On locally finite modular lattice varieties of finite height

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Abstract

It is shown that the variety generated by a finite modular lattice has only finitely many locally finite covers.

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1 Introduction

Theorem 1 *For any natural number, h, there are only finitely many varieties which are of height $\leq h$ in the lattice of varieties and generated by modular lattices of finite height. Each such variety is generated by a finite lattice.*

Of course, this includes all locally finite modular lattice varieties of finite height. The theorem does not extend to non-modular varieties: Nation [6] has constructed a variety generated by a finite lattice which has infinitely many covers each generated by a finite lattice. It remains open whether a variety generated by a finite modular lattice M can have a cover which contains an infinite subdirectly irreducible modular lattice L. Due to Jónsson [4] and Hong [3] this cannot occur if M is a subdirect product of lattices of height ≤ 3. On the other hand, Nation [6] has constructed a non-modular example of this type.

The present note has been prompted by Nation [5] which aimed at lattices in general. We just take advantage of the more powerful descriptions available in the modular case. In particular, his dependency relation D on the set of join irreducibles specializes to the perspectivity graph. Instead of bounds on path length we establish bounds for degree and diameter of these graphs. Bounds on the degree are obtained just in terms of bounds on size of height 2 and of certain width 4 sublattices and these can be extended to spatial algebraic modular lattices. In contrast, finite height of the generating lattices is crucial for the bound on diameter.

2 Preliminaries

We follow Crawley and Dilworth [1] for general concepts of lattice theory but write $a+b$ for joins and ab for meets. A quotient a/b *transposes up* to c/d and c/d *down* to a/b, in signs $a/b \nearrow c/d$ and $c/d \searrow a/b$, if $ad = b$ and $a + d = c$.

The *height* of a lattice L is $\mathsf{ht}(L) = \sup\{|C| - 1 \mid C \text{ a chain in } L\}$. The *varietal height* $\mathsf{vht}(L)$ of a lattice is the height of the variety generated by L - in the lattice of all lattice varieties. Let M_n denote the height 2 lattice with n atoms. Of course, $\mathsf{vht}(M_n) = n - 1$. M_3 is a 1-*snake*. An $n+1$-snake is a modular lattice which is the union of an n-snake $[0, a]$ and a 1-snake $[b, 1]$ with $ab \prec a$. An n-snake has at most $5n$ elements and is a subdirect product of a distributive lattice and a simple n-snake of height $n + 1$. In particular, its varietal height is $n + 1$, too. Moreover, any modular lattice containing an n-snake in its variety has an n-snake sublattice even with all its M_3-sublattices being cover preserving.

For a modular lattice, M, let P_M the set of completely join irreducibles, shortly *points*. For $p \in P_M$, the unique lower cover is denoted by $\underline{p}$. We shall be concerned with those M in which P_M is join dense, referred to as *spatial* lattices. In the sequel, M will always denote a spatial algebraic modular lattice. Observe that any interval sublattice $[a, b]$ of M is spatial.

On P_M one has a collinearity relation: $C(p, q, r)$ if and only if p, q, r are pairwise incomparable and $p + q = p + r = q + r$ - in which case p and q are *perspective*. Clearly, $p \neq q$ are perspective if and only if $p/\underline{p} \nearrow a/b$ and $q/\underline{q} \nearrow a/b$ for some a/b. The *perspectivity graph* of M is the graph with vertex set P_M and an edge pq if p and q are perspective. The *diameter* $d(M)$ of M is defined as the diameter of its perspectivity graph: the least number n such that every two points belonging to some connected component are connected by a path consisting of at most n edges. If no such bound exists, then $d(M) = \infty$.

The subdirectly irreducibles among spatial modular lattices have been characterized in Herrmann, Pickering, and Roddy [2] as those with P_M connected under perspectivity and it has been shown that every modular lattice variety is generated by subdirectly irreducible spatial algebraic members.

The connection between the lattice and the geometric structure is established through the following concept: An at least 5-element height 2 interval $l/\underline{l}$ in M is called a *line interval* if $x+\underline{l} = l$ implies $x = l$ for all x in M, equivalently, if $x \geq \underline{l}$ for all $x \prec l$ - observe that l is compact. According to [2] Lemma 2.1, an at least 5-element interval $l/\underline{l}$ is a line interval if and only if there are incomparable points p, q such that $l = p + q$ and $\underline{l} = \underline{p} + \underline{q}$. Writing $p \nearrow l$ if $p \leq l, p \not\leq \underline{l}$ then for a line interval $l/\underline{l}$ one has $l = x+y$, $\underline{l} = \underline{x}+\underline{y}$ for all pairs $x, y \nearrow l$ of points such that $x+\underline{l} \neq y+\underline{l}$. In particular, $\underline{l} = \prod\{x \in M | x \prec l\}$ and p, q with $l = p + q$ are perspective and $p, q \nearrow l$.

