IV. Classification of Subspaces

The material in this chapter has been presented by Christian Herrmann (above all), Remo Moresi, Reto Schuppli, Martin Walter, Christine Widmer, and Marcel Wild. All figures are shown at the end of the chapter.

1 Introduction

1.1 Statement of the problems and the lattice method

Let E and E' be non-degenerate ε-hermitean spaces over the same data $(k, \varepsilon, -)$ (see I.1.1) with linear subspaces F and F', respectively. The pairs (E, F) and (E', F') are *isometric* if there is an isometry ϕ of E onto E' such that $\phi F = F'$. Isometry classification of pairs means reduction to the classification of spaces, the latter being a classical and often difficult problem even in finite dimensions.

So one considers $E = E'$ and says that subspaces F and F' of E are *congruent* if they give rise to isometric pairs (E, F) and (E, F') , in other words, if they belong to the same orbit under the action of the orthogonal group. The *Congruence Problem* for a class of pairs asks for a complete set of invariants of those orbits - these invariants may include the isometry type of $\hat{F}$ and other subspaces naturally related to F.

For congruent F and F' one can ask under which conditions a given isometry of F onto F' can be extended to an isometry of E (*Extension Problem*). Witt's Theorem says that such an extension always exists in the case of trace valued finite-dimensional E (see [Wil87] Satz 4 and Anmerkung p.31; [Gro79,I.7]. So, in this case the Congruence Problem is solved using $\hat{F}$, only. The analogous result for finite-dimensional quadratic forms in characteristic 2 follows from Arf's Theorem [Arf41]. Observe that spaces over k of characteristic $\neq 2$ are trace valued, automatically cf [Gro79,I.3].

Beyond this, i.e. in infinite dimensions or for characteristic 2 finite-dimensional spaces, additional invariants are required describing the embedding of F into E. For trace valued spaces of countable dimension the Congruence Problem has been solved in [Gro79,V] under the proviso that there are sufficiently many isotropic vectors. Without this hypothesis, the problem comprises intractable parts even over the reals. Though, using infinite lattices, Bäni [Bän83/85] solved the problem for substantial classes of subspaces. In characteristic

2 and at most countable dimension solutions of the Congruence problem are given in [Gro79,VIII,XVI] provided that F is totally isotropic or E a quadratic space. There and in the cases to be discussed, below, these invariants and the classification results have been based on the *lattice method* (cf [Gro79,I.9], [Gro82], [Gro87]). Its beginnings can be traced back to Kaplansky [Kap50].

Step A. This method starts with selecting the relevant operations on the lattice $\mathcal{L}(E)$ of all linear subspaces of E which will include the sum $+$, the intersection $\cap$, the constants 0 and 1 for the subspaces (0) and E, the unary operation $^{\perp}$ of taking the orthogonal, and maybe additional unary operations and constants defined in terms of the sesquilinear form. If, in a given context, we consider lattices with such additional operations we speak of *quadratic lattices* associated with the spaces.

Step B. Next, one determines the possible quadratic sublattices $\mathcal{V}_E(F)$ generated by F in $\mathcal{L}(E)$ or certain relevant parts of those lattices. Depending on the setting chosen, these may be computed in rather geometric terms working in the concrete $\mathcal{V}_E(F)$ or, more abstractly, as quadratic lattices generated by some element a and subject to a suitable additional axiomatization. Also, a mixture of both approaches may be applied.

In all cases dealt with, below, it is possible to capture the "general form" of these lattices in a free object $\mathcal{V}(a)$: for each pair (E,F) there is a quadratic lattice homomorphism π_{EF} of $\mathcal{V}(a)$ into $\mathcal{V}_E(F)$ mapping a onto F and $\mathcal{V}(a)$ can be implemented as actual $\mathcal{V}_E(F)$. When working with an axiomatization this indicates the full adequacy. So one calculates the quadratic lattice $\mathcal{V}(a)$ freely generated by a under this axiomatization. A second approach finds subdirectly irreducible factors of $\mathcal{V}(a)$, heuristically, proves the completeness of the list by a method developed in section 2, and then computes $\mathcal{V}(a)$ based on that.

Step C. With this (hopefully finite) lattice at hand, (*cardinal*) invariants are provided by the *indices*, i.e. the dimensions of the quotient spaces $\pi_{EF}x/\pi_{EF}y$, $x > y$ in $\mathcal{V}(a)$. They include the information about the actual lattice $\mathcal{V}_E(F)$ - via the indices which are 0. *Metric* invariants are the isometry types $\hat{X}$, with X in $\mathcal{V}_E(F)$ - there are other invariants where X is a freely chosen supplement for a certain quotient of $\mathcal{V}_E(F)$. Other types of invariants, called *arithmetic*, are composed from field elements or classes thereof. To solve the Congruence Problem, it remains to prove the completeness of the chosen list of invariants: For any quadratic lattice isomorphism $\eta : \mathcal{V}_E(F) \to \mathcal{V}_E(F')$ with $\eta F = F'$ and preserving the invariants one has to construct an isometry inducing η. This might be achieved by a *Mapping* or *Congruence-Theorem* which, for a class of spaces and quadratic lattices, states that every index preserving isomorphism between quadratic sublattices $\mathcal{V}$ of $\mathcal{V}(E)$ and $\mathcal{V}'$ of $\mathcal{V}_{E'}(F')$ is induced by an isometry whenever the isometric spaces E, E' belong to the class of spaces, and $\mathcal{V}, \mathcal{V}'$ to the class of lattices.

A second aspect of classification is to provide decompositions of pairs (E,F) into "typic" components. Here, (E,F) is an orthogonal sum $\bigoplus_{i\in I}(E_i,F_i)$ if $E = \bigoplus_{i\in I} E_i$ and $F = \sum F_i$. A *Decomposition Theorem* for a class consist in a finite list of series $\mathcal{S}_i$ of pairs in the class, each parameterized by parameters e.g. of arithmetical, cardinal, or metric type, such that each (E,F) is isometric to some $\bigoplus_{i\in I}(E_i,F_i)$ with summands from $\mathcal{S}_i$.

This may also be the basis for a solution of the Congruence Problem, using invariants for the components and metric invariants coming up with the decomposition process (related to supplements to be chosen).

Since an orthogonal decomposition of (E,F) induces a subdirect decomposition of the quadratic lattice $\mathcal{V}_E(F)$, one has a guideline for finding the orthogonal decompositions and can base the search for the possible summands on the list of subdirectly irreducible homomorphic images of $\mathcal{V}(a)$. The decomposition may be obtained directly or via an isometry, provided by a solution of the Congruence Problem, onto a suitably chosen orthogonal sum of "elementary examples". The latter approach, taken in [Gro87], requires knowledge of the relations between the indices.

1.2 Outline of results

Our classification results are divided in the trace-valued case (sections 3 and 4) and the non trace-valued case (section 5 and 6) - cf I.i.3. An indispensable requirement in both cases is that the (E,F) are such that E is non-degenerate (this will be always assumed unless mentioned, otherwise) and that every infinite dimensional subspace of E contains an isotropic vector (the latter is the case if the field has finite u-invariant cf [Gro79,p.83]). Moreover, in uncountable dimensions we require that the spaces E be diagonal (orthogonal sums of finite dimensional ones - cf I.iii and III). Section 2 provides the necessary lattice theoretic background. In Section 3 we prove a general Mapping Theorems for distributive and, more generally, "acyclic lattices" and alternate spaces. Then, in the trace-valued case and dimensions at most $\aleph_3$ we derive a Congruence Theorem involving 54 cardinal and 2 metric invariants (section 4.5) modifying, for the particular lattices, the general Mapping Theorem - of course, in the alternate case no further work is needed. Applying this, over fields of finite u-invariant, a decomposition into 56 (arithmetically parametrized) series of "elementary examples" is proven. Actually, the two metric invariants are needed only when the corresponding cardinal invariants are finite - and then we have finite-dimensional orthogonal summands occurring in that part of the decomposition.

In the non-trace valued case, dealt with in section 5, we have to restrict to at most countable dimensions. A central rôle is played by the kernel E^* of the *value map*

$$E \to S/T : x \mapsto ||x|| := \phi(x,x) + T\,.$$

Accordingly, for an orthogonal decomposition of a pair we require, in addition, that

$$E^* = \sum_{i \in I} E^* \cap E_i\,.$$

We suppose that the dimension E/E^* is finite and that each infinite-dimensional subspace of E is isotropic. Then a Decomposition Theorem is obtained for each of the following cases:

$$I. \quad E^* = E^{*\perp\perp} \text{ and } F = F^{\perp\perp}$$

$$II. \quad E^* = E^{*\perp\perp} \text{ and } E^{*\perp *} = (0) \text{ or } E^{*\perp} \subset E^*$$

$$III. \quad k \text{ is a perfect field and } E \text{ admits an orthogonal basis.}$$

I includes the case of finite-dimensional E [GHM87] and II the case where the field is perfect and $E^* = E^{*\perp\perp}$. Based on this, a solution of the Congruence Problem is proposed for each of the cases, involving all types of invariants, of course. The decomposition is non-trivial even in the case of finite-dimensional E: there are 13 series, one of them including spaces of every even dimension.

Finally, in section 6 quadratic forms Q are studied for (commutative!) fields of characteristic 2 satisfying $[k : k^2] < \aleph_0$. A Mapping Theorem is presented for the case of sufficiently well behaved lattices leading to a Congruence Theorem for pairs (E, F) in dimension $\aleph_1$ where the subspace F is either totally isotropic or non-degenerate.

1.3 Bibliographic remarks

The indispensable prerequisite is [Gro79] which should be consulted for the history of the subject until 1979. In particular, all information on quadratic and trace-valued hermitean forms in countable dimension is to be found there. Section 2 contains a fairly complete account of abstract lattice theory relevant for the study of forms in infinite dimensions. It is based mainly on [GLS85], [Wil87], [HW91], [Wal88], and [Her94]. The axiomatic treatment of indexed lattices (2.7) elaborates on [Mor94]. The Split-Lemma (2.4) for lattices with operators (having applications e.g. to Boolean algebras with operators, too) appears here for the first time. Section 3 comprises [Gro82] (Thm.1), [Sch83b] (Thm.2), and [Wil87],[Wil91] (Thm.3). The computation of the free lattice $\mathcal{V}_3(a)$ (4.1) had been first done in [GLS85] after the lattice structure and the rôle of side-lattices had been clarified based on computer calculations done with the program described in [Lom82]. The relations between the indices (4.4), the elementary examples (4.5) and corresponding subdirect factors of $\mathcal{V}_3$ then have been obtained in [Sch83a]. Here, we choose the approach to prove the completeness of the list of irreducibles, first, which allowed to include a complete proof. The Congruence Theorem in (4.5) is taken from [Wil87] while the corresponding Decomposition Theorem is new. The relevant lattice and Congruence Theorem for dimension $\leq \aleph_2$ had already appeared in [Gro82].

The results of section 5 concerning case I and II substantially extend those of [Mor86], [Gro87], and [GW89]. They are extracted from work of R.Moresi still in progress. The results on case III come from [Wid89] and [GW89]. Section 6 is based on [Wal88].

1.4 Geometric preliminaries

We recall some facts about spaces which are of particular importance for this chapter. By an *ε-hermitean space* over the data $(k, \varepsilon, -)$ we mean a k-vector space E equipped with a form Φ sesquilinear with respect to $-$ and satisfying $\Phi(y, x) = \varepsilon\overline{\Phi(x, y)}$. In particular, Φ is orthosymmetric. Sometimes, *sesquilinear space* is used as synonymon. We write $E \cong E'$ to indicate isometry.

The subspaces $E_i, i \in I$, provide an *orthogonal decomposition* of E if $E = \bigoplus_{i\in I} E_i$ and $\Phi(x,y) = 0$ for all $x \in E_i, y \in E_j$ with $i \neq j$. Also, there is an external orthogonal sum of the E_i which is (canonically) isometric to E if and only if E has an internal decomposition into the E_i. In both cases we write $E = \bigoplus_{i\in I} E_i$. If F is a finite dimensional non-degenerate subspace of E then $E = F \oplus F^\perp$ according to [Gro79,I.5].

Recall that an ε-hermitean space $E = \bigoplus_{i\in I} E_i$ is non-degenerate if and only if all E_i are non-degenerate. And if so, then

$$X^\perp = (X^\perp \cap E_j) + \sum_{i\in I, i\neq j} E_i \text{ and } E_j^{\perp\perp} = E_j \quad \text{for all } X \subset E_j, j \in I\,.$$

Namely, $E_i^\perp \cap E_i \subset E^\perp$. Thus, if E is non-degenerate so are the E_i. Conversely, let the E_i be non-degenerate. If $x \in E_j$ and $y = \sum y_i$ with $y_i \in E_i$ then $\Phi(x,y) = \sum \Phi(x,y_i) = \Phi(x,y_j)$. It follows $X^\perp = X^\perp \cap E_j + \sum_{i\neq j} E_i$ and $E^\perp = 0$. Since the decomposition $E = E_j \oplus \sum_{i\neq j} E_i$ is orthogonal, too, one derives $E_j^{\perp\perp} = E_j$.

The space E is *diagonal* if it is an orthogonal sum of non-degenerate finite dimensional spaces, i.e. κ-diagonal for every $\kappa \geq \aleph_0$ in the terminology of III. In particular, it is non-degenerate. For such, the linear topologies σ_ι, $\iota < \omega$, with 0-neighbourhood basis consisting of the $X^\perp$ where $X \subset E$ and $\dim X < \aleph_\iota$ have been introduced in III (there denoted by σ_κ where $\kappa = \aleph_\iota$). Each gives rise to a closure operator σ_ι on E mapping subspaces to subspaces. Moreover, $\sigma_\iota X \subset \sigma_\gamma X$ for $\gamma < \iota$. If $E = \bigoplus_{i\in I} E_i$ then, for each ι, the topology induced on the subspace E_j is the topology of the space E_j: given basis vectors x_λ of X let $x_\lambda = y_\lambda + z_\lambda$ with $y_\lambda \in E_j$ and $z_\lambda \in E_j^\perp$ to obtain $X^\perp \cap E_j = Y^\perp \cap E_j$ where $Y \subset E_j$ is generated by the y_λ's. Consequently

$$\sigma_\iota X = \sigma_\iota^{E_j} X \quad \text{for } X \subset E_j\,.$$

Lemma 1. *For subspaces $X \subset Y$ of a diagonal ε-hermitean space E and $\iota \leq \gamma$*

(*a*) $$\sigma_0 X = X^{\perp\perp}\,,$$

(*b*) $$\sigma_\gamma \sum_{i\in I} X_i = \sum_{i\in I} \sigma_\gamma X_i \quad \textit{if } |I| < \aleph_\gamma \textit{ and } 0 < \gamma\,,$$

(*c*) $$\sigma_\gamma Y = Y \quad \textit{if } \sigma_\gamma X = X \textit{ and } \dim Y/X < \aleph_\gamma\,,$$

(*d*) $$\dim Y^{\perp\perp} = \dim \sigma_\iota Y = \dim Y = \dim E/Y^\perp\,,$$

(e) $\dim Y^{\perp\perp}/X^{\perp\perp} = \dim X^{\perp}/Y^{\perp} \leq \dim \sigma_\iota Y/\sigma_\iota X \leq \dim \sigma_\gamma Y/\sigma_\gamma X \leq \dim Y/X$.

PROOF. (a) and (b) were shown in [Gro79,I.8(32)] and III.2 Lemma 3. Now $Y = X + \sum_{i\in I} Y_i$ with finite dimensional Y_i and $|I| < \aleph_\gamma$. Hence $\sigma_\gamma Y_i = Y_i^{\perp\perp} = Y_i$ and (c) follows with (b) if $\gamma > 0$. For $\gamma = 0$ (c) is in [Gro79,I.6]. Next, we show $\dim E/F^\perp \leq \dim F$ for all subspaces F of E. For $\dim F < \infty$ this is [Gro79,I.6] Lemma 6. Otherwise, following [Haf70], since E is diagonal we have $E = \bigoplus_{i\in I} E_i$ with finite dimensional E_i's and there is $J \subset I$ and $F_0 = \bigoplus_{j\in J} E_j \supset F$ with $\dim F_0 = \dim F$. It follows $\dim E/F^\perp \leq \dim E/F_0^\perp = \dim F_0^\perp = \dim F$. Now, if $G \subset F$ then $F = G + H$ with $\dim H = \dim F/G$ whence $\dim G^\perp/F^\perp = \dim G^\perp/(G^\perp \cap H^\perp) = \dim(G^\perp + H^\perp)/H^\perp \leq \dim E/H^\perp \leq \dim H = \dim F/G$. Substituting X, Y and $Y^\perp, X^\perp$ as well as $X^{\perp\perp}, Y^{\perp\perp}$ we get $\dim X^{\perp\perp}/Y^{\perp\perp} = \dim Y^\perp/X^\perp \leq \dim X/Y$. With $X = 0$ we derive $\dim Y \leq \dim \sigma_\iota Y \leq \dim Y^{\perp\perp} \leq \dim E/Y^\perp \leq \dim Y$. For F, G as above and $\iota > 0$ we obtain $\dim \sigma_\iota F/\sigma_\iota G = \dim(\sigma_\iota G + \sigma_\iota H)/\sigma_\iota G \leq \dim \sigma_\iota H = \dim H = \dim F/G$. Hence we have $\dim \sigma_\beta F/\sigma_\beta G \leq \dim F/G$ for all β. Substituting $\sigma_\iota X, \sigma_\iota Y, 0$ and $\sigma_\gamma X, \sigma_\gamma Y, \iota$ we get (e).

For a space E with a family $(F_k | k \in K)$ of subspaces, an *orthogonal decomposition* is provided by E_i such that $E = \bigoplus_{i\in I} E_i$ and $F_k = \sum_{i\in I} F_k \cap E_i$ for all k. In the non trace-valued case we usually include the condition $E^* = \sum_{i\in I} E^* \cap E_i$, i.e. E^* has to be decomposed, too.

LEMMA 2. *In a diagonal ε-hermitean space, if $E = \bigoplus_{i\in I} E_i$ is an orthogonal decomposition of the pair (E, F) then*

$$F^\perp = \sum_{i\in I} F^\perp \cap E_i = \sum_{i\in I} (F \cap E_i)^\perp \cap E_i ,$$

$$\sigma_\gamma F = \sum_{i\in I} \sigma_\gamma (F \cap E_i) = \sum_{i\in I} \sigma_\gamma^{E_i} (F \cap E_i) .$$

PROOF. $F^\perp = (\sum_i F \cap E_i)^\perp = \bigcap_i (F \cap E_i)^\perp = \bigcap_i ((F^\perp \cap E_i) + \sum_{j\neq i} E_j) = \sum_i F^\perp \cap E_i = \sum_i (F \cap E_i)^\perp \cap E_i$. So the E_i provide an orthogonal decomposition of $F^\perp$, too, and we get the claimed decomposition of $F^{\perp\perp} = \sigma_0 F$. For $\gamma > 0$ we use finite additivity and $\sigma_\gamma E_j = E_j^{\perp\perp} = E_j$. Consider $y = \sum_i y_i \in \sigma_\gamma F$ with $y_i \in E_i$ and $y_i \neq 0$ only for i in the finite subset J of I. By additivity, $y = z + \sum_{i\in J} z_i$ with $z_i \in \sigma_\gamma(F \cap E_i)$ and $z \in \sigma_\gamma(F \cap \sum_{i\notin J} E_i)$ whence $z = 0$. This proves $\sigma_\gamma F \subset \sum \sigma_\gamma(F \cap E_i) \subset \sigma_\gamma F$.

For a commutative field k, a *quadratic form* (a much more general concept is studied in Appendix I) on the k-vector space E is a map $Q : E \to k$ such that

$$Q(\lambda x) = \lambda^2 Q(x) \text{ for all } x \in E,\ \lambda \in k$$

$$\Phi(x,y) = Q(x+y) - Q(x) - Q(y) \text{ is a bilinear form }.$$

For $\mathrm{char}(k) \neq 2$ one can recover Q from Φ, so we assume $\mathrm{char}(k) = 2$ which renders Φ an alternate form. The orthogonality relation and derived concepts then refer to Φ. However, the *kernel operator*

$$\kappa X = \{x \in X \cap X^{\perp} \mid Q(x) = 0\}$$

has to be taken into account, e.g. in the concept of orthogonal decomposition.

2 Quadratic lattices

2.1 General concepts

Considering quadratic lattices we deal with some kind of algebraic structure or *algebra* $\mathcal{A}$ given by a ground set A and a family of finitary operations on A, denoted by $f = f^{\mathcal{A}}$, $f \in F$, of fixed arities. We have to recall some of the most basic concepts (cf [Mal73], [BS81])

A *homomorphism* (*isomorphism*) $\varphi : \mathcal{A} \to \mathcal{B}$ is given by a (bijective) map $\varphi : A \to B$ which is compatible with the operations:

$$\varphi f^{\mathcal{A}}(a_1, \ldots, a_{n_f}) = f^{\mathcal{B}}(\varphi a_1, \ldots, \varphi a_{n_f}).$$

$\mathcal{B}$ is a *subalgebra* of $\mathcal{A}$ if $B \subset A$ and the identity map id_B is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$. For any subset G of A there is a smallest subalgebra of $\mathcal{A}$ containing G, the subalgebra *generated* by G.

The *direct product* $\mathcal{A} = \Pi\mathcal{A}_i$ is given by the direct product of the ground sets with operations carried out component wise. The projection maps $\pi_i : \mathcal{A} \to \mathcal{A}_i$ are homomorphisms. An algebra $\mathcal{B}$ embedded via ϕ into $\mathcal{A}$ is a *subdirect product* of the $\mathcal{A}_i$ if $\pi_i \phi B = A_i$ for all i. In other terms, a subdirect decomposition of $\mathcal{B}$ into the $\mathcal{A}_i$ is provided by a family of homomorphisms ψ_i with domain $\mathcal{B}$ and image $\mathcal{A}_i$ which is separating: $\psi_i(x) = \psi_i(y)$ for all $i \in I$ only if $x = y$. Call $\mathcal{B}$ *subdirectly irreducible* if in any subdirect product representation at least one of the maps $\pi_i \phi$ is an isomorphism.

An equivalence relation θ on A is a *congruence relation* of $\mathcal{A}$ if it is compatible with the operations, i.e.

$$a_1 \theta b_1, \ldots, a_{n_f} \theta b_{n_f} \Rightarrow f^{\mathcal{A}}(a_1, \ldots, a_{n_f}) \theta f^{\mathcal{A}}(b_1, \ldots, b_{n_f})$$

Then, the factor set A/θ can be turned in a homomorphic image (the *factor* or *quotient* algebra $\mathcal{A}/\theta$) of $\mathcal{A}$, naturally, under the homomorphism ϕ_θ mapping each element to its class. Conversely, with each homomorphism φ there is an associated congruence $ker\varphi$ identifying those elements which have the same image. Given homomorphisms $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{C}$, where φ is onto, there is a map $\chi : B \to C$ with $\psi = \chi \circ \varphi$ if and only if $ker\varphi \subset ker\psi$; if so, this map is unique and a homomorphism of $\mathcal{B}$ into $\mathcal{C}$.

Ordered by inclusion, the set of all congruences of $\mathcal{A}$ forms a lattice $Con(\mathcal{A})$, the *congruence lattice* of $\mathcal{A}$. If $\bigcap_i \theta_i$ is the identity congruence then $\mathcal{A}$ is a subdirect product of the $\mathcal{A}_i$, canonically, and conversely. So, $\mathcal{A}$ is subdirectly irreducible if and only if it has a smallest nontrivial congruence μ which is generated by each pair $(a, b) \in \mu$, $a \neq b$ - called a *critical pair* of the subdirectly irreducible algebra. And $\mathcal{A}/\theta$ is subdirectly irreducible if and only if θ is completely meet irreducible, i.e. there is a smallest congruence μ properly containing θ. So these θ are the maximal congruences not containing some pair $a \neq b$ and one has the theorem of Birkhoff that every algebra is a subdirect product of subdirectly irreducible homomorphic images. $\mathcal{A}$ is *simple* if it admits only the trivial congruences. We mention that the modularity (cf. 2.2) of $Con(\mathcal{A})$ can be decided in polynomial time [HW].

A *variety* is a class of algebras defined by identities. A class is a variety if and only if it is closed under subalgebras, direct products, and homomorphic images. The decomposition into a subdirect product of subdirectly irreducibles can be carried out inside the variety.

A *presentation* (E, R) consists of a set E of generator symbols and a set R of relations $s = t$ where s, t are terms over the alphabet E. In particular, a presentation can consist of a partial algebraic structure on E. Given a class $\mathbf{C}$ we say that an algebra $\mathcal{A}$ and a map $\alpha : E \to \mathcal{A}$ constitute a *model* $\mathcal{A}, \alpha$ (of (E, R)) if $\mathcal{A} \in \mathbf{C}$ is generated by $\alpha(E)$ and for all $s = t$ in R the terms s and t evaluated at the $\alpha(e)$, $e \in E$, yield the same result. Two models $\mathcal{A}, \alpha$ and $\mathcal{A}', \alpha'$ are *isomorphic* if there is an isomorphism ω from $\mathcal{A}$ to $\mathcal{A}'$ such that $\alpha' = \omega \circ \alpha$. A model $\gamma : E \to \mathcal{F}$ is *free* (the $\mathbf{C}$-algebra with presentation (E, R) or the *free* $\mathbf{C}$-algebra with generators E and relations R) if it is the "most general such", i.e. if for every model $\alpha : E \to \mathcal{A}$ there is a homomorphism $\bar{\alpha} : \mathcal{F} \to \mathcal{A}$ with $\alpha = \bar{\alpha} \circ \gamma$. For every variety and every presentation, such a free model exists and is unique up to isomorphism. It can be constructed from the term algebra by carrying out the identifications forced by the axioms and relations. Alternatively, it can be obtained as the subalgebra of $\prod_{\alpha \in I} \mathcal{A}_\alpha$ generated by the $(\alpha(e) \mid \alpha \in I)$, $e \in E$, where the $\alpha : E \to \mathcal{A}_\alpha$ range over all isomorphism types of models of (E, R) in $\mathbf{C}'$ where $\mathbf{C} = ISP\mathbf{C}'$ (if the presentation is 'projective', i.e. can be retracted into homomorphic preimages, then it suffices to consider models from a generating subclass of the variety $\mathbf{C}$). Thus, the models are up to isomorphism just the $\mathcal{F}/\theta, \phi_\theta$ where θ is a congruence on $\mathcal{F}$. The *word problem* for $\mathbf{C}$ asks for an algorithm deciding for every presentation and every pair s, t of terms whether $s = t$ in the free model.

2.2 Modular lattices

We recall the basic facts needed later on. As a general reference use [Bir79] and [CD73]. A (meet)*semilattice* $\mathcal{L}$ is a set with operation meet $\cap$ such that $x \leq y \Leftrightarrow x \cap y = x$ defines a partial order on L turning $x \cap y$ into the greatest lower bound of x and y. We also write $a \cap b = ab$. Algebraically, we have the laws of associativity, commutativity, and idempotency. Observe that an inequality $s \leq t$ may be replaced by the identity $s = s \cap t$. For $s \leq t$ the subset $[s, t] := \{x \in \mathcal{L} | \ s \leq x \leq t\}$ of $\mathcal{L}$ is called an *interval*. $\mathcal{L}$ is a lattice, if there is an additional operation $+$ such that $a+b$ is the smallest upper bound of a and b - i.e. $+$ provides a join semilattice and we have the absorption laws $x \cap (x+y) = x$, $x + x \cap y = x$. We use the convention that $\cap$ binds stronger than $+$. A trivial but useful observation is the following: If $\mathcal{S}$ and $\mathcal{L}$ are lattices and α and π a join and meet preserving map from $\mathcal{S}$ into $\mathcal{L}$, respectively, such that $\alpha x \leq \pi x$ for all $x \in \mathcal{S}$, then one has a sublattice of $\mathcal{L}$ which is the union of the intervals

$$\bigcup_{x \in \mathcal{S}} [\alpha x, \pi x].$$

$\mathcal{L}$ is *distributive* resp. *modular* if

$$x \cap (y + z) = (x \cap y) + z, \quad \text{resp. } x \geq z \ \Rightarrow \ x \cap (y + z) = (x \cap y) + z.$$

In the sequel, all lattices will be modular unless stated otherwise. These form a variety since the modular law can be stated as $x \cap (y + (x \cap z)) \leq (x \cap y) + z$. In a 0-1-*lattice* the constants 0, 1 denote the smallest and greatest element of $\mathcal{L}$. For elements a, b of a lattice $\mathcal{L}$ one says that b *covers* a, and writes $a \prec b$, if $a < b$ and there is no x with $a < x < b$. If $\mathcal{L}$ is finite, then for any two maximal chains $0 \prec a_1 \prec a_2 \prec \ldots \prec a_{n-1} \prec x$ and $0 \prec b_1 \prec b_2 \prec \ldots \prec b_{m-1} \prec x$ one has $n = m$. The number $\delta(x) := n$ is called the *height* of x and one puts $\delta(\mathcal{L}) := \delta(1)$. A pair $a \geq b$ of elements is called a *quotient* and written a/b. It is *prime* if $b \prec a$. Quotients a/b and c/d are *transposed* to each other if $c = a + d, b = a \cap d$ or dually. If so, then the interval sublattices $[b, a]$ and $[d, c]$ are isomorphic via $x \mapsto x + d$. Quotients are *projective* to each other if they are connected by a sequence of transpositions. c/d is a *subquotient* of a/b if $b \leq d \leq c \leq a$. An element p of $\mathcal{L}$ is *join irreducible* if $p = a + b$ implies $p = a$ or $p = b$ and *join prime* if $p \leq a + b$ implies $p \leq a$ or $p \leq b$. The set of all join irreducibles is denoted by $J(\mathcal{L})$ and $J_0(\mathcal{L}) = J(\mathcal{L}) \cup \{0\}$. If $\mathcal{L}$ is of finite height, then every join irreducible p has a unique lower cover $p_* \prec p$, each element is a finite join of join irreducibles and each prime quotient transposes to some p/p_* with $p \in J(\mathcal{L})$. Dually, $M(\mathcal{L})$ is the set of *meet irreducibles*.

(a) Let $a, b, c \in \mathcal{L}$ be such that some distributivity relation holds, e.g. $(a + b) \cap c = (a \cap c) + (b \cap c)$. Then the sublattice $\langle a, b, c \rangle \subset \mathcal{L}$ generated by a, b, c is distributive, whence *all* possible distributivity relations hold, e.g. $(b + c) \cap a = (b \cap a) + (c \cap a)$ or $(a + b) \cap (a + c) = a + (b \cap c)$. This is immediate by inspection of the free modular lattice on 3 generators.

Consider a finite modular lattice $\mathcal{L}$ and a vector space W with $\dim(W) = \delta(\mathcal{L})$. Let $f : \mathcal{L} \to \mathcal{L}(W)$ be a rank preserving order embedding, i.e. $a \leq b$ iff $f(a) \leq f(b)$ and $\dim(f(a)) = \delta(a)$ for all $a, b \in \mathcal{L}$. Then f is a lattice embedding. Namely, $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathcal{L}$ such that $ab \prec a, b$ and, by induction, for all others (see e.g. [Wil92, Lemma 1]).

(b) *Congruences.* Clearly, a lattice congruence is determined by it quotients, and by its prime quotients if $\mathcal{L}$ is finite. A set of quotients is that of a congruence if and only if it is closed under transposition, subquotients, and transitivity. The (set of quotients of the) congruence generated by a set of quotients is obtained by closing under subquotients, then under transpositions, and finally transitivity. In particular, (the quotient set of) the join of two congruences is the transitive closure of the union whence one readily sees that the congruence lattice $Con(\mathcal{L})$ is distributive. Also, the congruence $\theta(a, b)$ genererated by a prime quotient is an atom in $Con(\mathcal{L})$. Now, let $\mathcal{L}$ be finite. Then the set of prime quotients partitions into classes corresponding to these atomic congruences and the congruences correspond to unions of such classes. In particular, $Con(\mathcal{L})$ is isomorphic to the Boolean lattice of all subsets of a finite set and $\mathcal{L}$ is simple if and only if it is subdirectly irreducible. For a congruence θ let $\phi_\theta : \mathcal{L} \to \mathcal{L}/\theta$ be the canonical epimorphism. Then the *lower adjoint* $\alpha_\theta : \mathcal{L}/\theta \to \mathcal{L} : a \mapsto \bigcap \phi^{-1}(a)$ is easily seen to be a one-to-one join homomorphism. Also $J^\theta(\mathcal{L}) := \alpha_\theta(J(\mathcal{L}/\theta))$ is a subset of $J(\mathcal{L})$. If $\mathcal{L}/\theta$ is subdirectly irreducible then θ is maximal and the complementary congruence is the atomic one generated by some/all prime quotients p/q with $p \in J^\theta(\mathcal{L})$. Conversely, to each prime quotient p/q (and so to each $p \in J^\theta(\mathcal{L})$) there corresponds a unique maximal congruence not containing p/q (p/p_*). Moreover, for distinct maximal θ the $J^\theta(\mathcal{L})$ are disjoint. More generally, for any set Θ of congruences we have

$$J^{\cap\Theta}(\mathcal{L}) = \bigcup_{\tau\in\Theta} J^{\tau}(\mathcal{L}), \quad J^{\Sigma\Theta}(\mathcal{L}) = \bigcap_{\tau\in\Theta} J^{\tau}(\mathcal{L}).$$

Note that $\mathcal{L}$ is distributive iff each subdirectly irreducible factor lattice $\mathcal{L}/\theta$ is isomorphic to the two element chain $\mathcal{D}_2$. In this case $p \mapsto \{(a,b) \in \mathcal{L}^2 |\ a,b \geq p \text{ or } a,b \not\geq p\}$ is a bijection from $J(\mathcal{L})$ onto Θ. Equivalently, each $p \in J(\mathcal{L})$ is join prime.

(c) *Algebraic lattices.* An element c of a lattice $\mathcal{L}$ is *compact* if for each existing supremum $\sum S$ the inequality $\sum S \geq c$ implies $\sum S_0 \geq c$ for some finite $S_0 \subset S$. A lattice $\mathcal{L}$ is *complete* if each subset $S \subset \mathcal{L}$ has a supremum $\sum S$ and an infimum $\bigcap S$. In particular $\mathcal{L}$ has a unit $1 = \sum \mathcal{L}$ and a zero $0 = \bigcap \mathcal{L}$. Each finite lattice is complete. We shall be interested in *complete sublattices* $\mathcal{L} \subset \mathcal{L}(E)$, i.e. not only is $\mathcal{L}$ complete but also closed under $\sum$ and $\bigcap$. A lattice $\mathcal{L}$ is *algebraic* if it is complete and each $x \in \mathcal{L}$ is a join of (perhaps infinitely many) compact elements. A complete lattice $\mathcal{L}$ is *completely distributive*, if for every doubly indexed family of elements $x_{i,j}$ $(i \in I,\ j \in J)$ in $\mathcal{L}$ we have $\bigcap_{i\in I} \sum_{j\in J} x_{i,j} = \bigcap_{f\in J^I} x_{i,f(i)}$, where J^I denotes the set of all functions from I to J. Each finite distributive lattice is completely distributive. According to [Her79] an algebraic lattice is completely distributive if and only if the following holds:
(i) join irreducible compact elements are join prime,
(ii) compact elements are (finite) joins of join irreducibles.

(d) *Geometric structure.* The following facts are found in [HW]. Three distinct join irreducibles $\{p,q,r\}$ of a finite lattice $\mathcal{L}$ form a *triangle*, if they satisfy the *collinearity relation* $p+q = p+r = q+r$. It easily follows that $p \not\leq q_* + r$ and that $p/p*$ is projective to q/q_*. Moreover, p,q,r is a triangle if and only if it is of the form $\alpha_\theta p', \alpha_\theta q', \alpha_\theta r'$ where p',q',r' is a triangle in some subdirectly irreducible $\mathcal{L}/\theta$. The lattice $\mathcal{L}$ can be recovered from the order on $J(\mathcal{L})$ and the triangles: Call an order ideal I a subspace if with each two points from a triangle the third belongs to I, too. Then $I \mapsto \sum I$ is an isomorphism of the lattice of subspaces onto $\mathcal{L}$. More generally, if ϕ is a map from $J(\mathcal{L})$ into M preserving the order and with $\phi x \leq \phi y + \phi z$ for all collinear x,y,z then $\phi x = \sum\{\phi y \,|\, x \geq y \in J(\mathcal{L})\}$ is a join homomorphism from $\mathcal{L}$ into M.

A *triangle sequence* of $\mathcal{L}$ is a listing $J(\mathcal{L}) = \{p_0, q_1, p_1, \ldots, q_{n-1}, p_{n-1}\}$ such that for each $1 \leq i \leq n-1$ there is a $r_i \in \{p_0, \ldots, q_{i-1}, p_{i-1}\}$ yielding a triangle $\{p_i, q_i, r_i\}$. If a triangle sequence exists at all, then *each* $p_0 \in J(\mathcal{L})$ occurs as the first element of some triangle sequence. The number $n-1$ of triangles $\{p_i, q_i, r_i\}$ is just one less than the length $n = \delta(\mathcal{L})$. A finite lattice is *acyclic* if each simple factor $\mathcal{L}/\theta$ $(\theta \in \Theta)$ admits a triangle sequence. In particular, this includes all finitely generated members of the variety generated by the height 2 lattice $\mathcal{M}_3$ with 3 atoms. Let us mention that each modular lattice $\mathcal{L}$ of finite length satisfies the inequality $|J(\mathcal{L})| \geq 2\delta(\mathcal{L}) - s(\mathcal{L})$, where $s(\mathcal{L})$ denotes the number of subdirectly irreducible factors, cf. [Wil96]. It is easy to see that equality takes place for acyclic lattices. Probably they are characterized by this equality.

