SMALL NON-ARGUESIAN LATTICES

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INTRODUCTION

A lattice theoretic version of Desargues law from Projective Geometry has been introduced by Jónsson [11] in form of the Arguesian identity as a condition necessary and sometimes sufficient for representations in lattices of vector subspaces or, more generally, permuting equivalences. A detailed analysis of configurations associated with the failure of the Arguesian law has been given by Day and Jónsson [2, 3, 4]. In the present paper we construct small examples where such failures occur, each type including minimal non-Arguesian modular lattices. On the other hand, for lattices having a gluing structure as in the first three of our examples , the Arguesian law is shown to be sufficent for a representation in a lattice of submodules . As a consequence we get that all Arguesian lattices of length at most five have a permuting equivalence respesentation. This reopens the approach to representation from the gluing point of view, of course requiring stronger laws in general - as shown by Haiman [6] .

ARGUESIAN LAW In a lattice $\mathbf{L}$, given two "triangles" $\mathbf{a} = (a_0, a_1, a_2)$ and $\mathbf{b} = (b_0, b_1, b_2)$ of elements define

$$p = p(\mathbf{a}, \mathbf{b}) = (a_0 + b_0)(a_1 + b_1)(a_2 + b_2)$$

$$c_i = c_i(\mathbf{a}, \mathbf{b}) = (a_j + a_k)(b_j + b_k) \text{ for } \{i, j, k\} = \{0.1.2\}.$$

The triangles **a** and **b** are *centrally perspective* if

$$(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2),$$

and *axially perspective* if

$$c_2 \leq c_0 + c_1.$$

The lattice **L** is *Arguesian* if every two centrally perspective triangles in **L** are also axially perspective. For a projective plane this is is case if and only if it can be coordinatized over a division ring **R**. Thus, our list of non-Arguesian lattices begins with

EXAMPLE 0. Non-Arguesian projective planes.

EXAMPLE 1

Hall-Dilworth [7] introduced the gluing of two modular latties identifying a filter of one with an ideal of the other (via a "gluing map") always resulting in a modular lattice; their objective was to produce modular lattices not embeddable into complemented modular lattices. Jónsson [12] showed the following.

Theorem 1 *For a lattice* **L** *glued from (the subspace lattices of) two Arguesian projective spaces over a two dimensional quotient the following are equivalent:*

(i) **L** *is Arguesian*

(ii) *For all/some coordinatizations of the spaces there is a semilinear map between a quotient of the one and a subspace of the other inducing the gluing map*

(iii) **L** *can be (isometrically) embedded into a vector space lattice.*

So here arise two ways of generating non-Arguesian examples: Firstly, glue projective planes over non-isomorphic division rings of the same (infinite) cardinality, arbitrarily; secondly, for two planes over the same division ring **R** match a coordinate axis of the lower plane through the gluing with an axis of the upper plane such that zero, unit, and infinity correspond but the induced permuation of $\mathbf{R}$ is not an automorphism.

TWIST.

The further examples will be built according to the following scheme: Given a finite dimensional $\mathbf{R}$-vector space $\mathbf{V}$ and a finite lattice $\mathbf{S}$ let σ and π be join and meet embeddings of $\mathbf{S}$ into the lattice $L(\mathbf{V})$ of subspaces of $\mathbf{V}$ such that $\sigma x \leq \pi x$ for all x in $\mathbf{S}$. Then, of course the union of all intervals $\mathbf{L}_x = [\sigma x, \pi x]$ will be a sublattice of $L(\mathbf{V})$ which can be understood as arising from these intervals by gluing any two neighbouring ones in a natural way. By twisting such a natural gluing of an interval $[\sigma y, \pi x]$, y a lower cover of x in $\mathbf{S}$, we mean the ungluing of the interval into one copy in $\mathbf{L}_x$ and another in $\mathbf{L}_y$ and identifying again over an isomorphism γ_{xy}. In [8] there have been stated conditions under which the union of the $\mathbf{L}_x$ will result in a modular lattice $\mathbf{L}$ the order relation of which is just the transitive closure of those of the $\mathbf{L}_x$'s taken together. These conditions are obviously satisfied, if all the gluings are at most two-dimensional - which is the case in our examples. Of course, if we consider a subspace of $\mathbf{V}$ as an element of $\mathbf{L}$ there might be an ambiguity in case it belongs to two components $\mathbf{L}_x$ glued via a twist. We then have to state in which component this subspace is to be taken and to provide a new representing subspace when passing to the other component. In our examples we will have each gluing map γ_{xy} induced by an α_{xy}-semilinear map χ_{xy} on the $\mathbf{R}$-vector subquotient $\pi x / \sigma y$ of $\mathbf{V}$, α_{xy} an automorphism of $\mathbf{R}$. The lattice $\mathbf{S}$ will be also referred to as the "skeleton" ; in our examples it will be the skeleton of $\mathbf{L}$ as defined in [8] . For a more general approach to gluings compare [1] .

EXAMPLE 2.

This is the example of the gluing of four projective planes with skeleton $\mathbf{2}^2$. Let $\mathbf{V}$ an $\mathbf{R}$-vector space with basis $e_1, \ldots e_5$. Let $\mathbf{U}$ the subspace generated by e_1, e_2, e_4 and let

$$\sigma 0 = 0, \sigma u = \mathbf{R}e_2, \sigma v = \mathbf{R}e_4, \sigma 1 = \mathbf{R}e_2 + \mathbf{R}e_4$$

$$\pi 0 = \mathbf{U}, \pi u = \mathbf{U} + \mathbf{R}e_3, \pi v = \mathbf{U} = \mathbf{R}e_5, \pi 1 = \mathbf{V}$$

where u, v are the atoms of $\mathbf{S}$. Let all the gluing maps identity but γ_{v1} which is induced by an automorphism α of $\mathbf{R}$:

$$\mathbf{U} + \mathbf{R} \text{ fixed }, \mathbf{U} + \mathbf{R}(e_1 - re_5) \mapsto \mathbf{U} + \mathbf{R}(e_1 + \alpha(r)e_5) \text{ for } r \in \mathbf{R}.$$

As observed above, this results in a modular lattice which is an union of intervals $\mathbf{L}_x = [\sigma x, \pi x]$, x in the skeleton $\mathbf{S}$.

Lemma 1 $\mathbf{L}$ *is Arguesian if and only if* α *is the identity map on* $\mathbf{R}$.

Proof. Clearly, if α is identity, then so is the associated gluing map, whence $\mathbf{L}$ is a sublattice of $L(\mathbf{V})$ and therefore Arguesian. Conversely, assume $\mathbf{L}$ Arguesian and consider $r \in \mathbf{R}, r \neq 0$. Let $\alpha s = r$ and

$$a_0 = \mathbf{R}e_2 + \mathbf{R}e_3, a_1 = \mathbf{R}e_2 + \mathbf{R}(e_1 + e_3 + e_4), a_2 = \mathbf{R}e_2 + \mathbf{R}(e_1 - re_3)$$

$$b_0 = \mathbf{R}e_4 + \mathbf{R}e_5, b_1 = \mathbf{R}e_4 + \mathbf{R}(e_1 + e_2 + e_5), b_2 = \mathbf{R}e_4 + \mathbf{R}(e_1 - se_5)$$

.