The lattice element l is called a *line* (*top*). A line k *transposes up* to another, l, if k/b transposes up to $a/\underline{l}$ for some lower covers b of k and a of l. We write $k \nearrow l$. Observe that for two lines $k < l$ either $k \leq \underline{l}$ or $k \nearrow l$. The transitive closure of the relation "transposes up" defines the *transposition order* on the set of lines.

Lemma 2 *Let k, l are distinct lines of M and c is a point such that $c \nearrow k, l$. Then exactly one of the following takes place*

(1) k and l are transposed

(2) $kl = k\underline{l} + l\underline{k}$.

(3) $k\underline{l} = l\underline{k}$

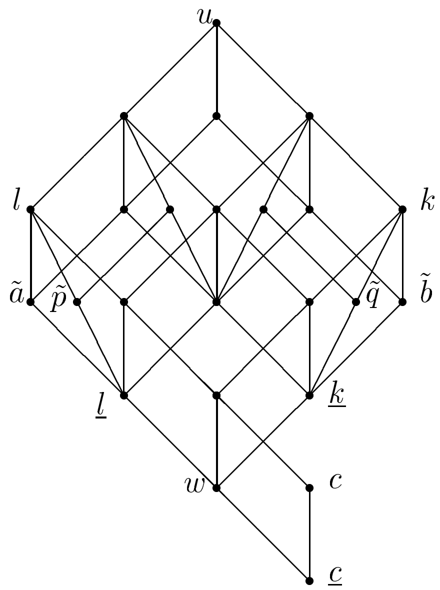
Moreover, (2) and (3) are equivalent to (2') and (3'), respectively:

(2') k and l are incomparable and there is a line h with $c \nearrow h \nearrow k, l$

(3') k and l are incomparable and any two points p, q with $p \nearrow l$, $q \nearrow k$, $p + \underline{l} \neq c + \underline{l}$, and $q + \underline{k} \neq c + \underline{k}$ are perspective.

If case (3) applies we say that k and l are *triangle related* - see Fig.1, *Proof.* In the proof of Thm.3.3 of [2] cases (1)-(3), (4), and (5) correspond to (1), (2), and (3), respectively. □

Figure 1: Triangle relatedness



3 Clusters

A set C of lines is a *cluster associated* with the point p if C consists of lines l (but maybe not all) which are minimal in the transposition order with respect to the property $p \nearrow l$. For any set of lines, let $\overline{C}$ the join of all l in C, $\underline{C}$ the join of all $\underline{l}$, l in C.

Lemma 3 *For an cluster C of M any two of its lines are triangle related. Moreover, if M is algebraic, the interval $\overline{C}/\underline{C}$ is (the subspace lattice of) an irreducible projective geometry.*

Proof. $\overline{C}/\underline{C}$ is a projective geometry since M is spatial and $\overline{C}$ is the join of p with $\underline{p} \leq \underline{C}$. Also, any two k, l in C are triangle related. For clusters, minimality excludes cases (1) and (2') hence by Lemma 2 any two members of the cluster are triangle related.

As observed above, $\overline{C}$ is a join of atoms of $\overline{C}/\underline{C}$ which are of the form $\tilde{p} + \underline{C}$ where $\tilde{p}$ an atom in $l/\underline{l}$ with $l \in C$. Consider a second such, $\tilde{q} + \underline{C}$, associated with some $k \in C$. If $k = l$ choose $\tilde{r}$ a third atom of $l/\underline{l}$. Then $\tilde{r} \not\leq \underline{C}$ since otherwise $\tilde{p} + \underline{C} = l + \underline{C} = \tilde{q} + \underline{C}$. Thus, $\tilde{p} + \underline{C}$ and $\tilde{q} + \underline{C}$ are persepctive via $\tilde{r} + \underline{C}$. Assume $k \neq l$. Then k and l are triangle related and $l + k/\underline{l} + \underline{k}$ is an irreducible projective plane - cf [2], Case 5 of Thm.3.3. In particular, there is a third point x on the line $\tilde{p} + \tilde{q}$ of this plane. As in the preceeding argument, we have $x \not\leq \underline{C}$ and the required perspectivity. Thus $\overline{C}$ is a join of pairwise perspective atoms of $\overline{C}/\underline{C}$ showing that this interval is an irreducible projective geometry. $\square$