(e) *Artinian lattices.* One says that $\mathcal{L}$ is *Artinian* if there are no infinite descending chains. Each Artinian lattice $\mathcal{L}$ with 1 is automatically complete. For Artinian lattices $\mathcal{L}$ one has an induction principle: Let P be a property holding for 0 and such that it holds for x whenever it holds for all $y < x$. Then it holds for all $x \in \mathcal{L}$. Also for each meet $\bigcap S$ there is a finite $S_0 \subset S$ with $\bigcap S = \bigcap S_0$. In particular, as in (b), for each epimorphism $\phi : \mathcal{L} \to \mathcal{L}/\theta$ the existence of smallest preimages $\alpha_\theta : \mathcal{L}/\theta \to \mathcal{L} : a \mapsto \bigcap \phi^{-1}(a)$ is guaranteed. Assume that $\mathcal{L}$ is Artinian and $\mathcal{L} \to \Pi_{\theta\in\Theta'}\mathcal{L}/\theta$ is a subdirect decomposition with finite simple factors $\mathcal{L}/\theta$. Clearly for each prime quotient $a \prec b$ there is a $\theta \in \Theta'$ with $(a,b) \notin \theta$. Suppose there is another $\psi \in \Theta'$ with $(a,b) \notin \psi$. Then the image of (a,b) in $(\mathcal{L}/\theta) \times (\mathcal{L}/\psi)$ would be a prime quotient in a finite lattice which is separated by distinct maximal congruences. This contradicts (b) and shows that $\theta \in \Theta'$ is uniquely determined. Put

$$\Theta := \{\theta \in Con(\mathcal{L}) |\ (\exists a,b \in \mathcal{L})\ a \prec b \text{ and } (a,b) \notin \theta\}.$$

By the above $\Theta \subset \Theta'$ for each subdirect decomposition $\mathcal{L} \to \Pi_{\theta\in\Theta'}\mathcal{L}/\theta$ with finite simple factors. On the other hand, since $\mathcal{L}$ is Artinian, for all $a < c$ in $\mathcal{L}$ there is a b with $a \prec b \leq c$. This implies that $\mathcal{L} \to \Pi_{\theta\in\Theta}\mathcal{L}/\theta$ is itself a subdirect decomposition. A lattice $\mathcal{L}$ with 1 is *Artinian-acyclic* if it is Artinian and admits a subdirect decomposition $\mathcal{L} \to \Pi_{\theta\in\Theta}\mathcal{L}/\theta$ with finite simple acyclic factors $\mathcal{L}/\theta$. As seen above there is a canonical (smallest) set Θ of such congruences.

(f) *Representation Theory.* Dealing with sublattices $\mathcal{L}$ of $\mathcal{L}(E)$, E a vector space, we need some basics of representation theory, cf [GP74] and [P]. Let $E = \bigoplus_{i\in I} E_i$ a finite direct decomposition. We say that $X \in \mathcal{L}(E)$ *decomposes* if $\sum_i X \cap E_i = X$. Trivially, if X and Y decompose then so does $X+Y = \sum_i (X\cap E_i)+(Y\cap E_i)$. On the other hand, X decomposes if and only if $X = \bigcap_i (X + E_i)$ which, in turn, carries over to intersections. Thus, if every X in a generating set of $\mathcal{L}$ decomposes, then so do all the X and the maps $X \mapsto X \cap E_i$ provide a subdirect decomposition of $\mathcal{L}$ into sublattices of the $\mathcal{L}(E_i)$. A *representation* of a partially ordered set S is an order preserving map ρ into some vector space lattice $\mathcal{L}(E)$ and two such, ρ_i, are *isomorphic* if there is a linear isomorphism f from E_1 into E_2 such that $\rho_2 x = f(\rho_1 x)$ for all $x \in S$. The representation is *indecomposable* if only the trivial direct decomposition of E decomposes all ρx, $x \in S$. By the above, these concepts carry over to the modular lattice $FM(S)$ freely generated by S with a representation defined as a lattice homomorphism from $FM(S)$ into $\mathcal{L}(E)$. The requirement $\rho 0 = 0$ and $\rho 1 = E$ for 0-1-representations can be achieved by directly splitting off 1-dimensional summands σ with $\sigma(\mathcal{L}) = E$ respectively $\sigma(\mathcal{L}) = 0$. The poset S is of finite *representation type* if over any fields there are only finitely many non-isomorphic finite dimensional indecomposables. These posets are characterized as not having any subposet $1+1+1+1$, $2+2+2$, $1+3+3$, $1+2+5$, nor $1+2+N$ where n stands for the n-element chain, $+$ for disjoint union, and N for the poset $a < b > c < d$. The isomorphism types of indecomposable finite dimensional representations ρ of partially ordered sets have been determined for all posets of finite representation type and $1+1+1+1$ - see [Gab72],[Naz67], [KLe72a], [Kle72b], [NR72], [GP70], [Bre74].

(g) *Finitely presented modular lattices.* Call elements $a_1, \ldots a_n$ of a lattice *independent* if $a_{k+1} \cap \sum_{i=1}^{k} a_i = \prod_i a_i$ for $k < n$. Such, generate a Boolean sublattice with a_i an atom or

0. Also, any modular lattice generated by two chains is distributive. The following is due to [Wi73b]

LEMMA 3. *$\mathcal{D}_2$ and $\mathcal{M}_3$ are the only subdirectly irreducible modular lattices which can be generated by a poset not containing* $1+1+1+1$ *nor* $1+2+2$.

Stronger results are obtained when working within the lattice varieties $\mathbf{V}_p$ relevant here, namely those generated by the subspace lattices of vector spaces over fields of characteristic p (including $p=\infty$ for the case of arbitrary characteristic). According to [HH75a] $\mathbf{V}_p$ is generated by the subspace lattices of finite dimensional vector spaces over prime fields of characteristic p. Thus, the free lattice $\mathcal{F}_{V_p}(S)$ in the variety is the subdirect product over all sublattices generated by $\rho(S)$ in $\mathcal{L}(E)$, ρ a indecomposable finite dimensional representation in a vector space E over a (prime) field of characteristic p. In particular, for a poset of finite representation type it follows from the results mentioned in (f) that the subdirectly irreducibles are among the lattices $\mathcal{D}_2$, $\mathcal{M}_3$, and the $\mathcal{L}(k^d)$ where $3 \leq d \leq 6$ and k is a prime field of characteristic p cf [HH74]. The result behind is that for $d \geq 3$ the $\mathcal{L}(k^d)$ are just the subdirectly irreducibles in $\mathbf{V}_p$ generated by a *frame* of order d. Such, is a presentation with generators $a_0, a_1, \ldots, a_n$ any n of which are independent and have the same join (for $d=2$ this states that we have a sublattice $\mathcal{M}_3$ or trivial). For modular lattices in general, assuming $d \geq 4$ or the Arguesian identity a list of subdirectly frame generated irreducibles has been given in [Her84], cf also [Day82]. The generating frames always are canonical, up to isomorphism. The following has been shown in [HKW75].

LEMMA 4. *The subdirectly irreducible modular lattices generated by a poset $a; b \leq c; d \leq e$ are $\mathcal{D}_2$ and $\mathcal{M}_3$, the 3-frame generated ones with $a=a_0, b=a_1, c=a_1+a_2, d=a_3, e=a_2+a_3$ and the duals of such.*

The 4-generated subdirectly irreducibles in $\mathcal{V}_p$ have been determined in [Her82]. Besides the $\mathcal{L}(k^d)$ there is an infinite series of finite acyclic lattices, the height 2 lattice $\mathcal{M}_4$ with four atoms and two homomorphic preimages thereof having infinite dimensions. The structure of the free generated lattice in $\mathbf{V}_p$ over the $1+1+1+1$ and other tame posets of finite growth has been analyzed in [Cyl92].

For calculations in finitely presented lattices in $\mathbf{V}_p$, for most purposes the modular law suffices. But, when frames occur stronger axioms are required to have only one model generated by a frame of order d. This can be accomplished combining the Arguesian identity with the identity for characteristic p given in [HH75]. For $p=2$ this is the Fano-identity

$$(w+x)\cap(y+z) \leq (w+y)\cap(x+z)+(w+z)\cap(x+y).$$

Lattices satisfying this identity are called *Fano-lattices*. Modularity follows and we have, easily, (cf [Pog75])

Lemma 5. *The only nontrivial Fano-lattice generated by a frame of order 3 is (up to isomorphism)* $\mathcal{L}(GF(2)^3) = \mathbb{P}(\mathbb{F}_2^3)$ *generated by the canonical frame. Every k-linear 0-1-representation of this lattice* $\mathcal{L}$ *is isomorphic to a direct sum of representations* $k \otimes_{\mathbb{F}_2} \mathbb{F}_2^3$ *- and there are only trivial representations if* $char(k) \neq 2$.

For the varieties $\mathcal{V}_p$, identities are decidable and the word problem in 4 generators is solvable. However, for 5 or more generators there are presentations with unsolvable word problem. For the variety of all modular lattices even the set of 4-variable valid identities is undecidable.

2.3 Lattices with operators

An *operator* on a lattice is a map f such that $f(x+y) = fx + fy$ for all x, y (*isotone*) respectively g such $g(x+y) = gx \cap gy$ for all x, y (*antitone*) In the sequel we work within a fixed variety of modular operator lattices, i.e. with a specified number of isotone and antitone operators and maybe some constants. (Though, the results of this paragraph remain valid if the f and g are just order preserving or inverting. Everything, but the description of congruences via quotients, even extends to the case of arbitrary additional operations).

For a set of quotients Q to be that of a congruence, in addition to the conditions in the lattice case we need that with x/y also fx/fy and gy/gx belong to Q, where the f and g range over all isotone and antitone operators, respectively. We say that x/y is obtained from a/b under *weak projectivity* if it arises by taking subquotients and transpositions and applying operators, in arbitrary order and allowing repetitions.

Lemma 6. *The set of quotients of the congruence generated by a set* Q *of quotients is the transitive closure of the set of all quotients obtained from members of* Q *under weak projectivity.*

In particular this implies that congruences generated by a prime quotient are completely join primes in the congruence lattice. Since the join of congruences θ and τ is obtained as the union of all relational products $\theta \circ \tau \circ \ldots \circ \tau$ the congruence lattice is a sublattice of the congruence lattice of the underlying lattice. For finite $\mathcal{L}$ it follows

$$J^{\theta+\tau}(\mathcal{L}) = J^{\theta}(\mathcal{L}) \cap J^{\tau}(\mathcal{L}), \quad J^{\theta\cap\tau}(\mathcal{L}) = J^{\theta}(\mathcal{L}) \cup J^{\tau}(\mathcal{L}).$$

However, subdirectly irreducibles need not be simple and the *components* $J^{\theta}(\mathcal{L})$ corresponding to them may overlap. More precisely, if Θ is the set of all meet irreducible congruences then the above yields for any θ_i that

$$J^{\theta_1}(\mathcal{L}) \cap J^{\theta_2}(\mathcal{L}) = \bigcup \{J^{\tau}(\mathcal{L}) \mid \tau \in \Theta, \ \tau \supset \theta_1 + \theta_2\}.$$

Also, the congruence lattice is distributive. Let Θ_m consist of the minimal members of Θ which correspond to the *maximal* subdirectly irreducibles not being proper homomorphic images of others. Then Θ_m provides the unique irredundant subdirect decomposition of $\mathcal{L}$ as a subdirect product of subdirectly irreducibles.

For $\tau \supset \theta$ in Θ let $\phi_{\theta\tau}$ be the canonical homomorphism of $\mathcal{L}/\theta$ onto $\mathcal{L}/\tau$ and $\alpha_{\theta\tau}$ its lower adjoint. Then $J(\mathcal{L})$ can be considered the set obtained by identifying the elements of the $J(\mathcal{L}/\theta)$, $\theta \in \Theta$ along the maps $\alpha_{\theta\tau}$. Of course, the triangles of $\mathcal{L}$ are given by those of the $\mathcal{L}/\theta$ and the order relation on $J(\mathcal{L})$ is known once one has the order on the $J^{\theta\cap\tau}(\mathcal{L})$ for every pair θ, τ in Θ_m - .i.e. if one computes the subdirect product of every two maximal subdirectly irreducibles. Defining for subdirectly irreducible $\mathcal{S}$ the set of *critical join irreducibles*

$$C(\mathcal{S}) = \{p \in J(\mathcal{S}) \,|\, p/p_* \text{ critical } \}$$

we finally get

$$C(\mathcal{L}/\theta) = J(\mathcal{L}/\theta) \setminus \bigcup \{J(\mathcal{L}/\tau) \,|\, \tau \in \Theta,\, \tau \supset \theta\}.$$

2.4 The splitting method

The purpose of this section is to establish a method for proving that a given (guessed) list of subdirectly irreducible models is complete. It succeeds if $\mathcal{F}$ is finite: using the relations provided by the Lemma derive $0 = 1$ or the operation table of a small model. The analogue for the case of modular lattices is well known, cf [Wi73a],[Her84].

Let $\mathbf{C}$ be a variety of lattices with operators and (E, R) a presentation with free model $\mathcal{F}$ (or an arbitrary model admitting a homomorphism onto $\mathcal{S}$ below). For terms s, t over E we write $s = t$ resp $s \leq t$ if this (in)equality holds in $\mathcal{F}$, i.e. if it can be derived from the axioms of $\mathbf{C}$ and the relations in R. A *binding pair* for a model $\mathcal{S}, \phi$ consists in two maps α and π associating terms αx and πx with each $x \in \mathcal{S}$ such that the following are satisfied for all $x, y \in \mathcal{S}$, isotone operators f, and antitone operators g (consider ϕ as the canonical homomorphism of $\mathcal{F}$ onto $\mathcal{S}$ and α, π as maps into $\mathcal{F}$):

$$(1)\ \alpha x \leq \pi x \quad \textit{and} \quad (2)\ \phi\alpha x = x = \phi\pi x$$

$$(3)\ \alpha\phi e \leq e \leq \pi\phi e \ \textit{ for all } e \in E \textit{ or } e \textit{ a constant}$$

$$(4)\ \alpha(x + y) = \alpha x + \alpha y \ \textit{ and } \ \pi(xy) = \pi x \cap \pi y$$

$$\alpha 0 = 0,\ \pi 1 = 1 \textit{ if } 0, 1 \textit{ are constants}$$

$$(5)\ \alpha f x \leq f\alpha x,\ f\pi x \leq \pi f x,\ \alpha g x \leq g\pi x,\ g\alpha x \leq \pi g x$$

LEMMA 7. *α, π constitute a binding pair for $\mathcal{S}, \phi$ if and only if for all $x \in \mathcal{S}$*

$$\alpha x = \bigcap \{t \in \mathcal{F} \,|\, \phi t = x\} \textit{ and } \pi x = \sum \{t \in \mathcal{F} \,|\, \phi t = x\}.$$

PROOF. By (1) and (2) the interval $I_x = [\alpha x, \pi x]$ in $\mathcal{F}$ is mapped onto x under ϕ. On the other hand, the union of these intervals forms a subalgebra of $\mathcal{F}$ in view of (4) and (5). By (3) this subalgebra contains all generators, so it equals $\mathcal{F}$. But this implies $I_x = \phi^{-1}x$, immediately, these are just the classes of the kernel θ of ϕ. This proves the Lemma.

We now provide modifications of the hypotheses easier to verify. Assume that $0, 1$ are constants.

In order to have (1)-(4) it suffices to define α on $J(\mathcal{S})$ and π on $M(\mathcal{S})$ such that

$$(1)\ \alpha x \leq \pi y \ \textit{ for all } x \in J(\mathcal{S}), x \leq y \in M(\mathcal{S})$$

$$(2)\ \phi\alpha x = x \text{ and } \phi\pi y = y \textit{ for all } x \in J(\mathcal{S}), y \in M(\mathcal{S})$$

$$(3)\ \alpha x \leq e \leq \pi y \textit{ for all } x \in J(\mathcal{S}), x \leq \phi e \leq y \in M(\mathcal{S}), e \in E \textit{ or a constant}$$

$$(4)\ \alpha 0 = 0, \pi 1 = 1,\ \alpha \textit{ is order preserving on } J(\mathcal{S}),\ \pi \textit{ on } M(\mathcal{S})$$

$$\alpha x \leq \alpha y + \alpha z \textit{ for collinear } x, y, z \in J(\mathcal{S})$$

$$\pi x \geq \pi y \cap \pi z \textit{ for dually collinear } x, y, z \in M(\mathcal{S}).$$

Then, put $\alpha 0 = 0$, $\pi 1 = 1$,

$$\alpha x = \Sigma\{\alpha y | x \geq y \in J(\mathcal{S})\} \text{ and } \pi x = \Pi\{\pi y | x \leq y \in M(\mathcal{S})\}.$$

Now, assume that the σ_i are join preserving closure operators with $\sigma_i x \leq \sigma_j x$ for $j > i$. Then (5) can be replaced by the following

$$(5\alpha)\ \alpha y \leq \sigma_i \alpha x \text{ if } \sigma_i x = y \neq x \in J_0(\mathcal{S}) \text{ and } i \text{ maximal with } \sigma_i x = y$$

$$(5\pi)\ \sigma_i \pi x \leq \pi x \textit{ for all } x \in \sigma_i \mathcal{S}, i \text{ minimal , and } x \in M(\sigma_i \mathcal{S})$$

Let $^\perp$ be an antitonic operator. Then (5) can be replaced by

$$(5\alpha)\ \alpha(x^\perp) \leq (\pi x)^\perp \textit{ for } x \in \mathcal{S} \textit{ with } x^\perp > 0 \quad (5\pi)\ (\alpha x)^\perp \leq \pi(x^\perp) \textit{ for all } x \in J(\mathcal{S})$$

COROLLARY 8. *Let $\mathcal{F}$ be a finite modular lattice with $0, 1$, ϕ a homomorphism of $\mathcal{F}$ onto $\mathcal{S}$, and α an order preserving map of $J(\mathcal{S})$ into $\mathcal{F}$ such that $\phi\alpha x = x$ for all $x \in J(\mathcal{S})$, $\alpha\phi e \leq e$ for all e in a generating set E of $\mathcal{F}$, and $\alpha x \leq \alpha y + \alpha z$ for all collinear x, y, z in $J(\mathcal{S})$. Then for any $y \in \mathcal{S}$ the smallest ϕ-preimage is*

$$\sum\{\alpha x \mid x \in J(\mathcal{S}), x \leq y\}.$$

Proof. The claim about (1)-(4) is obvious in view of 2.2(d). Clearly, $\alpha\sigma_i x \leq \sigma_i \alpha x$ for all $x \in J_0(\mathcal{S})$ (using that $a \leq \sigma_i a$) and so for all x since α and σ_i preserve joins. If $\sigma_i x = y$ then $\sigma_i y = y$ and $\sigma_i \pi x \leq \sigma_i \pi y \leq \sigma_i y$. Thus, in the second half of (5) it suffices to consider σ_i-closed $x \in \mathcal{S}$. But, $x = \Pi x_j$ with $x_j \in M(\sigma_i \mathcal{S})$ whence $\sigma_i \pi x \leq \Pi \sigma_i \pi x_j \leq \Pi \pi \sigma_i x_j = \Pi \pi x_j = \pi x$. The restriction to maximal resp. minimal i is justified by the pointwise ordering of the σ_i. For an antitonic operator, if (5π) holds for x, y then it does so for $x+y$. Namely, $\pi((x+y)^\perp) = \pi(x^\perp y^\perp) = \pi(x^\perp)\pi(y^\perp) \geq (\alpha x)^\perp (\alpha y)^\perp \geq (\alpha(x+y))^\perp$. In order to obtain Corollary 8 let α on $\mathcal{S}$ be defined via sums and let π be the upper adjoint.

The preimage maps α and π can be computed, systematically, adapting a method in [McK72]. Namely, define sequences α_k and π_k of maps from $\mathcal{S}$ into $\mathcal{F}$, inductively

$$\alpha_0 x = \bigcap\{e \in \bar{E} \mid x \leq \phi e\}, \quad \pi_0 x = \sum\{e \in \bar{E} \mid x \geq \phi e\}$$

$$\alpha_{k+1} x = \alpha_k x \cap \bigcap_{x \leq y+z} (\alpha_k y + \alpha_k z) \cap \bigcap_{x \leq fy} f\alpha_k y \cap \bigcap_{x \leq gy} g\pi_k y$$

$$\pi_{k+1} x = \pi_k x + \sum_{x \geq y \cap z} (\pi_k y \cap \pi_k z) + \sum_{x \geq fy} f\pi_k y + \sum_{x \geq gy} g\alpha_k y$$

where $\bar{E}$ consists of the generators and constants. The bounds exist and are given by $\alpha = \alpha_k$ and $\pi = \pi_k$ if and only if $\alpha_k = \alpha_{k+1}$ and $\pi_k = \pi_{k+1}$ for some k. Namely, observe that $\alpha_0 \phi e \leq e \leq \pi_0 e$, that the α_k and π_k are order preserving, $\alpha_{k+1} x \leq \alpha_k x \leq \pi_k x \leq \pi_{k+1} x$, and $\phi \alpha_k x = x = \phi \pi_k x$ for all k and x. And, the sequences become stationary at k if and only if the union of the intervals $[\alpha_k x, \pi_k x]$ is a subalgebra. Of course, if at least one sequence is not stationary then the free object is infinite.

When working with mere lattices, for $\mathcal{S} = \mathcal{D}_2$ we have $\alpha 1 = \bigcap\{e \in E \mid \phi e = 1\}$ and for $\mathcal{S} = \mathcal{M}_3$ with atoms a_1, a_2, a_3 we have $\alpha a_i = b_i(b_j + b_k)$ where $b_i = \bigcap\{e \in E \mid \phi e \geq a_i\}$. More generally, by modularity, $\alpha x_i \leq \alpha x_j + \alpha x_k$ if $\alpha x_i = t_i(t_j + t_k)$.

A *split covering* for a subdirectly irreducible model $\mathcal{S}, \phi$ is a binding pair α, π such that the following are satisfied for all $x, y, z \in \mathcal{S}$, isotone operators f, antitone operators g, prime quotients p/q in the minimal congruence μ on $\mathcal{S}$:

$$(6)\ f\alpha p \leq \alpha fp + f(\alpha p \cap \pi q) \ \text{ and } \ g\alpha p \geq \pi gp \cap g(\alpha p \cap \pi q)$$

$$(7)\ \alpha fp \cap \pi z \leq \alpha z + f(\alpha p \cap \pi q) \text{ if } fq \leq z < fp$$

$$\alpha z + \pi gp \geq \pi z \cap g(\alpha p \cap \pi q) \text{ if } gp < z \leq gq.$$

Lemma 9 (Split). *Given a presentation E, R and finite subdirectly irreducible model $\mathcal{S}, \phi$ with split covering α, π, any model having no subdirect factor (isomorphic to) $\mathcal{S}, \phi$ satisfies all relations $\alpha p \leq \pi q$ where p/q is a prime quotient in the minimal congruence of $\mathcal{S}$.*

Corollary 10 ($\mathcal{D}_2$-Lemma). *In the absence of operators, for $\mathcal{S} = \mathcal{D}_2$ one has $\Pi\{e \in E|\phi e = 1\} \leq \Sigma\{e \in E|\phi e = 0\}$ for any subdirectly irreducible model not isomorphic $\mathcal{D}_2, \phi$.*

Proof. A quotient a/b of $\mathcal{F}$ is a prime quotient not in θ if and only if there is a prime quotient x/y in $\mathcal{S}$ such that $a = b + \alpha x$ and $b = a\pi y$ - and then $x = \phi a, y = \phi b$. Namely, consider such a/b and $b \leq c \leq a$. Then $x = \phi\pi x \geq \phi c =: z$ and, similarly, $z \geq y$. Therefore, $z = x$ and $c \geq b + \alpha x = a$ or $z = y$ and, similarly, $c = b$. Moreover, $x = \phi a, y = \phi b$ and $a/b \notin \theta$ since $a \in I_x$ and $b \in I_y$. Conversely, given a prime quotient a/b not in θ one has $x := \phi a > \phi b =: y$ and if there were $x > z > y$ one had $b < b + \alpha z < a$.

Let Q be the set of quotients a/b of $\mathcal{F}$ such that $a = b$ or there is a chain $a = a_1 \succ a_2 \succ \ldots \succ a_n = b$ with $\phi a_i/\phi a_{i+1}$ a prime quotient in μ for $1 \leq i < n$. Clearly, Q is closed under transitive composition and, by modularity, under transposition. The Jordan-Hölder Refinement Theorem yileds closersure under subquotients. So it remains to show, that with a/b also fa/fb and gb/ga belong to Q. For that, one has to consider the case of prime $\phi a/\phi b$, only.

Let $p = \phi a, q = \phi b$, in particular $\alpha p \cap \pi q \leq b$. By (6) $fb + \alpha fp \geq fb + f\alpha p = f(b + \alpha p) = fa$. If $fp = fq$ then $fa = fb$ immediately (namely, $\phi fb = f\phi b = fq$: but $\alpha fp = \alpha fq$ being the smallest ϕ-preimage of fq one gets $\alpha fp \leq fb$ whence $fb = fb + \alpha fp \geq fa$ and $fb = fa$). Otherwise, if $fq \leq z \leq fp$ then $\alpha fp \cap \pi z \leq \alpha z + f(\alpha p \cap \pi q)$ by (7). Using modularity one derives $fa \cap \pi z = (\alpha fp + fb)\pi z = \alpha fp \cap \pi z + fb \leq \alpha z + f(\alpha p \cap \pi q) + fb = \alpha z + fb$ whence $fa \cap \pi z = fb + \alpha z$.

Let $fp = u_1 \succ u_2 \succ \ldots \succ u_n = fq$. Define $c_k = fb + \alpha u_k = fa \cap \pi z$. Then for $1 \leq k < n$ one has $c_k = c_{k+1} + \alpha u_k$ and $c_{k+1} = c_k \cap \pi u_{k+1}$ which means that c_k/c_{k+1} is a prime quotient in $\mathcal{F}$ with $\phi c_k/\phi c_{k+1} = u_k/u_{k+1}$ a prime quotient in μ.

Similarly, one gets $gb \cap \pi gp = gb \cap g(\alpha p \cap \pi q)\pi gp = gb \cap g\alpha p = g(b + \alpha p) = ga$ and $\alpha z + ga = \alpha z + gb \cap \pi gp = gb \cap (\alpha z + \pi gp) = gb \cap \pi z \cap g(\alpha p \cap \pi q) = gb \cap \pi z$ and $gb/ga \in Q$. Thus, defining $a\psi b$ if and only if $(a + b)/ab \in Q$, one obtains a congruence on $\mathcal{F}$ with $\theta \cap \psi = id$ where $\theta = \ker\phi$. W.l.o.g. the model $\mathcal{L}$ is subdirectly irreducible, i.e. $\mathcal{L} = \mathcal{F}/\delta$ with $\delta \not\subset \theta$ meet irreducible. By congruence distributivity, one has $\theta \subset \delta$ or $\psi \subset \delta$. Therefore, $\mathcal{F}/\delta$ satisfies $\alpha p \leq \pi q$ for $p/q \in \mu$ - in the first case one has $\phi^{-1}(\mu) \subset \delta$ whence $\alpha p \,\delta\, \pi q$, in the second $\alpha p/(\alpha p \cap \pi q) \in Q \subset \delta$. The Corollary is due to [Wi73a].

Now, assume that the σ_i are join preserving closure operators with $\sigma_i x \leq \sigma_j x$ for $j > i$. Then (6),(7) can be replaced by the following

$$(6)\ \sigma_i \alpha p \leq \alpha\sigma_i p + \sigma_i(\alpha p \pi p_*) \text{ for all } p \in C(\mathcal{S})$$

$$(7a)\ \alpha\sigma_i p \cap \pi\sigma_i y \leq \sigma_i(\alpha p \cap \pi y) \text{ for } p \in C(\mathcal{S}), \sigma_i p_* \prec \sigma_i p, y \geq p_* \text{ maximal}$$

$$\text{with } \sigma_i y \not\geq \sigma_i p, \text{ and } \sigma_i p \neq p \text{ or } \sigma_i y \neq y$$

$$(7b)\ \alpha\sigma_i x \cap \pi\sigma_i y \leq \sigma_i(\alpha x \pi y) \text{ for all prime } x/y \in \mu, \sigma_i x \prec \sigma_i y,$$

with no $p \in J(\mathcal{S})$ *such that* $x = y + p, \sigma_i p_* \prec \sigma_i p$

(7c) $\alpha\sigma_i x \cap \pi z \leq \alpha z + \sigma_i(\alpha x \cap \pi y)$ *for prime* $x/y \in \mu, \sigma_i y < z < \sigma_i x$.

Proof. In (6), (7) consider a prime quotient u/v in μ. Then there is $p \in C(\mathcal{S})$ with $p + v = u$ and $pv = p_*$. It follows $\sigma_i\alpha u = \sigma_i\alpha p + \sigma_i\alpha v \leq \alpha\sigma_i p + \sigma_i(\alpha p \cap \pi p_*) + \sigma_i\alpha v \leq \alpha\sigma_i u + \sigma_i(\alpha u \cap \pi v)$. This proves (6) for u/v. If $\sigma_i u = \sigma_i v$ then (7) holds for u/v, trivially. Assume $\sigma_i p_* \prec \sigma_i p$ for suitable p (otherwise, (7b) applies) and choose $y \geq v$ maximal with $\sigma_i y \not\geq \sigma_i p$. One derives $\alpha\sigma_i u \cap \pi\sigma_i v = (\alpha\sigma_i p + \alpha\sigma_i v)\pi\sigma_i v = \alpha\sigma_i p \cap \pi\sigma_i v + \alpha\sigma_i v \leq \alpha\sigma_i p \cap \pi\sigma_i y + \sigma_i\alpha v \leq \sigma_i(\alpha p \cap \pi y) + \sigma_i\alpha v \leq \sigma_i(\alpha u \cap \pi u \cap \pi y) + \sigma_i\alpha v = \sigma_i(\alpha u \cap \pi v)$. The remaining cases are covered by (7c).

Let $^\perp$ be an antitonic operator. Then (6),(7) can be replaced by

(6) $$(\alpha p)^\perp \geq \pi(p^\perp) \cap (\alpha p \cap \pi p_*)^\perp \text{ for } p \in C(\mathcal{S})$$

(7) $$\alpha z + \pi(p^\perp) \geq \pi z \cap (\alpha p \cap \pi p_*)^\perp \text{ for } p \in C(\mathcal{S}), p^\perp < z \leq (p_*)^\perp.$$

Proof. Concerning (6) we compute $(\alpha(p + x))^\perp = (\alpha p + \alpha x)^\perp = (\alpha p)^\perp(\alpha x)^\perp \geq \pi(p^\perp)(\alpha p\pi p_*)^\perp(\alpha x)^\perp \geq \pi((p+x)^\perp)(\alpha(p+x)\pi x)^\perp$ since $p \leq p+x$ and $\alpha p\pi p_* \leq \alpha(p+x)\pi x$. Now, consider $(p + xz)^\perp < z \leq x^\perp$. Then $p^\perp < z + p^\perp \leq (p_*)^\perp$ and, by modularity, $\alpha z + \pi((p+x)^\perp) = \alpha z + \pi(p^\perp)\pi(x^\perp) = \pi(x^\perp)(\alpha z + \pi(p^\perp)) = \pi(x^\perp)(\alpha(z + p^\perp) + \pi(p^\perp)) \geq \pi(x^\perp)\pi(z+p^\perp)(\alpha p\pi p_*)^\perp \geq \pi z(\alpha(p+x)\pi x)^\perp$ using the hypothesis and $\alpha p\pi p_* \leq \alpha(p+x)\pi x$.

Remarks. 1. If the computation of bounds succeeds but one of the hypotheses (6),(7) fails then the model under consideration must have a proper homomorphic preimage. Such can be found determining the set of relations failed by α, π, choosing a maximal logically closed subset, and computing the free model under these additional relations. In practice, though, the main source has been insight into the geometrical problems behind the lattices. Once a list of subdirectly irreducibles (including those to be obtained as homomorphic images) is given, the Splitting Method is applied to the maximal ones, i.e. those having no preimage in the list. Then, the associated relations are included into the presentation and the process is iterated with the reduced list.

2. At some point it might happen that the finitely presented quadratic lattice $\mathcal{F}$ under consideration is (suspected to be) the free modular lattice $\mathcal{M}$ (with 0, 1 as constants and, maybe, in a subvariety) with a suitable set of generators and relations between them which have been derived, already. If the subdirectly irreducible factors $\mathcal{S}_i$ of this modular lattice (known e.g. from 2.2(g)) are exactly the lattice factors of the (maximal) quadratic lattices in the list then we may consider $\mathcal{M}$ as a sublattice of $\mathcal{F}$. In order to prove completeness of the list, it suffices to show $\mathcal{M} = \mathcal{F}$, i.e. that $\mathcal{M}$ is closed under the quadratic operations,

too. In view of 2.2(d) this means to compute the smallest preimages $\alpha_i x \in \mathcal{M}$ for $x \in J(\mathcal{S}_i)$ and the $f\alpha x$ and $g\alpha x$.

2.5 Galois semilattices

A *Galois semilattice*, shortly GS, is a meet semilattice $\mathcal{L}$ with 0 and a unary operation $x \mapsto x^\perp$ such that

$$x \leq (x^\perp \cap y)^\perp \text{ for all } x, y \in \mathcal{L}.$$

A *Galois lattice*, shortly GL, is a lattice with $0, 1$ which is a Galois (meet)semilattice with $1 = 0^\perp$.

LEMMA 11. *Let $\mathcal{L}$ be a Galois semilattice. Then $^\perp$ is a Galois connection: $x \leq y \Rightarrow y^\perp \leq x^\perp$ and $x \leq x^{\perp\perp}$ resp. $x \leq y^\perp \Rightarrow y \leq x^\perp$. Also, $0^\perp =: 1$ is the greatest element of $\mathcal{L}$ and $x \mapsto x^{\perp\perp}$ is a closure operator with closed elements $x^\perp$, i.e. $x \leq x^{\perp\perp}$, $x \leq y \Rightarrow x^{\perp\perp} \leq y^{\perp\perp}$, $x^{\perp\perp\perp} = x^\perp$, and $(x^\perp \cap y^\perp)^{\perp\perp} = x^{\perp\perp\perp} \cap y^{\perp\perp\perp}$. In a Galois lattice, $^\perp$ is an antitonic operator*

$$(x + y)^\perp = x^\perp \cap y^\perp.$$

PROOF. With $x = a, y = x^\perp$ we get $a \leq a^{\perp\perp}$. If $a \leq b$ then $b^\perp \leq a^\perp$ with $x = b^\perp, y = a$. Thus, if $a \leq b^\perp$ then $b \leq b^{\perp\perp} \leq a^\perp$. The remaining claims on GS follow, easily. In a GL, if $c \leq a^\perp, b^\perp$, then $a, b \leq c^\perp$ whence $a + b \leq c^\perp$ and $c \leq c^{\perp\perp} \leq (a + b)^\perp$. Consequently, $a^\perp \cap b^\perp \leq (a + b)^\perp$. The converse inequality is clear.

Having a Galois connection $^\perp$ on a semilattice is equivalent to having a GS. Another view, particularly useful for the construction of examples, is that $^\perp$ is given by a complete meet subsemilattice $\mathcal{S}$ of $\mathcal{L}$ and an involutory order anti-isomorphism $x \mapsto x'$ of $\mathcal{S}$; namely $x^\perp = (\inf\{y \in \mathcal{S} | y \geq x\})'$ defines a Galois connection.

An important operation is that of forming the *radical* of an element

$$rx = r(x) = x \cap x^\perp.$$

We write $rx^\perp = r(x^\perp) = rx^{\perp\perp} = r(x^{\perp\perp})$.

In a GL, if all elements are closed then $(x \cap y)^\perp = x^\perp + y^\perp$ and we speak of *polarity lattices*. Polarity lattices behave basically like members of the lattice variety they generate, just with a doubled generator set. In contrast, distributive Galois lattices resemble general lattices.

We need some notation first. In a partially ordered set, an *order ideal* I contains with x also all $y \leq x$. The order ideal $\downarrow X$ generated by X consists of all $y \leq x \in X$. Call an element a of a lattice *completely join irreducible* if $a = \sup X$ implies $a \in X$.

Let $\mathcal{I}(\mathcal{S})$ be the lattice of all finitely generated order ideals of the Galois semilattice S and denote by $\downarrow X$ the order ideal generated by X. Endow $\mathcal{I}(\mathcal{S})$ with the Galois connection

$$I^\perp = \bigcap\{x^\perp | x \in I\}.$$

Indeed, if $J \subset I^{\perp}$ and $x \in I$ then for all $y \in J$ we have $y \leq x^{\perp}$ whence $x \leq y^{\perp}$ and $x \in J^{\perp}$. Identify x with $\downarrow x$ to have S a Galois subsemilattice of $\mathcal{I}(\mathcal{S})$. Then, $\mathcal{I}(\mathcal{S})$ is the distributive Galois lattice freely generated by the Galois semilattice S and S consists just of the completely join irreducibles of $\mathcal{I}(\mathcal{S})$ and contains all closed elements.

Proposition 12. *A Galois semilattice with closed generators consists entirely of closed elements. Defining $x + y = (x^{\perp} \cap y^{\perp})^{\perp}$ provides a rational (or term) equivalence between Galois semilattices satisfying $x^{\perp\perp} = x$ and polarity lattices.*

Corollary 13. *The Galois semilattice S freely generated by $a = a^{\perp\perp} \leq b = b^{\perp\perp}$ is the infinite lattice $FL(2+2)$ freely generated by two chains $a \leq b$ and $b^{\perp} \leq a^{\perp}$ with diagram given in [Rol58]. It satisfies $x \cap x^{\perp} \leq b$ for all x and so does $\mathcal{I}(\mathcal{S})$.*

Proof. By Lemma 11, the meet of closed elements is closed and $x^{\perp}$ is closed in any case. Now, verify that $x + y$ is a supremum: if $z \leq x, y$ then $z^{\perp} \leq x^{\perp}, y^{\perp}$ whence $z^{\perp} \leq x^{\perp} \cap y^{\perp}$ and $z = z^{\perp\perp} \geq (x^{\perp} \cap y^{\perp})^{\perp}$; on the other hand, $x^{\perp} \geq x^{\perp} \cap y^{\perp}$ and so $x = x^{\perp\perp} \leq (x^{\perp} \cap y^{\perp})^{\perp}$. For the proof of the Corollary observe that the lattice $FL(2+2)$ is self dual since its presentation is so. A dual automorphism α is given by interchanging a and $a^{\perp}$ and b and $b^{\perp}$. But $\alpha \circ \alpha = id$ since this is the case on the generators. So, α is a polarity and $FL(2+2)$ the polarity lattice freely generated by $a \leq b$.

Proposition 14. *The word problem for Galois semilattices is solvable .*

Proof. The proof relies on the concept of a *partial Galois semilattice* $\mathcal{A}$ which is a partially ordered set together with two partial operations, a binary written as $x \cap y$ which is $\inf(x, y)$ where defined and a unary $x^{\perp}$ satisfying the following axioms as far as they can be evaluated

$$x \leq y \Rightarrow y^{\perp} \leq x^{\perp},\ x \leq x^{\perp\perp},\ y \leq x^{\perp} \Rightarrow x \leq y^{\perp}$$

In particular, $x^{\perp\perp\perp} = x^{\perp}$ if defined. A *filter* of A is an order filter closed under all defined meets. Let $\uparrow X$ denote the order filter generated by X and $X^{\perp} = \uparrow \{x^{\perp} | x \in X\}$. Ordered by the dual of set inclusion, the filters form a meet semilattice $\mathcal{F}(\mathcal{A})$ extending A (if one identifies x with $\uparrow x$) and a partial Galois semilattice with $\perp$ inherited from $\mathcal{A}$.