Then the triangles $\mathbf{a}, \mathbf{b}$ are centrally perspective since

$$a_0 + b_0 = \mathbf{U} + \mathbf{R}e_3 + \mathbf{R}e_5, a_1 + b_1 = \mathbf{U} + \mathbf{R}(e_1 + e_3) + \mathbf{R}(e_1 + e_5)$$

$$(a_0 + b_0)(a_1 + b_1) = \mathbf{U} + \mathbf{R}(e_3 - e_5),$$

not depending whether we think of them as elements of $\mathbf{L}_v$ or $\mathbf{L}_1$, and since

$$\mathbf{U} + b_2 = \mathbf{U} + \mathbf{R}(e_1 - se_5) \in \mathbf{L}_v, \mathbf{U} + b_2 = \mathbf{U} + \mathbf{R}(e_1 - re_5) \in \mathbf{L}_1$$

$$a_2 + b_2 = \mathbf{U} + \mathbf{R}(e_1 - re_3) + \mathbf{R}(e_1 - re_5) \in \mathbf{L}_1$$

.

Therefore, $\mathbf{a}, \mathbf{b}$ are axially perspective. Calculating the c_i as in $L(\mathbf{V})$ we have $c_1 = \mathbf{R}e_1$, $c_2 = \mathbf{R}(e_1+e_2+e_4)$ and the elements of $c_0 = (a_1+a_2)(b_1+b_2)$ have coordinate vectors of the form

$$(x+y, z, x-yr, x, 0) = (k+m, k, 0, n, k-ms)$$

where, consequently, $x = yr, k = ms, x+y = k+m$, and $y(r+1) = m(s+1)$. Having $e_1+e_2+e_4$ in c_2 it is also in c_1+c_0 which means that the vector $(1,1,0,1,0)$ is of the above form, except the first coordinate, and then $x = k = 1$, $y = r^{-1}$, $m = s^{-1}$ whence $1+r^{-1} = 1+s^{-1}$ and $r = s$.

EXAMPLE 3.

This is the gluing of eight planes with skeleton $\mathbf{2}^3$. Let $\mathbf{V}$ an $\mathbf{R}$-vector space with basis $e_1, e_2, e_3, f_1, f_2, f_3$. Consider $\mathbf{S}$ as the lattice of subsets of $T = \{0,1,2\}$ with bottom $\perp$ and top T writing the proper subsets just as lists. Define embeddings σ and π of $\mathbf{S}$ into $L(\mathbf{V})$ with

$$\sigma X = \sum(\mathbf{R}e_i | i \in X),\ m = \sigma T = \mathbf{V}_\perp\ \ , \pi X = m + \sum(\mathbf{R}f_i | i \in X).$$

Let $e_i^X = e_i + \sigma X$ if $i \notin X$ and $e_i^X = f_i + \sigma X$ if $i \in X$. So the $e_i^X (0 \leq i \leq 2)$ form a basis of the subquotient space $\pi X / \sigma X$ of $\mathbf{V}$. We assume that each gluing map $\gamma_{X,Y}$, where $Y = X \cup \{i\}$, is induced by a linear map and fixes the $\pi Y + \mathbf{R}e_j^X$ with $j \neq i$. So it is uniquely determined by the invertible scalar $r_{X,i}$ such that

$$\gamma_{X,Y}(\pi Y + \mathbf{R}(e_j^X - e_k^X) = \pi Y + \mathbf{R}(e_j^Y - r_{X,i}e_k^X),$$

$j, k \neq i$, and (j,k) one of $(0,1), (1,2), (2,0)$. In other words, the coefficients $r_{X,i}$ are defined with reference to the system of bases $e_i^X (0 \leq i \leq 2)$ derived from the given basis of $\mathbf{V}$, canonically, using the cyclic order on the index set. Let $\mathbf{L}$ denote the resulting glued lattice.

Lemma 2 *If* $\mathbf{L}$ *is Arguesian then in* $\mathbf{R}$ *the following relation (*) holds.*

$$r_{01,2}^{-1} r_{1,0} r_{\perp,0}^{-1} r_{0,1} r_{02,1}^{-1} r_{0,2} r_{\perp,2}^{-1} r_{2,0} r_{12,0}^{-1} r_{2,1} r_{\perp,1}^{-1} r_{1,2} = 1$$

.

Proof. Define $a_i, b_i \in \mathbf{L}_i$ by

$$a_i = \mathbf{R}e_i + \mathbf{R}f_i, b_0 = \mathbf{R}e_0 + \mathbf{R}(e_1 + r_{\perp,0}e_2 + f_0)$$

$$b_1 = \mathbf{R}e_1 + \mathbf{R}(e_2 + r_{\perp,1}e_0 + f_1), b_2 = \mathbf{R}_2 + \mathbf{R}(e_0 + r_{\perp,2}e_1 + f_2).$$

Observe that the construction of $\mathbf{L}$ and the definition of the a_i, b_i behave symmetrically under cyclic permutation of indices. Straightforeward calulation yields

$$a_0 + b_0 = \mathbf{R}e_0 + \mathbf{R}f_0 + \mathbf{R}(e_1 + r_{\perp,0}e_2) \in \mathbf{L}_0$$

$$m(a_0+b_0) = \mathbf{R}e_0+\mathbf{R}(e_1+r_{\perp,0}e_2) \in \mathbf{L}_0, m(a_0+b_0) = \mathbf{R}e_0+\mathbf{R}(e_1+e_2) \in \mathbf{L}_\perp.$$

By symmetry we get

$$m(a_1 + b_1) = \mathbf{R}e_1 + \mathbf{R}(e_0 + e_2) \in \mathbf{L}_\perp, m(a_2 + b_2) = \mathbf{R}e_2 + \mathbf{R}(e_0 + e_1) \in \mathbf{L}_\perp.$$