Lemma 4 *$p \not\leq \underline{C}$ for any cluster C assoiated with p.*

Proof. By definition $p \not\leq \underline{l}$ for all $l \in C$. Assume $p \leq \underline{C}$. By compactness there is a finite subset D of C with $p \leq \sum(\underline{l}; l \in D)$ and $|D| > 1$. Choose $l \in D$, let E consist of the remaining k in D, and $e = \sum(\underline{k}; k \in E)$. Thus $p \leq \underline{l} + e$ and $p \not\leq \underline{l}$. If $p \leq e$, a contradiction follows by induction. Otherwise, by Lemma 2.2 of [2] there are points $s \leq \underline{l}$ and $t \leq e$ such that p, s, t are collinear. The $p + s \leq p + \underline{l} < l$ and $p + s$ is a line is a line in C contradicting the minimality of l. $\square$

Lemma 5 *Let C be a cluster of M associated with the point p and $a = \sum_{i=1}^{n} a_i \not\geq p$ where $a_i \leq \underline{k}_i$ for some line $k_i \in C$. Then, in the interval $[a, 1]$, the $a + l$ ($l \in C$) form an cluster associated with the point $a + p$.*

Proof. First, we consider the case that $a \leq \underline{k}$ for a fixed $k \in C$, in particular $p \not\leq a$. First, we show that for $l \in C$ we have $(a+l)/(a+\underline{l})$ a line interval. Indeed, if $a \leq l$ then $a \leq \underline{k}l = k\underline{l} \leq \underline{l}$ and nothing has to be done. Otherwise, by triangle relatedness and modularity, $l(\underline{l}\,\underline{k}+a) = \underline{l}\,\underline{k}+la = \underline{l}\,\underline{k}$ which means that $l/\underline{l}\,\underline{k}$ transposes up to $(a+l)/(a+\underline{l}\,\underline{k})$. In particular, $[a+\underline{l}, a+l]$ is complemented of height 2. Moreover, if $a \leq x \prec a+l$, then $x \not\geq l$ whence $xk \prec l$. Since l is a line , it follows $xl \geq \underline{l}$ whence $x \geq a+\underline{l}$. which shows that $(a+l)/(a+\underline{l})$ is a line interval in $[a,1]$. Moreover, $a+p$ is a point $\leq a+l$ in $[a,1]$ and $a+p \not\leq a+\underline{l} \leq \underline{C}$ by Lemma 4.

We are left to prove minimality. Consider any l in C (including k) and a line interval $h/\underline{h}_a$ in $[a,1]$ such that $a+p \nearrow h \nearrow a+l$.

First, assume $a \leq l$. There is an atom b of $h/\underline{h}_a$ distinct from $p+\underline{h}_a$ and $h\underline{l}$. Since $h = b+h\underline{l}$ there are points $q \leq b$ and $r \leq hl$ in M such that p, q, r are collinear. Then $p+q$ is a line in M and $p \nearrow p+q \nearrow l$, contradicting the minimality of l.

In general, from $a \leq \underline{k}$ we have $al \leq \underline{l}$ - by triangle relatedness if $k \neq l$. Also $a+l \geq h$ by hypothesis. Thus, the canonical isomorphism of $[a,\, a+l]$ onto $[al, l]$ maps the line interval $h/\underline{h}_a$ of $[a,1]$ onto the line interval $hl/\underline{h}_a l$ of $[al,1]$. Now, if $x \prec l$ in M then $x \geq \underline{l} \geq al$ whence $x \geq hl \geq al$ or $al \leq hlx \prec hl$. In any case, $hlx \geq \underline{h}_a l$ since $hl/\underline{h}_a l$ is a line interval in $[al,1]$. It follows $\underline{h}_a l \leq \underline{l}$ and $p \nearrow hl/\underline{h}_a l \nearrow l/\underline{l}$. This contradicts the minimality of l in $[al,1]$ which was already established.

This proves the special case where $a \leq \underline{k}$ for some $k \in C$. Now, a is a finite join of a_i with $a_i \leq \underline{k}_i$ for some line $k_i \in C$ and the general case follows by an obvious induction on the number of summands. □

4 Size of clusters

Lemma 6 *Let C be a cluster of M. Then $k+\underline{C} = l+\underline{C}$ implies $k = l$ for all $k, l \in C$.*

Proof. Observe that $\underline{k}+l = k$ implies $k = l$ since k, l are triangle related. Thus, by the Lemma 5, we may assume $\underline{k} = 0$ and, similarly, $\underline{l} = 0$. Asssume $k \neq l$. It follows, that $p = kl$ is an atom and $[0, l+k]$ an irreducible projective plane. Since k and l are compact and M algebraic, there are finitely many $h_i \in C$ such that $k + \sum \underline{h}_i = l + \sum \underline{h}_i$. Induction and the preceeding lemma reduces this to the case where $k + \underline{h} = l + \underline{h}$ for some $h \in C$ with $\underline{h} > 0$. Since $p \leq h$ and $p \prec l$ we have $h \prec l + h = l + k$ and it follows that $p \prec h(l+k) \prec l+k$, Thus, $h(l+k)$ is a line in the projective plane $[0, l+k]$ whence in M, too, contradicting the minimality of h. $\square$

Lemma 7 *There is a bound on the size of clusters of M in terms of varietal height.*

Proof. Let $h = \mathsf{vht}(M)$ and C a cluster of M. By Lemma 3, the interval sublattice $[\underline{C}, \overline{C}]$ is the subspace lattice N of an irreducible projective geometry. If N has height $\geq n$ then it contains a snake of height n as a sublattice. Thus $\mathsf{ht}(N) \leq h$. It follows that the geometry has dimension $\leq h - 1$ and order $\leq h - 2$. Together with Lemma 6 this establishes the bound on the size of C. $\square$.