On the other hand, for s in A such that $s^{\perp}$ is not defined adjoin a new element $s^{\perp}$ and define $p \leq_s q$ if an only if $p = q$ or $p \leq q$ in $\mathcal{A}$ or one of the following

$$p = s^{\perp}, q \in X^{\perp}, X \subset \downarrow s \text{ for some finite } X \subset A$$

$$q = s^{\perp}, s = y^{\perp}, p \leq y \text{ for some } y \in A$$

$$q = s^{\perp}, p \leq y^{\perp}, s \leq y \text{ for some } y \in A.$$

Then, the Galois semilattice freely generated by $\mathcal{A}$ is obtained by a (possibly countable) iteration of extension steps of these two kinds.

2.6 Quadratic lattices

The closure operators σ_i considered in section 1.4 lead to the notion of an *α-quadratic lattice*: such is a modular Galois lattice with additional unary operations σ_ι, $1 \leq \iota \leq \alpha$ satisfying

$$x \leq \sigma_\iota x, \;\; \sigma_\iota x = \sigma_\iota \sigma_\iota x, \;\; x \leq y \Rightarrow \sigma_\iota x \leq \sigma_\iota y$$

$$\sigma_\iota x \leq \sigma_\gamma x \text{ for } \gamma < \iota$$

$$\sigma_\iota(x^\perp) = x^\perp = (\sigma_\iota x)^\perp$$

$$\sigma_\iota(x+y) = \sigma_\iota x + \sigma_\iota y \text{ for } 0 < \iota .$$

For convenience, one defines $\sigma_0 x = x^{\perp\perp}$ and $\sigma_{\alpha+1} x = x$. On easily derives

$$\sigma_\iota \sigma_\gamma = \sigma_{\min\{\iota,\gamma\}}, \;\; \sigma_\iota 0 = 0$$

and that the σ_i are closure operators on $\mathcal{L}$. In the non trace-valued case a crucial rôle is played by the subspace

$$E^* = \{x \in E \mid \Phi(x,x) \in T\}$$

of trace valued vectors. Observe that $X \cap X^\perp \subset E^*$ for all X. This gives rise to the concept of *α-hermitean* lattices which are α-quadratic lattices with an additional constant b satisfying $x \cap x^\perp \leq b$. α-quadratic lattices can be understood as α-hermitean lattices with $b = 1$. Also, an α-hermitean lattice can be considered a β-hermitean lattice for each $\beta > \alpha$ putting $\sigma_\iota = id$ for $\iota > \alpha$. In any case, we deal with modular lattices with operators. We use the term *quadratic lattice* when α is not specified or obvious from the context. Also just modular lattices will be subsumed. In particular, in section 3 quadratic means α-quadratic, in section 5 hermitean means 0-hermitean.

In proofs we use, occasionally, the notations $s \simeq t$, $s \approx t$, $s \cong t$ if the identity $s = t$ is to be derived using the modular law, the law $(a^\perp b^\perp)^\perp = (a+b)^{\perp\perp}$ respectively the law $(a(b+c))^\perp = (a^{\perp\perp}(b+c)^{\perp\perp})^\perp = (a^\perp + (b+c)^\perp)^{\perp\perp} = (a^\perp + b^\perp c^\perp)^{\perp\perp}$ where a and $b+c$ are closed. We add a subscript $_i$ if it is also used that t (or s) is σ_i-closed. In particular, we have $(c^\perp d^\perp)^\perp \approx (c+d)^{\perp\perp}$ if c and d are closed and $(c^\perp d^\perp)^\perp \approx_0 c+d$ if $c+d$ is closed.

PROPOSITION 15. *The α-hermitean/quadratic lattices form a variety. The lattice $\mathcal{L}(E)$ of a diagonal, ε-hermitean space E is hermitean (in particular, quadratic if E is trace-valued). If $\dim E \leq \aleph_\gamma$ then $\sigma_\iota = id$ for all $\iota > \gamma$. If E is finite dimensional then $\mathcal{L}(E)$ is a polarity lattice.*

Here, the lattice $\mathcal{L}(E)$ of subspaces of E is endowed with the operations $X \mapsto X^\perp$ and $X \mapsto \sigma_\iota X$, the closure in the topology σ_ι, and the constants $0 = (0)$, $1 = E$, and $b = E^*$.

Proof. Monotonicity of σ_ι can be expressed by $\sigma_\iota(x \cap y) \leq \sigma_\iota x$ and an inequality $s \leq t$ by the identity $s = s \cap t$. Most of the other claims follow from 1.4, immediately. Moreover, $X^\perp = X^{\perp\perp\perp} \leq (\sigma_\iota X)^\perp \leq X^\perp$ and $X^\perp \leq \sigma_\iota(X^\perp) \leq X^{\perp\perp\perp} = X^\perp$. Finally, by definition $E^* = E$ in the trace-valued case.

Lemma 16. *If $\mathcal{L}$ is a hermitean sublattice of $\mathcal{L}(E)$, $E = \bigoplus_{i\in I} E_i$ is an orthogonal decomposition, and $X = \sum_{i\in I} X \cap E_i$ for all X in a generating set of $\mathcal{L}$ then the maps $X \mapsto X \cap E_i$, $i \in I$, provide a subdirect decomposition of $\mathcal{L}$ into hermitean sublattices of the $\mathcal{L}(E_i)$.*

Proof. For each X in $\mathcal{L}$ the E_i provide an orthogonal decomposition of the pair (E, X). Indeed, for the generators they do, by hypothesis, and if X and Y are decomposed then so are $X + Y$, $X \cap Y$, $X^\perp$, and $\sigma_\iota X$ according to the linear case 2.2(g) and to 1.4. ¿From the linear case, we know that the maps $X \mapsto X \cap E_i$ provide a subdirect decomposition of the lattice $\mathcal{L}$. From Lemma 2 in 1.4 we derive by modularity that they are homomorphisms with respect to $^\perp$ and σ_i, too: $f^{\mathcal{L}(E)}(X) \cap E_i = f^{\mathcal{L}(E_i)}(X)$.

2.7 Indexed lattices

The non-trace-valued case is accessible only if certain indices are finite. We shall discuss how to capture such finiteness conditions within lattice theory. Given a α-quadratic lattice or a (not necessarily modular) lattice $\mathcal{L}$, an *index function* is a map δ associating with each quotient x/y of $\mathcal{L}$ a cardinal $\delta(x/y) \leq \aleph_\alpha$ ($\leq \aleph_0$ in the case of lattices) such that

(8) $$\delta(x/y) \geq \delta(x \cap z/y \cap z) \text{ for all } x \geq y \in \mathcal{L}, z \in \mathcal{L}$$

(9) $$\delta(x/y) \geq \delta(y^\perp/x^\perp) \text{ for all } x \geq y \in \mathcal{L}$$

(10) $$\delta(x/z) = \delta(x/y) + \delta(y/z) \text{ for all } x \geq y \geq z \in \mathcal{L}$$

(11) $$\delta(x/y) = 0 \Leftrightarrow x = y \text{ for all } x \geq y \in \mathcal{L}$$

(12) $$\delta(x/y) \geq \delta(x + z/y + z) \text{ for all } x \geq y \in \mathcal{L}, z \in \mathcal{L}$$

(13). $$\delta(\sigma_j x/\sigma_j y) \geq \delta(\sigma_i x/\sigma_i y) \text{ for all } x \geq y \in \mathcal{L}, 0 \leq i \leq j \leq \alpha + 1$$

Conditions (9) and (13) are void when dealing with lattices. For finite $\mathcal{L}$, every index function is determined by its values on prime quotients, of course. If in (10) we have $\geq$, only, we speak of a *weak index function*. Observe, that this still implies that $\delta(u/v) \leq \delta(x/y)$ if u/v is a subquotient of x/y. By Lemma 1 we get

PROPOSITION 17. *For a diagonal, ε-hermitean space E of dimension $\aleph_\alpha$ an index function on the α-hermitean lattice $\mathcal{L}(E)$ is defined by $\delta(X/Y) = \dim(X/Y)$.*

LEMMA 18. *In any quadratic lattice with weak index function $\delta(p/q) \leq \delta(a/b)$ if p/q is obtained from a/b under weak projectivity. In particular, in a subdirectly irreducible algebra, $\delta(p/q) > 0$ is minimal if p/q is critical. A lattice admitting an index function with finite values, only, has to be modular. Any weak index function on a finite modular lattice $\mathcal{L}$ is determined by its values $\delta(p/p_*)$, $p \in J(\mathcal{L})$.*

PROOF. The first claim follows from Lemma 6 and the definition, immediately. In order to derive modularity from the existence of a finite index function just recall that a non-modular lattice contains the 5-element non-modular lattice as a sublattice but the latter does not admit a finite index function. Finally observe that in a finite modular lattice each prime quotient transposes to a quotient p/p_*.

LEMMA 19. *Let $\mathcal{L}$ be any quadratic lattice with weak index function δ and $\delta(a/b) < \aleph_0$. Let u/v be obtained from a/b under weak projectivity. Then $\sigma_i u = u + \sigma_i v$ for all $i = 0, \ldots, \alpha$.*

PROOF. Put $y = \sigma_i v$ and $x = u + y \leq \sigma_i u$. By Lemma 1 we have $\aleph_0 > \delta(x/y) \leq \delta(\sigma_i x/\sigma_i y) = \delta(\sigma_i x/y) = \delta(\sigma_i x/x) + \delta(x/y)$. It follows $\delta(\sigma_i x/x) = 0$ whence $\sigma_i x = x \leq \sigma_i u$ by (11). Since $u \leq x$ we derive $\sigma_i u \leq \sigma_i x$ and so equality.

LEMMA 20. *For each quadratic lattice admitting a weak index function there is a pointwise smallest such - and this is an index function.*

PROOF. Given weak index functions δ_i define $\delta(x/y) = \min_i \delta_i(x/y)$. Then δ is a weak index function. Namely, (8), (9), (12), (13) are obvious and if $\delta(x/y) = 0$ then $\delta_i(x/y) = 0$ for some i whence $x = y$; and for each i we have $\delta(x/y) + \delta(y/z) \leq \delta_i(x/y) + \delta_i(y/z) \leq \delta_i(x/z)$. Now, given a weak index function define

$$\delta'(x/y) = \min\{\sum_{i=1}^{m} \delta(x_i/x_{i+1}) \,|\, x = x_1 \geq \ldots \geq x_m = y\}.$$

Clearly, $\delta'(x/y) \leq \delta(x/y)$. Again, (8), (9), (12), (13) are obvious. If $\delta'(x/y) = 0$ then there are $x = x_1 \geq \ldots \geq x_m = y$ with all $\delta(x_i/x_{i+1}) = 0$, i.e. $x_i = x_{i+1}$ and $x = y$. The inequality $\leq$ in (10) holds by definition. To prove the inverse, consider $x \geq y \geq z$

and $x = x_1 \geq \ldots \geq x_m = z$. By the Jordan-Hölder-Ore Theorem there are refinements $v_1 \geq \ldots \geq v_l = y \geq \ldots \geq v_k$ and $u_1 \geq \ldots \geq u_k$ of these chains and a permutation π such that v_j/v_{j+1} and $u_{\pi j}/u_{\pi j+1}$ are projective via two transposition steps, in particular they have the same index. Hence $\delta'(x/y)+\delta'(y/z) \leq \sum_{1\leq j<l}\delta(v_j/v_{j+1})+\sum_{l\leq j<k}\delta(v_j/v_{j+1}) = \sum_{j=1}^{k}\delta(u_j/u_{j+1}) \leq \sum_{i=1}^{m}\delta(x_i/x_{i+1})$ and it follows equality in (10). Thus, $\delta = \delta'$, actually.

Proposition 21. *The class of α-quadratic lattices admitting an index function is closed under subalgebras, finite products, and homomorphic images*

Proof. For finite I and index functions δ_i on $\mathcal{L}_i$ we get an index function

$$\delta((x_i)_{i\in I}/(y_i)_{i\in I}) = \sum_{i\in I}\delta_i(x_i/y_i) \text{ on } \prod_{i\in I}\mathcal{L}_i \ .$$

Given a homomorphism ϕ of $\mathcal{L}$ onto $\mathcal{L}_1$, define

$$\delta_1(u/v) = \min\{\delta(x/y)\,|\,\phi x = u,\ \phi y = v\}.$$

Indeed, for $u \geq v \geq w$ in $\mathcal{L}_1$ one has $\delta_1(u/v)+\delta_1(v/w)) \leq \min\{\delta(x/y)+\delta(y/z)\,|\,x \geq y \geq z, \phi x = u, \phi y = v, \phi z = w\} \leq \min\{\delta(x/z)\,|\,x \geq z, \phi x = u, \phi z = w\} = \delta_1(u/w)$. If $\delta_1(u/v) = 0$ then $\delta(x/y) = 0$ for some x/y with $\phi x = u$, $\phi y = v$ whence $x = y$ and $u = v$. Conversely, if $u = \phi x$ then $\delta_1(u/u) \leq \delta(x/x) = 0$. Thus, δ_1 is a weak index function and we can apply Lemma 20.

Speaking about *indexed* quadratic lattices we always consider them endowed with the smallest index function δ according to Lemma 20. Our main task is to find, in a variety, for certain presentations (E, R) and specified term $s \geq t$ all indexed subdirectly irreducible models with $\delta(s/t) < \aleph_0$. If there is such a model having every other as (canonical) homomorphic image then this model may be considered the *free indexed object* of the variety with generators E and relations R and $\delta(s/t) < \aleph_0$. This happens if and only if there are only finitely many subdirectly irreducible indexed models separating s and t.

Analogously, for Galois semilattices we have a concept of index functions and Lemmas 18 and 19. Also, the indexed ones form a quasivariety. Though, a GS may admit a weak index function but not an index function. Nethertheless, in the above sense we have

Proposition 22. *The free indexed Galois semilattice $\mathcal{L}$ with generators $a = a^{\perp\perp} \leq b$ such that $\delta(a/b) < \aleph_0$ has diagram as depicted in Fig.1.*

Proof. Observe that any index function on $\mathcal{L}$ is also an index function of the polarity lattice on $\mathcal{L}$ defined according to Prop. 12 of 2.5. Thus, by Lemma 18, all intervals x/y with $\delta(x/y) < \aleph_0$ have to be modular sublattices. In particular, this applies to $x = a + a^{\perp}$, $b + b^{\perp}$ and $y = a + b^{\perp} + b \cap a^{\perp}$. Observe that $b = b^{\perp\perp}$ by Lemma 19.

Thus, there is a homomorphism from the GS freely generated by $a = a^{\perp\perp} \leq b = b^{\perp\perp}$ onto $\mathcal{L}$, cf Cor. 13 in 2.5. Using the diagram of $FL(2+2)$ given in [Rol58] we see that $a + b^{\perp} + b \cap a^{\perp} \prec (a + a^{\perp}) \cap (b + b^{\perp})$. The dual then holds as well. $\mathcal{L}$ has 10 subdirectly irreducible factors, 4 of which are non-modular.

A different way to incorporate finiteness of indices will be used in Section 6. Namely, one requires $\sigma_j u = u + \sigma_j v$ for all u/v obtained from s/t under weak projectivity and all $j = 0, \ldots, \alpha$. This actually amounts to a (infinite) set of identities (use weak transpositions e.g. x/y to $x + z/y + z$ in place of transpositions and subquotients). Lemma 19 tells that this variety contains all indexed members of the original variety satisfying $\delta(s/t) < \aleph_0$.

2.8 Side lattices

In this paragraph we deal with α-quadratic lattices $\mathcal{L}$ having generators a and c such that

$$(14) \qquad c \leq r a^{\perp\perp} \text{ and } \sigma_0(c \cap r\sigma_i a) = \sigma_0 r \sigma_i a \text{ for all } i = 0, \ldots, \alpha + 1\,.$$

One obtains the case of one generator a putting $c = ra^{\perp\perp}$. The basic examples of subdirectly irreducibles are

$\mathcal{R}_1$. $0 = a = c \prec 1 = a^{\perp} = p$, $\sigma_i = id$.

$\mathcal{R}_2$. $0 = a^{\perp} = c \prec 1 = a = p$, $\sigma_i = id$.

$\mathcal{R}_{\alpha i}$. $0 = a^{\perp} = c \prec a = \sigma_j a \prec 1 = \sigma_k a = p$ for $0 \leq k \leq i < j \leq \alpha + 1$.

$\mathcal{R}_3$. $0 = c \prec a \prec b \prec 1 = p, 0 \prec a^{\perp} \prec b$, $\sigma_i = id$.

$\mathcal{R}_4$. $0 \prec a = a^{\perp} = c = p_1 \prec 1 = p_2$, $\sigma_i = id$.

$\mathcal{R}_5$. $0 = c \prec a = a^{\perp} = p_1 \prec 1 = p_2$, $\sigma_i = id$.

See Figure 2. The homomorphisms are $\mathcal{R}_{\alpha i} \to \mathcal{R}_2$, $\mathcal{R}_3 \to \mathcal{R}_1$, $\mathcal{R}_3 \to \mathcal{R}_2$.

Lemma 23. *Any subdirectly irreducible α-quadratic lattice with generators a, c satisfying (14) and not isomorphic to one of $\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_{3i}$ satisfies $a + a^{\perp} = a^{\perp\perp}$.*

Proof. We apply the Split Lemma 9 with $\alpha 0 = 0$ and $\pi 1 = 1$ in all instances. We have to consider the operators σ_j, $1 \leq j \leq \alpha$ and $^{\perp}$. (1)-(3) will be always obvious as well as (4) for comparable x, y.

$\mathcal{R}_{3i}$. $\alpha a = a$, $\alpha 1 = \sigma_i a$, $\pi 0 = a^{\perp}$, $\pi a = a^{\perp} + \sigma_{i+1} a$. Concerning (5) we have $\alpha\sigma_j 0 = 0 \leq \sigma_j \alpha 0$. $\alpha\sigma_j a = \alpha 1 = \sigma_i a \leq \sigma_j \sigma_i a = \sigma_j \alpha a$ for $j \leq i$, $\alpha \sigma_j a = a \leq \sigma_j a = \sigma_j \alpha a$ for $j > i$, $\alpha\sigma_j 1 = \sigma_i a \leq \sigma_j \sigma_i a = \sigma_j \alpha 1$; $\sigma_j \pi 0 = \sigma_j a^{\perp} = a^{\perp} = \pi \sigma_j 0$, $\sigma_j \pi a = \sigma_j(a^{\perp} + \sigma_{i+1} a) = \sigma_j a^{\perp} + \sigma_j \sigma_{i+1} a = a^{\perp} + \sigma_{i+1} a = \pi\sigma_j a$ if $j > i$, while $\pi\sigma_j a = 1$, otherwise. Also $\pi\sigma_j 1 = 1$. Moreover, $\alpha 0^{\perp} = 1 = a^{\perp\perp} = (\pi 0)^{\perp}$, $\alpha a^{\perp} = \alpha 0 = 0 \leq (\pi a)^{\perp}$, $\alpha 1^{\perp} = 0 \leq (\pi 1)^{\perp}$, $(\alpha 0)^{\perp} \leq 1 = \pi 0^{\perp}$, $(\alpha 1)^{\perp} = a^{\perp} = 0 \leq \pi a^{\perp}$, and $(\alpha 1)^{\perp} = (\sigma_i a)^{\perp} = a^{\perp} = \pi 0 = \pi 1^{\perp}$.

In (6),(7) the only critical quotient to be considered is $p/q = 1/a$. Now, for $j > 0$, $\sigma_j \alpha 1 = \sigma_j \sigma_i a \leq \sigma_i a = \alpha \sigma_j 1$ if $j \geq i$. Otherwise, $\sigma_j \sigma_i a = \sigma_j \sigma_{i+1} a \leq \sigma_j(a^{\perp}\sigma_i a) + \sigma_j \sigma_{i+1} a =$

$\sigma_j(a^\perp\sigma_i a+\sigma_{i+1}a)=\sigma_j(\sigma_i a(a^\perp+\sigma_{i+1}a))=\sigma_j(\alpha 1\pi a)$. Also, $(\alpha 1)^\perp=(\sigma_i a)^\perp=a^\perp=\pi 0=\pi 1^\perp$ proving the second part of (6).

In (7) we have to consider $\sigma_j q=a=z<1=\sigma_j p$ for $j>i$, only. Here, $\alpha\sigma_j 1\cap\pi a=\sigma_i a\cap(a^\perp+\sigma_{i+1}a)\leq\sigma_j(\sigma_i a\cap(a^\perp+\sigma_{i+1}a))\leq\alpha a+\sigma_j(\alpha 1\pi a)$, indeed. For $^\perp$ the condition is satisfied, trivially, since $1^\perp=a^\perp$.

For all subdirectly irreducible models not isomorphic $\mathcal{R}_{3i}$ it follows $\alpha p\leq\pi q$, i.e. $\sigma_i a\leq a^\perp+\sigma_{i+1}a$ ($0\leq i\leq\alpha$). Hence $a^\perp+\sigma_i a\leq a^\perp+\sigma_{i+1}a$ and, by induction, $a^{\perp\perp}=\sigma_0 a\leq a^\perp+\sigma_{\alpha+1}a=a^\perp+a$.

$\mathcal{R}_3$. $\alpha a=a$, $\alpha a^\perp=a^\perp=\pi a^\perp$, $\alpha b=a+a^\perp$, $\alpha 1=(a+a^\perp)^{\perp\perp}=(ra^{\perp\perp})^\perp$, $\pi 0=ra^{\perp\perp}$, $\pi a=a^{\perp\perp}$, $\pi b=a^\perp+a^{\perp\perp}$. Then (4) holds by definition. Moreover, $\alpha\sigma_j x=\alpha x\leq\sigma_j\alpha x$ and $\sigma_j\pi x=\pi x=\pi\sigma_j x$ since all πx are closed. Concerning the orthogonal we get $\alpha 0^\perp=\alpha 1=(ra^{\perp\perp})^\perp=(\pi 0)^\perp$, $\alpha a^\perp=a^\perp=(\pi a)^\perp$, $\alpha a^{\perp\perp}=\alpha a=a\leq a^{\perp\perp}=(\pi a^\perp)^\perp$, $\alpha 1^\perp=\alpha b^\perp=\alpha 0=0$ as well as $(\alpha 0)^\perp=1=\pi 0^\perp$, $(\alpha a)^\perp=a^\perp=\pi a^\perp$, $(\alpha a^\perp)^\perp=a^{\perp\perp}=\pi a=\pi a^{\perp\perp}$, $(\alpha b)^\perp=(a+a^\perp)^\perp=r(a^{\perp\perp})=\pi 0=\pi b^\perp$, $(\alpha 1)^\perp=ra^{\perp\perp}=\pi 0=\pi 1^\perp$.

In (6),(7) we have $p/q=1/b$ and $\alpha p\geq\pi q$. Now, $\sigma_j\alpha 1=(ra^{\perp\perp})^\perp=\alpha\sigma_j 1$ and $\pi 1^\perp\cap(\alpha 1\pi b)^\perp=ra^{\perp\perp}=(\alpha 1)^\perp$ prove (6). Considering $z=b$ we have $\alpha\sigma_j 1\cap\pi b=\pi b\leq\sigma_j\pi b=\sigma_j(\alpha 1\pi b)$ whence (7) since $1^\perp=b^\perp=0$ in $\mathcal{R}_3$.

It follows that $\alpha p\leq\pi q$, i.e. $(ra^{\perp\perp})^\perp=a^\perp+a^{\perp\perp}$ in every subdirectly irreducible model not isomorphic $\mathcal{R}_3$. So we now may proceed assuming the additional relations $a^{\perp\perp}\leq a^\perp+\sigma_{\alpha+1}a=a^\perp+a$ and $(ra^{\perp\perp})^\perp=a^\perp+a^{\perp\perp}$ working within a more specialized free model, actually.

$\mathcal{R}_1$. $\alpha 1=a^\perp$, $\pi 0=a^{\perp\perp}$. Dealing with the σ_i part of (5)-(7), first, observe that $\alpha\sigma_i 0=\alpha 0=0$, $\alpha\sigma_i 1=\alpha 1=a^\perp=\sigma_i a^\perp=\sigma_i\alpha 1$, $\sigma_i\pi 0=\sigma_i a^{\perp\perp}=a^{\perp\perp}=\pi 0=\pi\sigma_i 0$, $\sigma_i\pi 1=\sigma_i 1=1=\pi\sigma_i 1$, and $\alpha\sigma_i 1\cap\pi 0=ra^{\perp\perp}=\sigma_i ra^{\perp\perp}=\sigma_i(\alpha 1\pi 0)$ where $z=\sigma_i 0=0$. The remaining relations are implied by the following: $\alpha 0^\perp=\alpha 1=a^\perp=(a^{\perp\perp})^\perp=(\pi 0)^\perp$, $\alpha 1^\perp=\alpha 0=0$, $(\alpha 0)^\perp=0^\perp=1=\pi 0^\perp$, $(\alpha 1)^\perp=a^{\perp\perp}=\pi 0=\pi 1^\perp$. Finally, we have $\alpha 1+\pi 1^\perp=a^\perp+a^{\perp\perp}=(ra^{\perp\perp})^\perp=(\alpha 1\pi 0)^\perp$ where $z=0^\perp=1$.

It follows $a^\perp\leq a^{\perp\perp}$ and $a+a^\perp=a^{\perp\perp}$.

The remaining subdirectly irreducibles can be more easily obtained via their ‘side lattices’. Here, an α-*side-lattice* $\mathcal{S}$, is a modular lattice with join preserving closure operators $\sigma_0\geq\sigma_1\geq\ldots\geq\sigma_{\alpha+1}=id$, and constants r_i, $0\leq i\leq\alpha+1$, satisfying $r_{\alpha+1}\leq x\leq r_0$ for all x, $r_i\geq r_j$ for $i<j$, and $\sigma_i r_i=r_i$, whence $\sigma_j r_i=r_i$ for $j\geq i$. We consider side lattices with generator c satisfying the relations

(15) $\quad c\leq r_0,\ \sigma_0(c\cap r_i)=\sigma_0 r_i,\ i=0,\ldots\alpha+1.$ we put $g=c\cap r_{\alpha+1}$.

Then one has $\mathcal{S}=\{g\}\cup[\sigma_\alpha g,r_0]$ since this is a subalgebra. Also, one easily derives, $\sigma_j\sigma_i x=\sigma_i x$ for $j\geq i$. A side lattice can be conveniently described via its completely meet closed sublattices C_i of σ_i-closed elements. Observe that $r_i\in C_i\subset C_j$ for $j\leq i$. Conversely, given a modular lattice with such C_i's and chain of r_i's (where $r_{\alpha+1}=0$, $r_0=1$) one obtains an α-side lattice where $\sigma_i x=\inf\{y\in C_i\,|\,y\geq x\}$.

Theorem 24. *Let $\mathcal{L}$ be an α-quadratic lattice with generators a, c satisfying (14) and $\mathcal{S}$ the subalgebra of $\mathcal{L}$ generated by c and the $r_i = r(\sigma_i a) = a^{\perp}\sigma_i a$, $i = 0, \ldots \alpha+1$ under the lattice operations and the σ_i, $i = 0, \ldots, \alpha+1$. Then $\mathcal{S}$ is an α-side lattice with generator c satisfying (15) and is a union of intervals (cf Fig.3.)*

$$\mathcal{S} = \bigcup_{i=0}^{\alpha+1} [c \cap r_i,\, \sigma_0 r_i].$$

Also, $\mathcal{L}$ is a union of intervals

$$\{0\},\ \mathcal{S} = [c \cap ra,\, ra^{\perp\perp}],\ \sigma_i a + \mathcal{S} = [\sigma_i a,\, ra^{\perp\perp} + \sigma_i a],\ i = 0, \ldots, \alpha+1$$

$$\{a^{\perp}, a^{\perp} + \sigma_{\alpha+1} a, \ldots, a^{\perp} + \sigma_0 a\},\ \mathcal{S}^{\perp} = [(ra^{\perp\perp})^{\perp},\, 1].$$

Proof. First, the given subset of $\mathcal{S}$ is a sublattice. Since $c \cap r_k \leq \sigma_j(c \cap r_i) \leq \sigma_0 r_k$ for $k = \min\{i, j\}$ it is closed under the σ_i. Containing the generators, it is just $\mathcal{S}$. Thus, each $\sigma_0 x$, $x \in \mathcal{S}$ is in the chain of the $\sigma_0 r_i$'s. This means, that σ_0 preserves joins and $\mathcal{S}$ is a side lattice. The claims about $\mathcal{L}$ can be derived in the same fashion or obtained from Theorem 25 below, immediately.

Given a side lattice $\mathcal{S}$ as above, form the direct product $\mathbf{2} \times \mathcal{S}$ with $\mathbf{2} = \{0_{\mathbf{2}}, 1_{\mathbf{2}}\}$ and then glue on top of that a chain A where $x \mapsto x'$ is an anti-isomorphism of the chain $\{\sigma_0 r_i \,|\, \alpha+1 \geq i \geq 0\}$ in $\mathcal{S}$ onto A - i.e. $(1_{\mathbf{2}}, r_0) = r_0'$ is the bottom of A and $y < z$ for $y \in \mathbf{2} \times \mathcal{S}$ and $z \in A$, otherwise - see Fig.4. Define

$$\sigma_i(0_{\mathbf{2}}, x) = (0_{\mathbf{2}}, \sigma_i x),\ \sigma_i(1_{\mathbf{2}}, x) = (1_{\mathbf{2}}, \sigma_i x) \text{ for } x \in \mathcal{S},\ \sigma_i x = x \text{ for } x \in A$$

$$(0_{\mathbf{2}}, x)^{\perp} = (\sigma_0 x)',\ (1_{\mathbf{2}}, x)^{\perp} = (0_{\mathbf{2}}, r_0) \text{ for } x \in \mathcal{S},\ z^{\perp} = x \text{ iff } z = x' \text{ for } z \in A.$$

Case 1: There are $i < j$ with $\sigma_i r_j < r_i$. a) If $\sigma_i g = g$ for all i then we define

$$\overline{S} := \{0_{\mathbf{2}}\} \times \mathcal{S} \cup \{1_{\mathbf{2}}\} \times [r_{\alpha+1}, r_0]_{\mathcal{S}} \text{ and } \hat{\mathcal{S}} := \overline{S} \cup A.$$

Otherwise, b), we add to that an additional bottom 0 and top 1.

Case 2: If $\sigma_i r_j = r_i$ for all $i < j$ then we put

$$\overline{S} = \{0_{\mathbf{2}}\} \times \mathcal{S} \text{ and } \hat{\mathcal{S}} = \overline{S} \cup A \cup \{0, 1\}.$$

In all cases we have an obvious α-quadratic lattice structure on $\hat{\mathcal{S}}$ forced by the given data. The additional generator is $a = (1_{\mathbf{2}}, r_{\alpha+1})$ and $a = r_{\alpha+1}$, respectively, and we have $r_i = r\sigma_i a$ for $0 < i \leq \alpha$, $r_0 = r(a^{\perp\perp})$, $r_{\alpha+1} = ra$. In all cases, $\mathcal{S}$ embeds naturally into $\hat{\mathcal{S}}$ and each homomorphism of $\hat{\mathcal{S}}$ onto $\hat{\mathcal{T}}$ restricts to a homomorphism taking $\mathcal{S}$ to $\mathcal{T}$.

Theorem 25. *The subdirectly irreducible α-quadratic lattices with generators a, c satisfying (14) are precisely the $\mathcal{R}_k$, $1 \le k \le 5$, $\mathcal{R}_{\alpha i}$, and the $\hat{\mathcal{S}}$ where $\mathcal{S}$ is a subdirectly irreducible α-side-lattice with generator c satisfying (15). One has $a \le a^\perp$ if and only if $\sigma_i r_j = r - i$ for all $i < j$. If θ is a congruence of $\mathcal{S}$ generating the $\hat{\mathcal{S}}$-congruence $\hat{\theta}$ then $\hat{\mathcal{S}}|\mathcal{S} = \theta$. If $c = ra^{\perp\perp}$ then the critical elements of $\hat{\mathcal{S}}$ are those of $\mathcal{S}$ and the $(p_*)^\perp$ where $p \not\le \sigma_0 p_*$ is critical in $\mathcal{S}$.*

Proof. Let $\mathcal{L}$ be a subdirectly irreducible α-quadratic lattice with generators a, c satisfying (14). First, assume $a^\perp \le a$. Then $c \le c^{\perp\perp} = ra = a^\perp a = a^{\perp\perp} \le c^\perp$ and $\mathcal{L} = \{0\} \cup [c, c^\perp] \cup \{1\}$ admits the two complementary congruences collapsing c with $c^\perp$ respectively 0 with c and 1 with $c^\perp$. Thus, either $\mathcal{L}$ is isomorphic to $\mathcal{R}_4$ or $c = 0$. In the latter case, $\mathcal{L} = \{0\} \cup [a^\perp, a^{\perp\perp}] \cup \{1\}$ is isomorphic $\mathcal{R}_2$ or $\mathcal{R}_5$. So assume $a^\perp \not\le a$. Observe $(\sigma_i a)^\perp = a^\perp$ and define $r_i = r(\sigma_i a) = a^\perp \sigma_i a$ for $i \le \alpha$ and $r_{\alpha+1} = ra$. From the axioms of an α-quadratic lattice we get $\sigma_i r_i = r_i$, immediately. Let $\mathcal{S}$ be the associated side lattice as in Thm. 24.

Now, assume that $a + a^\perp = a^{\perp\perp}$ - in view of Lemma 23 it suffices to consider such $\mathcal{L}$. Let $\tilde{\mathcal{S}}$ be the subset of $\mathcal{L}$ consisting of $0, 1$, $\mathcal{S}$, $\mathcal{S} + a = \{x + a | x \in \mathcal{S}\}$, and $A = \{r_i^\perp | 0 \le i \le \alpha + 1\}$. By modularity, $\mathcal{S} \cup (\mathcal{S} + a)$ forms a sublattice isomorphic to $\overline{S}$ and it holds $a + r_i = \sigma_i a$. For $x \in \mathcal{S}$ it follows $\sigma_i(a + x) = \sigma_i a = \sigma_i x = a + r_i = \sigma_i x \in \mathcal{S} + a$ if $i > 0$ and $a^{\perp\perp} \le \sigma_0(a + x) \le \sigma_0(a + a^\perp) = \sigma_0 a^{\perp\perp} = a^{\perp\perp}$ using the hypothesis. So this sublattice is closed under the σ_i. The chain A is glued on top of $\overline{S}$ and consists entirely of closed elements, so the union is closed under the σ_i. Also, $x^\perp = (\sigma_0 r_i)^\perp \in A$ and $(x + a)^\perp = a^\perp$ for $x \in \mathcal{S}$ showing that $\tilde{\mathcal{S}}$ is closed under $^\perp$. Moreover, $\tilde{\mathcal{S}}$ is generated by a and c under these operations which means that $\mathcal{L} = \tilde{\mathcal{S}}$. Suppose $\mathcal{L}$ to be subdirectly irreducible and not isomorphic $\mathcal{R}_4$. First, assume $a \not\le a^\perp$. If $g > 0$ (equivalently, $g^\perp < 1$) then there is an i such that $g = cr_{\alpha+1}$ is not σ_i-closed. Namely, otherwise one had the congruence identifying just 0 with g and 1 with $g^\perp$ which intersects trivially with the congruence collapsing the interval $[g, g^\perp]$. This means that in $\hat{\mathcal{S}}$ a 0 and 1 have been added. In contrast, if $g = 0$ then it is closed for all σ_i. So in both cases we have $\mathcal{L}$ isomorphic $\hat{\mathcal{S}}$. If $a \le a^\perp$, then $a = r_{\alpha+1} \in \mathcal{S}$ whence $\mathcal{S} = \mathcal{S} + a$ and $\mathcal{L} = \mathcal{S} \cup A$. Also, $r_i = \sigma_i a$ whence $\sigma_i r_j = r_i$ for all $i \le j$. So we have $\mathcal{L}$ isomorphic $\hat{\mathcal{S}}$, too.

Now, let $\mathcal{S}$ be given and $\mathcal{L} = \hat{\mathcal{S}}$. For a congruence θ of $\mathcal{S}$ let Q consist of all quotients transitively composed from the x/y, $y^\perp/x^\perp$, and $a+x/a+y$ with $x/y \in \theta$. The Q is the set of quotients of a congruence of $\hat{\mathcal{S}}$. Namely $x^{\perp\perp}/y^{\perp\perp} = \sigma_0 x/\sigma_0 y \in \theta$ and $\sigma_i(a+x)/\sigma_i(a+y) = a + \sigma_i x/a + \sigma_i y$, and $(a + y)^\perp = (a + x)^\perp$ if $a \ne r_{\alpha+1}$. Of course, this congruence is $\hat{\theta}$ and $\theta = \hat{\theta}|\mathcal{S}$.

On the other hand, if τ is a proper congruence of $\hat{\mathcal{S}}$ then $\tau|\mathcal{S}$ is proper, too. Namely, let x/y be a prime quotient of τ not transposed to a quotient of $\mathcal{S}$. We have to provide a proper quotient of $\tau|\mathcal{S}$. If x/y is in A then $y^\perp/x^\perp$ is in τ and proper since x and y are closed. If y is an additional 0, then $x = g$ and $\sigma_i x > x$ for some i, so $\sigma_i x/x$ will do. Otherwise, we are in Case 1a) and have x/y transposed to $a/r_{\alpha+1} \in \tau$ and $\sigma_i r_j < r_i$ for some $i < j$. It follows $\sigma_i a/a + \sigma_i r_j \in \tau$ whence $r_i/\sigma_i r_j \in \tau$ by modularity.

Thus, if $\mathcal{S}$ is subdirectly irreducible with critical quotient p/q then p/q is in every proper congruence of $\hat{\mathcal{S}}$ and is critical in $\hat{\mathcal{S}}$, too. Conversely, let $\hat{\mathcal{S}}$ be subdirectly irreducible with minimal congruence μ; in particular, $\mu|\mathcal{S}$ is proper. Consider any proper $\theta \subset \mu|\mathcal{S}$. Then $\hat{\theta} \subset \mu$, whence $\hat{\theta} = \mu$. So, $\theta = \mu|\mathcal{S}$ is unique and the minimal congruence of $\mathcal{S}$. Finally, observe that $\hat{\mathcal{S}} = \mathcal{R}_2$ if $\mathcal{S}$ is trivial.