Hence, the triangles $\mathbf{a}, \mathbf{b}$ are centrally perspective and, by the Arguesian law, axially perspective, too. Having

$$e_i^i = f_i + \mathbf{R}e_i,\ e_i^{01} = f_i + \mathbf{R}e_0 + \mathbf{R}e_1 \text{ for } i = 0, 1$$

one calculates

$$b_0 + \sigma 01 = \mathbf{R}e_0 + \mathbf{R}e_1 + \mathbf{R}(r_{0,1}f_0 + r_{\perp,0}e_2) \in \mathbf{L}_{01}$$

$$b_1 + \sigma 01 = \mathbf{R}e_0 + \mathbf{R}e_1 + \mathbf{R}(f_1 + r_{1,0}e_2) \in \mathbf{L}_{01}$$

$$c_2 = \mathbf{R}e_0 + \mathbf{R}e_1 + \mathbf{R}(r_{\perp,0}^{-1}r_{0,1}f_0 - r_{1,0}^{-1}f_1) \in \mathbf{L}_{01}$$

$$m + c_2 = m + \mathbf{R}(r_{\perp,0}^{-1}r_{0,1}f_0 - r_{1,0}^{-1}r_{01,2}f_1) = m + \mathbf{R}(rf_0 - f_1) \in \mathbf{L}_T$$

$$\text{where } r = r_{01,2}^{-1}r_{1,0}r_{\perp,0}^{-1}r_{0,1}.$$

By symmetry,

$$m + c_0 = m + \mathbf{R}(sf_1 - f_2) \in \mathbf{L}_T \text{ and } m + c_1 = m + \mathbf{R}(tf_2 - f_0) \in \mathbf{L}_T$$

where

$$s = r_{12,0}^{-1} r_{2,1} r_{\perp,1}^{-1} r_{1,2}, \quad t = r_{02,1}^{-1} r_{0,2} r_{\perp,2}^{-1} r_{2,0}.$$

Now, $m + c_2 \leq m + c_0 + c_2$ implies $rf_0 - f_1 \in \mathbf{R}(sf_1 - f_2) + \mathbf{R}(tf_2 - f_0)$ whence $rts = 1$.

EXAMPLE 4. The following example is a length six modular non-arguesian Hjelmslev plane.

Let $\mathbf{R}$ be a field in which 2 is invertible (for example $\mathbf{R} = \mathbf{Z}_3$) and let the $\mathbf{R}$-vector space $\mathbf{V}$ have basis $e_0, e_1, e_2, f_0, f_1, f_2$. We define

$$\mathbf{U} = \mathbf{R}e_0 + \mathbf{R}e_1 + \mathbf{R}e_2, \quad \mathbf{W} = \mathbf{R}f_0 + \mathbf{R}f_1 + \mathbf{R}f_2,$$
$$\phi : \mathbf{U} \longmapsto \mathbf{W} \text{ by } \phi(e_i) = f_i (0 \leq i \leq 2).$$

For $\mathbf{S} = [0, \mathbf{U}]$, a projective plane, and each $x \in \mathbf{S}$ let $\mathbf{L}_x = [x, \mathbf{U} + \phi(x)]$.

Now define the linear transformation $\mathbf{T} : \mathbf{V} \mapsto \mathbf{V}$ by $\mathbf{T}(e_i) = e_i$ and $\mathbf{T}(f_i) = -f_i$ For $x \prec y$ in $\mathbf{S}$,

$$\chi_{xy} : [y, \mathbf{U} + \phi x] \mapsto [y, \mathbf{U} + \phi(x)]$$

is given by:

$$\chi_{xy}(z) = \begin{cases} \mathbf{T}(z), & \text{if } y \neq \mathbf{R}e_1 + \mathbf{R}e_2 \text{ and } x \leq \mathbf{R}e_1 + \mathbf{R}e_2. \\ z, & \text{otherwise.} \end{cases}$$

Observe that for the instances where a twist occurs the bounds of the gluing interval are sums of subspaces of $\mathbf{U}$ and $\mathbf{W}$ so the twist maps the interval onto itself and we get a glued modular lattice $\mathbf{L}$, indeed.

To show that $\mathbf{L}$ is not arguesian take

$$a_i = \mathbf{R}e_i + \mathbf{R}f_i, \ b_0 = \mathbf{R}(e_1 + e_2 - e_0) + \mathbf{R}(f_1 + f_2 - f_0),$$

$$b_1 = \mathbf{R}(e_0 + e_2 - e_1) + \mathbf{R}(e_1 + f_0 + f_2 - f_1)$$

and

$$b_2 = \mathbf{R}(e_0 + e_1 - e_2) + \mathbf{R}(f_0 + f_1 - f_2).$$

Then

$$p = \mathbf{R}(e_0 + e_1 + e_2) + \mathbf{R}(f_0 + f_1 + f_2),\ c_2 = \mathbf{R}(e_0 - e_1) + \mathbf{R}(e_1 + 2(f_0 - f_1)),$$

$$c_1 = \mathbf{R}(e_0 - e_2) + \mathbf{R}(f_0 - f_2),\ \text{and}\ c_0 = \mathbf{R}(e_1 - e_2) + \mathbf{R}(e_1 + 2(f_1 - f_2)).$$

EXAMPLE 5.

In this section we provide an example of a non-Arguesian lattice that consists of twenty projective planes and two projective geometries of dimension 3 glued together as described by Day and Jónsson in their series of papers [2, 3, 4]. This indicates that their theory of non-Arguesian lattices probably cannot be sharpened.

Let $e_1, e_2, ..., e_8$ be the canonical basis for the vector space $V = \mathcal{Z}_2{}^8$ of dimension 8 over the field $\mathbf{R} = k = \mathcal{Z}_2$ of order 2, and let $\mathcal{L}$ be the lattice of subspaces of this vector space.

Define :

$a_0 = ke_1 + ke_3 + ke_7$
$b_0 = ke_2 + ke_3 + ke_8$
$a_1 = ke_1 + ke_4 + k(e_5 + e_7)$
$b_1 = ke_2 + ke_4 + k(e_5 + e_8)$
$a_2 = ke_1 + k(e_2 + e_3 + e_4) + k(e_6 + e_7)$
$b_2 = ke_2 + k(e_1 + e_3 + e_4) + k(e_6 + e_8)$
$p = ke_1 + ke_2 + k(e_7 + e_8)$
$c_0 = ke_4 + k(e_1 + e_2 + e_3) + k(e_5 + e_6)$
$c_1 = ke_3 + k(e_1 + e_2 + e_4) + ke_6$
$c_2 = ke_3 + ke_4 + ke_5$

Easy calculations yield the following in $\mathcal{L}$.