5 Degree of associated graphs

The *point-line graph* of M is the bipartite graph having all points and lines as vertices and an (undirected) edge from p to l if and only if $p \nearrow l$.

Lemma 8 *There are finite bounds in terms of h on the height of the transposition order, the degree of the point-line graph, and the degree of the perspectivity graph for all M of varietal height $\leq h < \infty$.*

Proof. The union of the line intervals associated to any n-element chain in the transposition order forms an n-snake sublattice of M. This gives a bound on the height of the transposition order. Also observe that there is a bound on the size of height 2 intervals and, by Lemma 7, the size of clusters.

To get a bound on the number of lines l with $p \nearrow l$, where p is a fixed but arbitrary point, we proceed by induction on the transposition height t. So we may assume that such bound b is available for all lattices of transposition height less than t. The l with $p \nearrow l$ and minimal with respect to the transposition order form an cluster and we have the bound c on the size available. The number of g with $l \nearrow g$ is bounded by $b \cdot a$, a the number of atoms x in the line interval of l so $a \leq h + 1$. Indeed, we have $l \nearrow g$ in $[x, 1]$ for some atom x of $[\underline{l}, l]$ and $[x, 1]$ has transposition height $< t$. Thus, we get the bound $b \cdot c \cdot (h + 1)$ where h is the varietal height.

For a given line l, the bound for the number of points p such that such that $p \nearrow l$ is obtained in terms of the height t of the transposition order and the bound s on the size of height 2 intervals: s^{t+1} will do. Indeed, if $p \nearrow l$ then we have a maximal chain l_i of $k \leq t+1$ of lines with $p \nearrow l_k \nearrow \ldots l_2 \nearrow l_1 = l$. Going downwards we have at most s choices at each step. This yields the bound.

The degree of the perspectivity graph is bounded by the square of the degree of the point-line graph. □

6 Finiteness

Lemma 9 *Let M be subdirectly irreducible of diameter $d < \infty$ and degree $k < \infty$ of the perspectivity graph. Then M is finite and $|M| \leq 2^{(k+1)^d}$.*

Proof. The perspectivity graph is connected and has diameter $\leq d$, so it has at most $(k+1)^d$ elements. □

Lemma 10 *There is a bound $d(h) < \infty$ in terms of $h < \infty$ such that every finite height modular lattice M generating a variety of height $\leq h$ has diameter $d(M) \leq d(h)$.*

Proof. We define $d(1) = 1$ and, recursively, $d(h) \leq 2 \cdot d(h-1) + 3$. We show $d(M) \leq d(\mathsf{vht}(M))$ by induction. Of course, we may assume M subdirectly irreducible. Choose a coatom a and apply the inductive hypothesis to $[0, a]$. Observe that $\mathsf{vht}([0, a]) < h := \mathsf{vht}(M)$ by Jónsson's Lemma. Thus, by inductive hypothesis, $d([0, a]) \leq d(h-1)$. Any two points $\not\leq a$ form an edge in the perspectivity graph. Thus, any shortest path connecting two points of P_M contains no, only one, or two neighbouring points not $\leq a$. In the first case, induction applies directly. In the two other cases the two paths obtained by deleting the one or two vertices have length $\leq d(h-1)$. Thus, the given path has length $\leq d(h)$. □

Proof of the Theorem. By Lemmas 8 and 9 every subdirectly irreducible spatial algebraic modular lattice of finite varietal height is finite. Together with Lemma 10, these Lemmas provide a bound in terms of h on the size of subdirectly irreducibles in varieties of height $\leq h$ generated by finite height modular lattices. □. Actually, the proof yields

Corollary 11 *A subdirectly irreducible spatial algebraic modular lattice is finite provided it is of finite diameter and admits bounds on the size of its height 2 intervals and snake sublattices.*

It should be remarked that one easily constructs a height 8 and breadth 2 acyclic simple modular lattice such that all its proper sublattices are subdirect products of lattices of height at most 6. We conjecture, that for every k there is an n and a finite simple modular lattice of height n having a modular cover containing a simple lattice of height $n+k$.

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