_T° is unbounded. Conversely, suppose that X_T° is unbounded. As the coordinate directions $e_1, e_2, \ldots, e_d$ positively span $\mathbb{T}^{d-1}$, there must be one coordinate direction e_k which is unbounded. This implies that $T_k = \emptyset$. □

EXAMPLE 4.9. Consider the four points from Example 4.4. The corresponding arrangement $\mathfrak{A}_V$ of max-tropical lines is shown in Figure 4.3. For instance, the type T of the point $(0, 4, 2)$ equals $(\emptyset, 134, 2)$, and its relatively open cell X_T° is unbounded.

Let $\preceq$ be the partial ordering on the set of types with respect to V which is induced by componentwise inclusion: $T \preceq T'$ if and only if $T_k \subseteq T'_k$ for all $k \in [d]$. For a cell X_T° the set

$$X_T \ := \ \left\{p \in \mathbb{T}^{d-1} : \ T \preceq \operatorname{type}_V(p)\right\}$$

is the corresponding *closed cell*.

EXAMPLE 4.10. The closure of the 2-dimensional relatively open cell $X^\circ_{(\emptyset,134,2)}$ from Example 4.9 contains the four 1-dimensional cells with the types $(\emptyset, 23, 123)$, $(3, 34, 12)$, $(2, 134, 2)$, and $(\emptyset, 1234, 2)$, as well as the three 0-dimensional cells $(3, 134, 12)$, $(23, 134, 2)$, and $(2, 1234, 2)$.

LEMMA 4.11. *For each type T the closed cell X_T is the topological closure of the cell X_T°. Each (closed) cell is bi-tropically convex and hence convex in the ordinary sense.*

PROOF. Each closed cell is the intersection of finitely many closed sectors of the max-tropical hyperplanes in the arrangement $\mathfrak{A}_V$. Hence the claim follows from Corollary 4.7. □

As a direct consequence the decomposition into closed types with respect to V is an ordinary polyhedral complex covering $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}$.

LEMMA 4.12 ([**4**, Corollary 11]). *Let X_S and X_T be two closed cells. Then $X_S \cap X_T = X_{S\cup T}$.*

EXAMPLE 4.13. Consider the types $S = (2, 134, \emptyset)$ and $T = (\emptyset, 134, 2)$ in Example 4.9 and Figure 4.3. The intersection of the closed cells $X_S \cap X_T$ is the line segment $[(0, 3, 1), (0, 4, 1)]_{\text{trop}}$, which is of type $(2, 134, 2)$.

LEMMA 4.14. *A point p is contained in the* min*-tropical convex hull* $\operatorname{tconv}(V)$ *if and only if for each $k \in [d]$ there is some index i such that $v_i \in p + \mathrm{S}_k^{\min}$.*

PROOF. Without loss of generality $p = 0$. Each point in $\mathbb{T}^{d-1}$ has a unique *canonical* coordinate representation as a vector in $\mathbb{R}^d$ with non-negative coordinates such that at least one coordinate is zero. Then 0 is in the tropical convex hull of V if and only if for each coordinate direction k there is a point v_i whose k-th canonical coordinate is zero. The latter property is equivalent to v_i being contained in $\mathrm{S}_k^{\min}$. □

The following is a first main result about tropical polytopes.

THEOREM 4.15 ([**4**, Theorem 15 and Proposition 16]). *The* min-*tropical polytope* $\operatorname{tconv}(V)$ *is the union of the bounded (closed) cells of the type decomposition with respect to the* max-*tropical hyperplane arrangement* $\mathfrak{A}_V$.

PROOF. Let p be a point in $\mathbb{T}^{d-1}$ with $\operatorname{type}_V(p) = (T_1, T_2, \ldots, T_d)$. By Lemma 4.14 the tropical polytope $\operatorname{tconv}(V)$ contains p if and only if $T_k \neq \emptyset$ for all $k \in [d]$. Lemma 4.8 says this is the case if and only if the cell X_T is bounded. □

It follows that, as a finite union of compact sets, a tropical polytope is compact.

EXAMPLE 4.16. We continue to use our four points V from Example 4.4. The arrangement $\mathfrak{A}_V$ has precisely four maximal bounded cells. Three of these are 2-dimensional, and their types are $(23, 4, 1)$, $(3, 4, 12)$, and $(3, 14, 2)$. The remaining maximal bounded cell is 1-dimensional; its type is $(2, 134, 2)$. The arrangement $\mathfrak{A}_V$ of max-tropical lines and the induced type decomposition of $\mathbb{T}^2$ are shown in Figure 4.3.