If x/y is a prime quotient in $\mathcal{S}$ with $x \not\leq \sigma_0 y$, then $y^\perp/x^\perp$ is a prime quotient generating the same congruence. Thus, to show the claim about critical elements it suffices to observe that neither $(1_{\mathbf{2}}, r_{\alpha+1})$ nor $r_{\alpha+1}$ may be critical in $\hat{\mathcal{S}}$.

Obviously, $\hat{\mathcal{S}}$ satisfies (14) if and only if $\mathcal{S}$ satisfies (15).

Corollary 26. *If the free α-quadratic lattice $\mathcal{F}$ with generator a is finite then its poset $\mathcal{I}_\alpha$ of join irreducibles is obtained from the poset $\mathcal{J}_\alpha$ of join irreducibles of the free 0-generated α-side-lattice just by adding the chain $\{ra\} \cup \{\sigma_i a | \alpha+1 \geq i \geq 0\} \cup \{r_i^\perp | 0 \leq i \leq \alpha+1\} \cup \{1\}$ (with all these being distinct) the element $a^\perp$ and the covering relations $ra \prec \sigma_\alpha ra$, $r_i \prec \sigma_i a$, and $r_0 a \prec a^\perp \prec r_0^\perp$. Terms representing elements of $\mathcal{J}_\alpha$ also represent the corresponding elements of $\mathcal{I}_\alpha$. Also, $\mathcal{I}_\alpha$ has just the collinearities of $\mathcal{J}_\alpha$.*

This remains true in the infinite case in the sense that the quadratic lattice is the ideal lattice of the partial lattice consisting of the side-lattice and the additional elements.

Proof. First of all, the claimed order relations are obvious by the axioms. Any α-quadratic lattice with generator a is a union of the interval $[0, a^\perp + a^{\perp\perp}]$ and the chain $\{a^\perp + a^{\perp\perp}\} \cup \{r_i^\perp | 0 \leq i \leq \alpha+1\} \cup \{1\}$ since this is a subalgebra, obviously. The elements in the chain are pairwise distinct. Indeed, $a^\perp + a^{\perp\perp} < r_0^\perp$ in $\mathcal{R}_3$ and $(ra)^\perp < 1$ in $\mathcal{R}_4$. Now, let $\mathcal{A}_{-1i} = \mathcal{D}_2$ with $r_i = 0$, $r_{i-1} = 1$, and $\sigma_j = id$ for all j. Then $\sigma_0 r_i = r_i < r_{i-1} = \sigma_0 r_{i-1}$ in $\mathcal{A}_{-1i}$ for $\alpha+1 \geq i \geq 1$ whence by construction $r_{i-1}^\perp = (\sigma_0 r_{i-1})^\perp < (\sigma_0 r_i)^\perp = r_i^\perp$ in $\hat{A}_{-1i}$. In particular, all the elements in the chain, besides the first one, are join irreducible.

Therefore, we have to consider the interval $[0, a^\perp + a^{\perp\perp}]$, only. The smallest preimages of the join irreducibles in $\mathcal{R}_1, \mathcal{R}_3$ and the $\mathcal{R}_{\alpha i}$ have been computed in the proof of Lemma 23: they are $a^\perp$, ra and the $\sigma_i a$ - all these being separated from their lower covers in that chain, too. With $\mathcal{R}_2$ and $\mathcal{R}_4$ we get a and ra - separated from each other and 0. By Theorem 25, all of the remaining subdirect factors are of the form $\hat{\mathcal{S}}$. Since $\mathcal{F} = \{0\} \cup [ra, a^\perp] \cup [a, 1]$ we see that the smallest preimage of $a \in \mathcal{S}$ is $a \in \mathcal{F}$. The remaining join irreducibles of $\hat{\mathcal{S}}$ are those of $\mathcal{S}$. Now, think of $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ composed from their components as described in 2.3. In view of Theorem 25, the identifications to be made between the elements under consideration are the same in both cases so the remaining elements of $\mathcal{I}_\alpha$ form a poset isomorphic to $\mathcal{J}_\alpha$ and we may use the same terms to represent them. It is only in this part that collinearities may occur; namely, the $\mathcal{R}$'s are all distributive and collinearities in an $\hat{\mathcal{S}}$ happen inside $\mathcal{S}$, only. Excess relations are excluded by $a^{\perp\perp} \not\leq a^\perp$, $a \not\leq a^\perp$, $a^\perp \not\leq ra^{\perp\perp}$ - all witnessed by $\mathcal{R}_1$ and $\mathcal{R}_2$.

Actually, we may consider the free 0-generated α-side-lattice as a subalgebra $\{ra\} \cup [\sigma_\alpha ra, r_0]$ of F and $\mathcal{J}_\alpha$ a subset of $\mathcal{I}_\alpha$.

2.9 Arfian lattices

In the classification of subspaces in quadratic spaces one needs an additional unary operation κ satisfying

$$\kappa x \leq rx, \ \ rx \leq ry \Rightarrow \kappa x = rx \cap \kappa y.$$

An α-quadratic lattice with such an operation will be called α-arfian if it satisfies the finiteness conditions

$$\sigma_j u = u + \sigma_j v, \ j = 0, \dots, \alpha$$

for all u/v obtained from some $rx/\kappa x$ under weak projectivity in the α-quadratic lattice.

One easily derives $\kappa x = \kappa rx$ and $\kappa(x^\perp) = \kappa(x^{\perp\perp})$. The defining implication can be equivalently replaced by the identity $\kappa(rx \cap ry) = rx \cap \kappa y$. So one has a variety.

Proposition 27. *For each quadratic space (E, Q) over a commutative field k with char$(k) = 2$ and $[k : k^2] < \aleph_0$, the α-quadratic lattice $\mathcal{L}(E)$ is turned into an α-arfian lattice when endowed with*

$$\kappa X = \{x \in X \cap X^\perp \,|\, Q(x) = 0\}.$$

Proof. The condition $[k : k^2] < \aleph_0$ ensures that all indices dim $rX/\kappa X$ are finite. Now use Prop. 17 and Lemma 19 in 2.7.

Consider the relations

$$\sigma_j u = u + \sigma_j v, \ j = 0, \dots, \alpha \tag{16}$$

where u/v is obtained from r_0/c under weak projectivity in the α-side lattice generated by $c \leq r_0$. Observe, that this implies (15) and even $\sigma_j r_i = r + i + \sigma_j(cr_i)$ for all i, j. If the side lattice is that derived from a quadratic lattice then, in view of Theorem 25 in 2.8, these relations imply (14).

Proposition 28. *The α-arfian lattices with generator a are just the α-quadratic lattices with generators a and $c \leq ra^{\perp\perp}$ satisfying (16). And the arfian congruences are just the quadratic ones. The correspondence is given by*

$$c = \kappa r(a^{\perp\perp}), \ \ \kappa x = c \cap rx.$$

Proof. Given an α-arfian lattice $\mathcal{A}$ with generator a let $\mathcal{L}$ be the α-quadratic lattice generated by a and $c = \kappa r(a^{\perp\perp})$. The finiteness conditions yield (16) and (14). Thus, $\mathcal{L}$ splits into intervals according to Thm. 24 in 2.8 and it follows $x^\perp \leq ra^{\perp\perp}$ for each $x \in \mathcal{L}$ whence $\kappa x = c \cap rx \in \mathcal{L}$. Therefore, $\mathcal{A} = \mathcal{L}$. The converse transition is obvious. Now, if θ is a congruence of the quadratic lattice and $x\theta y$, then $rx\theta ry$, whence $\kappa x\theta\kappa y$.

With Theorem 25 in 2.8 we conclude

COROLLARY 29. *The subdirectly irreducible α-arfian lattices with generator a are precisely the ones obtained from $\mathcal{R}_k$, $1 \le k \le 5$, $\mathcal{R}_{\alpha i}$, and the $\hat{\mathcal{S}}$ where $\mathcal{S}$ is a subdirectly irreducible α-side-lattice with generator $c \le r_0$ satisfying (16).*

3. Mapping theorems for alternate spaces

3.1 Isometries in countable dimensions

Step C of the lattice method requires the construction of isometries from lattice isomorphisms. Here preservation of indices is an obvious necessary condition. Simple counterexamples [Wil87,p.39] show that also all joins must be preserved. Moreover, the constructions below require that all meets be preserved. This motivates the next definition.

A class $\mathbf{C}$ of quadratic lattices is *isometrically induced* for a class of spaces if for any (nd) isometric spaces E and E' in the class the following holds: Let $\mathcal{V}$ and $\mathcal{V}'$ be quadratic lattices belonging to $\mathbf{C}$, which happen to be complete sublattices of $\mathcal{L}(E)$ respectively $\mathcal{L}(E')$. If $\eta : \mathcal{V} \to \mathcal{V}'$ is an index preserving isomorphism of quadratic lattices, then there is an isometry φ of E onto E' with $\eta(A) = \varphi(A)$ for all A in $\mathcal{V}$. In the sequel it shall be convenient to put $A' := \eta(A)$ for all $A \in \mathcal{V}$.

Considering 0-1-lattices, vector spaces, and linear isomorphisms yields the concept of being *linearly induced*. Here, of course it suffices to have E, E' over the same division ring since then isomorphy follows from the indices.

THEOREM 30. *Finite distributive quadratic lattices are isometrically induced for alternate diagonal spaces of dimension $\le \aleph_0$.*

PROOF. Typically, this requires the construction of a nested family $\varphi_\iota : X_\iota \to X'_\iota$ $(\iota \in J)$ of partial isometries such that

$$(17) \qquad \bigcup_{\iota \in J} X_\iota = E \text{ and } \bigcup_{\iota \in J} X'_\iota = E',$$

as well as

$$(18) \qquad \varphi_\iota(A \cap X_\iota) = A' \cap X'_\iota \quad \text{for all} \quad A \in \mathcal{V} \text{ and } \iota \in J.$$

The subsequent construction also requires for all $\iota \in J$ the satisfaction of the following conditions (we drop the subscripts on X_ι and X'_ι):

$$(19) \qquad \bigcap_{\mu \in I}(X + A_\mu) = X + \bigcap_{\mu \in I} A_\mu \text{ for all families } \{A_\mu | \mu \in I\} \subseteq \mathcal{V},$$

$$(19') \qquad \bigcap_{\mu\in I}(X'+A'_\mu) = X' + \bigcap_{\mu\in I} A'_\mu \text{ for all families } \{A'_\mu|\mu\in I\}\subseteq \mathcal{V}'.$$

To fulfil (17) choose any base $B := (w_i|i\in J)$ of E respectively $B' := (w'_i|i\in J)$ of E' and ensure that for all $\iota\in J$ one has $\langle w_i|i\leq\iota\rangle\subseteq X_\iota$ and $\langle w'_i|i\leq\iota\rangle\subseteq X'_\iota$. Thus we are left with the following *key problem*: Given an isometry $\varphi: X\to X'$ satisfying (18), (19), (19′) and any vector $w\in E-X$, find an isometry $\widetilde{\varphi}: X\oplus W\to X'\oplus W'$ with $w\in X\oplus W$ that satisfies again (18), (19), (19′).

Let us first show how to define W and W' in order to get a *linear* map $\widetilde{\varphi}$ of the above kind. Consider the set $M(w,X) := \{A\in\mathcal{V}|\ w\in X+A\}$ of all elements $A\in\mathcal{V}$ that "catch" the vector w. Because of (19) the set $M(w,X)$ is a filter in $\mathcal{V}$ which is generated by one element $D(w,X)\in M(w,X)$. By switching from $w = x+w_0\in X+D(w,X)$ to the component w_0 we can assume that $w\in D(w,X)$. If $D := D(w,X)$ is a join reducible element of $\mathcal{V}$, say $D = D_1+D_2$ with $D_1,D_2\subset D$ from $\mathcal{V}$, decompose w in $w = w_1+w_2\in D_1+D_2$ and try to adjoin w_1, w_2. Since $\mathcal{V}$ is finite, it suffices to solve the key problem for vectors w with a join irreducible $D(w,X)$ (this is shown below for more general lattices $\mathcal{V}$). So suppose that $D = D(w,X)$ is join irreducible. Then $w\in D-(X+D_*)$; recall that P_* denotes the unique lower cover of a join irreducible $P\in\mathcal{V}$. We claim that for any vector $w'\in D'-(X'+D'_*)$ the linear map $\widetilde{\varphi}: X\oplus W\to X'\oplus W'$ defined by $W := \langle w\rangle$, $W' := \langle w'\rangle$, $\widetilde{\varphi}|X := \varphi$, $\widetilde{\varphi}(w) := w'$, satisfies again (18), (19), (19′). Observe that $(X\oplus W)\cap A = (X\cap A)\oplus W$ for all $A\in M(w,X)$, and $(X\oplus W)\cap A = X\cap A$ for all $A\in\mathcal{V}-M(w,X)$. Since the analogous equations hold on the right side, $\widetilde{\varphi}$ satisfies (18). That also (19) and (19′) are perpetuated shall be proven in Theorem 33; here the *distributivity* of $\mathcal{V}$ is crucial.

Now let us refine the above argument in order to get an *isometric* extension $\widetilde{\varphi}$ of φ. Because E, E' are alternate the equation $\Phi(w,w) = \Phi(w',w')$ is trivially satisfied. Thus the linear extension $\widetilde{\varphi}: X\oplus\langle w\rangle\to X'\oplus\langle w'\rangle$ is an isometry whenever

$$(20) \qquad \varphi(x,w) = \Phi(\varphi(x),w') \quad \text{for all } x\in X.$$

To find an appropriate w' with (20) we invoke the so called "Kaplansky-Lemma" [Gro79, p.25]:

(21) Let $g_1,\dots,g_n$ be linearly independent vectors in a sesquilinear space E and let $Y\subseteq E$ be a subspace. Then for any choice $\alpha_1,\dots,\alpha_n\in k$ the condition $Y^\perp\cap\langle g_1,\dots,g_n\rangle = \langle 0\rangle$ is sufficient for the existence of $y\in Y$ with $(\forall 1\leq i\leq n)\ \Phi(y,g_i) = \alpha_i$.

Because $\varphi: X\to X'$ satisfies (18) one has in particular $\varphi(X\cap D^\perp) = X'\cap D'^\perp$. Let $f_1,\dots,f_n$ be linearly independent vectors such that $X = (X\cap D^\perp)\oplus\langle f_i|1\leq i\leq n\rangle$. Setting $g_i := \varphi f_i$ one concludes $X' = (X'\cap D'^\perp)\oplus\langle g_i|1\leq i\leq n\rangle$. By (21) there is a $w' \in D'$ with $\Phi(w', g_i) = \Phi(w, f_i)$ for all $1 \leq i \leq n$. Hence for all $f \in X$ we have $f = a + b \in (X \cap D^{\perp}) \oplus \langle f_i | 1 \leq i \leq n \rangle$, $\varphi f = \varphi a + \varphi b \in (X' \cap D'^{\perp}) \oplus \langle g_i | 1 \leq i \leq n \rangle$ and $\Phi(f, w) = \Phi(b, w) = \Phi(\varphi(b), w') = \Phi(\varphi(f), w')$ which is (20).

Unfortunately, it may happen that $w' \in D'$ also lies in $X' + D'_*$. In order to move w' out of $X' + D'_*$ without destroying property (20), replace w' by $w' + t$ where t comes from $(D' \cap X'^{\perp}) - (X' + D'_*)$. However, the proof that this difference set is *nonempty* is quite complicated: It follows from $\dim(D' \cap X'^{\perp} / (D' \cap X'^{\perp}) \cap (X' + D'_*)) = \dim(D \cap X^{\perp} / (D \cap X^{\perp}) \cap (X + D_*))$ [Gro79,p.116(5)] and from $(D \cap X^{\perp}) - (X + D_*) \neq \emptyset$ [Gro79, p.118].

3.2 Isometries in uncountable dimensions

Theorem 33 and Theorem 35 below are quite sophisticated generalizations of Theorem 30. The more technical arguments are comprised in four Lemmas whose proofs are deferred to section 3.3.

First some introductionary remarks on Theorem 33 are in place. We shall admit spaces E, E' of dimension $\aleph_\alpha > \aleph_0$. Then the spaces X_ι, X'_ι of the partial isometries $\varphi_\iota : X_\iota \to X'_\iota$ $(\iota < \omega_\alpha)$ are (almost all) *infinite dimensional*. This means that matching w with w' amounts to solving an infinite system of linear equations. Since E is diagonal, one fortunately can restrict oneself to partial isometries of the form

$$(22)\ \ \varphi : X_h \oplus^{\perp} U \to X'_h \oplus^{\perp} U'$$

$$E = X_h \oplus^{\perp} X_h^{\perp},\ E' = X'_h \oplus^{\perp} X_h'^{\perp},\ \dim(U) = \dim(U') < \aleph_0,$$

$$\varphi(X_h) = X'_h, \text{ and } \varphi(U) = U'.$$

Concerning the "key problem", it is then possible to apply the Kaplansky-Lemma (21) to the finite dimensional space U' and to find again a $w' \in D' - (X' + D'_*)$ with (20). However, much more difficulties than in the finite dimensional case will arise (Lemma 32).

With $\varphi(= \varphi_\iota)$ also $\widetilde{\varphi}(= \varphi_{\iota+1})$ is automatically of form (22). The problem is the construction of $\varphi_\lambda : X_\lambda \to X'_\lambda$ for *limit ordinal* $\lambda < \omega_\alpha$. Although $\varphi_\lambda := \bigcup_{\iota<\lambda} \varphi_\iota : \bigcup_{\iota<\lambda} X_\iota \to \bigcup_{\iota<\lambda} X'_\iota$ is clearly isometric and inherits (18), (19), (19'), it is generally not of form (22)! In order to overcome this problem, one becomes involved in difficult set theoretic constructions (Lemma 31).

Also we shall admit *infinite* completely distributive lattices $\mathcal{V}$ in Theorem 33. Then only *compact* join irreducibles $D \in \mathcal{V}$ have a unique antecedent $D_* \prec D$.

We now introduce the basic concepts to be used in the proof of Theorem 33. A *partitioning* of a vector space E with $\dim(E) = \aleph_\alpha > \aleph_0$ is a family $S \subseteq \mathcal{L}(E)$ such that
(23) $S = \bigcup\{S_\gamma |\ \gamma \leq \alpha\}$,
(24) $(\forall \gamma \leq \alpha)\ E = \bigoplus S_\gamma$ and $(\forall F \in S_\gamma)\ \dim(F) = \aleph_\gamma$,

(25) $(\forall\beta \leq \gamma < \alpha)\ F \in S_\gamma \ \Rightarrow F = \bigoplus\{G \in S_\beta |\ G \subseteq F\}$.

The pair (E, S) is a *partitioned space*. Call $X \subseteq E$ *homogeneous* if it is a sum of spaces from S_0, and call it a γ-*space* if it is a sum of $\leq \aleph_\gamma$ spaces from S_γ. For $X \subseteq E$ with $\dim(X) \leq \aleph_\gamma$ the space $A(X,\gamma) := \bigoplus\{F \in S_\gamma |\ \exists x \in X : x_F \neq 0\}$ is the γ-space generated by X (whereby $x = \sum\{x_F |\ F \in S_\gamma\}$ with almost all $x_F = 0$). A *partitioned subspace* X of (E, S) is a direct sum

(26) $$X = X_h \oplus U = (X_{\gamma_m} \oplus \ldots \oplus X_{\gamma_1}) \oplus U$$

of finitely many γ_i-spaces X_{γ_i} $(0 \leq \gamma_1 < \gamma_2 < \ldots < \gamma_m \leq \alpha)$ and a finite dimensional $U \subseteq E$ such that $A(U,0) \cap X_h = \langle 0\rangle$, where $X_h := X_{\gamma_m} \oplus \ldots \oplus X_{\gamma_1}$ is the homogeneous part of X. A bijective linear map

(27) $$\varphi : X := X_{\gamma_m} \oplus \ldots \oplus X_{\gamma_1} \oplus U \to X' := X'_{\gamma_m} \oplus \ldots \oplus X'_{\gamma_1} \oplus U'$$

between partitioned subspaces X and X' of the same type $(\gamma_1, \ldots, \gamma_m)$ is a *partitioned isomorphism* if $(\forall 1 \leq i \leq m)\ \varphi(X_{\gamma_i}) = X'_{\gamma_i}$ and $\varphi(U) = U'$.

For each $w \in E$ consider the decomposition $w = w_h + w_r \in X_h \oplus \bigoplus\{F \in S_0 |\ F \cap X_h = \langle 0\rangle\}$. Assume that $\bar\varphi : X \oplus W \to X' \oplus W'$ is any linear bijection with

(28) $$\bar\varphi|X = \varphi,\ \dim(W) < \aleph_0,\ \bar\varphi(W) = W',\ (\forall w \in W)\varphi(w_h) = (\tilde\varphi(w))_h.$$

Then it follows that $\bar\varphi$ is also a partitioned isomorphy: ¿From $\bar\varphi(w) = (\bar\varphi(w))_h + (\bar\varphi(w))_r$, $\bar\varphi(w) = \bar\varphi(w_h + w_r) = \varphi(w_h) + \bar\varphi(w_r)$, and $\varphi(w_h) = (\tilde\varphi(w))_h$ follows $(\forall w \in W)\ (\bar\varphi(w))_r = \bar\varphi(w_r)$. Now $X \oplus W$ and $X' \oplus W'$ decompose as $X \oplus W = X_h \oplus (U \oplus \{w_r | w \in W\})$ respectively $X' \oplus W' = X'_h \oplus (U' \oplus \{w'_r | w' \in W'\})$ and $\bar\varphi(u + w_r) = \bar\varphi(u) + \bar\varphi(w_r) = \varphi(u) + (\bar\varphi(w))_r \in U' \oplus \{w'_r | w' \in W'\}$.

As mentioned before, the *limit* map of a family of partitioned isomorphisms (even of the form $\varphi : U \to U'$) needs not to be a partitioned isomorphism. The following setup fits these set theoretical difficulties.

Let $B = (e_\iota |\ \iota < \omega_\alpha)$ respectively $B' = (e'_\iota |\ \iota < \omega_\alpha)$ be bases of two partitioned spaces (E, S) and (E', S'). A partitioned isomorphy $\varphi : X \to X'$ is (B, B')-*saturated* if either $d := \dim(X) < \aleph_0$ or if $\langle e_\iota |\ \iota < d\rangle \subseteq X$ and $\langle e'_\iota |\ \iota < d\rangle \subseteq X'$.

Let $\mathcal{F}$ be a family of partitioned isomorphisms φ between (E, S) and (E', S'). Say that $(\varphi : X \to X') \in \mathcal{F}$ satisfies the *ping pong property* if

(PP) For each $w \in E$ (or $w' \in E'$) there is a $(\bar\varphi : X \oplus W \to X' \oplus W') \in \mathcal{F}$ with $\bar\varphi \supseteq \varphi$ and $w \in dom\bar\varphi$ (or $w' \in Im\bar\varphi$).

Also $\mathcal{F}$ satisfies the *chain condition* if

(CC) For each ascending chain $\{\varphi_\iota |\ \iota < \lambda\} \subseteq \mathcal{F}$ for which $\bigcup_{\iota<\lambda}\varphi_\iota$ happens to be a partitioned isomorphy, one has $\bigcup_{\iota<\lambda}\varphi_\iota \in \mathcal{F}$.

Recall that the *cofinality* $cf(\kappa)$ of a cardinal κ is the smallest cardinal γ such that there is an unbounded subset $S \subseteq \kappa$ whose order type equals γ. For each *successor cardinal* $\kappa = \aleph_{\alpha+1}$ one has $cf(\kappa) = \kappa$. But for *limit cardinals* $\kappa = \aleph_\alpha$ (where α is a limit ordinal) one "usually" has $cf(\kappa) < \kappa$, since always $cf(\kappa) \leq \alpha$. A limit cardinal κ with $cf(\kappa) = \kappa$ is called *weakly inaccessible*. Let u_0 be the first weakly inaccessible cardinal. Weakly inaccessible cardinals are "so big" that some models of ZFC contain none of them [Kun80,p.34,133]. In such a model Lemma 31 holds without cardinality restriction on E.

Lemma 31. *Let (E, S) and (E', S') be partitioned spaces of equal dimension $\aleph_\alpha < u_0$. Fix bases $B = (e_\iota |\ \iota < \omega_\alpha)$ and $B' = (e'_\iota |\ \iota < \omega_\alpha)$ of E and E'. Assume that $\mathcal{F}$ is a family of partitioned isomorphisms which satisfies (CC) and such that each (B, B')-saturated $\varphi \in \mathcal{F}$ satisfies (PP). Then each $(\varphi_0 : X_0 \to X'_0) \in \mathcal{F}$ has an extension $(\varphi_* : E \to E') \in \mathcal{F}$.*

Assume that (E, S) is a partitioned space of dimension $\aleph_\alpha$ with an underlying alternate diagonal space (E, φ). Then S is an *orthogonal partitioning* if $(\forall \gamma \leq \alpha)\ E = \bigoplus S_\gamma$. A partitioned isomorphism which happens to be isometric is called a *partitioned isometry*.

Lemma 32. *Let E, E' be alternate diagonal spaces of dimension $\aleph_\alpha < u_0$. Let $\mathcal{V}$ and $\mathcal{V}'$ be denumberable quadratic lattices which are complete sublattices of $\mathcal{L}(E)$ respectively $\mathcal{L}(E')$, and let $\eta : \mathcal{V} \to \mathcal{V}'$ be an index preserving isomorphism. Then there are orthogonal partitionings S of E and S' of E', as well as bases B of E and B' of E', such that the following (and its dual) hold:*
If $D' \in \mathcal{V}'$ is compact and join-irreducible, then for each (B, B')-saturated partitioned isometry $\varphi : X \to X'$ and each $w' \in D' - (X' + D'_*)$ there is a $w \in D - (X + D_*)$ satisfying (20) and such that $\varphi(w_h) = w'_h$.

Theorem 33. *Countable completely distributive quadratic lattices are isometrically induced for alternate diagonal spaces of dimension $< u_0$*

Proof. Corresponding to our denumberable completely distributive lattices $\mathcal{V} \subseteq \mathcal{L}(E)$, $\mathcal{V}' \subseteq \mathcal{L}(E')$ choose orthogonal partitionings S, S' and bases B, B' as in Lemma 32. Consider the family

$$\mathcal{F} := \{\varphi : X \to X' |\ \varphi \text{ is partitioned isometry with (18), (19), (19')}\}.$$

Clearly $\mathcal{F}$ satisfies (CC), independent from our particular choice of S, S'. We shall show that each (B, B')-saturated $\varphi \in \mathcal{F}$ satisfies (PP). Then Lemma 31, applied to $(\varphi_0 : \langle 0 \rangle \to \langle 0 \rangle) \in \mathcal{F}$, finishes the proof.

So pick a (B, B')-saturated $(\varphi : X \to X') \in \mathcal{F}$ and any $w \in E$. Consider the set $M = M(w, X) := \{A \in \mathcal{V} |\ w \in X + A\}$. Since the embedding $\mathcal{V} \subseteq \mathcal{L}(E)$ is $\bigcap$-preserving, it follows from (19) that M is a principal filter generated by $D(w, X) := \bigcap M \in \mathcal{V}$. As in the proof of Theorem 30 we may assume $w \in D$. Next show that $D \in \mathcal{V}$ is compact:

Consider any subset $S \subseteq \mathcal{V}$ with $D \leq \sum_{\mathcal{V}} S$. One has $\sum_{\mathcal{V}} S = \sum S$ since the embedding $\mathcal{V} \subseteq \mathcal{L}(E)$ is $\sum$-preserving. Whence $w \in \sum_{\mathcal{V}} S$ implies $w \in \sum S_0$ for some finite $S_0 \subseteq S$, i.e. $D \leq \sum_{\mathcal{V}} S_0$.

First case: D is join-irreducible. Being compact it has a unique antecedent $D_* \prec D$ and clearly $w \in D - (X + D_*)$. By Lemma 32 there is a $w' \in D' - (X' + D'_*)$ with $\varphi(w_h) = w'_h$ and $\Phi(x, w) = \Phi(\varphi(x), w')$ for all $x \in X$. Put $W := \langle w \rangle$, $W' := \langle w' \rangle$, and define $\bar{\varphi} : X \oplus W \to X' \oplus W'$ by $\bar{\varphi}|X := \varphi$ and $\bar{\varphi}(w) := w'$. Then $\bar{\varphi} \supseteq \varphi$ is isometric as well as a partitioned isomorphism (see the remarks after (28)). As in the proof of Theorem 30 one sees that $\bar{\varphi}$ still satisfies (18).

To see (19) let $\{A_\iota \mid \iota \in I\} \subseteq \mathcal{V}$ be an arbitrary family. If we should have that $A_\iota \in M$ for all $\iota \in I$ then $\bigcap_{\iota \in I}((X \oplus W) + A_\iota) = \bigcap_{\iota \in I}(X + A_\iota) = X + \bigcap_{\iota \in I} A_\iota = (X \oplus W) + \bigcap_{\iota \in I} A_\iota$, using the fact that φ satisfies (19). Assume therefore that $A_0 \notin M$ for at least one index $0 \in I$. Let $v = x_\iota + \lambda_\iota w + a_\iota$ $(\iota \in I)$ be an arbitrary vector of $\bigcap_{\iota \in I}((X \oplus W) + A_\iota)$. If $A_\mu \notin M$ for some $\mu \in I$ then $\lambda_\mu = \lambda_0$ for otherwise we could solve for w and obtain $w \in X + (A_0 + A_\mu)$; but this contradicts (i) of 2.2(c) whereby D is join prime. Thus $\lambda_\mu = \lambda_0$ for all $\mu \in I$ with $A_\mu \notin M$. This implies $v - \lambda_0 w \in \bigcap_{\iota \in I}(X + A_\iota) = X + \bigcap_{\iota \in I} A_\iota$, i.e. $v \in (X \oplus W) + \bigcap_{\iota \in I} A_\iota$. The other inclusion $(X \oplus W) + \bigcap_{\iota \in I} A_\iota \subseteq \bigcap_{\iota \in I}((X \oplus W) + A_\iota)$ is trivial. The proof of $(19')$ is mutatis mutandis the same. Hence $\bar{\varphi} \in \mathcal{F}$, i.e. φ satisfies (PP).

Second case : D is join-reducible. Being compact, it is by (ii) of 2.2(c) a finite sum of $m > 1$ join irreducibles $D_1, \ldots, D_m \in \mathcal{V}$. Choose a representation where m is minimal and write w as sum of components $w_i \in D_i$. Putting $W_0 := \langle 0 \rangle$, $W_i := \langle w_1, \ldots, w_i \rangle$, one has $D(w_i, X + W_{i-1}) \subseteq D(w_i, X) \subseteq D_i$ for all $1 \leq i \leq m$. Clearly $D(w_i, X + W_{i-1}) + \sum_{j \neq i} D_j \in M(w, X)$. Hence $D(w_i, X + W_{i-1}) + \sum_{j \neq i} D_j = D_i + \sum_{j \neq i} D_j$, which yields $D(w_i, X + W_{i-1}) + (D_i \cap \sum_{j \neq i} D_j) = D_i$ by modularity. Therefore $D(w_i, X + W_{i-1}) = D_i$ by the join-irreducibility of D_i and the minimality of m. The $D(w_i, X + W_{i-1})$ being join-irreducible one concludes by the first case that there is a partitioned isometry $(\bar{\varphi} : X \oplus W_m \to X' \oplus W'_m) \in \mathcal{F}$ with $\bar{\varphi} \supseteq \varphi$ and $w \in dom(\bar{\varphi})$. Thus φ satisfies (PP).

Corollary 34. *Completely distributive lattices are linearly induced for all vector spaces.*

Proof. In the affine case the machinery of partitioned spaces is superfluous, and thereby the dimension bound u_0 inflicted by Lemma 31. Also the restriction that $\mathcal{V}$ be denumberable was only used for "quadratic" purposes (in Lemma 32).

In Theorem 35 below the distributivity assumption of Theorem 33 will be dropped. Recall from 2.2(e) that an Artinian-acyclic lattice L is defined as an Artinian complete lattice which is a (canonical) subdirect product $L \to \Pi_{\theta \in \Theta} L/\theta$ of finite simple acyclic lattices L/θ. It is said to have *property* Δ if for each $\theta \in \Theta$ of L either $L/\theta \simeq D_2$ or L/θ admits a triangle sequence whose α_θ-image $\{P_0, Q_1, P_1, \ldots, Q_{n-1}, P_{n-1}\} \subseteq L$ is such that for all $1 \leq i \leq n-1$ one has:

$$\varepsilon_i := \min\{\gamma | \sigma_\gamma P_i = P_i, \sigma_\gamma Q_i = Q_i, \sigma_\gamma R_i = R_i\} \geq 1$$

and $\sigma_\delta(P_i \cap Q_i) \geq P_i$ for all $\delta < \varepsilon_i$. In particular, this means that not all three spaces P_i, Q_i, R_i are $\perp\perp$-closed. This is essential in view of the counterexample in [Wil87,p.50] or [Wil91,p.395].

In Theorem 33 we had completely distributive lattices. They need *not* be Artinian, but if they are then condition Δ is trivially satisfied. Insofar the next result generalizes Theorem 33.

THEOREM 35. *Countable Artinian-acyclic quadratic lattices with property Δ are isometrically induced for alternate diagonal spaces of dimension $< \aleph_0$.*

PROOF. We choose orthogonal partitionings S, S' and bases B, B' as in Lemma 32. Again it suffices to show that for

$$\mathcal{F} := \{\varphi : X \to X' |\ \varphi \text{ is partitioned isometry with (18), (19), (19')}\}$$

each (B, B')-saturated $(\varphi : X \to X') \in \mathcal{F}$ satisfies (PP). So pick any $w \in E$. As in the proof of Theorem 33 the set $M = M(w, X) := \{A \in \mathcal{V} |\ w \in X + A\}$ is a principal filter of $\mathcal{V}$, generated by the compact element $D(w, X) := \bigcap M \in \mathcal{V}$. Furthermore one may assume $w \in D := D(w, X)$.

First case: D is join-irreducible. As before $w \in D - (X + D_*)$. But in contrast to Theorem 33, the required extension $(\tilde{\varphi} : X \oplus W \to X' \oplus W') \in \mathcal{F}$ of φ will generally have $n := \dim(W) = \dim(W') > 1$. We start by defining W. By 2.2(e) there is a unique maximal congruence θ of $\mathcal{V}$ with $(D_*, D) \notin \theta$ and such that $\mathcal{V}/\theta$ is a simple acyclic lattice. Because $\mathcal{V}$ is Artinian, the canonical epimorphism $\phi : \mathcal{V} \to \mathcal{V}/\theta$ admits smallest preimages $\alpha := \alpha_\theta : \mathcal{V}/\theta \to \mathcal{V}$. Put $p_0 := \phi(D)$. Then $p_0 \in J(\mathcal{V}/\theta)$ since $(D_*, D) \notin \theta$ implies $D = \alpha(p_0)$ and since α preserves joins. Choose a triangle sequence $\{p_0, q_1, p_1, \ldots, q_{n-1}, p_{n-1}\} \subseteq J(\mathcal{V}/\theta)$, i.e. for all $1 \leq i \leq n-1$ there is a $r_i \in \{p_o, \ldots, p_{i-1}\}$ with $r_i + q_i = r_i + p_i = q_i + p_i$ (2.2(d)). For a join irreducible $r \in J(\mathcal{V}/\theta)$ the corresponding capital letter R shall henceforth denote the element $\alpha(r) \in J(\mathcal{V})$. Thus $\{P_0, Q_1, P_1, \ldots, Q_{n-1}, P_{n-1}\} \subseteq J(\mathcal{V})$ is the α-image of our triangle sequence. Also $P_0 = D$ and $P_0 \subseteq Q_1 + P_1$. Accordingly decompose $w(r_1) = w(p_0) := w \in P_0$ as $w(r_1) = w(q_1) + w(p_1) \in Q_1 + P_1$. Then we may decompose $w(r_2) \in \{w(p_0), w(q_1), w(p_1)\}$ into $w(r_2) = w(q_2) + w(p_2) \in Q_2 + P_2 \supseteq R_2$ and $w(r_3) \in \{w(p_0), \ldots, w(p_2)\}$ into $w(r_3) = w(q_3) + w(p_3) \in Q_3 + P_3 \supseteq R_3$ and so on. Put $W := \langle w(p_0), \ldots, w(p_{n-1})\rangle$.

LEMMA 36. *Under the above hypotheses $X + W = X \oplus W$ and $w(p_0), \ldots, w(p_{n-1})$ is a base of W. Furthermore $X \oplus W$ satisfies (19), as well as $(X \oplus W) \cap A = (X \cap A) \oplus (W \cap A)$ for all $A \in \mathcal{V}$. More specifically $W \cap A = \langle w(r) |\ r \in J(\mathcal{V}/\theta), r \leq \phi(A)\rangle$.*

In the proof of Lemma 36 it is crucial that $w(p_o) \in P_0 - (X + P_{0*})$. By Lemma 32 we know that also on the "right side" $P'_0 - (X' + P'_{0*}) \neq \emptyset$. Pick a $w'(p_0) \in P'_0 - (X' + P'_{0*})$ and decompose it as before along the sequence $\{P'_0, Q'_1, P'_1, \ldots, Q'_{n-1}, P'_{n-1}\} \subseteq J(\mathcal{V}')$. Hence $W' := \langle w'(p_0), \ldots, w(p'_{n-1})\rangle$ satisfies an analogous Lemma 36$'$. Define a linear bijection $\bar\varphi : X \oplus W \to X' \oplus W'$ by $\bar\varphi|X := \varphi$ and $\bar\varphi(w(p_i)) := w'(p_i)$ for $p_0, \ldots, p_{n-1} \in J(\mathcal{V}/\theta)$. Then clearly $\bar\varphi(w(r)) = w'(r)$ for all $r \in J(\mathcal{V}/\theta)$. This together with Lemma 36, 36$'$ and the fact that φ satisfies (18), implies that $\bar\varphi((X \oplus W) \cap A) = \bar\varphi((X \cap A) \oplus (W \cap A)) = (X' \cap A') \oplus \bar\varphi(\langle w(r)|\ r \in J(\mathcal{V}/\theta), r \leq \phi(A)\rangle) = (X' \cap A') \oplus \langle w'(r)|\ r \in J(\mathcal{V}/\theta), r \leq \phi(A)\rangle = (X' \cap A') \oplus (W' \cap A') = (X' \oplus W') \cap A'$ for all $A \in \mathcal{V}'$. Thus (18) is perpetuated by $\bar\varphi$.