$p + a_0 = p + b_0 = a_0 + b_0 = ke_1 + ke_2 + ke_3 + ke_7 + ke_8$
$p + a_1 = p + b_1 = a_1 + b_1 = ke_1 + ke_2 + ke_4 + k(e_5 + e_7) + k(e_5 + e_8)$
$p + a_2 = p + b_2 = a_2 + b_2 = ke_1 + ke_2 + k(e_3 + e_4) + k(e_6 + e_7) + k(e_6 + e_8)$
$p(a_0 + a_1) = a_0(p + a_1) = a_0(p + a_2) = a_1(a_0 + a_2) = a_2(a_0 + a_1) = ke_1$
$p(b_0 + b_1) = b_0(p + b_1) = b_0(p + b_2) = b_1(b_0 + b_2) = b_2(b_0 + b_1) = ke_2$

Let $m = ke_1 + ke_2 + ke_3 + ke_4$ then
$\underline{a} = a_0 a_1 a_2 = ke_1$
$\underline{b} = b_0 b_1 b_2 = ke_2$
$a_0 b_0 = ke_3$
$a_1 b_1 = ke_4$
$a_2 b_2 = k(e_1 + e_2 + e_3 + e_4)$
$a_0 + a_1 + a_2 = m + ke_5 + ke_6 + ke_7$
$b_0 + b_1 + b_2 = m + ke_5 + ke_6 + ke_8$
$p + a_0 + a_1 = m + ke_5 + ke_7 + ke_8$
$p + a_0 + a_2 = m + ke_6 + ke_7 + ke_8$
$p + a_1 + a_2 = m + k(e_5 + e_7) + k(e_5 + e_8) + k(e_5 + e_6)$

Let L be the union of the intervals $m/0$, $1/m$ and the intervals listed below.
$I_1 : m + k(e_7 + e_8)/ke_1 + ke_2$
$I_2 : m + ke_7/ke_1 + ke_3$
$I_3 : m + ke_8/ke_2 + ke_3$
$I_4 : m + k(e_5 + e_7)/ke_1 + ke_4$
$I_5 : m + k(e_5 + e_8)/ke_2 + ke_4$
$I_6 : m + ke_5/ke_3 + ke_4$
$I_7 : m + k(e_6 + e_7)/ke_1 + k(e_2 + e_3 + e_4)$
$I_8 : m + k(e_6 + e_8)/ke_2 + k(e_1 + e_3 + e_4)$
$I_9 : m + ke_6/ke_3 + k(e_1 + e_2 + e_4)$
$I_{10} : m + k(e_5 + e_6)/ke_4 + k(e_1 + e_2 + e_3)$
$J_1 : m + ke_7 + ke_8/ke_1 + ke_2 + ke_3$
$J_2 : m + k(e_5 + e_7) + k(e_5 + e_8)/ke_1 + ke_2 + ke_4$
$J_3 : m + ke_5 + ke_7/ke_1 + ke_3 + ke_4$
$J_4 : m + ke_5 + ke_8/ke_2 + ke_3 + ke_4$
$J_5 : m + k(e_7 + e_8) + k(e_6 + e_7)/ke_1 + ke_2 + k(e_3 + e_4)$
$J_6 : m + ke_6 + ke_7/ke_1 + ke_3 + k(e_2 + e_4)$
$J_7 : m + k(e_6 + e_7) + k(e_5 + e_6)/ke_1 + ke_4 + k(e_2 + e_3)$
$J_8 : m + ke_6 + ke_8/ke_2 + ke_3 + k(e_1 + e_4)$
$J_9 : m + k(e_5 + e_6) + k(e_6 + e_8)/ke_2 + ke_4 + k(e_1 + e_3)$
$J_{10} : m + ke_5 + ke_6/ke_3 + ke_4 + k(e_1 + e_2)$

We will use the notation 0_{I_s} to indicate the least element of the interval I_s and 1_{I_s} the greatest element.

Claim 1. L is a sublattice of $\mathcal{L}$.

In proving the claim, we will only check meets. Joins are dual and can be easily verified.

(1) if $x \in m/0$ and $y \in I_s, J_s$ or $1/m$ then $xy \in m/0$.

(2) If $x \in I_s$ and $y \in 1/m$ then $xy \in I_s$.

(3) If $x \in I_s$ and $y \in I_t$ then $xy \in m/0$.

(4) If $x \in I_s$ and $y \in J_t$ and $1_{I_s} \not\leq 1_{J_t}$ then $xy \in m/0$.

(5) If $x \in I_s$ and $y \in J_t$ and $1_{I_s} \leq 1_{J_t}$ then $xy \in I_s$.

(6) If $x \in J_s$ and $y \in 1/m$ then $xy \in J_s$.

(7) If $x \in J_s$ and $y \in J_t$ and $s \neq t$ then $xy \in (1_{J_s}1_{J_t})/(0_{J_s}0_{J_t})$ and this latter interval is either I_r for some r, or a subinterval of $m/0$.

We note first that $\underline{a}, \underline{b}, a_0b_0, a_1b_1, a_2b_2$, form a 4-diamond in the interval $m/0$ and that $a_0+a_1+a_2$, $b_0+b_1+b_2$, $p+a_0+a_1$, $p+a_1+a_2$, and $p+a_0+a_2$ are a dual 4-diamond in the interval $1/m$.

Each interval I_s and each interval J_t is isomorphic to the projective plane with seven points and seven lines.

The skeleton of this lattice is isomorphic to the lattice consisting of a least element, a greatest element, and all two and three element subsets of a five element set, with the natural order.

From the lattice L we now construct a non-Arguesian lattice L' by redefining the gluing between the blocks $I = I_6$ and $J = J_{10}$.

The interval $I \cap J$ is the interval $m + ke_5/ke_3 + ke_4 + k(e_1 + e_2)$, and is isomorphic to M_3.

Claim 2. If $x \in I \cap J$ and $x \in I_s$ and $I \neq I_s$, then $x \leq m$. If $x \in I \cap J$ and $x \in J_s$ and $J \neq J_s$, then $x \geq m$.

First, if $x \in J_s \cap J_t$ for distinct s and t, then $x \in 1/m$, and if $x \in I_s \cap I_t$ for distinct s and t, then $x \in m/0$.

If the first condition in the claim is true then $x \in I_s \cap I_T$ for distinct s and t. If the second holds then $x \in J_s \cap J_t$ again for $s \neq t$.

The interval $I \cap J$ consists of the elements

$ke_3 + ke_4 + k(e_1 + e_2)$ $ke_1 + ke_2 + ke_3 + ke_4 + ke_5$ and the three atoms (in the interval), $s = ke_3 + ke_4 + ke_5 + k(e_1 + e_2)$ $q = ke_1 + ke_2 + ke_3 + ke_4$, and $r = ke_3 + ke_4 + k(e_1 + e_2) + k(e_1 + e_5)$

and $\quad a_2b_2 + c_2 \ = \ ke_3 + ke_4 + ke_5 + k(e_1 + e_2) \ = \ s.$

Using the construction technique of Herrmann [4], define the gluing map σ from $I \cap J$ onto $I \cap J$ by:

$\sigma(x) = x$ with the exceptions $\sigma(s) = r$ and $\sigma(r) = s$.

Let L' be the "lattice" obtained from L by using σ in place of the identity gluing map between the intervals I and J, and all other gluings as in L.

Claim 3. L' is a modular lattice.

