On n-distributive modular ortholattices.

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In [5] the third author has constructed a 3-generated finitely presented modular ortholattice with unsolvable word problem and indicated how to modify the proof in order to obtain the following

Theorem 1 *For any $n \geq 14$, the free algebra on 3 generators in the variety $\mathcal{D}_n$ of n-distributive modular ortholattices has an unsolvable word problem.*

In this note we shall provide a proof via a structural result on algebras in $\mathcal{D}_n$ having a spanning n-frame. Recall that a lattice is n-distributive if it satisfies the equation

$$x_0 \sum_{i=1}^{n+1} x_i = \sum_{j=1}^{n+1} x_0 \sum_{1 \leq i \neq j} x_i$$

Let $n \geq 2$ in the sequel. Elements $c_0, \ldots, c_n$ form an n-frame if for all $j \neq k$ and all l

$$c_j \leq \sum_{i \neq j} c_i, \quad c_j \sum_{i \neq j,k} c_i \leq c_l$$

The frame is spanning if $\sum_i c_i = 1$ and $\prod_i c_i = 0$, trivial if $c_i = c_j$ for all i, j, and orthogonal if $c_j \leq c_k{}'$ for all $1 \leq j \neq k \leq n$. In a modular lattice, the elements $c_i \, (i \neq j)$ of a nontrivial n-frame are the atoms of a boolean sublattice. For the following see Huhn [3] and Mayet and Roddy [4]: A modular (ortho)lattice is n-distributive, if and only if it does contain only trivial (orthogonal) $n+1$-frames. The subdirectly irreducibles in $\mathcal{D}_n$ are irreducible, at most $n-1$-dimensional projective geometries; in particular, they are simple.

Theorem 2 *Within $\mathcal{D}_n$ every finitely presented algebra containing a spanning n-frame is a direct factor of a free algebra on the same number of generators.*

A convenient way to deal with intervals in modular ortholattices and to turn factors in subalgebras is to consider relative modular ortholattices: these are modular lattices with a ternary operation $k(x,y,z)$ such that $k(x,b,c)$ lies in $[bc,\ b+c]$ and within this interval $x \mapsto k(x,b,c)$ defines an orthocomplementation. Moreover, if $uv \leq bc$ and $b+c \leq u+v$ holds, then we require

$$k(x,b,c) = (k(x,u,v)+bc)(b+c)$$

It is easily seen that these form a variety. From each modular ortholattice L we derive its relativization L^* as the underlying lattice together with

$$k(a,b,c) := (a'+bc)(b+c)$$

and observe, that each homomorphism $\phi : L \to M$ is a homomorphism $\phi : L^* \to M^*$ as well. Conversely, any relative modular ortholattice with 0 and 1, e.g. any finitely generated one, can be obtained as an L^*. For such, a direct product $L_0 \times L_1$ can be understood as a direct sum of the subalgebras $L_0 \times \{0\}$ and $\{0\} \times L_1$.

Also, with each variety $\mathcal{V}$ of modular ortholattices one has the variety $\mathcal{V}^*$ of relative modular ortholattices the finitely generated members of which are the L^* where L is finitely generated in $\mathcal{V}$. ¿From this we get, immediately,

Proposition 3 *For a free algebra F in $\mathcal{V}$ with finite generating set X, the subalgebra generated by X in F^* is free in $\mathcal{V}^*$.*

Now, Thm.2 extends to $\mathcal{D}_n^*$, obviously, and we may convert factors into subalgebras to obtain

Corollary 4 *In $\mathcal{D}_n^*$, finitely presented algebras containing a spanning n-frame are projective.*

In order to prove the theorems we shall associate central elements with finitely generated congruences. Recall, that an element u in a modular ortholattice L is central, if and only if it satisfies the following equivalent conditions: the sublattice generated by u,x,y is distributive for all $x,y \in L$; the map $x \mapsto ux$ resp. $x \mapsto u+x$ is a lattice endomorphism (and an endomorphism of L^*); L is isomorphic to $L_0 \times L_1$ with $u \mapsto (0,1)$ (and $u' \mapsto (1,0)$); for some/every subdirect decomposition $\phi_i : L \to M_i$ $(i \in I)$ into subdirectly irreducibles we have $\phi_i u \in \{0_i, 1_i\}$ for all i. Then, the kernel of the projection of L onto L_0 is the smallest congruence identifying u with 0. Also, $L_0 \cong [u,1] \cong [0,u']$.

To make use of the spanning frame we consider the variety $\mathcal{D}_n^+$ of n-distributive modular ortholattices L with constants $c_0, \ldots c_n$ forming a spanning n-frame of L.

Lemma 5 *In $\mathcal{D}_n^+$ there is a binary term $d(x,y)$ such that*

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$$

holds in any subdirectly irreducible member of $\mathcal{D}_n^+$.