Lemma 37. *The vectors $w'(p_i)$ in Lemma 36$'$ can be chosen in such a way that the above $\bar\varphi$ is a partitioned isometry.*

It follows that $\bar\varphi \in \mathcal{F}$, i.e. $\bar\varphi$ is the required extension of φ with $w \in dom(\bar\varphi)$.

Second case: D is join-reducible. Since $\mathcal{V}$ is Artinian we may use induction to show the following (and its dual): For all $D \in \mathcal{V}$, if $D = D(X, w)$ for some (B, B')-saturated $(\varphi : X \to X') \in \mathcal{F}$ and $w \in E$, then there is a $\bar\varphi \in \mathcal{F}$ with $\bar\varphi \supseteq \varphi$ and $w \in dom(\bar\varphi)$. The claim trivially holds for $D := \langle 0\rangle$. Now let $D \in \mathcal{V}$ be arbitrary. By the first case we may assume that D is join-reducible. Decompose it as $D = D_1 + D_2$ with $D_1, D_2 \in \mathcal{V}$ and $D_1, D_2 \subset D$. Accordingly put $w = w_1 + w_2 \in D_1 + D_2$. Since $D(w_1, X) \subseteq D_1$, induction yields a (trivially (B, B')-saturated) $(\varphi_1 : X_1 \to X'_1) \in \mathcal{F}$ with $\varphi_1 \supseteq \varphi$ and $w_1 \in dom(\varphi_1)$. Because of $D(X_1, w_2) \subseteq D(X, w_2) \subseteq D_2$ induction also yields a $\bar\varphi \in \mathcal{F}$ with $\bar\varphi \supseteq \varphi_1 \supseteq \varphi$ and $w_2 \in dom(\bar\varphi)$. Hence also $w \in dom(\bar\varphi)$.

Corollary 38. *Artinian-acyclic lattices are linearly induced for all vector spaces.*

3.3 Proofs of the Lemmas

Proof of Lemma 31. In the case $\dim(E) = \aleph_\alpha < \aleph_{\omega_1}$ Lemma 31 was proven in [Gro82]. The cardinal $\aleph_{\omega_1}$ has the enjoyable property that each limit ordinal $\lambda < \aleph_{\omega_1}$ has cofinality ω. In [Wil87,Satz 33] the upper bound $\aleph_{\omega_1}$ was raised to u_0 (however,the first weak Mahlo cardinal, postulated in [Wil87,Satz 33], seems to be a too optimistic upper bound).

Proof of Lemma 32. We first establish two preliminaries (a) and (b).

(a) There are orthogonal partitionings S of E and S' of E' such that

(29) $(\forall\gamma \leq \alpha)(\forall A \in \mathcal{V})\ A = \sigma_{\gamma+1}(A) \Rightarrow A = \bigoplus\{A \cap G|\ G \in S_\gamma\}$

(29)' $(\forall\gamma \leq \alpha)(\forall A' \in \mathcal{V}')\ A' = \sigma_{\gamma+1}(A') \Rightarrow A' = \bigoplus\{A' \cap G|\ G \in S'_\gamma\}$.

Proof of (a). Since (E, φ) is diagonal, there is an orthogonal decomposition $E = \bigoplus S_{0,0}$ with $\dim(G) \leq \aleph_0$ for all $G \in S_{0,0}$. We mention that if $A \subseteq E$ happens to be decomposed, i.e. $A = \bigoplus\{A \cap G|\ G \in S_{0,0}\}$, then A is *necessarily* σ_1-closed: Indeed, pick a $G \in S_{0,0}$ with $x_G \notin A \cap G$. Then for each $x = \sum\{x_F|\ F \in S_{0,0}\} \in E - A$ the set $x + G^\perp$ yields a σ_1-neighbourhood of x disjoint from A. Conversely, a σ_1-closed $A \subseteq E$ needs not to be decomposed with respect to $E = \bigoplus S_{0,0}$. But there is a coarser orthogonal decomposition $S_{0,1}$ (i.e. $(\forall F \in S_{0,0})(\exists G \in S_{0,1})\ F \subseteq G$), such that $(\forall G \in S_{0,1})\ \dim(G) \leq \aleph_0$ and $A = \bigoplus\{A \cap G|\ G \in S_{0,1}\}$. This is the content of the "Gattersäge-Lemma" in [Bän77]. Another σ_1-closed $B \in \mathcal{V}$ might not yet be decomposed by $E = \bigoplus S_{0,1}$. Choose a coarser orthogonal decomposition $E = \bigoplus S_{0,2}$ which decomposes B (and a fortiori A), etc. Put $S_0 := lim_{\iota<\omega} S_{0,\iota}$. Analogously define $S_1, S_2, \ldots$ by transfinite induction (more details in [Wil87,p.59]). Instead of $|\mathcal{V}| \leq \aleph_0$, here only the weaker restrictions $|\{A \in \mathcal{V}|\sigma_{\gamma+1}(A) = A\}| \leq \aleph\gamma$ $(\gamma \leq \alpha)$ must be met.

(b) There are bases B of E and B' of E' such that for all (B, B')-saturated partitioned isomorphisms $\varphi : X \to X'$ with $\aleph_0 \leq d := \dim(X)$ one has:

(30) $D \in \mathcal{V}$ *compact and join-irreducible and* $\dim(D/D_*) \leq d \Rightarrow D \subseteq D_* + X$,

(30)' $D' \in \mathcal{V}'$ *compact and join-irreducible and* $\dim(D'/D'_*) \leq d \Rightarrow D' \subseteq D'_* + X'$.

Proof of (b). Define $\mathcal{V}_{<0} := \{D \in \mathcal{V}|\ D$ compact, join-irreducible, and $\dim(D/D_*) < \aleph_0\}$ and $\mathcal{V}_\gamma := \{D \in \mathcal{V}|\ D$ compact, join-irreducible, and $\dim(D/D_*) = \aleph_\gamma\}$ and similarly $\mathcal{V}'_{<0}$, $\mathcal{V}'_\gamma \subseteq \mathcal{V}'$ $(\gamma \leq \alpha)$. By hypothesis $|\mathcal{V}|, |\mathcal{V}'| \leq \aleph_0$, but here we only need $|\mathcal{V}_{<0}|$, $|\mathcal{V}'_{<0}| \leq \aleph_0$ and $|\mathcal{V}_\gamma|$, $|\mathcal{V}'_\gamma| \leq \aleph_\gamma$ for all $\gamma \leq \alpha$. Then it is easy to find a base $B := (e_\iota \mid \iota < \omega_\alpha)$ of E such that $D \subseteq D_* + \langle e_\iota|\iota < \omega\rangle$ for all $D \in \mathcal{V}_{<0}$ and $D \subseteq D_* + \langle e_\iota|\iota < \omega_\gamma\rangle$ for all $D \in \mathcal{V}_\gamma$ and all $\gamma < \alpha$ (more details in [Wil87,p.58]). The base B' of E' is defined analogously. Let $\varphi : X \to X'$ be a (B, B')-saturated isomorphism with $\aleph_0 \leq d := \dim(X)$ and $D \in \mathcal{V}_{<0}$ or $D \in \mathcal{V}_\gamma$. Since φ is (B, B')-saturated one has $\langle e_\iota|\iota < d\rangle \subseteq X$, whence $D \subseteq X + D_*$. This proves (30), and $(30)'$ is mutatis mutandis the same.

Now let $D' \in \mathcal{V}'$ be compact and join-irreducible. Consider any (B, B')- saturated partitioned isometry $\varphi : X \to X'$ and a $w' \in D' - (X' + D'_*)$. We have to exhibit a $w \in D - (X + D_*)$ with $\varphi(w_h) = w'_h$ and (20).

First case: $d := \dim(X) < \aleph_0$. Using the Kaplansky-Lemma (21) one finds as in the proof of Theorem 30 a $w \in D$ with (20) (here $w_h = w'_h = 0$). If also $w \in X + D_*$, choose a

$t \in (X^\perp \cap D) - (X + D_*)$ and replace w by $w + t$. It remains to show that $(X^\perp \cap D) - (X + D_*)$ is nonempty. For $\dim(D/D_*) < \aleph_0$ one basically proceeds as mentioned in Theorem 30 (a complete proof, using the diagonality of E, can be found in [Wil87,p.64-66]). Assume now $\dim(D/D_*) \geq \aleph_0$. One has $D \supseteq D \cap X^\perp \supseteq (D \cap X^\perp) \cap (X + D_*) = (D_* + (X \cap D) \cap X^\perp \supseteq D_* \cap X^\perp$. It follows from $\dim(D/D_* \cap X^\perp) \geq \aleph_0$ and $\dim(D/D \cap X^\perp) \leq d$ (since $\dim(E/X^\perp) = d$ by diagonality) and $\dim((D_* + (X \cap D)) \cap X^\perp / D_* \cap X^\perp) \leq d$ that $\dim(D \cap X^\perp / (D \cap X^\perp) \cap (X + D_*)) \geq \aleph_0$.

Second case: $d = \dim(X) \geq \aleph_0$. Thus φ is of form (27) with nonzero $X_{\gamma_i},\ X'_{\gamma_i}$ $(1 \leq i \leq m)$. Consider the decompositions

$$E = X_h \oplus X_h^\perp = X_h \oplus (A(U, \gamma_1) \oplus A(U, \gamma_1)^\perp)) \tag{31}$$

$$E' = X'_h \oplus X_h'^\perp = X'_h \oplus (A(U', \gamma_1) \oplus A(U', \gamma_1)^\perp)) \tag{31$'$}$$

Let $w' = w'_h + w'_o + y'$ be the decomposition of w' with respect to $(31)'$. We shall find a $w \in D - (X + D_*)$, decomposed as $w = w_h + w_0 + y$ with respect to (31), which satisfies

(i) $\varphi(w_h) = w'_h$ and

(ii) $\Phi(x_0, w_0) = \Phi(\varphi(x_0), w'_0)$ for all $x_0 \in U$.

This will imply (20): $\Phi(x, w) = \Phi(x_h, w_h) + \Phi(x_0, w_0) + \Phi(0, y) = \Phi(\varphi(x_h), w'_h) + \Phi(\varphi(x_0), w'_0) + \Phi(0, y') = \Phi(\varphi(x), w')$ for all $x = x_h + x_0 \in X_h \oplus U$.

By (29) and $(29)'$ the spaces $\sigma_{\gamma_1+1}(D)$ and $\sigma_{\gamma_1+1}(D')$ are decomposed with respect to (31) and $(31)'$. In particular $w'_h \in \sigma_{\gamma_1+1}(D') \cap X'_h$. Since φ is a partitioned isomorphism and because of (18) one has $\varphi(\sigma_{\gamma_1+1}(D) \cap X_h) = \sigma_{\gamma_1+1}(D') \cap X'_h$. Hence $w_h := \varphi^{-1}(w'_h) \in \sigma_{\gamma_1+1}(D) \cap X_h$. More specifically one can show that for $w_h = \sum_{i=1}^m w_i$ $(w_i \in X_{\gamma_i})$, respectively $w'_h = \sum_{i=1}^m w'_i$ $(w'_i \in X'_{\gamma_i})$, one has $w_i = \varphi^{-1}(w'_i)$ as well as $w_i \in \sigma_{\gamma_i+1}(D) \cap X_{\gamma_i}$ $(1 \leq i \leq m)$.

In order to define w_0 consider a decomposition $U' = (U' \cap \sigma_{\gamma_1+1}(D')^\perp) \oplus \langle f'_1, \ldots, f'_n \rangle$ with linearly independent f'_i. By the same token as above $\varphi((U \cap \sigma_{\gamma_1+1}(D)^\perp) = (U' \cap \sigma_{\gamma_1+1}(D')^\perp)$. Hence $U = (U \cap \sigma_{\gamma_1+1}(D)^\perp) \oplus \langle f_1, \ldots, f_n \rangle$ for $f_i := \varphi^{-1}(f'_i)$. Since $\sigma_{\gamma_1+1}(D)^\perp \cap \langle f_1, \ldots, f_n \rangle = \langle 0 \rangle$ the Kaplanski-Lemma (21) yields a $w_0 \in \sigma_{\gamma_1+1}(D)^\perp$ with $\Phi(f_i, w_o) = \Phi(f'_i, w'_o)$ $(1 \leq i \leq n)$. This implies (ii). Switching to the second component of w_0 with respect to decomposition (31) we may assume $w_0 \in \sigma_{\gamma_1+1}(D)^\perp \cap A(U, \gamma_1)$.

(iii) There is a $y \in A(X, \gamma_1)^\perp$ with $w_h + w_o + y \in D$.

Proof of (iii). We know that $w_h + w_0 = (w_m + \ldots + w_1) + w_0 \in (X_{\gamma_m} \oplus \ldots \oplus X_{\gamma_1}) \oplus U$ is in $\sigma_{\gamma_1+1}(D)$. First show by induction that for all $1 \leq i \leq m$ there are vectors $y_j \in A(X_{\gamma_j} \oplus \ldots \oplus U, \gamma_j) \cap A(X, \gamma_1)^\perp$ $(1 \leq j \leq i)$ with $v(i) := \sum_{j=0}^i w_j + \sum_{j=1}^i y_j \in \sigma_{\gamma_i+1}(D) \cap A(X_{\gamma_i} \oplus \ldots \oplus U, \gamma_i)$. For $i = 1$ put $y_1 := 0$. Assume that the claim holds for a fixed i. By (29) the spaces $\sigma_{\gamma_{i+1}+1}(D)$ and $\sigma_{\gamma_{i+1}}(D)$ are decomposed with respect to

$$E = E_1 \oplus E_2 \oplus E_3 := (X_{\gamma_m} \oplus \ldots \oplus X_{\gamma_{i+1}}) \oplus A(X_{\gamma_i} \oplus \ldots \oplus U, \gamma_{i+1}) \oplus A(X, \gamma_{i+1})^\perp$$

Because of $\sigma_{\gamma_i+1}(D)\cap E_2 = \sigma_{\gamma_i+1}(\sigma_{\gamma_{i+1}+1}(D)\cap E_2)$ the vector $v(i)$ is a σ_{γ_i+1}-accumulation point of $\sigma_{\gamma_{i+1}+1}(D)\cap E_2$. Hence there is a $v(i)+y_{i+1} \in (v(i)+A(X_{\gamma_i}\oplus\ldots\oplus U,\gamma_i)^\perp)\cap(\sigma_{\gamma_{i+1}+1}(D)\cap E_2)$. From $y_{i+1}\in A(X_{\gamma_i}\oplus\ldots\oplus U,\gamma_{i+1})$ and $y_{i+1}\in A(X_{\gamma_i}\oplus\ldots\oplus U,\gamma_i)^\perp$ one derives $y_{i+1}\in A(X,\gamma_1)^\perp$. Therefore $v(i+1) := w_{i+1}+(v(i)+y_{i+1})\in\sigma_{\gamma_{i+1}+1}(D)\cap A(X_{\gamma_{i+1}}\oplus\ldots\oplus U,\gamma_{i+1})$, i.e. the claim holds for $i+1$. It follows that $y^* := \sum_{i=1}^m y_i \in A(X,\gamma_m)\cap A(X,\gamma_1)^\perp$ and $\sum_{i=0}^m w_i + y^* = v(m)\in\sigma_{\gamma_m+1}(D)$. Finally pick a $v(m)+y_{m+1}\in(v(m)+A(X,\gamma_1)^\perp)\cap D$ and put $y := y^*+y_{m+1}$.

It might happen that $w := w_h + w_0 + y \in D$ also lies in $X + D_*$. Because of $D' \not\subseteq X' + D'_*$ and $(30)'$ one has $\dim(D'/D'_*) > d = \dim(X)$, whence also $\dim(D/D_*) > d$. A fortiori $\dim(D/D_*\cap X^\perp) > d$ and the same argument as in the first case yields $\dim(D\cap X^\perp/(D\cap X^\perp)\cap(X+D_*)) > d$. Thus one can pick a $t\in(D\cap X^\perp)-(X+D_*)\neq\emptyset$ and replace y by $y+t\in A(X,\gamma_1)^\perp\subseteq X^\perp$. Then $w\in D-(X+D_*)$.

Proof of Lemma 36. Recall that $\phi:\mathcal{V}\to\mathcal{V}/\theta$ admits smallest preimages $\alpha:\mathcal{V}/\theta\to\mathcal{V}$ and that $R := \alpha(r)$ for all $r\in J(\mathcal{V}/\theta)$.

$$(32)\quad (\forall r\in J(\mathcal{V}/\theta))\ w(r)\in R-(X+R_*)$$

From the proof of Theorem 35 we know that (32) holds for $r=p_0$. By induction on the triangle sequence $\{p_0,q_1,p_1,\ldots,q_{n-1},p_{n-1}\}$ it suffices to show (32) for $r=q_1$. Assume to the contrary that $w(q_1)\in X+Q_{1*}$. Then $w(p_0)=w(q_1)+w(p_1)\in(X+P_0)\cap(X+Q_{1*}+P_1) = X+(P_0\cap(Q_{1*}+P_1))$ by (19). Now $P_0\not\subseteq Q_{1*}+P_1$, since otherwise $p_0=\phi(P_0)\le\phi(Q_{1*}+P_1)=\hat{q}_1+p_1$, which is false by 2.2(d). Hence one gets the contradiction $w(p_0)\in X+(P_0\cap(Q_{1*}+P_1))\subseteq X+P_{0*}$. This proves (32).

The notation $W_c := \langle w(r)|\ r\in J(\mathcal{V}/\theta), r\le c\rangle$ for $c\in\mathcal{V}/\theta$ is convenient.

(33) $(\forall c,d\in\mathcal{V}/\theta)\ c\le d \Leftrightarrow W_c\subseteq W_d \Leftrightarrow$
$(\forall A\in\phi^{-1}(c))(\exists B\in\phi^{-1}(d))\ (X+A)\cap W\subseteq(X+B)\cap W.$

Let $c\le d$. Trivially $W_c\subseteq W_d$. Also for a given $A\in\phi^{-1}(c)$ put $B := A+\alpha(d)$. Then $B\in\phi^{-1}(d)$ and $(X+A)\cap W\subseteq(X+B)\cap W$. Conversely, let $c\not\le d$. Fix some $r\in J(\mathcal{V}/\theta)$ with $r\le c$, $r\not\le d$. Then $w(r)\in W_c\subseteq(X+\alpha(c))\cap W$. It suffices to show $w(r)\notin(X+B)\cap W\supseteq W_d$ for all $B\in\phi^{-1}(d)$. Assuming the contrary one gets $w(r)\in(X+B)\cap(X+R) = X+(B\cap R)$. From $r\not\le d$ follows $R\not\subseteq B$, whence $w(r)\in X+(B\cap R)\subseteq X+R_*$. This contradicts (32) and thereby proves (33).

¿From (33) one gets $\dim(W_c)\ge\delta(c)$ for all $c\in\mathcal{V}/\theta$. On the other hand $\dim(W)\le n=\delta(\mathcal{V}/\theta)$ by 2.2(d). Hence the inequalities are equalities and $\mathcal{V}/\theta\to\mathcal{L}(W): c\mapsto W_c$ is a rank preserving order embedding. Since $\mathcal{V}/\theta$ is *modular*, it is by 2.2(a) also join and meet preserving. Clearly $W_c\subseteq A\cap W\subseteq(X+A)\cap W$ for all $c\in\mathcal{V}/\theta$ and $A\in\phi^{-1}(c)$. The assumption $W_c\subset(X+A)\cap W$ for some $A\in\phi^{-1}(c)$ immediately leads to a contradiction in view of (33). Thus equality holds throughout. In particular $W\cap(A+B) = W_{\phi(A+B)} = W_{\phi A+\phi B} = W_{\phi A}+W_{\phi B} = (W\cap A)+(W\cap B)$ for all $A,B\in\mathcal{V}$. Putting $A := \langle 0\rangle$ one sees $X\cap W=\langle 0\rangle$, whence also $(X+A)\cap W = (X\cap W)+(A\cap W)$ for all $A\in\mathcal{V}$. By 2.2(a) this is equivalent to $(X\oplus W)\cap A = (X\cap A)\oplus(W\cap A)$ for all $A\in\mathcal{V}$.

Finally $X \oplus W$ satisfies (19): $(X \oplus W) \cap (A+B) = (X \cap (A+B)) \oplus (W \cap (A+B)) = (X \cap A)+(X \cap B)+(W \cap A)+(W \cap B) = ((X \oplus W) \cap A)+((X \oplus W) \cap B)$ for all $A, B \in \mathcal{V}$. By 2.2(a) this is equivalent to $((X \oplus W)+A) \cap ((X \oplus W)+B) = (X \oplus W)+(A \cap B)$, i.e. to (19) for *finite* index set I_0. Let $\{A_\mu | \mu \in I\} \subseteq \mathcal{V}$ be arbitrary. By 2.2(e) there is a finite $I_0 \subseteq I$ with $\bigcap_{\mu \in I} A_\mu = \bigcap_{\mu \in I_0} A_\mu$. Hence $\bigcap_I (X+A_\mu) \subseteq \bigcap_{I_0}(X+A_\mu) = X+\bigcap_{I_0} A_\mu = X+\bigcap_I A_\mu$. The other inclusion is clear.

Proof of Lemma 37. One can choose the vectors $w'(p_0), \ldots, w'(p_{n-1})$ of Lemma 36' in such a way that our given $\varphi : X = X_h \oplus U \to X' = X'_h \oplus U'$ happens to map $w(p_i)_h$ onto $w'(p_i)_h$ $(0 \leq i \leq n-1)$. This is the content of [Wil87,Lemma 31]. By (28) the extension $\bar{\varphi} : X \oplus \langle w(p_0), \ldots, w(p_{n-1}) \rangle \to X' \oplus \langle w'(p_0), \ldots, w'(p_{n-1}) \rangle$ (defined after Lemma 36') is then a *partitioned isomorphism*. Hereby $w'(p_0) \in P'_0 - (X' + P'_{0*})$ is found as in (the dual of) Lemma 32. Hence $\bar{\varphi}$ restricted to $X \oplus \langle w(p_0) \rangle$ is also isometric.

Assume that $\psi : X \oplus \langle w(p_0), \ldots, w(p_{n-1}) \rangle \to X' \oplus \langle w'(p_0), \ldots, w'(p_{n-1}) \rangle$ is a partitioned isomorphism. Call it *i-isometric* if the restriction to $X \oplus \langle w(p_0), \ldots, w(p_i) \rangle$ is isometric. By the above there is a 0-isometric ψ. By [Wil87,Lemma 32] the existence of a i-isometric $\psi : X \oplus W \to X' \oplus W'$ implies the existence of a $i+1$-isometric $\overline{\varphi} : X \oplus W \to X' \oplus \overline{W}$. Thus induction completes the proof.

The splitting of the proof into [Wil87,Lemma 31] and [Wil87,Lemma 32] seems necessary in view of the remark in [Wil87,p.70]. Also observe that condition $\triangle$ is crucial in *both* parts.

If all subdirectly irreducible factors $\mathcal{V}/\theta$ are isomorphic to $\mathcal{D}_2$ or $\mathcal{M}_3$, then condition $\triangle$ simply requires that for each $\theta \in \Theta$ with $\mathcal{V}/\theta \simeq \mathcal{M}_3$ the triangle $\alpha_\theta(J(\mathcal{V}/\theta))$ admits a permutation $\{R, Q, P\}$ such that $\sigma_{\varepsilon-1}(Q \cap P) \supseteq P$. If furthermore $\dim(E) = \aleph_0$, the topology $\sigma_{\varepsilon-1}$ equals σ_0. Since $\sigma_0(X) = X^{\perp\perp}$ the above condition reads $(Q \cap P)^{\perp\perp} \supseteq P$. A demonstration of this easy special case can be found in [Wil87,p.66] or [Wil91,p.395].

3.4 Classification of prediagonal alternate spaces

Proposition 39. *Let E be an alternate diagonal space of dimension $\aleph_\alpha$, $\alpha < \omega$, with subspace F such that $E = F^{\perp\perp}$. Then a complete set of isometry invariants for (E, F) is given by the indices*

$$\dim F, \;\dim \sigma_i F/\sigma_{i+1} F, \; i = 0, \ldots, \alpha.$$

For an alternate prediagonal space F of dimension $\aleph_\alpha$ with diagonal hull E a complete set of isometry invariants is given by the indices

$$\dim F, \;\dim \sigma_i^E F/\sigma_{i+1}^E F, \; i = 1, \ldots, \alpha.$$

Proof. The quadratic lattice $\mathcal{V}_E(F)$ is a chain

$$0 = F^\perp \subset F = \sigma_{\alpha+1} F \subset \sigma_\alpha F \ldots \subset \sigma_1 F \subset \sigma_0 F = E$$

and the first claim follows with Thm. 33. For prediagonal spaces we need the diagonal hull considered in III.3. It is the uniquely determined smallest nd diagonal extension E of F. In view of Thm.1 in [Gro79,II.2] 0- and 1- (pre-)diagonality are equivalent concepts. So the second Theorem in III.3 yields $\sigma_1 F = E$ and $\dim E = \dim F$ by Lemma 1d in 1.4. The alternating property of the form on F extends to E due to the compatiblity with filter convergence stated in Lemma 1 of [Ogg70].

Remark. It follows, in particular, that the isometry type of an alternate diagonal space is determined by the dimension (and the base field). The proposition extends to trace-valued spaces of dimension at most $\aleph_m$, $m < \omega$, with every infinite dimensional subspace being isotropic (for $m \leq 3$ use the Congruence Theorem 47 below, otherwise use Scholion 39 in [Wil87]). For quadratic spaces (in the narrow sense of 1.4) over fields of characteristic 2 it is valid if $\dim[k : k^2] < \infty$. Namely, the hypotheses of Theorem 60 in 6.1 are satisfied since $\kappa X = 0$ for all X, whence $R = \kappa R = 0$. An alternative proof is given in V.7.

4 Classification of subspaces of trace-valued spaces

4.1 The structure of the free 1-generated 3-quadratic lattice and its subdirect factors

THEOREM 40. *Every 1-generated 3-quadratic lattice is finite, acyclic, and enjoys property Δ.*

THEOREM 41. *The free 1-generated α-quadratic lattices $\mathcal{V}_\alpha(a)$, $\alpha = 1, 2$, have order diagrams according to Figure 5. For $\alpha = 3$ the poset $\mathcal{I}$ of join irreducibles is given by the terms in Table 1 with order as in Figure 6 and the indicated additional structure. $\mathcal{I}$ is closed under meets. $x \mapsto x^{\perp\perp} = \sigma_0 x$ is join preserving.*

Here, we used the abbreviations $r(x) := xx^\perp$ and $r_i = r(\sigma_i a) = a^\perp \sigma_i a$ - and read $rx^\perp = r(x^\perp)$. We denote elements of the set $\mathcal{I}$ of terms just by their labels. As a geometry on a poset, $\mathcal{I}$ carries the partial order given in Figure 6 and the collinearity relations

$$22 + 32 = 22 + 33 = 32 + 33, \;\; 23 + 35 = 23 + 36 = 35 + 36.$$

The elements of $\mathcal{I}$ which are σ_i-closed (i=0,1,2,3) are indicated in the figure by a large disk, a sqaure, a small disk, and a small circle, respectively - just $a = 54$ and $R_4 = 2$ are not σ_3-closed. For $x \in \mathcal{I}$ we obtain $\sigma_i x$ as the meet of the σ_i-closed $y \in \mathcal{I}, y \geq x$ with the exceptions $\sigma_1 22 = 33+37$ and $\sigma_1 23 = 37+38$. In view of join preservation, this completely describes the σ_i on $\mathcal{V}_3(a)$. The orthogonal can be computed using $x^\perp = (\sigma_0 x)^\perp$ and the following

$$48^\perp = 63, \; 49^\perp = 62, \; 50^\perp = 61, \; 51^\perp = 60, \; 52^\perp = 59.$$

		2	$r_4 = ra$	3	$\sigma_3 r_4$
4	$8 \cap 5$	5	$\sigma_2 r_4$	6	$26 \cap 8$
7	$48 \cap 8$	8	$r_3 = r\sigma_3 a$	9	$\sigma_2(26 \cap 8)$
10	$26 \cap 11$	11	$\sigma_2(48 \cap 8)$	12	$26 \cap 15$
13	$27 \cap 15$	14	$48 \cap 15$	15	$\sigma_2 r$
16	$26 \cap 25$	17	$27 \cap 25$	18	$28 \cap 25$
19	$48 \cap 37 \cap 25$	20	$37 \cap 25$	21	$48 \cap 25$
22	$(37 + 33) \cap 25$	23	$(48 + 37) \cap 25$	24	$49 \cap 25$
25	$r_2 = r\sigma_2 a$	26	$\sigma_1 r_4$	27	$\sigma_1(48 \cap 8)$
28	$\sigma_1(48 \cap 15)$	29	$\sigma_1(48 \cap 37 \cap 25)$	30	$37 \cap 33$
31	$(33 + 25) \cap 37$	32	$(37 + 25) \cap 33$	33	$\sigma_1(48 \cap 25)$
34	$48 \cap 37$	35	$(48 + 25) \cap 37$	36	$(37 + 25) \cap 48$
37	$\sigma_1 r_3$	38	$\sigma_1 36$	39	$48 \cap 40$
40	$\sigma_1(49 \cap 25)$	41	$48 \cap 43$	42	$49 \cap 43$
43	$\sigma_1 r\sigma_2 a$	44	$48 \cap 47$	45	$49 \cap 47$
46	$50 \cap 47$	47	$r_1 = r\sigma_1 a$	48	$\sigma_0 r_4$
49	$\sigma_0 r_3$	50	$\sigma_0 r_2$	51	$\sigma_0 r_1$
52	$r_0 = ra^{\perp\perp}$	53	$a^\perp$	54	a
55	$\sigma_3 a$	56	$\sigma_2 a$	57	$\sigma_1 a$
58	$\sigma_0 a = a^{\perp\perp}$	59	$r_0^\perp$	60	$r_1^\perp$
61	$r_2^\perp$	62	$r_3^\perp$	63	$r_4^\perp = (ra)^\perp$
64	1				

Table 1: The join-irreducible elements of $\mathcal{V}_3(a)$

Theorem 42. *There are exactly 54 subdirectly irreducible 1-generated 3-quadratic lattices, namely the lattices Q_2 and Q_{53} - Q_{59} and the 46 lattices $Q_n = \hat{S}_n$ arising from the list of subdirectly irreducible 3-side-lattices. The surjective homomorphisms between them are composed from those in the list.*

The numbering is such that $\mathcal{Q}_p$ ($\mathcal{S}_p$) has a critical element p with smallest preimage in $\mathcal{I}$ (resp. in $\mathcal{J} = \{3, \ldots, 52\}$) having label p. We explicitly state the cases where the same lattice occurs with different indexing. In the examples we list all elements of $\mathcal{Q}_p$ resp. $\mathcal{S}_p$ with their covering relations as well as the critical elements p resp. p_i if there are several ones. In the $\mathcal{Q}_p$ we also indicate the generator a and $a^\perp$ (which then yields the $x^\perp$ for the σ_0-closed x, too. Possible surjective homomorphisms are unique (since determined on the generators) so it suffices to give their domains and codomains.

The basic subdirectly irreducible quadratic lattices: $\mathcal{Q}_{53} = \mathcal{R}_1$; $\mathcal{Q}_{54} = \mathcal{R}_2$; $\mathcal{Q}_{58-i} = \mathcal{R}_{3i}$, $0 \leq i \leq 3$; $\mathcal{Q}_{59} = \mathcal{R}_3$, $\mathcal{Q}_2 = \mathcal{R}_4$, cf 2.8.

The subdirectly irreducible 3-side-lattices: In the $\mathcal{S}_t$ we indicate the values of the r_i. For convenience write $r_4 = 0$ and $r_{-1} = r_0 = 1$. The values of the σ_i are given only where $\sigma_i a = b \neq a$ and i is maximal for this a and b. $\sigma_{-1}a = b$ means $\sigma_i a = a$ for all $i \geq 0$. For parametrized families of S's we give a list of numbers associating with S the (unique) t such that $S = \mathcal{S}_t$ - varying through the parameters, lexicographically. See Figure 7.

$\mathcal{A}_{kn}$, $-1 \leq k < n \leq 4$, $1 \leq n$; 52,47,25,8,48,44,21,7,26,16,6,5,4,3. $0 = r_n \prec 1 = r_{n-1} = p = \sigma_k 0$. $\sigma_i = id$ for $i > k$,

$\mathcal{B}_{klmn}$, $-1 \leq k < l < m < n \leq 4$; 51,50,49,46,45,24,43,37,20,15,33,27,17,11,9. $0 = r_n \prec r_{n-1} = r_m \prec 1 = p = r_{m-1} = \sigma_l r_m = \sigma_k 0$, $\sigma_i = id$ for $i > l$.

$\mathcal{C}_{hklmn}$, $-1 \leq h < k < l < m < n \leq 4$; 41,34,12,14,19,10. $0 = r_n \prec r_{n-1} = r_m \prec r_{m-1} = 1 = \sigma_l r_m = \sigma_h p$, $0 \prec p = \sigma_k 0 \prec 1$. $\sigma_i = id$ for $i > l$.

$\mathcal{D}_k$, $k = 0, 1$; 42,13. $0 \prec b \prec r_{k+2} \prec 1$, $b \prec p = \sigma_k b = \sigma_{k-1} 0 \prec 1$, $1 = \bar{r}_{k+1} = \sigma_{k+1} \bar{r}_{k+2}$, $\bar{r}_{k+3} = b$ where $\bar{r}_i = r_i$ for $i < 4$ and $\bar{r}_4 := r_3 \sigma_0 0 = 7$. $\sigma_i = id$ for $i > k + 1$.

$\mathcal{E}_k$, $k = 1, 2, 3$; 29,28,18. $0 \prec r_3 \prec c \prec 1$, $0 \prec b \prec p = \sigma_1 b = \sigma_0 0 \prec 1$, $b \prec c$, $1 = r_1 = \sigma_1 r_3 = \sigma_1 c$. The k-th quotient in $T = \{r_3 \leq \sigma_2 r_3 \leq r_2 \leq r_1\}$ collapses in $\mathcal{E}_k$. $\sigma_3 = id$, $\sigma_2 = id$ for $k = 1$.

$\mathcal{S}_{22} = \mathcal{S}_{31} = \mathcal{S}_{32}$: $0 \prec r_3 \prec p_2 = \sigma_1 r_3 \prec e \prec 1$, $r_3 \prec c \prec e$, $0 \prec b \prec p_3 = \sigma_0 0 = \sigma_1 b \prec d \prec 1$, $b \prec c \prec d$, $c \prec r_2 = p_1 \prec 1 = r_1 = \sigma_1 r_2 = \sigma_1 c = \sigma_1 d = \sigma_1 e = \sigma_0 p_2 = \sigma_0 r_3$. $\sigma_i = id$ for $i > 1$.

$\mathcal{S}_{38}$: $0 \prec r_3 \prec r_2 \prec d \prec 1$, $r_3 \prec \sigma_1 r_3 \prec d \prec 1$, $r_3 \prec b \prec d$, $0 \prec e \prec \sigma_0 0 = \sigma_1 e = p \prec c \prec 1$, $e \prec b \prec c$, $r_1 = 1 = \sigma_0 r_3 = \sigma_0 p_2 = \sigma_1 r_2 = \sigma_1 b = \sigma_1 c = \sigma_1 d$. $\sigma_i = id$ for $i > 1$.

$\mathcal{S}_{23} = \mathcal{S}_{35} = \mathcal{S}_{36}$: $0 \prec r_3 \prec p_2 = \sigma_1 r_3 \prec 1$, $r_3 \prec r_2 = p_1 \prec 1$, $r_3 \prec b \prec 1$, $0 \prec p_3 = \sigma_0 0 \prec b$, $1 = r_1 = \sigma_0 r_3 = \sigma_0 p_2 = \sigma_1 r_2 = \sigma_1 b$. $\sigma_i = id$ for $i > 1$.

$\mathcal{S}_{30}$: $0 \prec r_3 \prec r_2 \prec 1$, $r_3 \prec \sigma_1 r_3 \prec 1$, $0 \prec p \prec \sigma_1 r_3$, $0 \prec b \prec r_2$, $b \prec \sigma_1 b = \sigma_0 p \prec 1$, $p \prec \sigma_1 b$, $r_1 = \sigma_0 r_3 = \sigma_0 \sigma_1 r_3 = \sigma_1 r_2 = 1$. $\sigma_i = id$ for $i > 1$.

$\mathcal{S}_{39}$: $0 \prec r_3 \prec r_2 \prec 1$, $0 \prec p = \sigma_0 0 \prec b \prec 1$, $r_3 \prec b$, $r_1 = \sigma_0 r_3 = \sigma_0 b = \sigma_1 r_2 = 1$. $\sigma_i = id$ for $i > 1$.

$\mathcal{S}_{40}$: $0 \prec r_3 \prec r_2 \prec 1 = r_1 = p = \sigma_0 r_3 = \sigma_1 r_2$. $\sigma_i = id$ for $i > 1$.

The immediate homomorphic relationships between the $\mathcal{S}_t$'s are given in the following list.

$\mathcal{B}_{klmn} \to \mathcal{A}_{kn}, n > k + 2$, $\mathcal{C}_{hklmn} \to \mathcal{A}_{hn}, n > h + 3$, $\mathcal{D}_k \to \mathcal{A}_{k-1,k+3}$, $\mathcal{D}_k \to \mathcal{A}_{k-1,k+2}$, $\mathcal{E}_k \to \mathcal{C}_{-1 0 \min(k,2) 3 4}$, $\mathcal{S}_{22} \to \mathcal{S}_8$, $\mathcal{S}_{30} \to \mathcal{S}_8$, $\mathcal{S}_{23} \to \mathcal{S}_8$, $\mathcal{S}_{22} \to \mathcal{S}_{21}$, $\mathcal{S}_{30} \to \mathcal{S}_{21}$, $\mathcal{S}_{38} \to \mathcal{S}_{23}$, $\mathcal{S}_{39} \to \mathcal{S}_{24}$, $\mathcal{S}_{40} \to \mathcal{S}_{24}$. To each such, there corresponds a homomorphism $\hat{S} \to \hat{T}$. The only immediate homomorphisms between the $\hat{S}$'s and R's are $\hat{A}_{kn} \to \mathcal{R}_4 = \mathcal{Q}_2$ for $k \geq 0$.

Proof. Inspection yields that the above are all subdirectly irreducible with critical elements as indicated. By Theorem 25 this extends to the associated quadratic lattices and together with Lemma 43 below we have the completeness of the list as stated in Thm. 42.