We must show (1) that for every covering sublattice of the skeleton:

If $x \in I$ and $y \in T$ then $(x + 0_J) + 0_T = (x + 0_{J_s}) + 0_T$
and $((y)1_{J_s})1_I + ((y)1_J)1_I$
and (2) that for every covering sublattice of the skeleton:

If $x \in B$ and $y \in J$ then $(x + 0_{I_s}) + 0_J = (x + 0_I) + 0_J$
and $((y)1_I)1_B + ((y)1_{I_s})1_B$

Suppose that $x \in I$ and $y \in T$ and that

is a covering sublattice of the skeleton. Then,
$m \leq (x + 0_J) + 0_T \leq m + ke_5 \quad$ and $\quad m \leq (x + 0_{J_s}) + 0_T \leq m + ke_5$
and

$$(x + 0_J) + 0_T = \begin{cases} m, & \text{if } x \leq m; \\ m + ke_5, & \text{if } x \not\leq m. \end{cases}$$

$$(x + 0_{J_s}) + 0_T = \begin{cases} m, & \text{if } x \leq m; \\ m + ke_5, & \text{if } x \not\leq m. \end{cases}$$

Similarly,
$m \leq (y1_J)1_I \leq m + ke_5 \quad$ and $\quad m \leq (y1_{J_s})1_I \leq m + ke_5$

and

$$(y1_J)1_I = \begin{cases} m & \text{if } m + ke_5 \not\leq y, \\ m + ke_5, & \text{if } m + ke_5 \leq y. \end{cases}$$

$$(y1_{J_s})1_I = \begin{cases} m & \text{if } m + ke_5 \not\leq y, \\ m + ke_5 & \text{if } m + ke_5 \leq y. \end{cases}$$

Now, suppose that $x \in B$, $y \in J$ and

is a covering sublattice of the skeleton.

Since $x \in B$ implies $x \leq m$,

$0_J \leq x + 0_I + 0_J \leq m$ and $0_J \leq x + 0_{I_s} + 0_J \leq m$

$$x + 0_I + 0_J = \begin{cases} 0_J & \text{if } x \leq 0_J, \\ m & \text{if } x \not\leq 0_J. \end{cases}$$

$$x + 0_{I_s} + 0_J = \begin{cases} 0_J & \text{if } x \leq 0_J; \\ m & \text{if } x \not\leq 0_J. \end{cases}$$

$0_J \leq (y1_I)1_B \leq m$ and $0_J \leq (y1_{I_s})1_B \leq m$

$$(y1_I)1_B = \begin{cases} m & \text{if } m \leq y; \\ 0_J & \text{if } m \not\leq y. \end{cases}$$

$$(y1_{I_s})1_B = \begin{cases} m & \text{if } m \leq y; \\ 0_J & \text{if } m \not\leq y. \end{cases}$$

Claim 4. The lattice L' is non-Arguesian.

By construction $c_2 = ke_3 + ke_4 + ke_5$ and $c_0 + c_1 = ke_3 + ke_4 + ke_5 + ke_6 + k(e_1 + e_2)$. However, since $c_2 \in I$ and $c_0 + c_1 \in J$, we must use σ in calculating $c_2(c_0+c_1)$. More precisely, $a_2b_2 \in I \cap J$ and $a_2b_2 + c_2 \leq c_0 + c_1$ iff $c_2 \leq c_0 + c_1$.

$\sigma(a_2b_2+c_2) = ke_3+ke_4+k(e_1+e_2)+k(e_1+e_5)$ and $ke_3+ke_4+k(e_1+e_2)+k(e_1+e_5) \not\leq ke_3+ke_4+ke_5+ke_6+k(e_1+e_2)$ since $k(e_1+e_5) \not\leq ke_5+k(e_1+e_2)$ but,

$(a_0 + b_0)(a_1 + b_1) = p \leq a_2 + b_2$.

It is easy to check that this is a prime perspectivity configuration in the sense of Day and Jónsson [3], since $c_2(c_0+c_1) = (ke_3+ke_4+ke_5)(c_0+c_1) = ke_3+ke_4 \prec ke_3+ke_4+ke_5$.

A *gluing setup* over the *skeleton* lattice $\mathbf{S}$ of finite length consists of a family of modular lattices $\mathbf{L}_x$ of finite length, $x \in \mathbf{S}$, and *gluing isomorphisms* γ_{xy}, $x < y$ in $\mathbf{S}$, between an ideal of $\mathbf{L}_x$ and a filter of $\mathbf{L}_y$ (which are proper for covering pairs) such that $\gamma_{xz} = \gamma_{yz} \circ \gamma_{xy}$ for all $x < y < z$ in $\mathbf{S}$ and, for all uncomparable x, y in $\mathbf{S}$ with $u = x \wedge y$, $z = x \vee y$,

$$dom\gamma_{uz} = dom\gamma_{ux} \cap dom\gamma_{uy}, \quad im\gamma_{uz} = im\gamma_{xz} \cap im\gamma_{yz}.$$

Alternatively, we may think of the gluing maps given for covering pairs only and the others defined by composition - and then satisfying the axioms .

According to [8] identifying the elements of the $\mathbf{L}_x$'s along the gluing maps yields a modular lattice $\mathbf{L}$ of finite length , the *glued sum* which is the union of interval sublattices (isomorphic to) $\mathbf{L}_x$, $x \in \mathbf{S}$, and is ordered by the transitive closure of the unions of the orders on these intervals. If the $\mathbf{L}_x$ are complemented, then they become the maximal complemented intervals of $\mathbf{L}$, so $\mathbf{S}$ is (isomorphic to) $S(\mathbf{L})$. Here, the *skeleton* $S(\mathbf{L})$ and the dual skeleton $S^*(\mathbf{L})$ of a finite length modular lattice $\mathbf{L}$ are defined as the sets of bottom resp. top elements of the maximal complemented intervals. These form a join and a meet subsemilattice, respectively, and are isomorphic as lattices. In the associated *natural* gluing setup the component lattices are these intervals and the gluing maps identity on the overlaps.

Given another gluing setup $\mathbf{L}'_x$, γ'_{xy} over the same skeleton $\mathbf{S}$, a *transformation* consists in a family of isomorphisms $\phi_x : \mathbf{L}_x \to \mathbf{L}'_x$,$x \in \mathbf{S}$, such that $\gamma'_{xy} \circ \phi_x = \phi_y \circ \gamma_{xy}$ for all $x < y$ in $\mathbf{S}$ - of course the $x \prec y$ suffice. Such a transformation induces an isomorphism of the glued sums $\mathbf{L}$ and $\mathbf{L}'$, obviously. Given just the $\mathbf{L}'_x$ and isorphisms ϕ_x of $\mathbf{L}_x$ onto $\mathbf{L}'_x$ we produce the *associated* gluing setup, to which we transform via the ϕ's defining

$$\gamma'_{xy} = \phi_x^{-1} \circ \gamma_{xy} \circ \phi_y.$$

Representing a glued lattice as a sublattice of a vector space lattice amounts just to establishing a transformation between the given gluing setup and the natural one of this sublattice.