Proof. We first show that for each n there is a lattice term $t_n(x, x_0, \ldots, x_n)$ such that for every $n-1$-dimensional projective geometry L, spanning n-frame $c_0, \ldots, c_n$, and $a \in L$ one has

$$t_n(a, c_0, \ldots, c_n) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0 \end{cases}$$

This is done by induction on n. For $n = 2$ the c_i are distinct atoms in a height 2 lattice. Thus, we define

$$t_2(x, x_0, x_1, x_2) = \sum_{j \neq k} (x + x_j)(x + x_k).$$

In the step from $n-1$ to n observe that the c_i are atoms, the $\sum_{i \neq 0,k} c_i$ are dual atoms of L, and for $k \geq 1$

$$(c_0 + c_k) \sum_{1 \leq i \neq k} c_i, \; c_1, \ldots, c_{k-1}, c_{k+1}, \ldots c_n$$

is a spanning $n-1$-frame of the interval $[0, \sum_{i \neq 0,k} c_i]$ of L. With this in mind we define

$$t_n(x, x_0, \ldots, x_n) = \prod_{k=1}^{n} (x + s_n^k) \; + \sum_{k=1}^{n} t_{n-1}^k + (t_{n-1}^k + x_0)x_k$$

where

$$s_n^k = \sum_{i \neq 0,k} x_i$$

$$t_{n-1}^k = t_{n-1}(x s_n^k, \; (x_0 + x_k) s_n^k, \; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$$

Now, observe that $ab' + ba' = 0$ if and only if $a = b$ and let

$$d(x,y) = t_n(xy' + yx', c_0, \ldots, c_n) \quad \square$$

Corollary 6 *$\mathcal{D}_n^+$ is a discriminator variety.*

Proof. The discriminator is given by $t(x,y,z) = z \cdot d(x,y)' + x \cdot d(x,y)$. $\square$
It follows that finitely generated congruences have complements (Fried and Kiss [2]) and that the word problem for a finitely presented algebra reduces to the word problem for the free algebra with the same number of generators (Blok and Pigozzi [1]). We will provide central elements for both purposes.

Lemma 7 *In $\mathcal{D}_n^+$, for each finitely generated congruence θ there is a central element v such that θ is the smallest congruence identifying v with 0.*

Proof. If θ is the smallest congruence identifying a^j with b^j for $1 \leq j \leq k$ then we define v as

$$v = \sum_j d(a^j, b^j)$$

By the property of d given in Lemma 7 we have $v\,\theta\,0$. In a decomposition $\phi_i : L \to M_i$ into subdirectly irreducibles we have $\phi_i v = 0$ if $\phi_i a^j = \phi_i b^j$ for all j and $\phi_i v = 1$, else. Thus, $L \cong L_0 \times L_1$ with $v \mapsto (0,1)$ where L_0 is the (induced) subdirect product of the first, L_1 of the second M_i's. It follows, that θ is contained in the kernel of the projection onto L_0. This congruence is generated by the pair $0, v$ whence equal to θ. $\square$

Lemma 8 *For each L in $\mathcal{D}_n$ and n-frame $a_0, \ldots, a_n$ in L there is a central element u such that for any homomorphism ϕ of L onto M one has $\phi a_0, \ldots, \phi a_n$ a spanning frame of M if an only if $\phi u = 1$.*

Proof. Let $u = (\prod a_i)' \sum a_i$ and notice the isomorphism of $[\prod a_i, \sum a_i]$ onto $[0, u]$. Then $\phi u = 0$ if and only if the frame $\phi a_0, \ldots, \phi a_n$ is trivial. Otherwise, the interval $[0, \phi u]$ has height at least n. Now, consider ϕ such that M is subdirectly irreducible, whence of height at most n. Then either $\phi u = 0$ or $\phi u = 1$ which amounts to a spanning frame $\phi a_0, \ldots, \phi a_n$ of M. In particular, u is central and the claim follows for all ϕ. $\square$

Lemma 9 *Suppose, $\phi : A \to B$ is a surjective homomorphism in $\mathcal{D}_n$ with finitely generated kernel and B contains a spanning n-frame. Then there is a central element $w \in A$ such that ϕ induces an isomorphism of the interval subalgebra $[0, w]$ of A^* onto B^*.*

Proof. According to Huhn [3], from the spanning n-frame b_i of B, one can construct a frame a_i in A with $\phi(a_i) = b_i$. By Lemma 8 there is a central element u of A such that $\phi u = 1$. Let C be the interval subalgebra $[0, u]$ of A^*. Then $\phi(C) = B$, the map $u \mapsto ux$ is an endomorphism of A^*, and the ua_i form a spanning frame of C. Moreover, the kernel of $\phi|C$ is finitely generated (by the pairs us, ut where s, t is a generating pair for ϕ). Choose the central element v of C corresponding to this congruence according to Lemma 5. Then $w = v'$ is central in C, whence in A and $[0, w]$ is mapped isomorphically onto B under ϕ. $\square$ Thm.2 follows, immediately.

Proof of Theorem 1. In [5] there has been constructed for each $n \geq 14$ a finite presentation $\phi : A \to B$ in $\mathcal{D}_n$ with unsolvable word problem such that A is free on 3 generators and B contains a spanning n-frame - use Lemma 2.7.2 to add dimensions exceeding 14. Applying Lemma 9 we get a central element $w \in A$. The relativization t^w of terms is defined, inductively: $x^w = xw$, $(s+t)^w = s^w + t^w$, $(st)^w = s^w t^w$, $(t')^w = w(t^w)'$. Now, $\phi\pi s = \phi\pi t$ if and only if $\pi s^w = \pi t^w$ where π is the canonical homomorphism of the term algebra onto A. This reduces the word problem for B to that for A. $\square$

References

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