Also, all lattices in the list are acyclic and enjoy property Δ. Since triangles occur in the side-lattice part, only, this carries over to the associated quadratic lattices, too, proving Thm. 40. The list of homomorphisms is verified by inspection. Observe, that the image of a lattice $\hat{S}$ must be of this form, too, or $\mathcal{R}_4$ and that by Theorem 25 $\hat{S} \to \hat{T}$ iff $S \to T$. The case $\alpha = 3$ of Thm. 41 is implied by Corollary 29 and the second part of Lemma 43 below. The cases $\alpha \leq 2$ then follow factoring out the congruences generated by the $\sigma_i x/x$ with $\alpha < i \leq 3$.

4.2 Calculating the side-lattices

LEMMA 43. *The subdirectly irreducible 3-side lattices with empty generator set are exactly the lattices $\mathcal{S}_p$, $p = 3, \ldots, 52$. The poset $\mathcal{J}$ of join irreducibles of the free 3-side lattice is the subset $\{3, \ldots, 52\}$ of $\mathcal{I}$, defined above, with the inherited structure.*

PROOF. Let $\mathcal{F}_3$ be the free 0-generated 3-side-lattice, and $\mathcal{J}$ the subset $\{3, \ldots, 52\}$ of the set $\mathcal{I}$ of terms - these indeed are 3-side-lattice terms. Consider $\mathcal{J}$ with the structure inherited from $\mathcal{I}$ but neglecting $^{\perp}$.

First we have to show that the set $\mathcal{J}$ of terms satisfies all the relations claimed in Theorem 41. We build up $\mathcal{J}$ in several steps, indicating in each what is added and which laws are used, in addition to the order established so far, to verify the order relations (it suffices to consider the coverings between terms already introduced). 2,8,25,47,52; generators and bounds. 3,5,15,26,37,48-51; properties of the σ_i. 6,7,14,19, 21,24; meet semilattice axioms. 36; lattice axioms. 9,11,27-29,33,38,40; properties of the σ_i. 4,10,12,13,16-20,30,34,39,41,42,44-46; meet semilattice axioms. 22,23,32,33,35; lattice axioms. The join relations are immediate by the following variant of the modular law: $z(x+y) \leq x(y+z) + y(x+z)$. The closure under meet is obvious. To verify that the elements of $\mathcal{J}$ are σ_i-closed as indicated, observe that the meet of elements of the form $\sigma_j x, j \leq i$ is σ_i-closed. The closure under the operators, up to the exceptions, is computed, easily, using that $x \leq y \leq \sigma_i x$ implies $\sigma_i y = \sigma_i x$. Finally, $33+37 = \sigma_1(21+8) \leq \sigma_1 22 = \sigma_1 25(33+37) \leq \sigma_1 25 \cap \sigma_1(33+37) = 43(33+37) = 33+37$ and $37+38 = \sigma_1(35+36) = \sigma_1(35+23) = \sigma_1 35 + \sigma_1 23 = \sigma_1 8 + \sigma_1 23 = \sigma_1 23 \leq \sigma_1 25(37+48) \leq \sigma_1(37+48(25+37)) = \sigma_1(37+36) = 37+38$.

The proof continues now by applying the Split Lemma 9 (more precisely, its first modification) to subdirectly irreducible S to prove relations of the form "p is the join of its lower covers in $\mathcal{J}$" where p is associated with S. As it turns out, it is easier to establish the hypotheses of the Split Lemma for a given S after the preimages of S have been dealt with (if there are any).

In proceeding so, out of the relations already derived we use only those associated with homomorphic preimages of S. Therefore, we get the smallest preimage $\alpha_S x$ of x in $\mathcal{F}_3$ under the canonical homomorphism $\phi_S : \mathcal{F}_3 \to S$, either directly or as $\alpha_T \alpha_{ST} x$ where $\alpha_{ST} x$ is the smallest preimage of x in T under the homomorphism onto S. Considering $J(S)$ as a subset of T we have $\alpha_{ST} x = x$ for $x \in J(S)$. We apply the Split Lemma to all

S except $\mathcal{S}_8$ and its homomorphic images $\mathcal{S}_{23}$ and $\mathcal{S}_{36}$. Here, we show that $\mathcal{F}_3$ factored by the relations associated with all the other subdirectly irreducibles is just $\mathcal{S}_8$, In addition we compute $\alpha_{\mathcal{S}_8}$. So, we have computed all α_S, indeed. An important fact to note is that $\alpha_S p \in \mathcal{J}$ for all $p \in C(S)$ and that $\phi_S s < p$ for all lower covers s of αp in $\mathcal{J}$. So, we can consider $\mathcal{J}$ as a subset of $J(\mathcal{F}_3)$. Also, all $t \in \mathcal{J}$ occur (which is a consequence of the completeness of the list of subdirectly irreducibles, too) and we have $J(\mathcal{F}_3) = \mathcal{J}$. Since $\mathcal{J}$ is meet closed, we even get that no additional order relations take place. Additional collinearities are excluded by the lack of more non-distributive subdirectly irreducibles. Therefore, $J(\mathcal{F}_3)$ has exactly the indicated structure of $\mathcal{J}$ and $\mathcal{F}_3$ is isomorphic to the subspace lattice of $\mathcal{J}$.

The definition of α is given for nonzero join irreducibles $x \in J(S)$, only. Observe that αx is in $\mathcal{J}$, always (we will refer to this important fact, later on). Then $\alpha 0 = 0$ and $\alpha x = \Sigma\{\alpha y | y \leq x, y \in J_0(S)\}$. π is defined on $M(S)$, $\pi 1 = 1$, and $\pi x = \Pi\{\pi y \in M(S) \,|\, x \leq y\}$. We use the convention $r_0 = r_{-1} = 1 = \sigma_{-1}$ for $\mathcal{F}_3$, too, and $r_4 = 0$ except in case $\mathcal{D}_k$. (1)-(4) are straightforwardly checked, only the case of (dually) collinear triples needs some attention. The checking of (5π) is partly incorporated into the definition of π: we write $\pi x =_i t$ to indicate that $\sigma_i \pi x = \pi x = t$.

Unless mentioned otherwise, we have $M(\sigma_i S) \subset M(S)$ and it suffices to consider the $x \in M(S)$. In (5α) and (6) we frequently use the isotonicity of the σ_i. If for given i we have $\alpha\sigma_i 0 \leq \sigma_i 0$, then we have (5$\alpha$) for this i and every $x \leq \sigma_i 0$ since α is order preserving, too, and so $\alpha\sigma_i x \leq \alpha\sigma_i\sigma_i 0 = \sigma_i 0 \leq \sigma_i \alpha x$. If, for $p \in C(S)$, $\alpha p =_j t$ then (6) holds for $i \geq j$ since $\alpha p \leq \sigma_i \alpha p$. If for given i, $\alpha p \leq \sigma_i 0$ then (6) holds for this i since $\sigma_i \alpha p = \sigma_0 0 = \sigma_i(\alpha p \cap \pi p_*)$. In general, it suffices to show $\alpha p \leq \sigma_i(\alpha p \cap \pi p_*)$. In the verification of (7a) we always consider y with $p_* \leq y$ and $\sigma_i p \not\leq y$. (7b,c) are valid by default except for the few cases dealt with, explicitly. All of (7) is trivially satisfied for the i with $\sigma_i = id$.

$\mathcal{S}_{40}$. $\alpha r_3 = 8$, $\alpha r_2 = 24$, $\alpha 1 = 40$, $\pi 0 =_0 48$, $\pi r_3 =_1 37 + 48$, $\pi r_2 =_2 25 + 37 + 48$. Ad (5): $\alpha\sigma_0 r_3 = \alpha 1 = 40 \leq 49 = \sigma_0 8 = \sigma_0 \alpha r_3$. $\alpha\sigma_1 r_2 = \alpha 1 = 40 = \sigma_1 24 = \sigma_1 \alpha r_2$. Ad (6): $\sigma_i \alpha p = \sigma_i 40 = \sigma_i \sigma_1 24 = 40 = \alpha p$ for $i \geq 1$. $\sigma_0 40 = 49 = \sigma_0 8 \leq \sigma_0(\alpha p \cap \pi p_*)$. Ad (7): $y = r_2$ and $\sigma_i y = \sigma_i p$ for $i \leq 1$. Now, by the Split Lemma it follows $40 \leq 25 + 37 + 48$ and by modularity $40 \leq 37 + 40(25 + 48) \leq 37 + 40 \cap 49(25 + 48) = 37 + 40(49 \cap 25 + 48) = 37 + 40(24 + 48) = 37 + 24 + 40 \cap 48 = 37 + 24 + 39$.

$\mathcal{S}_{39}$. $\alpha r_3 = 8$, $\alpha r_2 = 24$, $\alpha p = 39$, $\pi p =_0 48$ $\pi b =_1 37 + 48$, $\pi r_2 =_2 25 + 37 + 38$. Ad (5α): $\alpha\sigma_0 0 = \alpha p = 39 \leq 48 = \sigma_0 \alpha 0$. $\alpha\sigma_0 r_3 = \alpha 1 = 24 + 39 \leq 49 = \sigma_0 8 = \sigma_0 \alpha r_3$. $\alpha\sigma_1 r_2 = \alpha 1 = 24 + 39 \leq 40 = \sigma_1 24 = \sigma_1 \alpha r_2$. Ad (5$\pi$): the only additional meet irreducible is r_3 in $\sigma_1 S$. But, $\pi r_3 = 48(25+37+38)+37 \simeq 48(25+37)+38+37 = 36+38+37 =_1 37+38$. Ad (6): $\sigma_i 39 = 39 = \alpha p$ for $i \geq 1$, $\sigma_0 39 = 48 = \sigma_0(\alpha p \cap \pi 0)$. Ad (7): $\sigma_i p = p$, $\sigma_i y = y$ for all $i \leq 1$. Thus $39 \leq 48(25 + 37 + 38) = 38$ by the Split Lemma.

$\mathcal{S}_{30}$. $\alpha r_3 = 8$, $\alpha p = 30$, $\alpha b = 21$, $\pi r_2 =_2 25 + 29$, $\pi\sigma_1 r_3 =_1 37$, $\pi\sigma_1 b =_0 48$. Ad (5): $\alpha\sigma_1 r_3 = 8 + 30 \leq 37 = \sigma_1 8 = \sigma_1 \alpha r_3$. $\alpha\sigma_0 0 \leq \alpha\sigma_0 p = 21 + 30 \leq 48 = \sigma_0 \alpha 0 \leq \sigma_0 \alpha p$. $\alpha\sigma_1 b = 21 + 30 \leq 33 = \sigma_1 \alpha b$. Ad (6): $\sigma_i \alpha p = \sigma_i 30 = 30$ for $i \geq 1$. $\sigma_0 \alpha p = \sigma_0 30 = 48 \leq \sigma_0(\alpha p \pi p_*)$.

Ad (7): $\sigma_i y \geq \sigma_i p$ for $i \leq 1$ and $y \neq 0$ as well as for $i = 0$ and $y = 0$; otherwise $\sigma_i p = p$ and $\sigma_i y = y$. Therefore, by the Split Lemma $30 \leq \pi 0 \leq 48(25+29) = 48 \cap 25+29 = 21+29 = 29$ whence $30 = (21+29)30 = 29 + 21 \cap 30 = 29 + 19 = 29$.

$\mathcal{S}_{22}$. $\alpha r_3 = 8$, $\alpha p_2 = 31$, $\alpha b = 21$, $\alpha p_3 = 32$, $\alpha r_2 = 22$, $\pi p_2 =_1 37$, $\pi e =_2 21 + 37$, $\pi r_2 =_2 25 + 34$, $\pi p_3 =_0 48$, $\pi d =_2 20 + 48$. Ad (4): The only collinear triple is p_1, p_2, p_3, but $\alpha p_1 = 22, \alpha p_2 = 31, \alpha p_3 = 32$ are collinear. The only dually collinear triple is r_2, d, e. By modularity $(25+34)(20+48) = 20+34+25 \cap 48 = 20+34+21$, $(25+34)(21+37) = 34+21+25 \cap 37 = 34+21+20$, $(20+48)(21+37) = 20+21+48 \cap 37 = 20+21+34$. Ad (5): $\alpha\sigma_0 0 = 32 \leq 48 = \sigma_0 \alpha 0$. $\alpha\sigma_1 r_3 = 31 \leq 37 = \sigma_1 \alpha r_3$. $\alpha\sigma_1 b = 32 \leq 33 = \sigma_1 21 = \sigma_1 \alpha b$. $\alpha\sigma_1 r_2 = 31 + 32 \leq 37 + 33 = \sigma_1(8+21) \leq \sigma_1 22 = \sigma_1 \alpha r_2$. Ad (6): $\sigma_i \alpha r_2 = \sigma_i 22 = \sigma_i 25(33+37) = 22$ for $i \geq 2$. $\sigma_0 \alpha r_2 = 49 = \sigma_0 8 = \sigma_0(\alpha r_3) \leq \sigma_0(\alpha r_2 \pi c)$. $\sigma_1 \alpha r_2 = \sigma_1 22 \leq \sigma_1(33+37) = 33+37 = \sigma_1 21 + \sigma_1 8 \leq \sigma_1(\alpha r_2 \pi c)$ since $21 + 8 \leq 22 \cap 25(21+37)(20+48)$. $\sigma_i \alpha p_2 = \sigma_i 31 = 31$ for $i \geq 2$. $\sigma_1 \alpha p_2 = 37 = \sigma_1 8 \leq \sigma_1(\alpha p_2 \pi r_3)$. $\sigma_0 \alpha p_2 = 49 = \sigma_0(\alpha p_2 \pi r_3)$. $\sigma_i \alpha p_3 = \sigma_i 32 = 32$ for $i \geq 2$. $\sigma_1 \alpha p_3 = 33 = \sigma_1 21 \leq \sigma_1(\alpha p_3 \pi b)$. $\sigma_0 \alpha p_3 = 48 \leq \sigma_0(\alpha p_3 \pi b)$. Ad (7): $\sigma_i p_k = \sigma_i p_{k*}$ for $i \leq 1$. Consequently, $22 \leq (25+34)(21+37) \simeq 21+34+25 \cap 37 = 21+34+20$ whence $22 \simeq 20+21+22 \cap 34 = 20+21+19 = 20+21$. It follows $31 = 31(22+32) = 31(20+21+32) = 20+31 \cap 32 = 20+30$ and, similarly, $32 = 21+30$.

$\mathcal{D}_k$. $\alpha b = \bar{r}_{k+3}$, $\alpha \bar{r}_{k+2} = \bar{r}_{k+2}$, $\alpha p =_{k+1} \sigma_{k+1}\bar{r}_{k+2} \cap \sigma_k \bar{r}_{k+3}$, $\pi 0 =_k \sigma_k 0$, $\pi \bar{r}_{k+2} = \bar{r}_{k+2} + \sigma_k 0 + \sigma_{k+1}(\bar{r}_{k+2}\sigma_o \bar{r}_{k+3})$. Ad (3): $\bar{r}_{k+1} \geq \bar{r}_{k+2} + \sigma_{k+1}\bar{r}_{k+2} \geq \alpha 1$, $\bar{r}_{k+3} \leq \sigma_0 \bar{r}_{k+3} \cap \bar{r}_{k+2} \leq \pi b$. Ad ($5\alpha$): $\alpha\sigma_k b = \alpha p \leq \sigma_k \bar{r}_{k+3} = \sigma_k \alpha b$. $\alpha 1 = \bar{r}_{k+2} + \alpha p \leq \sigma_{k+1}\bar{r}_{k+2}$. $\alpha\sigma_{k-1} 0 = 13 \leq \sigma_0 \alpha 0 = 48$ for $k = 1$. Ad (5π): the only additional meet irreducible is b in $\sigma_{k+1} S$. But $\pi b = \pi p \cap \pi \bar{r}_{k+2} \simeq \bar{r}_{k+2}\sigma_0 \bar{r}_{k+3} + \sigma_k 0 + \sigma_{k+1}(\bar{r}_{k+2}\sigma_0 \bar{r}_{k+3}) =_{k+1} \sigma_k 0 + \sigma_{k+1}(\bar{r}_{k+2}\sigma_0 \bar{r}_{k+3})$. Ad (6): we have to consider $i = k = 1$, only. Here, $\sigma_1 \alpha p = \sigma_1 13 = 27 = \sigma_1 7$ and $7 \leq 13 \cap \sigma_2(8 \cap \sigma_0 7) \leq \alpha p \cap \pi b$ whence $\sigma_1 \alpha p \leq \sigma_1(\alpha p \cap \pi p_*)$. Ad (7): $\sigma_i p_* \prec \sigma_i p$ implies $i > k$ and $y = \bar{r}_{k+2}$. But, $\sigma_{k+1} y = 1$ and $\sigma_i = id$ for $i > k+1$. It follows $\alpha p = \alpha p \cap \pi b \simeq \sigma_k 0 \cap \sigma_{k+1}\bar{r}_{k+2} + \sigma_k \bar{r}_{k+3} \cap \sigma_{k+1}(\bar{r}_{k+2}\sigma_0 \bar{r}_{k+2})$, i.e. $42 = 40 + 41$ and $13 = 11 + 12$.

$\mathcal{E}_k$. Let s and t (u and v) be the minimal (maximal) preimages in T of c and 1 (c and r_3 in $\mathcal{E}_k$, respectively). Then $r_3 \leq v \leq s \leq u \leq r_2 \leq t \leq r_1$ and s, t, u are σ_2-closed. $\alpha r_3 = r_3$, $\alpha b = s\sigma_1 r_3 \cap \sigma_0 0$, $\alpha p =_2 t\sigma_1(s\sigma_1 r_3 \cap \sigma_0 0)$, $\pi r_3 =_3 v + \sigma_1(v\sigma_0 0)$ which is σ_2-closed for $k = 1$, $\pi c =_2 u + \sigma_1(v\sigma_0 0)$, $\pi p =_0 \sigma_0 0$. Ad (3): T contains all r_i. Ad (5α): $\alpha p \leq \sigma_1 \alpha b \leq \sigma_0 0$. $\alpha 1 \leq \sigma_1 r_3$. $\alpha c \leq s \leq \sigma_2 r_3$ for $k = 2, 3$. Ad (5π): $\pi r_4 \simeq v\sigma_0 0 + \sigma_1(v\sigma_0 0) =_1 \sigma_1(v\sigma_0 0)$. (6) and (7) being satisfied, trivially, we get $\alpha p = \alpha p \pi c \simeq u\sigma_1(s\sigma_1 r_3 \cap \sigma_0 0) + t\sigma_1(v\sigma_0 0)$ since $v\sigma_0 0 \leq \alpha b$ and $u \leq t$. This amounts to $29 = 19 + 28$, $28 = 18 + 27$, and $18 = 14 + 17$.

$\mathcal{B}_{klmn}$. $\alpha r_m = r_{n-1}\sigma_k 0$, $\alpha 1 =_{m-1} r_{m-1}\sigma_l(r_{n-1}\sigma_k 0)$, $\pi 0 =_{k+1} \sigma_{k+1} r_n$, $\pi r_m =_{l+1} \sigma_{k+1} r_n + \sigma_{l+1} r_m$. Ad ($5\alpha$): $\alpha\sigma_k 0 = \alpha 1 \leq \sigma_l \sigma_k 0 = \sigma_k 0 \leq \sigma_k \alpha 0$. $\alpha\sigma_l r_m = \alpha 1 \leq \sigma_l(r_{n-1}\sigma_k 0) = \sigma_l \alpha r_m$. Ad (6): The cases $i \geq m-1$ and $i = 0$ are clear. This leaves to consider $i = 1, m = 3, n = 4$. If $l = 2$ then $r_3\sigma_k 0 \leq \alpha 1 \cap \pi r_3$ whence $\alpha 1 = r_2\sigma_2(r_3\sigma_k 0) \leq \sigma_1(\alpha 1 \pi r_3)$. If $l = 1, k = 0$ then $\alpha 1 = 16 \leq 26 = \sigma_1 0 \leq \sigma_1(\alpha 1 \cap \pi r_3)$. If $l = 1, k = -1$ then $r_3 \leq \alpha 1 = 20 \leq 37 = \sigma_1 r_3 \leq \sigma_1(\alpha 1 \cap \pi r_3)$. Finally, we have to deal with $l = 0$. Here, according to the two preimages of $\mathcal{S}_{24} = \mathcal{B}_{-1034}$ we use the relations $39 = 38$ and

$40 = 24 + 37 + 39$ which have been derived, already. Indeed, $\sigma_1\alpha 1 = \sigma_1 24 = 40 = 24 + 37 + 38 \leq \alpha 1 + \sigma_1 23 = \alpha 1 + \sigma_1(24(48 + 37)) = \alpha 1 + \sigma_1(\alpha 1 \pi r_3)$. (7) being satisfied by default, using the split we get the relations $r_{m-1}\sigma_l(r_{n-1}\sigma_k 0) \leq \sigma_{k+1} r_n + \sigma_{l+1} r_m$ which say that each p critical for a $\mathcal{B}_{klmn}$ is the join of its two lower covers in $\mathcal{J}$.

$\mathcal{C}_{hklmn}$. $\alpha r_m = r_{n-1}\sigma_h r_n$, $\alpha p = r_{m-1}\sigma_l r_{n-1} \cap \sigma_k 0$, $\pi r_m =_{l+1} \sigma_{l+1} r_m + \sigma_{k+1}(\sigma_{l+1} r_m \cap \sigma_{h+1} r_n)$, $\pi p =_{h+1} \sigma_{h+1} r_n$. Ad (5$\pi$): $\pi 0 \simeq (\sigma_{h+1} r_n \cap \sigma_{l+1} r_m) + \sigma_{k+1}(\sigma_{l+1} r_m \cap \sigma_{h+1} r_n) =_{k+1} \sigma_{k+1}(\sigma_{l+1} r_m \cap \sigma_{h+1} r_n)$. Ad (6) $\sigma_i \alpha p = \alpha p$ for $i \geq m - 1$ and $\alpha p \leq \sigma 0$ for $i \leq k$. This leaves $k < i < m - 1$, whence $k = 0, m = 3, i = 1$ to consider. If $l = 1$ we have $\alpha p = 19$, $\sigma_1 \alpha p = 29$, $\pi 0 = 28$. This yields $\sigma_1 \alpha p = 19 + 28 \leq \alpha \sigma_1 p + \sigma_1(\alpha p \pi 0)$ applying the relation $29 = 19+28$ obtained by the exclusion of the preimage $\mathcal{S}_{29}$ of $\mathcal{S}_{19}$. If $l = 2$ we have $\alpha p = 14$, $\sigma_1 \alpha p = 28$, $\pi 0 = 27$ and we apply the relation $28 = 18 + 27 = 14 + 17 + 27 = 14 + 27$ obtained by the exclusion of the preimages $\mathcal{S}_{28}$ and $\mathcal{S}_{18}$ of $\mathcal{S}_{14}$. Ad (7a): The only case to be dealt with is $y = r_3, i < k$, whence $i = h = 0$ and $\sigma_i 0 \geq \alpha 1$. For each term $p \in \mathcal{J}$ critical for a $\mathcal{C}_{hklmn}$, having $\alpha p \leq \pi r_m$ we derive that p is identified with its unique lower cover in $\mathcal{J}$.

$\mathcal{A}_{kn}$ except $\mathcal{A}_{-14} = \mathcal{S}_8$. $\alpha 1 = r_{n-1}\sigma_k 0$, $\pi 0 =_{k+1} \sigma_{k+1} r_n$. Ad(6): $\sigma_i \alpha p = \alpha p$ for $i \geq n - 1$ and $\alpha p \leq \sigma_i 0$ for $i \leq k$. This leaves to consider the cases $n = k+3$ and $n = k+4$. We have to compute $\alpha p = r_{n-1}\sigma_k 0$ and $\pi 0 = \sigma_{k+1} r_n$ and to verify $s_i = \sigma_i \alpha p \leq t_i = \alpha p + \sigma_i(\alpha p \cap \pi 0)$ for $k + 1 \leq i < n$. Let $n = k + 3$, so $i = k + 1$. For $\mathcal{A}_{-12} = \mathcal{S}_{47}$ we have $\alpha p = 47$, $\pi 0 = 50$, $s_0 = 51$, $t_0 = 47 + \sigma_0(46) = 47 + 50$ and are done using the relation $51 = 47 + 50$ given by the exclusion of the preimage $\mathcal{S}_{51} = \mathcal{B}_{-1012}$. For $\mathcal{A}_{03} = \mathcal{S}_{21}$ we have $\alpha p = 21$, $\pi 0 = 37$, $s_1 = 33$, $t_1 = 21 + \sigma_1 19 = 21 + 29$ and get $33 \leq 32 \leq 22 + 31 \leq 20 + 21 + 30$ using the relations given by the exclusion of the preimages $\mathcal{S}_{33} = \mathcal{B}_{0123}$ and $\mathcal{S}_{22}$. By modularity, $33 = 33 \cap 20 + 21 + 30 = 19 + 21 + 30 = 21 + 30 = 21 + 29$ using the relation $30 = 29$ provided by the preimage $\mathcal{S}_{30}$. For $\mathcal{A}_{14} = \mathcal{S}_6$ we have $\alpha p = 6$, $\pi 0 = 3$, and $s_2 = 9 \leq 6 + 5 = 6 + \sigma_2 3 = t_2$ using the exclusion of $\mathcal{S}_9 = \mathcal{B}_{1234}$. In the case $n = k + 4$ we have to consider $i = k + 1$ and $i = k + 2$. In $\mathcal{A}_{-13} = \mathcal{S}_{25}$ we get $\alpha p = 25$, $\pi 0 = 49$, $s_0 = 50$, and $t_0 = 25 + \sigma_0 24 = 29 + 49 \geq 43,\ 46,\ 50$ using the preimages $\mathcal{B}_{-1lm3}$; moreover $s_1 = 43$ and $t_1 = 25 + \sigma_1 24 = 25 + 40 \geq 41,\ 42,\ 43$ using the preimages $\mathcal{C}_{-10123}$, $\mathcal{D}_0$, and $\mathcal{B}_{-1123}$. Finally, for $\mathcal{A}_{04} = \mathcal{S}_7$ we have $\alpha p = 7$, $\pi 0 = 26$, $s_1 = 27$, and $t_1 = 7 + \sigma_1 6 = 7 + 26 \geq 11,\ 13,\ 17,\ 27$ using the preimages $\mathcal{B}_{0lm4}$ and $\mathcal{D}_1$; moreover, $s_2 = 11$ and $t_2 = 7 + \sigma_2 6 = 7 + 9 \geq 10,\ 11$ using $\mathcal{C}_{01234}$ and $\mathcal{B}_{0234}$. For each p critical for such $\mathcal{A}_{kn}$ we get $\alpha 1 \leq \pi 0$ whence the relation identifying p with its unique lower cover in $\mathcal{J}$.

Having applied the Split Lemma to all $\mathcal{S}_t$ except for $t = 8, 23, 38$ we have to consider the free model M satisfying all relations of $\mathcal{J}$ plus all relations of the form "t is the join of its lower covers in $\mathcal{J}$" for $t \neq 8, 23, 35, 36, 38$. In particular we have $0 = 32 = 34$, $38 = 48$, $22 = 31 = 8$, $23 = 25$, $35 = 37$, and $40 = 1$. So the image of $\mathcal{J}$ in M consists just of $0, 1, e_1 = 38, e_2 = 36, e_3 = 23, e_4 = 35, e_5 = 8$ with the relations $e_1 \geq e_2$, $e_5 = e_3 e_4$, $e_1 e_3 = e_1 e_4 = 0$, and $e_i + e_j = e_i + e_k$ for $\{i, j, k\} = \{2, 3, 4\}$. Let L be the modular lattice (without operators) having this presentation - omitting e_5 as a generator. The

$\mathcal{D}_2$-Lemma 10 applied to $\phi e_1 = 1, \phi e_i = 0$ for $i > 1$ yields $e_1 \leq e_2 + e_3 + e_4 = e_2 + e_3$ whence $e_1 = e_1(e_2 + e_3) = e_2 + e_1 e_3 = e_2$. With $\phi e_1 = \phi e_2 = 0$ and $\phi e_3 = \phi e_4 = 1$ one gets $e_3 e_4 \leq e_1 + e_2$. Together, this implies $e_1 = e_2$, $e_i e_j = 0, e_i + e_j = 1$ for $i \neq j$ in $\{2, 3, 4\}$. So, what is left is a lattice $\mathcal{M}_3$. Now, as a lattice, $\mathcal{S}_{38}$ is the subdirect product of these factors - under the considered interpretation of generator symbols. Thus, it is the sublattice M' of M generated by $\mathcal{J}$. Moreover, due to the relations of $\mathcal{J}$, the operators are computed exactly as required - and inside M'. This shows $M{=}M'$.

Since $\mathcal{S}_{23}$ and $\mathcal{S}_8$ are the only homomorphic images of $\mathcal{S}_{38}$ the completeness of the list of subdirectly irreducible models is proved. Though, we have to compute the smallest preimages for these factors. But in doing so it suffices to verify (1)-(5) for α, π associated with $\mathcal{S}_{23}$ and observe that the preimage maps for the homomorphic images are obtained composing α and π with the preimage maps for these homomorphisms.

Define $\alpha r_3 = 8$, $\alpha r_2 = 23$, $\alpha p_2 = 35$, $\alpha p_3 = 36$, $\alpha\sigma_0 0 = 38$, $\pi r_2 =_2 25 + 33 + 34$, $\pi p_2 =_1 33 + 37$, $\pi d =_2 25 + 33 + 37$, $\pi\sigma_0 0 =_0 48$, $\pi c =_2 22 + 48$. Ad (4): $\alpha p_i \leq \alpha p_j + \alpha p_k$, for i, j, k all distinct. The only dually collinear triple is p_2, r_2, c. By modularity, $(33 + 37)(25 + 33 + 34) = 33 + 34 + (33 + 37)25 = 33 + 34 + 22$, $(33 + 37)(22 + 48) = 33 + 22 + 37 \cap 48 = 33 + 22 + 34$, $(25 + 33 + 34)(22 + 48) = 22 + 33 + 34 + 25 \cap 48 = 22 + 33 + 34 + 21 = 22 + 33 + 34$. Ad (5): $\alpha\sigma_0 0 = 38 \leq 48$. $\alpha\sigma_1 p_3 = 38 = \sigma_1 36 = \sigma_1 \alpha p_3$. $\alpha\sigma_0 r_3 = \alpha\sigma_0 p_2 = 35 + 38 \leq 49 = \sigma_0 8 = \sigma_0 \alpha r_3$. $\alpha\sigma_1 r_2 = 35 + 38 \leq 43 = \sigma_1 \alpha r_2$.

4.3 Relations between the indices

As observed in Lemma 18 of 2.7 , in case of a finite lattice $\mathcal{V}(a)$ the indices $\dim \pi_{EF}x/\pi_{EF}y$ of a pair (E, F) are determined by the indices $a_i := \dim \pi_{EF}i/\pi_{EF}i_*$ corresponding to join irreducibles i and their lower covers i_*. Here we deal with $\mathcal{V}_3(a)$ and denote i just by its label from Table 1.

Theorem 44. *Let (E, Φ) be a diagonal ε-hermitean space over $(k, \varepsilon, -)$, $\dim E \leq \aleph_3$, and F a subspace of E. Then the indices a_i satisfy the relations of Table 2 (the terms in brackets may be omitted).*

Deleting the terms in brackets one obtains the original relations by transitivity, immediately.

Proof. We give the proof of the relations in one representative case, to wit for (R7). We have $10 = 26 \cap 11$ whence $a_{11} = \dim 11/10 = \dim(26 + 11)/26 \leq \dim 27/26 = \dim \sigma_1 7/\sigma_1 6 \leq \dim 7/6 = a_7$ using the definition and structure of $\mathcal{I}$ and Lemma 1(e) in 1.4. Similarly, $a_{13}, a_{17}, a_{27} \leq a_7$. Also $a_{10} = \dim 10/9 \leq \dim 11/9 = \dim \sigma_2 7/\sigma_2 6 \leq \dim 7/6 = a_7$. On the other hand, from $a_7 < \aleph_1$ we get $7 = 6 + U$ with $\dim U < \aleph_1$ and now by (b) and (c) $27 = \sigma_1 7 = \sigma_1 6 + \sigma_1 U = \sigma_1 6 + U = \sigma_1 6 + 7 = 26 + 7$. This yields $a_{13} = 0$, $a_{17} = 0$, $a_{27} = 0$. Analogously, from $a_7 < \aleph_2$ we get $11 = \sigma_2 7 = \sigma_2 6 + 7 = 9 + 7$ and thus $a_{10} = a_{11} = 0$. All this results into (R7). ($R31$) and ($R35$) follow from the collinearity relations. Concerning $(R60) - R(64)$ observe that for $p = 2, 8, 25, 47, 63$ we

$$
\begin{aligned}
(R2)\qquad a_2 \geq{} & a_3\aleph_3 + a_4\aleph_2 + a_5\aleph_2 + a_6\aleph_1 + a_7\aleph_0 \\
& + (a_9\aleph_2 + a_{10}\aleph_2 + a_{11}\aleph_2) + a_{12}\aleph_1 + (a_{13}\aleph_1) \\
& + a_{14}\aleph_0 + a_{16}\aleph_1 + (a_{17}\aleph_1 + a_{18}\aleph_1 + a_{19}\aleph_0) \\
& + a_{21}\aleph_0 + (a_{22}\aleph_1) + a_{23}\aleph_0 + a_{26}\aleph_1 + (a_{27}\aleph_1) \\
& + (a_{28}\aleph_1 + a_{29}\aleph_1 + a_{30}\aleph_1 + a_{33}\aleph_1) + a_{34}\aleph_0 \\
& + a_{38}\aleph_1 + a_{39}\aleph_0 + a_{41}\aleph_0 + a_{44}\aleph_0 + a_{48}\aleph_0 \\
(R6)\qquad a_6 \geq{} & a_9\aleph_2 \\
(R7)\qquad a_7 \geq{} & a_{10}\aleph_2 + a_{11}\aleph_2 + a_{13}\aleph_1 + a_{17}\aleph_1 + a_{27}\aleph_1 \\
(R8)\qquad a_8 \geq{} & a_{12}\aleph_2 + a_{13}\aleph_2 + a_{14}\aleph_2 + a_{15}\aleph_2 \\
& + (a_{18}\aleph_2) + a_{19}\aleph_1 + a_{20}\aleph_1 + a_{22}\aleph_1 + a_{23}\aleph_1 \\
& + a_{24}\aleph_0 + (a_{28}\aleph_2 + a_{29}\aleph_1) + a_{30}\aleph_1 + a_{34}\aleph_1 \\
& + a_{37}\aleph_1 + (a_{38}\aleph_1 + a_{39}\aleph_1 + a_{40}\aleph_1) + a_{42}\aleph_0 \\
& + a_{45}\aleph_0 + a_{49}\aleph_0 \\
(R14)\qquad a_{14} \geq{} & a_{18}\aleph_1 + a_{28}\aleph_1 \\
(R19)\qquad a_{19} \geq{} & a_{29}\aleph_1 \\
(R21)\qquad a_{21} \geq{} & a_{22}\aleph_1 + a_{30}\aleph_1 + a_{33}\aleph_1 \\
(R23)\qquad a_{23} \geq{} & a_{38}\aleph_1 \\
(R24)\qquad a_{24} \geq{} & a_{39}\aleph_1 + a_{40}\aleph_1 \\
(R25)\qquad a_{25} \geq{} & a_{41}\aleph_1 + a_{42}\aleph_1 + a_{43}\aleph_1 + a_{46}\aleph_0 + a_{50}\aleph_0 \\
(R47)\qquad a_{47} \geq{} & a_{51}\aleph_0 \\
(R53)\qquad a_{53} \geq{} & a_{59}\aleph_0 \\
(R54)\qquad a_{54} \geq{} & a_4\aleph_3 + a_6\aleph_3 + a_7\aleph_3 + a_8\aleph_3 + (a_9\aleph_3 + a_{10}\aleph_3) \\
& + (a_{11}\aleph_3 + a_{12}\aleph_3 + a_{13}\aleph_3 + a_{14}\aleph_3 + a_{15}\aleph_3) + a_{16}\aleph_2 \\
& + (a_{17}\aleph_3 + a_{18}\aleph_3 + a_{19}\aleph_3) + a_{21}\aleph_2 + (a_{20}\aleph_3 + a_{22}\aleph_3) \\
& + (a_{23}\aleph_3 + a_{24}\aleph_3) + a_{25}\aleph_2 + (a_{27}\aleph_3 + a_{28}\aleph_3 + a_{29}\aleph_3) \\
& + (a_{30}\aleph_3 + a_{33}\aleph_2 + a_{34}\aleph_3 + a_{37}\aleph_3 + a_{38}\aleph_3 + a_{39}\aleph_3) \\
& + (a_{40}\aleph_3 + a_{41}\aleph_2 + a_{42}\aleph_3 + a_{43}\aleph_2 + a_{44}\aleph_1 + a_{45}\aleph_3) \\
& + (a_{46}\aleph_2) + a_{47}\aleph_1 + a_{49}\aleph_3 + a_{50}\aleph_2 + a_{51}\aleph_1) + a_{52}\aleph_0 \\
& + a_{55}\aleph_3 + a_{56}\aleph_2 + a_{57}\aleph_1 + a_{58}\aleph_0 + (a_{59}\aleph_0) \\
(R31)\qquad a_{31} ={} & a_{32} = a_{22} \\
(R35)\qquad a_{35} ={} & a_{36} = a_{23} \\
(R60)\qquad a_{60} ={} & a_{52} \\
(R61)\qquad a_{61} ={} & a_{47} \\
(R62)\qquad a_{62} ={} & a_{25} \\
(R63)\qquad a_{63} ={} & a_8 \\
(R64)\qquad a_{64} ={} & a_2
\end{aligned}
$$

Table 2: The relations between the indices

have $p \not\leq \sigma_0 p_*$ and whence the $(p_*)^\perp = 64, 63, 62, 61, 60$ are critical by Theorem 25 with p/p_* and $(p_*)^\perp/p^\perp$ generating the same congruence, in each instance, and so having the same index by Prop. 17.

4.4 Elementary examples

Let $(k, \varepsilon, -) \neq (k, -1, id))$ with arbitrary field k be given. Subspaces of the form $< a, a' >$ will denote *hyperbolic planes*, i.e. $\Phi(a, a) = \Phi(a', a') = 0$, $\Phi(a, a') = 1$; spaces of the form $< b >$ will denote non-isotropic lines, i.e. $\Phi(b, b) \neq 0$. $\sum_{\iota<\tau} e_\iota$ stands for the vector space spanned by the family $(e_\iota, \iota < \tau)$. The striking fact about the examples is that they are in 1-1-correspondence with the subdirectly irreducible factors $\mathcal{Q}_p$ of $\mathcal{V}_3(a)$ and that the index of the critical quotient p/p_* is $a_p = 1$ in each instance (notice the identities (R31)-(R64) above). In each example we also indicate the nonzero indices a_i, $i \in I = \{2, \ldots, 59\} \setminus \{31, 32, 35, 36\}$, and the shape of the lattice if it belongs to one of the series. Some examples come parametrized. For these, the p are listed lexicographically.