In view of Theorem 1 (and the Fundamental Theorem of Projective Geometry) we refine ourselves to considering gluing setups with each $\mathbf{L}_x$ the lattice of $\mathbf{R}_x$-vector subspaces of some $\mathbf{V}_x$ and gluing maps γ_{xy} induced by

α_{xy}-semilinear maps χ_{xy}, where α_{xy} is an isomorphism of $\mathbf{R}_x$ onto $\mathbf{R}_y$. For two such, a (semi)linear transformation will be a transformation induced by a family ϕ_x of (semi)linear maps. This allows us to pass to $\mathbf{R}_x = \mathbf{R}$ for all x.

Observation 1 *Having an $\mathbf{R}$-vector space $\mathbf{V}$ and join and meet embeddings σ and π of $\mathbf{S}$ into $L(\mathbf{V})$ such that $\sigma x \leq \pi x$ and the subquotient $\pi x/\sigma x$ of $\mathbf{V}$ has the same dimension as $\mathbf{V}_x$ for all x , we may assume that $\mathbf{V}_x$ is actually this subquotient and χ_{xy} just a semilinear map of π_x/σ_y onto itself. Of course, we can force $\sigma 0 = 0$ and $\pi 1 = \mathbf{V}$. Also, the isomorphisms α_{xy} and gluing maps γ_{xy} can be turned into identity along any fixed subtree (say with root 0) of the diagram of $\mathbf{S}$.*

Namely, adding a covering edge (x, y) let ϕ_y any α_{xy}^{-1}-semilinear transformation of $\mathbf{V}_y$ extending χ_{xy}^{-1} and let all other ϕ's be identity. The automorphism associated with γ'_{zy} is $\alpha_{xy}^{-1}\alpha_{zy}$, the one with γ'_{yz} is $\alpha_{xy}^{-1}\alpha_{yz}^{-1}$.

For the remainder of this section let all $\mathbf{V}_x$ have dimension 3, all gluing quotients (for covering pairs) have dimension 2. This implies, in particular, that the skeleton and its dual are cover preserving sublattices of $\mathbf{L}$. If, moreover, the skeleton is complemented then $S(\mathbf{L}) \subseteq \mathbf{L}_0$, $S^*(\mathbf{L}) \subseteq \mathbf{L}_1$, and $1_{S(\mathbf{L})} \leq 0_{S^*(\mathbf{L})}$.

Theorem 2 *For skeleton $\mathbf{S}$ of length 2 the following are equivalent:*

(i) $\mathbf{L}$ *is Arguesian*

(ii) $\alpha_{x1} \circ \alpha_{0x} = \alpha_{y1} \circ \alpha_{01}$ *for all atoms x, y of $\mathbf{S}$*

(iii) $\mathbf{L}$ *has an (isometric) embedding into a vector space lattice (over $\mathbf{R}$).*

Proof. Assume (i) . With $\mathbf{V}$ of dimension 5 and any cover preserving σ, π such that $\sigma 0 = 0$, $\pi 1 = \mathbf{V}$, and $\sigma 1 \leq \pi 0$, all the above mentioned reductions apply. Now, fix an atom u of $\mathbf{S}$. Using the tree with edges $(u, 1)$ and $(0, x)$ for all atoms x, the above procedure produces in effect a semilinear transformation to a gluing setup in which the γ_{v1} ,$u \neq v \prec 1$, are induced by an $\alpha_v = \alpha_{0u}^{-1} \circ \alpha_{u1}^{-1} \circ \alpha_{v1} \circ \alpha_{0v}$-semilinear map while all others are identity. By still another transformation we can provide all γ_{v1} with at least two fixpoints, one of them being $\sigma 1$. Applying Lemma 1 to the gluing over the sublattice $\{0, u, v, 1\}$ of $\mathbf{S}$ after choosing the basis, properly, we conclude that γ_{v1} is identity, too - for all v.

Proposition 1 *Every Arguesian lattice of length at most 5 has a permuting equivalence representation.*

Proof. Actually, with the exception of two planes of different characteristic glued over a 2-element interval considered in [11], in all other cases one has an isometric vector representation. The cases of lattices of length at most 4, of breadth at most 2, or skeleton which is a chain have been covered in [11, 9, 13, 14]. If the skeleton is not a chain then according to [10] it has to be of length 2. Consider an atom x of the skeleton such that $\mathbf{L}_x$ has length 2. Then the only atom of this interval which is join reducible in $\mathbf{L}$ is the top of the skeleton and of $\mathbf{L}_0$ at the same time. With the dual observation and the comparability derived from $\mathbf{S}$ being complemented we have that this is also the bottom of $\mathbf{L}_1$ and the dual skeleton and the only meet reducible atom of $\mathbf{L}_x$. Since the adding of doubly irreducibles to an isometric vector representation is just a matter of the field being large enough we may assume that all $\mathbf{L}_x$ have length 3. If $\mathbf{L}_0$ is reducible then $\mathbf{L}_x$ is irreducible for atmost one atom x of the skeleton and the pair $(0, x)$ generates a congruence decomposing $\mathbf{L}$ in a subdirect product of a length 2 and a 2-element lattice from where we get the representation. Having both $\mathbf{L}_0$ and $\mathbf{L}_1$ irreducible all other $\mathbf{L}_x$ are irreducible, too, and Theorem 2 applies.

Observation 2 *Having all gluing maps induced by linear maps we turn these into identities along any fixed directed edge subgraph of the diagram of* $\mathbf{S}$ *which has in-degree at most 2 and the vertices of which form a meet subsemilattice of* $\mathbf{S}$.

Namely, proceeding by induction on the distance from the root, add edges $(x, y), (z, y)$ and let $u = x \wedge z$. Then on $[\sigma y, \pi u]$ the linear maps χ_{xy} and χ_{zy} induce the same lattice automorphism γ_{uy} so on this subspace of $\mathbf{V}_y$ they differ just by a scalar factor and we can find a linear transformation ϕ_y inducing both γ_{xy}^{-1} and γ_{zy}^{-1} on their domains. Keeping all other ϕ's identity we get the transformation looked for.

Now, let $\mathbf{S}$ the direct product of a length 2 and a 2-element lattice. Then in view of Thm.2 we may, indeed, suppose that all gluing maps are induced by linear maps. Let ϵ an embedding of the lattice of of subsets of $T = \{0, 1, 2\}$ into $\mathbf{S}$. Having for each $\mathbf{V}_{\epsilon X}$, $X \subseteq T$, a basis $e_i^X (0 \leq i \leq 2)$ we speak of a base system if $e_i^{\perp} \in U_{\perp i}$ and for all $Y = X \cup \{i\}$, $i \notin X$,and $j \neq i$, we have

$$\gamma_{XY}(\mathbf{R}e_j^X + U_{XY}) = \mathbf{R}e_j^Y + \gamma_{XY}U_{XY},$$

where U_{XY} is the bottom of $dom\gamma_{XY}$.