$p =53,\ E =< g >,\ F = (0)$

$p =54,\ E =< g >,\ F = (g)$

$p = 58, 57, 56, 55;\ \mathcal{R}_{3i};\ 0 \leq \alpha \leq 3;\ a_{54} = \aleph_\alpha$
$$E =< g > \oplus [\bigoplus_{\iota<\omega_\alpha} < g_\iota >],\ F = \sum_{\iota<\omega_\alpha} (g + g_\iota)$$

$p = 59;\ a_{53} = a_{54} = \aleph_0$
$$E =< t > \oplus ([\bigoplus_{\iota<\omega_0} < s_\iota >] \oplus [\bigoplus_{\iota<\omega_0} < u_\iota >]),\ \ F = \sum_{\iota<\omega_0} (s_\iota)$$
where $\Phi(t, s_\iota) = \Phi(t, u_\kappa) = 1\ (\forall \iota, \kappa)$

$p =2,\ E =< a, a' >,\ F = (a)$

$p = 48, 26, 5, 3;\ \mathcal{A};\ 0 \leq \alpha \leq 3;\ a_2 = a_{54} = \aleph_\alpha$
$$E =< a, a > \oplus [\bigoplus_{\iota<\omega_\alpha} < e_\iota, e'_\iota >];\ F = \sum_{\iota<\omega_\alpha} (a + e_\iota)$$

$p = 52, 47, 25, 8;\ \mathcal{A};\ 0 \leq \beta \leq 3;\ a_{54} = \aleph_\beta$
$$E =< a, a' > \oplus [\bigoplus_{\kappa<\omega_\beta} < t_\kappa >],\ \ F = \sum_{\iota<\omega_\beta} (a + t_\kappa)$$

$p = 44, 21, 16, 7, 6, 4;\ \mathcal{A};\ 0 \leq \alpha < \beta \leq 3;\ a_2 = \aleph_\alpha,\ a_{54} = \aleph_\beta$
$E =< a, a' > \oplus [\bigoplus_{\iota<\omega_\alpha} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\kappa<\omega_\beta} < t_\kappa >]$
$F = \sum_{\iota<\omega_\alpha} (a + e_\iota) + \sum_{\kappa<\omega_\beta} (a + t_\kappa)$

$p = 51, 50, 49, 43, 37, 15;\ \mathcal{C};\ 0 \leq \alpha < \beta \leq 3;\ a_{54} = \aleph_\beta;\ a_{47}/a_{25}/a_8 = \aleph_\alpha$ for $\beta = 1/2/3$
$E =< a, a' > \oplus [\bigoplus_{\iota<\omega_\alpha} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_\alpha \\ \kappa<\omega_\beta}} < t_{\iota\kappa} >]$
$F = \sum_{\substack{\iota<\omega_\alpha \\ \kappa<\omega_\beta}} (a + e_\iota + t_{\iota\kappa})$

$p = 46, 45, 24, 20;\ \mathcal{B};\ 0 \leq \alpha < \beta < \gamma \leq 3;\ a_{25}/a_8/a_8/a_8 = \aleph_\alpha,\ a_{54} = \aleph_\gamma$
$E =< a, a' > \oplus [\bigoplus_{\iota<\omega_\alpha} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\kappa<\omega_\beta} < t_\kappa >] \oplus [\bigoplus_{\substack{\iota<\omega_\alpha \\ \kappa<\omega_\gamma}} < s_\kappa >]$
$F = \sum_{\kappa<\omega_\beta} (a + t_\kappa) + \sum_{\substack{\iota<\omega_\alpha \\ \kappa<\omega_\gamma}} (a + e_\iota + s_{\iota\kappa})$

$p =33, 27, 11, 9;\ \mathcal{B};\ 0 \leq \alpha < \beta < \gamma \leq 3;\ a_2 = \aleph_\beta,\ a_{54} = \aleph_\gamma,\ a_{21}/a_7/a_7/a_6 = \aleph_\beta$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_\alpha} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_\beta} < f_\iota, f'_\iota >]$$
$$\oplus [\bigoplus_{\substack{\iota<\omega_\beta \\ \kappa<\omega_\alpha}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_\beta \\ \kappa<\omega_\gamma}} < t_{\iota\kappa} >]$$
$$F = \sum_{\iota<\omega_\alpha} (a + e_\iota) + \sum_{\substack{\iota<\omega_\beta \\ \kappa<\omega_\gamma}} (a + f_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_\beta \\ \kappa<\omega_\alpha}} (f_\iota + h_{\iota\kappa})$$

$p = 17$; $\mathcal{B}$; $a_2 = a_7 = \aleph_1$, $a_{54} = \aleph_3$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >]$$
$$\oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus < t_\iota >_{\iota<\omega_2}]$$
$$\oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} < s_{\iota\kappa} >]$$
$$F = \sum_{\iota<\omega_0} (a + e_\iota) + \sum_{\kappa<\omega_2} (a + t_\kappa) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} (a + f_\iota + s_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} (f_\iota + h_{\iota\kappa})$$

$p = 41, 34, 14, 12$; $\mathcal{C}$; $0 \le \alpha < \beta < \gamma \le 3$; $a_2 = \aleph_\alpha$, $a_{25}/a_8/a_8/a_8 = \aleph_\beta$, $a_{54} = \aleph_\gamma$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_\alpha} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_\beta} < f_\iota, f'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_\beta \\ \kappa<\omega_\gamma}} < t_{\iota\kappa} >]$$
$$F = \sum_{\iota<\omega_\alpha} (a + e_\iota) + \sum_{\substack{\iota<\omega_\beta \\ \kappa<\omega_\gamma}} (a + f_\iota + t_{\iota\kappa})$$

$p = 19$; $\mathcal{C}$; $a_2 = \aleph_0$, $a_8 = \aleph_1$, $a_{54} = \aleph_3$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >]$$
$$\oplus [\bigoplus_{\iota<\omega_2} < t_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} < s_{\iota\kappa} >]$$
$$F = \sum_{\iota<\omega_0} (a + e_\iota) + \sum_{\iota<\omega_2} (a + t_\iota) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} (a + f_\iota + s_{\iota\kappa})$$

$p = 10$; $\mathcal{C}$; $a_2 = a_7 = \aleph_2$, $a_{54} = \aleph_3$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_1} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_2} < f_\iota, f'_\iota >]$$
$$\oplus [\bigoplus_{\substack{\iota<\omega_2 \\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_2 \\ \kappa<\omega_3}} < t_{\iota\kappa} >]$$
$$F = \sum_{\iota<\omega_1} (a + e_\iota) \sum_{\substack{\iota<\omega_2 \\ \kappa<\omega_3}} (a + f_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_2 \\ \kappa<\omega_0}} (f_\iota + h_{\iota\kappa})$$

$p = 42$; $\mathcal{D}$; $a_8 = \aleph_0$, $a_{25} = \aleph_1$, $a_{54} = \aleph_3$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_0 \\ \kappa<\omega_3}} < t_{\iota\kappa} >]$$
$$\oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_2}} < s_{\iota\kappa} >]$$
$$F = \sum_{\substack{\iota<\omega_0 \\ \kappa<\omega_3}} (a + e_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_2}} (a + f_\iota + s_{\iota\kappa})$$

$p = 13$; $\mathcal{D}$; $a_2 = \aleph_1$, $a_7 = \aleph_1$, $a_8 = \aleph_2$, $a_{54} = \aleph_3$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \oplus \bigoplus_{\iota<\omega_2} < g_\iota, g'_\iota >]$$
$$\oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} < t_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_2 \\ \kappa<\omega_3}} < s_{\iota\kappa} >]$$
$$F = \sum_{\iota<\omega_0} (a + e_\iota) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} (a + f_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_2 \\ \kappa<\omega_3}} (a + g_\iota + s_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} (f_\iota + h_{\iota\kappa})$$

$p = 18$; $\mathcal{E}$; $a_2 = \aleph_1$, $a_8 = \aleph_2$, $a_{14} = \aleph_1$, $a_{54} = \aleph_3$

$$E = < a, a' > \oplus [\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_1}} < g_{\iota\kappa}, g'_{\iota\kappa} >]$$

$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\iota<\omega_2} < t_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2\\ \tau<\omega_3}} < s_{\iota\kappa\tau} >]$$

$$F = \sum_{\iota<\omega_0}(a+e_\iota) + \sum_{\iota<\omega_1}(a+t_\iota) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_0}}(f_\iota + h_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_2\\ \tau<\omega_3}}(a + f_\iota + g_{\iota\kappa} + s_{\iota\kappa\tau})$$

$p = 28$; $\mathcal{E}$; $a_2 = \aleph_1,\ a_8 = \aleph_2,\ a_{14} = \aleph_1,\ a_{54} = \aleph_3$

$$E = < a, a' > \oplus[\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0}} < g_{\iota\kappa}, g'_{\iota\kappa} >]$$
$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2\\ \tau<\omega_3}} < t_{\iota\kappa\tau} >]$$

$$F = \sum_{\iota<\omega_0}(a+e_\iota) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_0}}(f_\iota + g_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_2\\ \tau<\omega_3}}(a + f_\iota + h_{\iota\kappa} + t_{\iota\kappa\tau})$$

$p = 29$; E; $a_2 = a_8 = a_{19} = \aleph_1,\ a_{54} = \aleph_3$

$$E = < a, a' > \oplus < b, b' > \oplus[\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >]$$
$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0}} < g_{\iota\kappa}, g'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_0\\ \kappa<\omega_1}} < h_{\iota\kappa}, h'_{\iota\kappa} >]$$
$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2}} < t_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_3}} < s_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_1\\ \tau<\omega_3}} < u_{\iota\kappa\tau} >]$$

$$F = \sum_{\iota<\omega_0}(a+e_\iota) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_2}}(a + f_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_3}}(a + b + f_\iota + s_{\iota\kappa}) +$$
$$\sum_{\substack{\iota<\omega_1\\ \kappa<\omega_0}}(f_\iota + g_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_1\\ \tau<\omega_3}}(e_\iota + h_{\iota\kappa} + u_{\iota\kappa\tau})$$

$p = 30$; $a_2 = a_8 = a_{21} = \aleph_1,\ a_{54} = \aleph_3$

$$E = < a, a' > \oplus[\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1}, < f_\iota, f'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < g_\iota, g'_\iota >]$$
$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_3}} < t_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2}} < s_{\iota\kappa} >]$$

$$F = \sum_{\iota<\omega_0}(a+e_\iota) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_2}}(a + f_\iota + s_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_3}}(a + g_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_0}}(f_\iota + h_{\iota\kappa})$$

$p = 38$; $a_2 = a_8 = a_{23} = \aleph_1,\ a_{54} = \aleph_3$

$$E = < a, a' > \oplus[\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < g_\iota, g'_\iota >]$$
$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \tau<\omega_1}} < d_{\iota\tau}, d'_{\iota\tau} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >]$$
$$\oplus[\bigoplus_{\substack{\iota,\tau<\omega_1\\ \kappa<\omega_3}} < t_{\iota\tau\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2}} < s_{\iota\kappa} >>]$$

$$F = \sum_{\iota<\omega_0}(a+e_\iota) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_2}}(a + g_\iota + f_\iota + s_{\iota\kappa}) + \sum_{\substack{\tau,\iota<\omega_1\\ \kappa<\omega_3}}(f_\iota + d_{\iota\tau} + t_{\iota\tau\kappa}) +$$
$$\sum_{\substack{\iota<\omega_1\\ \kappa<\omega_0}}(g_\iota + h_{\iota\kappa})$$

$p = 39$; $a_2 = \aleph_0,\ a_8 = a_{24} = \aleph_1,\ a_{54} = \aleph_3$

$$E = < a, a' > \oplus[\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_0} < f_\iota, f'_\iota >]$$
$$\oplus[\bigoplus_{\iota<\omega_1} < g_\iota, g'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >]$$
$$\oplus[\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_2}} < t_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_0\\ \kappa<\omega_3}} < s_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1\\ \kappa<\omega_0\\ \tau<\omega_3}} < u_{\iota\kappa\tau} >]$$

$$F = \sum_{\iota<\omega_0}(a+e_\iota) + \sum_{\substack{\iota<\omega_1\\ \kappa<\omega_2}}(a + g_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_0\\ \kappa<\omega_3}}(a + f_\iota + s_{\iota\kappa}) +$$

$$\sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_0 \\ \tau<\omega_3}} (g_\iota + h_{\iota\kappa} + u_{\iota\kappa\tau})$$

$p = 40$; $a_8 = a_{24} = \aleph_1,\ a_{54} = \aleph_3$

$$\begin{aligned} E =\ & < a, a' > \oplus [\textstyle\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \\ & \oplus [\textstyle\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_2}} < t_{\iota\kappa} >] \\ & \oplus [\textstyle\bigoplus_{\substack{\iota<\omega_0 \\ \kappa<\omega_3}} < s_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_0 \\ \tau<\omega_3}} < u_{\iota\kappa\tau} >] \end{aligned}$$

$$F = \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_2}} (a + f_\iota + t_{\iota\kappa}) + \sum_{\substack{\iota<\omega_0 \\ \kappa<\omega_3}} (a + e_\iota + s_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_0 \\ \tau<\omega_3}} (f_\iota + h_{\iota\kappa} + u_{\iota\kappa\tau})$$

$p = 22$; $a_2 = a_8 = a_{21} = \aleph_1,\ a_{22} = 1,\ a_{54} = \aleph_3$

$$\begin{aligned} E =\ & < a, a' > \oplus < b, b' > \oplus [\textstyle\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \\ & \oplus [\textstyle\bigoplus_{\iota<\omega_1} < g_\iota, g'_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} < h_{\iota\kappa}, h'_{\iota\kappa} >] \\ & \oplus [\textstyle\bigoplus_{\iota<\omega_2} < t_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_2}} < s_{\iota\kappa} >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} < u_{\iota\kappa} >] \end{aligned}$$

$$\begin{aligned} F =\ & \textstyle\sum_{\iota<\omega_0} (a + e_\iota) + \sum_{\iota<\omega_2} (a + b + t_\iota) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_2}} (a + f_\iota + s_{\iota\kappa}) \\ & + \textstyle\sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} (b + g_\iota + u_{\iota\kappa}) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_0}} (f_\iota + h_{\iota\kappa}) \end{aligned}$$

$p = 23$; $a_2 = \aleph_0,\ a_8 = \aleph_1,\ a_{54} = \aleph_3$

$$\begin{aligned} E =\ & < a, a' > \oplus < b, b' > \oplus [\textstyle\bigoplus_{\iota<\omega_0} < e_\iota, e'_\iota >] \oplus [\bigoplus_{\iota<\omega_1} < f_\iota, f'_\iota >] \\ & \oplus [\textstyle\bigoplus_{\iota<\omega_2} < t_\iota >] \oplus [\bigoplus_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} < s_{\iota\kappa} >] \end{aligned}$$

$$F = \sum_{\iota<\omega_0} (a + e_\iota) + \sum_{\iota<\omega_2} (a + b + t_\iota) + \sum_{\substack{\iota<\omega_1 \\ \kappa<\omega_3}} (b + f_\iota + s_{\iota\kappa})$$

The properties of the examples are easily verified applying the following Lemma.

LEMMA 45. *Let (E, Φ) be a diagonal, ε-hermitean space over $(k, \varepsilon, -) \neq (k, -1, id)$ of dimension $\aleph_\alpha$ where $\alpha < \omega$.*

(a) $$\text{If } E = \bigoplus_{\iota<\omega_\beta} < e_\iota, e'_\iota > \oplus E_0, \quad F = \sum_{\iota<\omega_\beta} (a + e_\iota) \quad (a \in E_0)$$

$$\text{then } \sigma_\gamma F = F \ (\gamma > \beta),\ \sigma_\gamma F = (a) + \sum_{\iota<\omega_\beta} (e_\iota)\ (\gamma \leq \beta)\,.$$

(b) $$\text{If } E = \bigoplus_{\iota<\omega_\beta} < t_\iota > \oplus E_0, \quad F = \sum_{\iota<\omega_\beta} (a + t_\iota) \quad (a \in E_0)$$

$$\text{then } \sigma_\gamma F = F \ (\gamma > \beta),\ \sigma_\gamma F = (a) + \sum_{\iota<\omega_\beta} (t_\iota)\ (\gamma \leq \beta)\,.$$

(c) $$\text{If } E = [\bigoplus_{\iota<\omega_\beta} < e_\iota, e'_\iota >] \oplus [\bigoplus < f_\iota, f'_\iota >_{\iota<\omega_\beta}] \oplus E_0,$$

$$F = \sum_{\iota<\omega_\beta} (e_\iota + f_\iota), \quad \text{then } F \text{ is } \sigma_0\text{-closed}\,.$$

(*d*)
$$\text{If } E=[\bigoplus_{\iota<\omega_\beta}<e_\iota,e'_\iota>]\oplus[\bigoplus_{\substack{\iota<\omega_\beta\\ \kappa<\omega_\gamma}}<h_{\iota\kappa},h'_{\iota\kappa}>]\oplus E_0,$$
$$F=\sum_{\substack{\iota<\omega_\beta\\ \kappa<\omega_\gamma}}(e_\iota+h_{\iota\kappa}),\ 0\le\gamma\le\beta,$$
$$\text{then } \sigma_\delta F=F\ (\delta>\gamma),\ \sigma_\delta F=\sum_{\iota<\omega_\beta}(e_\iota)+\sum_{\substack{\iota<\omega_\beta\\ \kappa<\omega_\gamma}}(h_{\iota\kappa})\ (\delta\le\gamma).$$

Proof. (a): $\sigma_\gamma F=F$ for $\gamma>\beta$ by (c) of Lemma 1 in 1.4. Let $U=W^\perp$, $\dim W<\aleph_\gamma\le\aleph_\beta$, be a typical zero-neighbourhood for $\sigma_\gamma(\gamma\le\beta)$. There must be an $\iota_0<\omega_\beta$ with $e_{\iota_0}\perp W$, hence $a+e_{\iota_0}\in(a+U)\cap F\neq\emptyset$. Therefore a is a σ_γ-cluster point of F and $(a)\oplus\bigoplus_{\iota<\omega_\beta}e_\iota\subset\sigma_\gamma F$. On the other hand $(a)\oplus\bigoplus_{\iota<\omega_\beta}(e_\iota)$ is σ_0-closed and $\sigma_\gamma F\subset\sigma_0 F\subset(a)\oplus\bigoplus_{\iota<\omega_\beta}(e_\iota)$. The proof of (b) is similar.

Proof of (c): Let $x\in\sigma_0F\subset[\bigoplus_{\iota<\omega_\beta}<e_\iota,e'_\iota>]\oplus[\bigoplus_{\iota<\omega_\beta}<f_\iota,f'_\iota>]$, $x=\sum_{\iota<\omega_\beta}\lambda_\iota e_\iota+\sum_{\iota<\omega_\beta}\mu_\iota f_\iota$.

Suppose $\lambda_\tau\neq\mu_\tau$ for some $\tau<\omega_\beta$. Without loss of generality we may assume $\lambda_\tau=0$ (since $x-\lambda_\tau(e_\tau+f_\tau)\in\sigma_0F$).

Let $W:=(e'_\tau,f'_\tau)$ and $t\in W^\perp$ such that $x+t\in F$. If follows $t=\sum_{\iota<\omega_\beta}\lambda^*_\iota e_\iota+\sum_{\iota<\omega_\beta}\mu^*_\iota f_\iota+g,\ g\in E_0$, with $\lambda^*_\tau=\mu^*_\tau=0$. Therefore the τ-th component of $x+t$ is $\mu_\tau f_\tau$ and $x+t\notin F$ a contradiction. Now $\lambda_\tau=\mu_\tau$ and $x\in F$.

Proof of (d): Using (b) of Lemma 1 in 1.4 and (a) from above one gets for $\delta>0$ and $\delta\le\gamma$, respectively,
$$\sigma_\delta F=\sum_{\kappa<\omega_\gamma}\sigma_\delta(\sum_{\iota<\omega_\beta}(e_\iota+h_{\iota\kappa}))=F$$
$$\sum_{\iota<\omega_\beta}(e_\iota)+\sum_{\substack{\iota<\omega_\beta\\ \kappa<\omega_\gamma}}(h_{\iota\kappa})=\sum_{\iota<\omega_\beta}\sigma_\delta(\sum_{\kappa<\omega_\gamma}e_\iota+h_{\iota\kappa})\subset\sigma_\delta(F)\subset\sigma_0F\subset\sum_{\iota<\omega_\beta}(e_\iota)+\sum_{\substack{\iota<\omega_\beta\\ \kappa<\omega_\gamma}}(h_{\iota\kappa}).$$

Theorem 46. *The list of relations (R2)-(R64) for the indices is complete: Given $(k,\varepsilon,-)\neq(k,-1,id)$ and cardinals α_i, $2\le i\le 64$, satisfying the relations there is aε-hermitean space E' with subspace F' such that for the indices we have $\alpha'_i=\alpha_i$ for all i. Namely, if each (E_{pj},F_{pj}) is a copy of the p-th elementary example and $I=\{2,\dots,59\}\setminus\{31,32,35,36\}$ then*
$$E'=\bigoplus_{p\in I}\bigoplus_{j<\alpha_p}E_{pj},\quad F'=\sum_{p\in p}\sum_{j<\alpha_p}F_{pj}.$$

Proof. For each $p \in I$ consider the p-th example and its indices a_i, $i \in I$. Verify that, if $a_i = \aleph_{k(i,p)}$ in the example then $a_i \geq \ldots + a_p \aleph_{k(i,p)} + \ldots$ occurs in the relation (Ri). Observe that $a_i = 1$ if and only if $i = p$ and that $a_i = 0$ for all other $i \in I$.

Then, in the orthogonal sum (E', F') we have the indices $a_i = \alpha_i + \sum_{p \in I} \alpha_p \aleph_{k(i,p)} = \alpha_i$ for $i \in I$. The remaining indices are obtained by (R31),(R35), and (R60)-(R64).

4.5 Classification results

Theorem 47 (Congruence). *Let (E, Φ) be a nd, diagonal, trace-valued ε-hermitean space over $(k, \varepsilon, -)$ with $\dim E \leq \aleph_3$ and let F, F' be two subspaces of E. The following three conditions are sufficient for F and F' to be congruent in E*

(a) There is a lattice isomorphism $\mathcal{V}_E(F) \longrightarrow \mathcal{V}_E(F')$ that respects indices.

(b) $\dim(F/F \cap F^\perp) < \infty$ implies $F \cong F'$
$\dim(F^\perp / F^\perp \cap F^{\perp\perp}) < \infty$ implies $F^\perp \cong F'^\perp$.

(c) Each infinite dimensional subspace of (E, Φ) contains isotropic vectors.

Proof. By 2.6 we know that $\mathcal{V}_E(F)$ is 3-quadratic, by Thm 40 that it is finite, acyclic, and has property Δ. So Thm. 35 applies to yield the Mapping Theorem in the *alternate* case. In the sequel "alternate" is weakened to "trace valued with properties (b) and (c)". The proof is based upon [Wil87,p.83-86].

We choose orthogonal partitionings S, S' and bases B, B' as in Lemma 32. Again it suffices to show that for

$$\mathcal{F} := \{\phi : X \to X' \mid \phi \text{ is partitioned isometry with (18), (19), (19)}'\}$$

each (B, B')-saturated $(\phi : X \to X') \in \mathcal{F}$ satisfies (PP). So let $w' \in E$ be given. We may w.l.o.g. assume that (the compact) $P' := D'(w', X')$ is irreducible and that $w' \in P' \setminus (X' + P'_*)$. If $P' \in (r{F'}^\perp / \langle 0 \rangle) \subseteq \mathcal{V}(F')$, then construct $\widetilde{\varphi} : X \oplus W \to X' \oplus W'$ as in the proof of Thm. 35 (here we have $\dim(W) \leq 2$). Since $W \subseteq rF^\perp$ and $W' \subseteq r{F'}^\perp$ are totally isotropic, the restriction $\widetilde{\varphi}|W$ is trivially isometric, i.e. $\widetilde{\varphi}$ is a partitioned isometry satisfying (18),(19),(19') of Section 3 and $w' \in \operatorname{im}\widetilde{\varphi}$. According to the "worst case" picture of $\mathcal{V}_E(F)$ in Figure 5 (with a replaced by F) it remains to see what happens if the irreducible element P (the "left" pendant of P') is from the set

$$\mathbf{M} := \{\sigma_\gamma F \mid 0 \leq \gamma \leq 4\} \cup \{(r\sigma_\gamma F)^\perp \mid 0 \leq \gamma \leq 4\} \cup \{F^\perp, E\} \tag{34}$$

This leads to the following case distinction.

Case 1: $(\forall P \in \mathbf{M})$ $\dim(P/P_*) = \infty$ (similar to [Gro79,p.123]). According to Lemma 32 there is a w with (i) $w \in P \setminus (X + P_*)$, ($ii$) $\varphi(w_h) = w'_h$, (iii) $(\forall x \in X)$ $\Phi(x, w) = \Phi(\varphi(x), w')$. It suffices to construct a y such that $w + y$ satisfies (i), (ii), (iii) and $\|w + y\| =$

$\|w'\|$. Let $rP \oplus Y = P$. Then $\dim(Y) \geq \dim(P/P_*)$. We claim that $\dim(P/P_*) > \dim(X \oplus \langle w \rangle) =: d$. This is clear for $d < \aleph_0$. Assume that $d \geq \aleph_0$ and $\dim(P/P_*) \leq d$. Then (30) implies $w \in X + P \subseteq X + P_*$, contradicting (i). Putting $Y_1 := Y \cap (X \oplus \langle w \rangle)^{\perp}$, it follows from $\dim(Y/Y_1) \leq \dim(E/(X \oplus \langle w \rangle)^{\perp}) \leq d$ (see Lemma 1) that $\dim(Y_1) = \dim(Y)$. Now $\dim(rY_1) \leq \dim(Y_1^{\perp}/Y^{\perp}) \leq d$ since Y is non-degenerate. Whence any supplement of rY_1 in Y_1 is regular and infinite-dimensional. Assumption (c) together with [Gro79,p.14] yields $\|Y_1\| = T$. So there is a $y_1 \in Y_1$ with $\|y_1\| = \|w'\| - \|w\|$, $\|w + y_1\| = \|w'\|$ (since $y_1 \perp w$). In case of $w + y_1 \in X + P_*$, pick $\bar{Y}$ with $P_* \oplus \bar{Y} = P$ and put $Y_2 := \bar{Y} \cap (X \oplus \langle w + y_1 \rangle)^{\perp}$. Since $\bar{Y}$ is non-degenerate, one concludes as above that $\dim(Y_2) = \dim(\bar{Y}) > d$ and $\dim(rY_2) \leq d$. According to (c) and [Gro79,p.14] there is a decomposition $rY_2 \oplus \langle y_\iota \mid \iota < \dim(Y_2) \rangle = Y_2$ with isotropic vectors y_ι and because of $\dim(X \oplus \langle w + y_1 \rangle + P_*/P_*) \leq d$ there is a $y_\iota \in X \oplus \langle w + y_1 \rangle + P_*$. Setting $y := y_1 + y_\iota$ one has $\|w + y\| = \|w'\|$ and $w + y$ satisfies $(i), (ii), (iii)$ (observe $(w + y)_h = w_h \mapsto w'_h$).

Case 2: $a_{54} := \dim(F/rF) = \infty$ and $a_{53} := \dim(F^{\perp}/rF^{\perp}) = \infty$. For $P' = F'$, $F'^{\perp}$ property (PP) follows from case 1. Otherwise we claim that

$$(35) \qquad (\forall P \in \mathbf{M} \setminus \{F, F^{\perp}\}) \quad \dim(P_*/rP_*) = \infty.$$

Proof of (35). For $P \in \{E\} \cup \{(r\sigma_\gamma F)^{\perp} \mid 0 \leq \gamma \leq 4\}$ one has $rP_* \subseteq rF^{\perp}$, whence $\dim(P_*/rP_*) \geq \dim(F + F^{\perp}/F^{\perp}) = a_{54} = \infty$. For $P \in \{\sigma_\gamma F \mid 0 \leq \gamma \leq 3\}$ we have $P_* = \sigma_{\gamma+1} F + r\sigma_\gamma F$ and $rP_* = (\sigma_{\gamma+1} F + r\sigma_\gamma F) \cap (\sigma_{\gamma+1} F)^{\perp} \cap (r\sigma_\gamma F)^{\perp} = r\sigma_\gamma F$, whence again $\dim(P_*/rP_*) \geq a_{54} = \infty$.

Now let $P \in \mathbf{M} \setminus \{F, F^{\perp}\}$. In case of $\dim(P/P_*) = \infty$ correct the vector $w \in P \setminus (X + P_*)$ as in the first case. Otherwise $\dim(P/P_*) < \infty$ forces $\dim(X \oplus \langle w \rangle) < \infty$ (see above), so $\dim(X \oplus \langle w \rangle) < \dim(P_*/rP_*)$ by (35). Thus, setting $rP_* \oplus Y = P_*$ and $Y_1 := Y \cap (X \oplus \langle w \rangle)^{\perp}$, it follows from $\dim(Y_1) = \infty$ and $\dim(rY_1) \leq \dim(Y_1^{\perp}/Y^{\perp}) \leq \dim(Y/Y_1) < \infty$ that $rY_1 \neq Y_1$. Thence (c) and [Gro79,p.14] yield a $y_1 \in Y_1$ with $\|y_1\| = \|w'\| - \|w\|$. Therefore $w + y_1 \in P \setminus (X + P_*)$ (recall $y_1 \in P_*$) and $\|w + y_1\| = \|w'\|$.

Case 3: $a_{54} = \infty$ and $a_{53} < \infty$. According to (b) there is an isometry $\psi : F^{\perp} \to F'^{\perp}$ and clearly $\psi(rF^{\perp}) = rF'^{\perp}$. Setting $F^{\perp} = rF^{\perp} \oplus X_0$ one has $F'^{\perp} = rF'^{\perp} \oplus X'_0$ where $X'_0 := \psi(X_0)$. Since $F^{\perp} \in \mathcal{V}(F)$ is prime, the map $\varphi_0 := \psi | X_0 : X_0 \to X'_0$ is an isometry which satisfies (18),(19),(19'). If one begins the recursive construction of $\phi : E \to E$ with $\varphi_0 : X_0 \to X'_0$ (instead of $\varphi_0 : \langle 0 \rangle \to \langle 0 \rangle$), then the case $D(w, X) = F^{\perp}$ respectively $D'(w', X') = F'^{\perp}$ does not occur anymore. The spaces $P \in \mathbf{M} \setminus \{F^{\perp}\}$ are dealt with as in the second case. (More on so called "initial triples" (X_0, φ_0, X'_0) can be found in [Wil87,p.70].)

Case 4: $a_{54} < \infty$. From $\sum_{i=54}^{63} a_i = \dim((rF)^{\perp}/F^{\perp}) = a_{54}$ (Lemma 1) follows $(\forall 55 \leq i \leq 63)$ $a_i = 0$ (Figure 8). Thus in $\mathcal{V}(F)$ the only join irreducibles from $\mathbf{M}$ are F, $F^{\perp}$ and E.

Subcase (a): $a_{53} = \infty$ and $a_2 := \dim(rF) < \infty$. According to (b) there is an isometry $\psi : F \to F'$. As in the third case $\psi(rF) = rF'$, $F = rF \oplus X_0$, $F' = rF' \oplus X'_0$ $(X'_0 := \psi(X_0))$, and $\varphi_0 := \psi | X_0 : X_0 \to X'_0$ is an isometry satisfying (18),(19),(19'). During the inductive construction of $\phi : E \to E$ the case $D(w, X) = F^{\perp}$ is trouble-free since $a_{53} = \infty$ (see case 1), and the case $D(w, X) = E$ only occurs as long as $\dim(X) < \infty$ (since $\dim(E/E_*) < \infty$). Because of $\dim(E_*/rE_*) = \dim((rF)^{\perp}/(rF)^{\perp\perp}) \geq a_{53} > \dim(X)$, we can proceed as in the second case.

Subcase(b): $a_{53} < \infty$ and $a_2 = \infty$. According to (b) one has $F \cong F'$ and $F^{\perp} \cong F'^{\perp}$. Therefore we may define $\varphi_0 : X_0 \to X_0'$ as in (a) and find decompositions $F^{\perp} = rF^{\perp} \oplus \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle$ and $F'^{\perp} = rF'^{\perp} \oplus \langle x_1' \rangle \oplus \cdots \oplus \langle x_n' \rangle$ with $(\forall 1 \leq i, j \leq n)\ \Phi(x_i, x_j) = \Phi(x_i', x_j')$. For $0 \leq i \leq n$ consider the mapping

$$\varphi_i : X_0 \ominus \langle x_1, \ldots, x_i \rangle \to X_0' \ominus \langle x_1', \ldots, x_i' \rangle, \quad \varphi_i | X_0 := \varphi_0,\ \varphi_i x_j := x_j'.$$

By construction φ_0 is an isometry satisfying (18),(19),(19'). Assume that φ_{i-1} has the same properties. Because of $x_i \notin (X_0 \ominus \langle x_1, \ldots, x_{i-1} \rangle) + rF^{\perp}$ we have $D(X_0 \ominus \langle x_1, \ldots, x_{i-1} \rangle, x_i) = F^{\perp}$ and analogously $D'(X_0' \ominus \langle x_1' \ldots, x_{i-1}' \rangle, x_i') = F'^{\perp}$. The spaces $F^{\perp}$, $F'^{\perp}$ being prime, we know that φ_i satisfies again (18),(19),(19'). Because of $x_i \perp X_0$, $x_i' \perp X_0'$ and $(\forall 1 \leq j \leq i)\ \Phi(x_i, x_j) = \Phi(x_i', x_j')$, also φ_i is isometric. By induction it follows that $\varphi_n : X_0 \ominus \langle x_1, \ldots, x_n \rangle \to X_0' \ominus \langle x_1', \ldots, x_n' \rangle$ takes care of the join irreducibles F and $F^{\perp}$. The case $D(w, X) = E$ is harmless because of $\dim(E/E_*) = a_2 = \infty$ (1. case).

Subcase (c): $a_{53} < \infty$ and $a_2 < \infty$. Then $\dim(F) = a_2 + a_{53} < \infty$ which, together with Lemma 1, implies $F = F^{\perp\perp}$ whence $\dim(E) = a_2 + a_{53} + a_{54} + a_{64} < \infty$. By assumption (b) the spaces F and F' are isometric. By Witt's Theorem (see [Gro79,p.376])) there is an isometry $\phi : E \to E$ with $\phi(F) = F'$.

Remark. Besides the alternate case, conditions (b) and (c) of the Congruence Theorem are always satisfied if k has finite u-invariant, i.e. if there is a finite bound u such that all forms over k of dimension $> u$ are isotropic. The latter is a rather big class of fields including the algebraically closed ones, cf [Lam73,p.315] and [Gro79,App.I].

Theorem 48 (Decomposition). *Let (E, Φ) be a diagonal, trace-valued, ε-hermitean, non-alternate space of $\dim E \leq \aleph_3$ over a field of finite u-invariant and let F be a subspace of E. Then there is an orthogonal decomposition of the pair (E, F) into elementary examples as introduced in the preceeding section. The multiplicity of the i-th example can be chosen as the index a_i of the pair (E, F).*

Proof. First observe that here any two spaces of the same infinite dimension (and the same data) are isometric - combine [Gro79]II.3 Ex.2, II.4.Thm.2 and X.8 Thm.9. Now, given (E, F) according to Thm. 44 the indices $\alpha_i = a_i$ satisfy the relations (R2)-R(64) and so by Thm. 46 we may construct (E', F') as orthogonal sum of elementary examples such that $a_i' = a_i$ for all i. In particular, the homorphisms π_{EF} and $\pi_{E'F'}$ identify the same quotients of $\mathcal{V}_3(a)$ which means that there is a (unique) isomorphism η of $\mathcal{V}_E(F)$ onto $\mathcal{V}_E'(F)$ and η preserves indices. This implies $\dim \eta X = \dim X$ for all X in $\mathcal{V}_E(F)$ including $X = E, F, F^{\perp}$. Therefore, we may identify E' with E and, if neither $\dim F$ nor

$\dim F^\perp$ is finite, the Congruence Theorem applies to provide an isometry taking F to F' and so a decomposition of the pair (E, F).

If $\dim F$ or $\dim F^\perp$ is finite, we get an orthogonal decomposition into examples of type $p = 2$ and $53 \leq p \leq 58$. Namely, if $\dim rF < \infty$ choose a supplement U of $(rF)^\perp$ in the interval $[rF, E]$. Then $rF = (rF)^{\perp\perp}$ and $U^\perp \cap rF = (U + rF)^\perp = rF$ whence $U^\perp \cap U = 0$. Also $\dim U = 2 \dim rF < \infty$ and so $E = U \oplus U^\perp$, cf [Gro79,I.5]. Now $F \subseteq (F + U^\perp) \cap (F + U) \subseteq (rF)^\perp \cap (F + U) = F$ using modularity, whence we got an orthogonal decomposition of (E, F) into a summand with $F = rF = (rF)^\perp$ and another with $rF = 0$. So we have to consider (E, F) with $F = F^\perp$ or $rF = 0$. For the first, any orthogonal basis $r_1, \ldots, r_n$ of F gives rise to a decomposition $\bigoplus_i < r_i, y_i >$ of (E, F) into summands of type $p = 2$. If $rF = 0$ and F has finite dimension, $E = F \oplus F^\perp$ is an orthogonal decomposition of (E, F) with summands satisfying $F = E$ and $F = 0$, respectively, which obviously decompose into type $p = 54$ and $p = 53$ examples (since E is diagonal). If $rF = 0$ and $\dim F^\perp < \infty$, then $E = F^\perp \oplus F^{\perp\perp}$ is an orthogonal decomposition into a summand with $F = 0$ (dealt with, already) and one with $F^\perp = 0$. For the latter, the case $\dim F < \infty$ is already done, so we may assume that $\dim F$ is infinite and argue as above using the Congruence Theorem and $F \cong F'$, $F^\perp = 0 = F'^\perp$.

Remarks. 1. Both theorems generalize to the case where $\dim(E) \leq u_0$, the first weakly inaccessible cardinal, and F is σ_4-closed. In this case $\mathcal{V}_E(F)$ is 3-quadratic, too. Also, the proof of Thm. 35 extends to include dimensions less than u_0, cf [Wil87].