Theorem 3 *Under the above hypotheses the following are equivalent:*

(i) $\mathbf{L}$ *is Arguesian*

(ii) *For each ϵ there exists a base system $e_i^X, X \subseteq T, i \in T$ such that (*) from Lemma 2 holds*

(iii) $\mathbf{L}$ *can be (isometrically) embedded into a vector space lattice.*

Proof. The reductions of Observation 1 apply with $\mathbf{V}$ of dimension 6. Let $\mathbf{L}$ Arguesian and consider any ϵ. Choose $e_i \neq 0$ in $\sigma\epsilon i$ and $f_i \in \mathbf{V}$ such that $\mathbf{R}f_i + \sigma\epsilon i$ is a complement of $m = \mathbf{V}_0$ in the interval $[\sigma\epsilon i, \sigma\pi i]$. Then the e_i, f_i give rise to a base system satisfying (*) as in Example 3.

Assume (ii) .Let a, b be an atom and a coatom directly decomposing $\mathbf{S}$. Let $a \prec u$, $d = u \wedge b$. By Thm. 2 we may assume all gluing maps γ_{xy}, $y \leq b$ to be identity. Also, by Observation 2 we can force γ_{0a} , γ_{du}, γ_{au}, γ_{b1} ,and γ_{u1} to be identity. Then following the proof of Thm.2 we can turn all γ_{xy}, $a \leq x$ into identity. Now cosider any atom $x \neq a, d$ and let $y = x \wedge a$. Then the sublattice $\{0, 1, a, b, d, u, x, y\}$ of $\mathbf{S}$ is the image under a suitable ϵ . Then, for the base system guaranteed by (ii) we have all but one coefficient r_{XY} equal 1 by construction; so by (*) the last one is 1 ,too, and we can conclude that γ_{xy} is identity. But this means that all gluing maps are identity.

MINIMALITY

Proposition 2 *Every non-Arguesian modular lattice* $\mathbf{L}$ *of length at most six contains a sublattice which has a non-Arguesian homomorphic image as in one of the Examples 0,1,2,3, or has skeleton a non-degenerate plane and all blocks isomorphic non-degenerate planes (which includes lattices as in Example 4).*

So Examples 0-4 present the typical non-Arguesian modular lattices up to length 6 - of course, in Example 4 we just presented the easiest variant of construction and there might exist non-Arguesian gluings of planes over a plane such that all primary sublattices are Arguesian. In the proof we make full use of the concepts and results in [2, 3, 4].

Proof. If the second case does not take place then by 3.4 and 3.6 in [4] there is a Boolean prime PC in $\mathbf{L}$ which by 4.3 and 5.1 of [3] can be choosen to be contained in a sublattice $\bigcup_{x\in\mathbf{2}^r}[\sigma x, \pi x]$ for an $r \leq 3$ and lattice homomorphisms σ and π of $\mathbf{2}^r$ into $\mathbf{L}$ such that all $[\sigma x, \pi x]$ are non-degenerate projective planes. Factoring out the congruence generated by the $\sigma y/\sigma y \pi x$, $x \prec y$ in $\mathbf{2}^r$ in view of Theorems 1,2,3 one obtains a lattice as in one of Example 0-3.

Proposition 3 *In Examples 1-5 all proper sublattices are Arguesian provided that* $\mathbf{R}$ *is a prime field, resp. a minimal extension of a prime field in Example 2.*

Proof. Consider a proper sublattice $\mathbf{L}'$ and assume it non-Arguesian. If $\mathbf{L}'$ contains a Boolean prime PC we can apply the above reduction to a union of projective planes which then are sublattices of a block $\mathbf{L}_x$ each and equal this block in all cases but Example 2. So the image of σ is a sublattice $\mathbf{S}'$ of the skeleton $\mathbf{S}$.If there are u, v in $\mathbf{S}'$ with $u \vee v$ covering v in $\mathbf{S}'$ but not in $\mathbf{S}$, then we may assume u an atom, v a coatom, and have that $\bigcup_{x\leq u}\mathbf{L}_x$ with $\bigcup_{x\geq v}\mathbf{L}_x$ are Arguesian (inducting on the size of $\mathbf{S}'$) and their intersection is a chain whence $\mathbf{L}'$ would be Arguesian by 2.2 in [4]. If $\mathbf{S}'$ is a cover preserving sublattice then by Theorems 1,2,3 $\mathbf{L}'$ would be Arguesian : In Example 2, by minimality of $\mathbf{R}$, the blocks of $\mathbf{L}'$ have to be coordinatized over the prime subfield, so the restrictions of the semilinear maps which induce the gluings are actually linear.

In Example 4 we have to verify condition (*) of Lemma 2 in the case of an isomorphism ϵ of the subset lattice of $\{0, 1, 2\}$ onto $\mathbf{S}'$. Construct a base system for $\mathbf{L}'$ as in the proof of Theorem 3 with $e'_i = e_i^{\perp}$ generating ϵi extended to a basis $e'_i, f'_i (0 \leq i \leq 2)$ by the $f'_i = \phi e'_i$. Then the coefficients r_{XY} are all 1 except in the case that ϵX is a point on $l = \mathbf{R}e_1 + \mathbf{R}e_2$ and $\epsilon Y \neq l$ a line where $r_{XY} = -1$. There are two such point line incidences if l is a line of $\mathbf{S}'$ or if l contains a single point of $\mathbf{S}'$, none otherwise. Hence the product in (*) equals 1, always, and $\mathbf{L}'$ is Arguesian by Theorem 3.

On the other hand assume that there is no Boolean prime PC contained in $\mathbf{L}'$. In the case of Example 4 the preceeding Proposition applies and $\mathbf{L}'$ has skeleton $\mathbf{S} = \mathbf{L}_0$ and all blocks are irreducible whence $\mathbf{L}' = \mathbf{L}$. In Example 5 we have by 4.3 and 4.4 of [3] a stable prime non-Boolean PC in $\mathbf{L}'$ and combining 4.4 of [3] with 3.1 of [4] the associated intervals I_μ, $\mu \neq \emptyset, \mathbf{5}$ are irreducible planes, those with $|\mu| = 2, 3$ all distinct from each other and not in a common block of $\mathbf{L}'$ with an interval $I_{\{i\}}$, $i \in \mu$ or I_ν, $|\nu| = 4, \mu \subseteq \nu$. Therefore, these I_μ have to be intervals I_k or J_k of $\mathbf{L}$ and it follows $\mathbf{L}' = \mathbf{L}$.

LATTICES OVER RINGS

In this section we describe how to construct other non-Arguesian lattices. The technique is to find rings $\mathbf{R}$ and $\mathbf{S}$ with submodule lattices $L({}_\mathbf{R}\mathbf{R}^2) \cong L({}_\mathbf{S}\mathbf{S}^2)$ but $\mathbf{R} \not\cong \mathbf{S}$. Then gluing the lattices $L({}_\mathbf{R}\mathbf{R}^3)$ and $L({}_\mathbf{S}\mathbf{S}^3)$ together yields a non-Arguesian lattice.

If (x_0, x_1, x_2, x_3) is a 3-diamond in a modular lattice - the lattice theoretic equivalent of a projective coordinate system in a plane, cf [5] - with least element u then $x_i + x_j/u \cong x_i + x_k/u$ for distinct i, j, k. We will call any of the above isomorphic intervals a *2-diamond-quotient* of the 3-diamond (x_0, x_1, x_2, x_3). The dimension (=length) of a 2-diamond-quotient is 2/3 the dimension of the 3-diamond quotient.

Suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are Arguesian lattices, $\mathbf{L}_1$ is spanned by the 3-diamond (z, t, x, y) and $\mathbf{L}_2$ is spanned by the 3-diamond (z', t', x', y'). Also suppose that $z + t/0 \cong z' + t'/0$ via a lattice isomorphism f with $f(z) = z'$ and $f(t) = t'$. Let $\mathbf{L}$ be the lattice obtained by the gluing of $\mathbf{L}_1$ and $\mathbf{L}_2$ over the intervals $1/x$ in $\mathbf{L}_1$ and $z' + t'/0$ of L_2 with gluing map g defined by

$$g(a) = f(a(z + t)) \text{ for } a \in 1/x.$$

Define

$$w = (x + y)(z + t),\ w' = (z' + t')(x' + y'),$$

$$\mathbf{R} = \{p\colon p + w = z + w,\ pw = 0\},\ \mathbf{R}' = \{q\colon q + w' = z' + w',\ qw' = x\}$$

$$a \oplus b = (z + t)(x + (y + a)(w + (z + y)(x + b)))$$

$a \odot b = (z+t)(x+(y+a)(z+(y+t)(x+b)))$
$a \oplus' b = (z'+t')(x'+(y'+a)(w'+(z'+y')(x'+b)))$
$a \odot' b = (z'+t')(x'+(y'+a)(z'+(y'+t')(x'+b)))$

Theorem 4 *If* $\mathbf{L}$ *is Arguesian then* f *satisfies the two conditions:*
(1) $f(a \oplus b) = f(a) \oplus' f(b)$ *for* $a, b \in \mathbf{R}$
(2) $f(a \odot b) = f(a) \odot' f(b)$ *for* $a, b \in \mathbf{R}$

If $a, b \in \mathbf{R}$ then $a \oplus b, a \odot b \in \mathbf{R}$ and $f(a), f(b) \in \mathbf{R}'$. The lattice expressions for $\oplus, \odot, \oplus', \odot'$ are von Staudts definition of addition and multiplication in geometry. $\mathbf{R}$ and $\mathbf{R}'$ are the rings obtained from the 3-diamonds in $\mathbf{L}_1$ and $\mathbf{L}_2$ respectively, (cf. [5]), and the two conditions say that f must be a ring isomorphism when restricted to $\mathbf{R}$.

Proof . To shorten the expressions define $\tilde{b} = (z+y)(x+b)$ and $\tilde{b}' = (z'+y')(x'+b) = (z'+y')(x'+b')$, $a' = f(a)$, $b' = f(b)$ for $a, b \in \mathbf{R}$. The triangles $\langle w, \tilde{b}, \tilde{b}' \rangle$ and $\langle a, y, y' \rangle$ are centrally perspective and if $\mathbf{L}$ is Arguesian then axially perspective. Therefore $(y+a)(w+\tilde{b}) \le (y'+a)(w+\tilde{b}') + (y+y')(\tilde{b}+\tilde{b}')$. $(y+y')(\tilde{b}+\tilde{b}') = (x'+y')(\tilde{b}+(z'+y')(x'+b)) = (x'+y')(x'+b)(z'+y'+b) = x'(z'+y'+b) \le x'$. Therefore $(y+a)(w+\tilde{b}) \le x' + (y'+a)(w+\tilde{b}')$ and so $(z+t)(x+(y+a)(w+\tilde{b})) \le (z'+t')(x'+(y'+a)(w+\tilde{b}'))$. This implies $f(a \oplus b) \le f(a) \oplus f(b)$. Since both $f(a \oplus b)$ and $f(a) \oplus f(b)$ are complements of w' in the interval $z'+t'/0$ and comparable complements in modular lattices are equal, $f(a \oplus b) = f(a) \oplus f(b)$.

For (2) define $b_1 = (y+t)(x+b)$ and $b_1' = (y'+t')(x'+b') = (y'+t')(x'+b)$ The triangles $\langle a, y, y' \rangle$ and $\langle z, b_1, b_1' \rangle$ are centrally perspective and by our assumption that $\mathbf{L}$ is Arguesian,
$(y+a)(z+b_1) \le (y+y')(b_1+b_1') + (y'+a)(z+b_1')$.
$(y+y')(b_1+b_1') = (x'+y')(b_1+(y'+t')(x'+b)) = (x'+y')(x'+b)(y'+t'+b_1) \le x'$ and therefore $(y+a)(z+b_1) \le x' + (y'+a)(z'+b_1')$ so that $a \odot b \le f(a) \odot' f(b)$. In the same manner as the last argument we obtain $f(a \odot b) = f(a) \odot' f(b)$, for all $a, b \in \mathbf{R}$.

We need now to find modular lattices with spanning 3-diamonds so that the rings obtained from the 3-diamonds are non-isomorphic but the 2-diamond-quotients are isomorphic. Then by the last theorem the resulting lattice will be non-Arguesian. Let $\mathbf{R}$ be the ring $\mathbf{Z}_2[x]/(x^2)$,$\mathbf{S}$ be the ring $\mathbf{Z}_4$. The lattices $L({}_{\mathbf{R}}\mathbf{R}^2)$ and $L({}_{\mathbf{S}}\mathbf{S}^2)$ are isomorphic, so the lattices $L({}_{\mathbf{R}}\mathbf{R}^3)$ and $L({}_{\mathbf{S}}\mathbf{S}^3)$

can be glued together so that the hypotheses of the theorem are satisfied. The resulting lattice **L** is non-Arguesian since the rings **R** and **S** are not isomorphic. It is minimal such since a stable Boolean PC with planes of order two is impossible by Theorem 3 and a stable non-Boolean PC has to involve 22 of the 32 planes in **L** and so to generate **L**.

Notice that there is no way to glue these two lattices together over this quotient and obtain an Arguesian lattice. Also, the 3-diamonds arising from the failure of the Arguesian law span one of the intervals $L({}_{\mathbf{S}}\mathbf{S}^3)$ or $L({}_{\mathbf{R}}\mathbf{R}^3)$, but the sublattices generated by these 3-diamonds are non-isomorphic.

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