2. The Mapping Theorem no longer holds for $\dim E = \aleph_5$. Namely, there is a nd, diagonal, alternate space E of dimension $\aleph_5$ and with two subspaces $F, F' \subseteq E$ such that there is an index-preserving isomorphism of the finite lattice $\mathcal{V}_E(F) \longrightarrow \mathcal{V}_E(F')$ (card $\mathcal{V}_E(F) = 32$) but F and F' are not congruent in E. The "trick" is to choose $F, F' \subseteq E$ in such a way that $\mathcal{V}_E(F)$ and $\mathcal{V}_E(F')$ contain a sublattice $\mathcal{M}_4$. It is well known that lattice isomorphisms between such sublattices of $\mathcal{L}(E)$ are usually not even induced by a linear automorphism $\varphi = E \xrightarrow{\sim} E$! Exact information about the case $\dim E = \aleph_4$ is still missing. In any case, there is a distributive 4-quadratic lattice with one generator which is not Artinian. See [Wil87,p.91,102] for these results.

5 Classification of subspaces in non trace-valued spaces

5.1 Assumptions to be made Let (E, Φ) be a ε-hermitean space over $(k, -)$ with trace-valued part E^*. For each subspace $X \subset E$ we put $X^* := X \cap E^*$.

Throughout the whole section we make the following assumptions:

(36) Φ *is non-degenerate and* $\dim E \leq \aleph_0$

(37) Any infinite-dimensional subspace of E *is isotropic*

(38) $\dim E/E^* < \infty$.

It is not difficult to see that there is a very large class of division rings $(k, \varepsilon, -)$ admitting ε-hermitean spaces (E, Φ) with (36), (37), (38). In particular, this contains the class of all fields k with $\mathrm{char}(k) = 2$ and $[k : k^2] < \infty$ or $\mathrm{char}(k) \neq 2$ and $u(k) < \infty$ ($u(k)$ means here the u-invariant of k, i.e. the maximal dimension of anisotropic quadratic forms over k).

Let $F \subset E$ be a subspace. We recall that the problem to be discussed is the classification of the pair (E, F) up to orthogonal isomorphism of the hermitean space (E, Φ).

In the sequel we will present a solution of this problem in the following cases:

I. $E^* = E^{*\perp\perp}$ *and* $F = F^{\perp\perp}$

II. $E^* = E^{*\perp\perp}$ *and* $E^{*\perp *} = 0$ *or* $E^{*\perp} \subset E^*$

III. k *is a perfect field and* E *admits an orthogonal basis.*

5.2 Reduction steps in cases I and II

In order to distinguish these cases from the third, it is appropriate to remember that

$$E^{*\perp\perp} = E^* \tag{39}$$

together with (38) means that (E, Φ) does not admit an orthogonal basis if Φ is symmetric (cf [Gro79,p.66]). Another important consequence of (38) and (39) concerns the value map: in fact, it holds for arbitrary subspaces $X, Y \subset E$.

$$||X|| = ||Y|| \iff X + E^* = Y + E^* \iff X^\perp \cap E^{*\perp} = Y^\perp \cap E^{*\perp}\,. \tag{40}$$

The first part of our considerations reduces the classification problem to the situation where

$$F \cap E^{*\perp} = F^\perp \cap E^{*\perp} = (0)\,. \tag{41}$$

This we achieve in 7 steps by "separating away" "typical" summands of (E, F).

1. Consider $F_1 := F \cap F^\perp \cap E^{*\perp}$. There is a supplement F_1' of $E \cap F_1^\perp$ in E and a decomposition

$$(E, F) = (E_1, F_1) \overset{\perp}{\oplus} (E_1', F_1')$$

where $E_1 := F_1 \oplus F_1'$, $E_1' = E_1^\perp$ and $F_1' := F \cap E_1'$. The isometry class of (E_1, F_1) is fully described by the isometry class $\hat{E}_1$ of E, and the indexed hermitean lattice of Fig. 9(a). This lattice depends only on the isometry class of the pair (E, F). Note that this is not the case for $\hat{E}_1$; the "true" invariant is in fact $||F_1^\perp||$. $\hat{E}_1$ is determined by the choice of a supplement of $||F_1^\perp||$ in $||E||$ as subspaces of S/T; of course, this is possible in infinitely many ways. This difficulty can be avoided by fixing once for all such a supplement.

2. We assume, without loss of generality, that $E = E_1'$. Consider $F_{21} := F \cap E^{*\perp *}$. Of course, $\dim F_{21} < \infty$ and $F/F \cap F_{21}^{\perp} \cong F + F_{21}^{\perp}/F_{21}^{\perp} \cong F_{21}/F_{21} \cap F^{\perp} \cong F_{21}$, so there is a supplement F_{22} of $F \cap F_{21}^{\perp}$ in F and a decomposition

$$(E,F) = (E_2,F_2) \overset{\perp}{\oplus} (E_2',F_2')$$

where $F_2 := E_2 := F_{21} \oplus F_{22}$, $E_2' := E_2^{\perp}$ and $F_2' := F \cap E_2'$. The isometry class of (E_2,F_2) is fully described by $\hat{E}_2$ and the indexed hermitean lattice of Fig. 9(b). Note that $\hat{E}_2$ is determined by the choice of a supplement of $\|F_{21}^{\perp}\|$ in $\|F_1^{\perp}\|$ (compare the remark at the end of the previous step).

3. We assume, without loss of generality, that $E = E_2'$. Consider $F_3 := E_3 := F \cap E^{*\perp}$. This is a finite-dimensional rigid subspace of E; hence there is a decomposition:

$$(E,F) = (E_3,F_3) \overset{\perp}{\oplus} (E_3',F_3')$$

where $E_3' := E_3^{\perp}$, $F_3' := F \cap E_3'$. The isometry class of (E_3,F_3) is fully described by $\hat{E}_3$ and the indexed hermitean lattice of Fig. 9(c). Note that in this case the set

$$\{\|x\| \in S \mid x \in F_3 \setminus \{0\}\} \subset S$$

is an invariant of $\widehat{(E,F)}$ and can adequately be chosen to span a supplement of $\|F_3^{\perp}\|$ in $\|F_{21}^{\perp}\|$.

4. W.l.o.g. we assume $E = E_3'$. Consider $D_4 := (F \cap F^{\perp})^{\perp\perp} \cap E^{*\perp}$. By the classification of the pair $(F, F \cap F^{\perp})$ (see [Gro79,ch.VIII]) one can find a subspace $F_4 \subset F \cap F^{\perp}$ such that $F_4^{\perp\perp} = F_4 + D_4$. Using [Mor88, Satz, p.119] one obtains a decomposition

$$(E,F) = (E_4,F_4) \overset{\perp}{\oplus} (E_4',F_4')$$

where $E_4 = F_4^{\perp\perp} \oplus F_4''$ for some supplement F_4'' of $D_4^{\perp}$ in E and $E_4' := E_4^{\perp}$, $F_4' := F \cap E_4'$. The isometry class of (E_4,F_4) in determined by $\hat{E}_4$ and the indexed hermitean lattice of Fig. 9(d).

5. W.l.o.g. $E = E_5'$. Consider $D_5 := F^{\perp} \cap F^{\perp\perp} \cap E^{*\perp}$. Using the classification of the pair (E, F^*) (see [Mor86, p.284]) one can find a subspace $F_5 \subset F^*$ such that $F_5^{\perp\perp} = F_5 + D_5$. Moreover, there is a F^*-decomposition

$$E = (F_5^{\perp\perp} \oplus F_5') \overset{\perp}{\oplus} E_5'.$$

Of course, $F = F_5 + (D_5 + E_5') \cap F$. Let B be a supplement of $E_5' \cap F$ in $E_5' \cap (D_5 + F)$. Surely $B^* = (0)$, so $E_5' = B \overset{\perp}{\oplus} E_5''$. It is now an easy matter to modify F_5' and B in order to obtain an F-decomposition of E. In other words, one finds a decomposition

$$E = (F_5^{\perp\perp} \oplus F_5'') \overset{\perp}{\oplus} E_5''$$

such that

$$F = F_5 + F \cap E_5'', \quad E^* = F_5^{\perp\perp} + E_5^{''*}.$$

Put $E_5 := F_5^{\perp\perp} + F_5^{\perp\perp}$. One can check that the isometry class of (E_5, F_5) is determined by $\hat{E}_5$ and the indexed hermitean lattice of Fig. 9(e).

6. W.l.o.g. $E = E_5'$. Consider $D_6 := F^\perp \cap E^{*\perp*}$. Then there is a F-decomposition

$$(E, F) = (E_6, F_6) \overset{\perp}{\oplus} (E_6', F_6')$$

where $F_6 := (0)$ and E_6 is of the shape $E_6 = D_6 \oplus D_6'$ for a supplement D_6' of $F^\perp \cap D_6^\perp$ in $F^\perp$. The isometry class of (E_6, F_6) is determined by $\hat{E}_6$ and the indexed hermitean lattice of Fig. 9(f).

7. W.l.o.g. $E = E_6'$. Consider $F_7 := F^\perp \cap E^{*\perp}$. Put $E_7 := F_7$. $\widehat{(E_7, F_7)}$ is determined by $\hat{E}_7$ and the indexed hermitean lattice of Fig. 9(g). Put $(E_r, F_r) := (E_7', F_7')$. For the original (E, F) we have now a decomposition

$$(E, F) = \overset{\perp}{\bigoplus_{i=1}^{7}} (E_i, F_i) \overset{\perp}{\oplus} (E_r, F_r) \tag{42}$$

where (E_i, F_i) $(i = 1, \ldots, 7)$ are the pairs separated through the above steps and (E_r, F_r) satisfies condition (41). Such a pair shall be termed *quasi reduced*.

Note that the subspaces $\|F_1^\perp\|, \|F_{21}^\perp\|, \|F_3^\perp\|, \|D_4^\perp\|, \|D_5^\perp\|, \|D_6^\perp\|, \|F_7^\perp\| \subset S/T$ are all invariants of $\widehat{(E, F)}$ and determine the above reduction (42). The sublattice they span in $\mathcal{L}(S/T)$ looks as Fig. 10.

In the sequel we will assume $E = E_r$ and discuss the classification of a quasi reduced pair. From (41) it is clear that

$$\dim F/F^* = \dim F^\perp/F^{\perp*} = \dim E/E^*$$

and also

$$\dim F/F \cap E^{*\perp*\perp} = \dim E/E^{*\perp*\perp}.$$

Let C be a *fixed* supplement of $F \cap E^{*\perp*\perp}$ in F. Then there is a decomposition

$$\begin{aligned} E &= H \overset{\perp}{\oplus} (D \oplus C) \overset{\perp}{\oplus} B \\ \text{with} \quad E^* &= H + D \\ E^{*\perp} &= D + B. \end{aligned} \tag{43}$$

Here $H = (C + E^{*\perp})^\perp$ depends only on the choice of C, but $\hat{H}$ is independent thereof. We remember that, in view of (37), H is hyperbolic if $\dim E = \aleph_0$ and, in particular, alternate.

Returning to our classification problem, we note that congruent subspaces $F, \overline{F}$ of E will possess congruent supplements $C, \overline{C}$ of $F \cap E^{*\perp*\perp}$ in F and $\overline{F} \cap E^{*\perp*\perp}$ in $\overline{F}$, respectively, so we will not essentially restrict our investigation if we will classify the triple (E, F, C) instead of the pair (E, F).

Before going any further, we first achieve one more reduction step: Consider $D_0 := (F^* + H) \cap E^{*\perp}$. Obviously, D_0 is an invariant of the congruence class of (F, H) in E. Let $F_8 \subset C$ be a supplement of $F \cap D^\perp$ in $F \cap D_0^\perp$; there is a supplement D_8 of D_0 in D such that $D_8 \oplus F_8$ is an orthogonal sum of metabolic planes. Putting $E_8 := D_8 \oplus F_8$, one obtains a decomposition of the kind

$$\begin{aligned} E &= E_8 \overset{\perp}{\oplus} (D_0 \oplus C_0) \overset{\perp}{\oplus} B \overset{\perp}{\oplus} H \\ \text{such that} \quad F &= F_8 + F \cap E_8^\perp \\ \text{and} \quad E^* &= D_8 + D_0 + H \\ E^{*\perp} &= D_8 + D_0 + B\,. \end{aligned}$$

The isometry class of (E_8, F_8) is completely determined by $\hat{F}_8$ and the indexed hermitean lattice of Fig. 11. So we may assume, without loss of generality, that $E = E_8^\perp$, and call such a triple (E, F, C) *reduced*.

Now, we put

$$(44) \qquad U := F \cap H, \qquad V := (F + D) \cap H, \qquad W := (F + E^{*\perp}) \cap H\,.$$

The relations (45) among all crucial subspaces we have introduced are best described by the diagram of Fig. 12.

Suppose now $(\overline{E}, \overline{F}, \overline{C})$ is another triple, with associated subspaces $\overline{H}, \overline{U}, \overline{V}, \overline{W}$. If (E, F, C)
$\cong (\overline{E}, \overline{F}, \overline{C})$, then clearly $(H, W, V, U) \cong (\overline{H}, \overline{W}, \overline{V}, \overline{U})$ and $(H^\perp, C) \cong (\overline{H}^\perp, \overline{C})$. Suppose conversely that the latter condition holds. How can F be recovered from the given subspaces? We explain this by supposing E fixed. Put $m := \dim E^{*\perp}$, and let $w_1, \ldots, w_m$ and $z_1, \ldots, z_m$ be basis of W modulo U and $E^{*\perp}$, respectively. Each $F \subset E$ with (45) will then have a unique basis $f_1, \ldots, f_m$ modulo U such that

$$f_i \equiv \sum_{j=1}^{m} \mu_{ij} w_j + z_i \pmod U$$

for some matrix $(\mu_{ij}) \in GL_m(k)$.
If we assume

$$(46) \qquad k(w_1, \ldots, w_m) \quad \text{is a totally isotropic supplement of} \quad U \quad \text{in} \quad W$$

then any such matrix will furnish a representative of the congruence class of F in E. We will see that in practice only few exceptions escape to the "rule" (46).

It is clear that an important task is the classification of the quadruplet (H, W, V, U). Using the lattice method, we associate the sublattice $\mathcal{V} = \mathcal{V}(H, W, V, U)$ of the quadratic lattice $(\mathcal{L}(H), \cap, +, \perp)$ generated by H, W, V, U. Although $\mathcal{V}$ is not known in general, it is possible to compute it in the cases I & II (i.e. $F = F^{\perp\perp}$, respectively $E^{*\perp*} = (0)$ or $E^{*\perp} \subset E^*$). In the next sections we determine $\mathcal{V}$ under this hypothesis in a more general lattice theoretic frame.

5.3 The classification in case I

For our purposes the most convenient lattice theoretic axiomatic context is that of indexed Galois Fano lattices. We recall that the Fano axiom is motivated by the fact that our lattices are represented over fields of characteristic 2. Hence we may formulate the following problem in this class of lattices:

Compute the free object $\mathcal{V}_I$ generated by a 3-chain $u \leq v \leq w$ under the relations

$$u = u^{\perp\perp} \text{ and } \delta(w/u) < \aleph_0 . \tag{47}$$

We define

$$p = wu^{\perp},\ p_1 = vu^{\perp} = pv,\ p_2 = wv^{\perp} = pv^{\perp},\ q = pp^{\perp}.$$

Theorem 49. *There are 21 subdirectly irreducible factors of $\mathcal{V}_I$. The diagrams and generators thereof are given in Figure 14 and Table 3. The index functions on factor no. i are in 1-1-correspondence to the index γ_i $(i = 0, \ldots, 20)$ of a prime quotient in the minimal congruence: This may be infinite only for $i \in \{18, 19\}$. The factors no. 0-19 are implemented by hermitean spaces.*

Remarks. 1. The closed elements of the Galois lattice $\mathcal{V}_i$ are represented in the diagram by large disks. The operation $^\perp$ is then uniquely determined by the given pairs $x, x^\perp$. One of the prime quotients generating the minimal congruence is marked by a γ. The corresponding index is later on referred to as γ_j, j the number from Table 3 of the model under consideration. We also indicate the prime quotients p/p_* with index $\aleph_0$ under the smallest index function δ_0. Actually, $\delta_0(x/y) = 1$ for every prime quotient x/y not transposed to one of those - and any such x/y is in the minimal congruence.

2. In order to understand the diagrams $\mathcal{V}_7$ and $\mathcal{V}_9$ we specify that the interval $[m, 1]$ is the lattice $\mathbb{P}(\mathbb{F}_2^3)$.

3. The factors 19 and 18 are redundant in an image of $\mathcal{V}$ of $\mathcal{V}_I$ if $\delta(u/0) = \aleph_0$ and $\delta(w^\perp/0) = \aleph_0$, respectively. They are homomorphic images of all factors no. 7-17 - and no other epimorphisms occur.

4. $\mathcal{V}$ is distributive iff $pp_1^\perp p_2^\perp = uu^\perp + ww^\perp$. $\mathcal{V}$ is of *finite type* if $\delta(1) < \aleph_0$ (equivalently $\delta(u) + \delta(w^\perp) < \aleph_0$) for every δ with $\delta(w/u) < \aleph_0$. This occurs necessarily in cases no. 1-6, possibly in no. 0,18,19 (or subdirect products, thereof), and is impossible otherwise.

$No.$	Geometric invariants	Metric i.	Arithmetic invariants
0	$\mathcal{V}_2$, $u = v = w = a$	$\hat{E} = \hat{H}$	
1	$\mathcal{V}_2$, $u = v = a, w = 1$	$\hat{E}$	
2	$\mathcal{V}_2$, $u = a, v = w = 1$	$\hat{E}, \hat{C}$	
3	$\mathcal{V}_2$, $u = 0, v = w = a$	$\hat{E}, \hat{C}$	
4	$\mathcal{V}_2$, $u = v = 0, w = a$	$\hat{E}$	
5	$\mathcal{V}_1$, $u = 0, v = w = 1$	$\widehat{F^*}, \hat{C}$	
6	$\mathcal{V}_1$, $u = v = 0, w = 1$	$\hat{E}$	$\mu \in GL_{\gamma_6}(k)/\mathcal{O}(H)$
7	$\mathcal{V}_5$, $u = a, v = b, w = c$	$\hat{E}, \hat{C}$	
8	$\mathcal{V}_5$, $u = a, v = w = c$	$\hat{E}, \hat{C}$	$\sigma(U + R)$
9	$\mathcal{V}_5$, $u = v = a, w = c$	$\hat{E}$	$\sigma(U + R)$
10	$\mathcal{V}_6$	$\hat{E}, \hat{C}$	$\sigma(U + R)$
11	$\mathcal{V}_7$	$\hat{E}, \hat{C}, \hat{P}$	$\sigma(U + P), \sigma(U + P_1), \sigma(U + P_2),$ $\sigma(U + R), \sigma(W \cap P_1^\perp)$
12	$\mathcal{V}_8$, $u = a, v = b, w = c$	$\hat{E}, \hat{C}$	$\sigma(U + P), \sigma(U + P_1), \sigma(W \cap P^\perp)$
13	$\mathcal{V}_8$, $u = c^\perp, v = b^\perp, w = a^\perp$	$\hat{E}, \hat{C}$	$\sigma(U + P), \sigma(U + P_2), \sigma(W \cap P^\perp)$
14	$\mathcal{V}_9$	$\hat{E}, \hat{C}, \hat{P}$	$\sigma(U + P), \sigma(W \cap P^\perp)$
15	$\mathcal{V}_4$, $u = a, v = w = b$	$\hat{E}, \hat{C}$	
16	$\mathcal{V}_4$, $u = v = a, w = b$	$\hat{E}$	
17	$\mathcal{V}_3$, $u = v = w = a$	$\hat{E} = \hat{H}$	
18	$\mathcal{V}_1$, $u = v = w = 1$	$\hat{E} = \hat{H}$	
19	$\mathcal{V}_1$, $u = v = w = 0$	$\hat{E} = \hat{H}$	
20	$\mathcal{V}_2$, $u = 0, v = a, w = 1$		

Table 3: Factors of $\mathcal{V}_I$ and classification of a reduced triple (E, F, C) in case I

5. Of course, the existence of a free object is part of our claim, so for the moment we consider $\mathcal{V}_I$ as presentation, rather.

6. Notice the symmetry $u \leftrightarrow w^\perp$, $v \leftrightarrow v^\perp$, $w \leftrightarrow u^\perp$ - recall that the u, v, w are closed in view of Lemma 19 in 2.7!

Lemma 50. *Let $\mathcal{L}$ be any model of $\mathcal{V}_I$ with no subdirect factor no. 0-6,17,20. Then $\mathcal{L}$ satisfies the relations*

$$uu^\perp = ww^\perp = 0\,;\ p_1 \le p_1^\perp\,;\ p_2 \le p_2^\perp\,;\ vv^\perp \le q$$

$$w + u^\perp = p^\perp + w = p^\perp + u^\perp = 1$$

$$p^\perp = (u + w^\perp)^{\perp\perp},\ p_1^\perp = v^\perp + p^\perp,\ p_2^\perp = v + p^\perp,\ q^\perp = p + p^\perp$$

and is a union of intervals and so is its set of closed elements

$$\mathcal{L} = [0,p] \cup [u,w] \cup [w^\perp, u^\perp] \cup [u + w^\perp, 1]$$

$$\mathcal{L}^{\perp\perp} = [0,p]_\mathcal{L} \cup [u,w]_\mathcal{L} \cup [w^\perp, u^\perp]_\mathcal{L} \cup [p^\perp, 1]_\mathcal{L}$$

Proof. Observe that $x = x^{\perp\perp}$ for every $x \in [u,w]$ or $x \in [w^\perp, u^\perp]$: this follows from the given relations and Lemma 19. Clearly, $\mathcal{L}$ splits into intervals, as indicated, since the union of these intervals is a subalgebra containing the generators. Now, we apply the Splitting Method of 2.4.

No. 0: $\alpha a = uw^\perp$, $\alpha 1 = 1$, $\pi 0 = 0$, $\pi a = (uw^\perp)^\perp$. Since $uw^\perp$ is closed, we have hypothesis (7) of 2.4 for $1/a$ and $z = a$. The remaining hypotheses are satisfied, trivially. It follows $uw^\perp = 0$.

No. 17: $\alpha a = u$, $\alpha a^\perp = w^\perp$, $\alpha 1 = 1$, $\pi a = w$, $\pi a^\perp = u^\perp$, $\pi(a + a^\perp) = w + u^\perp$. Ad (5): trivial since $(a + a^\perp)^\perp = 0$. Ad (6): The critical quotient is $1/(a + a^\perp)$: here $(\alpha 1 \cap \pi(a + a^\perp))^\perp = w^\perp u = 0$ as already known. (7) holds, trivially. It follows $w + u^\perp = 1$.

No. 1: $\alpha a = uv^\perp$, $\alpha 1 = w$, $\pi 0 = w^\perp$, $\pi a = (uv^\perp)^\perp$. Ad (7): For the quotient $1/a$ we have $z = a$ and $(\alpha 1 \cap \pi a)^\perp = (w(uv^\perp)^\perp)^\perp \approx_0 uv^\perp + w^\perp = \alpha a + \pi 0$ - since the latter is in $[w^\perp, u^\perp]$. Also, $\delta(1/(uv^\perp)^\perp) \leq \delta(uv^\perp/0) = \delta(uv^\perp/uw^\perp) \leq \delta(v^\perp/w^\perp) < \aleph_0$. Consequently, $w + (uv^\perp)^\perp \approx_0 (w^\perp uv^\perp)^\perp$ for the quotient $a/0$ and $z = 1$. It follows $w \leq (uv^\perp)^\perp$.

No. 2: $\alpha a = uu^\perp$, $\alpha 1 = v$, $\pi 0 = v^\perp$, $\pi a = (uu^\perp)^\perp$. Ad (7): Use that $uu^\perp + v^\perp \in [w^\perp, u^\perp]$ and $v + (uu^\perp)^\perp$ are closed. The latter follows from $\delta(1/(uu^\perp)^\perp) \leq \delta(1/u + u^\perp) = \delta(w + u^\perp/u + u^\perp) < \aleph_0$. Thus $uu^\perp + v^\perp \approx_0 ((uu^\perp)^\perp v)^\perp$ and $v + (uu^\perp)^\perp \approx_0 (uu^\perp v^\perp)^\perp$. The Split Lemma yields $v \leq (uu^\perp)^\perp$. Since $uu^\perp$ is closed it follows $uu^\perp \leq v^\perp$ whence $uu^\perp \leq uv^\perp \leq w^\perp$ using no. 1. Therefore, $uu^\perp \leq uw^\perp = 0$ by no. 0. By symmetry $ww^\perp = 0$.

Now, $\delta(1/p^\perp) \leq \delta(0/p) = \delta(wu^\perp/uu^\perp) \leq \delta(w/u) < \aleph_0$ whence by Lemma 19 from 2.7 the intervals $[0,p]$ and $[p^\perp, 1]$ consist of closed elements, too. Observe $p^\perp = (u + w^\perp)^{\perp\perp}$. It follows $v^\perp + p^\perp \approx_0 (vp)^\perp = p_1^\perp$ and $r^\perp \approx_0 p^\perp + p$ and $p^\perp + w \approx_0 (pw^\perp)^\perp = (wu^\perp w^\perp)^\perp = 0^\perp = 1$.

No. 5: $\alpha 1 = p_1$, $\pi 0 = (p_1)^\perp$. Ad (7): We have that $p^\perp \leq p_1 + p_1^\perp \approx_0 (p_1 p_1^\perp)^\perp$. We may conclude $p_1 \leq p_1^\perp$.

No. 20: $\alpha a = vv^\perp$, $\alpha 1 = p$, $\pi 0 = p^\perp$, $\pi a = (vv^\perp)^\perp$. Ad (7): $vv^\perp + p^\perp \approx_0 ((vv^\perp)^\perp p)^\perp$ and $p^\perp \leq p + (vv^\perp)^\perp \approx_0 (vv^\perp)^\perp p^\perp)^\perp$. The Split Lemma yields $vv^\perp \leq p^\perp$.

Lemma 51. *Let $\mathcal{L}$ be as in Lemma 50 and, moreover, with no subdirect factor no. 15, 16, 7, 8, 9. Then $\mathcal{L}$ satisfies the relations*

$$vp^\perp = u,\ v^\perp p^\perp = w^\perp,\ vv^\perp = 0$$

$$p+v=w,\ p+v^\perp=u^\perp,\ w+v^\perp=v+u^\perp=p^\perp+v+v^\perp=1.$$

Proof. No. 15: $\alpha a=u$, $\alpha b=vp^\perp$, $\alpha b^\perp=w^\perp$, $\alpha a^\perp=u^\perp p^\perp$, $\pi b=w$, $\pi a^\perp=u^\perp$, $\pi(b+b^\perp)=w+v^\perp$, $\pi(a+a^\perp)=u+u^\perp$. Ad (5α): $\alpha(0^\perp)\leq p^\perp=(\pi 0)^\perp$. The remaining instances are obvious. In particular, $x^\perp=0$ and $\alpha(x^\perp)=0$ for $x\geq a+b^\perp$. Ad (5π): $\pi(a^{\perp\perp})=w(u+u^\perp)\simeq u+p\approx_0 (u^\perp p^\perp)^\perp=(\alpha(a^\perp))^\perp$. $\pi(b^\perp)=u^\perp(w+v^\perp)\simeq p+v^\perp\approx_0 (vp^\perp)^\perp=(\alpha b)^\perp$. Ad (6): the critical elements are $b\succ a$ and $a^\perp\succ b^\perp$. Here, $(\alpha b)^\perp=(vp^\perp)^\perp\approx_0 v^\perp+p\simeq u^\perp(w+v^\perp)=\pi(b^\perp)$ and $(\alpha a^\perp)^\perp=(u^\perp p^\perp)^\perp\approx_0 u+p\simeq w(u+u^\perp)=\pi(a^{\perp\perp})$. Ad (7): We have $z=a^\perp$ resp. $z=b$ and $\alpha z+\pi(b^\perp)=u^\perp p^\perp+u^\perp(w+v^\perp)\simeq u^\perp(p^\perp+p+v^\perp)\approx_0 u^\perp(pp^\perp v)^\perp\geq u^\perp(vp^\perp w(u+u^\perp))^\perp=\pi z(\alpha b\pi a)^\perp$ (since $p\leq p+u\simeq w(u+u^\perp)$). Respectively $\alpha z+\pi(a^{\perp\perp})=vp^\perp+w(u+u^\perp)\simeq vp^\perp+u+p=(vp^\perp+u+p)^{\perp\perp}\geq(p+p^\perp v)^{\perp\perp}\cong(p^\perp(p+v^\perp))^\perp\simeq(u^\perp p^\perp(w+v^\perp))^\perp\geq\pi z(\alpha a^\perp\pi b^\perp)^\perp$. The Split Lemma yields $vp^\perp\leq u+u^\perp$. No. 16 is symmetric to no. 15 and yields $v^\perp p^\perp\leq w+w^\perp$.

No. 7: $\alpha a=u$, $\alpha c=w$, $\alpha(c^\perp)=w^\perp$, $\alpha(a^\perp)=u^\perp$, $\alpha d=vv^\perp$, $\alpha(d^\perp)=p^\perp$, $\pi c=w$, $\pi(a^\perp)=u^\perp$, $\pi(a+c^\perp)=u+w^\perp$, $\pi(d^\perp)=(vv^\perp)^\perp=(v+v^\perp)^{\perp\perp}$, $\pi(c+b^\perp)=w+u^\perp(vv^\perp)^\perp$, $\pi(b+a^\perp)=u^\perp+w(vv^\perp)^\perp$. Observe that $\pi 0=wu^\perp(u+w^\perp)\simeq u^\perp(u+ww^\perp)=uu^\perp=0$ and $\alpha 1=w+u^\perp=1$. Also, notice the symmetry of this model.. Ad (4): The collinearity relation holds due to the preceeding lemma, the dual collinearity due to the fact that $(w+u^\perp(vv^\perp)^\perp)(u^\perp+w(vv^\perp)^\perp)\simeq u^\perp(vv^\perp)^\perp+w(vv^\perp)^\perp+p\leq\pi(d^\perp)$ since $q\geq vv^\perp$ and $p\leq q^\perp\leq(vv^\perp)^\perp$. Ad (5$\alpha$): $(\pi b)^\perp\geq(w(u^\perp+w(vv^\perp)^\perp)^\perp\simeq(p+w(vv^\perp)^\perp)^\perp=p^\perp(w(vv^\perp)^\perp)^\perp\approx_0 p^\perp(w^\perp+vv^\perp)\geq w^\perp+vv^\perp=\alpha(b^\perp)$. For $a+c^\perp\leq x\leq d^\perp$ we have $\alpha(x^\perp)\leq\alpha((a+c^\perp)^\perp)=\alpha d=vv^\perp\leq p=(u+w^\perp)^\perp\leq(\pi x)^\perp$. The remaining instances are obvious and so are those of (5π), (6), and (7) - the critical elements are $d,c,a^\perp,d^\perp$ (observe that $^\perp$ takes $d^\perp/(b+b^\perp)$ to d/d). The Split Lemma yields $vv^\perp=0$.

No. 8: $\alpha a=u$, $\alpha c=v$, $\alpha(c^\perp)=w^\perp$, $\alpha(a^\perp)=u^\perp(v+p^\perp)=u^\perp p_2^\perp$, $\alpha d=p^\perp p_1$, $\alpha(d^\perp)=p^\perp(v+u^\perp)$, $\pi c=w+v^\perp p=w+p_2$, $\pi(a^\perp)=u^\perp$, $\pi(a+c^\perp)=u+v^\perp$, $\pi(d^\perp)=(p^\perp p_1)^\perp=(p+v^\perp+u)^{\perp\perp}$, $\pi(c+b^\perp)=w+u^\perp(p^\perp p_1)^\perp$, $\pi(b+a^\perp)=u^\perp+w(p^\perp p_1)^\perp$. Observe that $\pi 0\simeq wu^\perp(u+v^\perp)+p_2\simeq u^\perp(u+p_2)+p_2\simeq uu^\perp+p_2=p_2$ and $\alpha 1=v+u^\perp$. Also, $p^\perp\leq(p^\perp p_1)^\perp$ since $p\geq p_1$. Ad (4): The (dual) collinearities are obvious from $p^\perp+u^\perp=1$ and $(w+u^\perp(p^\perp p_1)^\perp)(u^\perp+w(p^\perp p_1)^\perp)\simeq w(p^\perp p_1)^\perp+u^\perp(p^\perp p_1)^\perp+p\leq\pi(d^\perp)$. Ad (5$\alpha$): $\alpha(b^\perp)=w^\perp+p^\perp p_1\approx_0 (w(p^\perp p_1)^\perp)^\perp=(\pi b)^\perp$, $\alpha b=u+p^\perp p_1\approx_0 (u^\perp(p^\perp p_1)^\perp)^\perp=(\pi(b^\perp))^\perp$, and $\alpha(x^\perp)\leq\alpha d=p^\perp p_1\leq(u+v^\perp)^\perp\leq(\pi x)^\perp$ for $a+c^\perp\leq x\leq d^\perp$. Ad (5$\pi$) and (6): $(\alpha(a^\perp))^\perp\approx_0 u+p_2\simeq\pi(a^{\perp\perp})$. From $u+w^\perp\leq\alpha(d^\perp)\leq p^\perp$ we get by Lemma 50 that $(\alpha(d^\perp))^{\perp\perp}=p^\perp$ whence $(\alpha(d^\perp)^\perp=p\simeq\pi d$. Also, $(\alpha(a^\perp)^\perp\approx_0 u+p_2\simeq\pi a$. Ad (7): For c/b and $z=b^\perp$ we have $\alpha z+\pi(c^\perp)\geq p^\perp p_1+v^\perp\approx_0 (v(p^\perp p_1)^\perp)^\perp=(\alpha c\pi b)^\perp$. For $a^\perp/b^\perp$ and $z=b$ we have $(\alpha(a^\perp)\pi(b^\perp))^\perp=(u^\perp p_2^\perp(p^\perp p_1)^\perp)^\perp\approx_0 u+p_2+p^\perp p_1\leq(p^\perp p_1)^\perp(u+u^\perp)\simeq u^\perp(p^\perp p_1)^\perp+u\leq\alpha z+\pi(a^\perp)$. The Split Lemma yields $p^\perp p_1\leq p_2$. No. 9 is symmetric to no. 8 and yields $p^\perp p_2\leq p_1$.

It follows $p^\perp p_1\leq vv^\perp=0$ and $u\leq vp^\perp\leq v(u+u^\perp)\simeq u+p_1$. With modularity we derive $vp^\perp=u+vp^\perp p_1=u+0=u$. We conclude $u^\perp=(vp^\perp)^\perp\approx_0 v^\perp+p\simeq u^\perp(v^\perp+w)$ whence $u^\perp\leq v^\perp+w$, $v^\perp+w\geq u^\perp+w=1$, and $p+v^\perp\simeq u^\perp(v^\perp+w)=u^\perp$. Also, $1=0^\perp=(pvv^\perp)^\perp\approx_0 p^\perp+v+v^\perp$. The remaining claims follow by symmetry.

Proof. (Theorem 49) Let $\mathcal{F}$ be the free object with presentation $\mathcal{V}_I$ and the relations of Lemma 50 and 51 and $\mathcal{M}$ the 0-1-sublattice of $\mathcal{F}$ generated by $w, v, u^\perp, v^\perp, p^\perp$. Then $\mathcal{M}$ is a Fano lattice generated by a poset $1+2+2$ whence, according to Lemma 4 and 5 of 2.2 a subdirect product of lattices $\mathcal{D}_2$, $\mathcal{M}_3$ and Fano planes $\mathbb{P}(\mathbb{F}_2^3)$- see Fig. 13.

In a factor $\mathcal{D}_2$ assume $p^\perp = 0$. Then $u^\perp = 1$ from $p^\perp + u^\perp = 1$ and $v \leq v^\perp$ from $p_1 \leq p_1^\perp \leq v^\perp + p^\perp$. By symmetry $w = 1$ and $v^\perp \leq v$. Thus $v = v^\perp = 0$ from $vv^\perp = 0$. This contradicts $p^\perp + v + v^\perp = 1$. Therefore, $p^\perp = 1$. If $v = 1$ then $u^\perp = 0$ from $vp^\perp u^\perp = uu^\perp = 0$. Similarly, if $v^\perp = 1$ then $w = 0$. Finally, if $v = v^\perp = 0$ then $w = u^\perp = 1$ from $w + v^\perp = v + u^\perp = 1$. Thus. there are 3 factors $\mathcal{D}_2$. In view of $w + v^\perp = u^\perp + v = 1$ we have 6 out of the 9 factors $\mathcal{M}_3$ generated by the poset $1+2+2$. These are exactly the subdirect irreducible factors of the (mere) 0-1-lattices no. 10-14 with the given generators - recall that no. 18 and 19 are factors of each of those. Observe that $\mathcal{M}$ contains $u, w^\perp, p, p_i, q, p_i^\perp, q^\perp$. Recall that $J(\mathcal{M})$ consists of the smallest preimages αx, $x \in J(\mathcal{S})$, ϕ a homomorphism of $\mathcal{M}$ onto a subdirectly irreducible lattice $\mathcal{S}$ and the α's may be computed as in 2.4. According to Remark 2 in 2.4 we are done after computing $(\alpha x)^\perp$ for $x \in J(\mathcal{S})$ with $\mathcal{S}$ running through these factors. The αx can be read off the generators as explained in 2.4. For each lattice, we deal with the $\mathcal{D}_2$-factors, first, listing the values of the generators in the above order, $\alpha 1$, and $y = (\alpha 1)^\perp$. For the $\mathcal{M}_3$-factors we list the generators mapped to the atoms $a_1|a_2|a_3$, the $x_i = \alpha a_i$ and $y_i = x_i^\perp$. Sometimes we use $z_i = x_i^{\perp\perp}$ calculated via the interval decomposition of $\mathcal{L}^{\perp\perp}$. Recall the symmetry which yields no. 13 and 19, in particular.

No. 10: 10101, q, $y = p + p^\perp$. Secondly, $v, w|v^\perp, u^\perp|p^\perp$; $v(p^\perp + v^\perp)$, $p^\perp$, $v^\perp(p^\perp + v)$; $y_1 \cong_0 v^\perp + p_1$, $y_2 \cong_0 v + p_2$, $y_3 = p$.

No. 11: $v|v^\perp|p^\perp$; $p_1(q + p_2)$, $p_2(q + p_1)$, $q(p_1 + p_2)$; $y_1 \cong_0 p_1^\perp + q^\perp p_2^\perp$, $y_2 \cong_0 p_2^\perp + q^\perp p_1^\perp$, $y_3 \cong_0 q^\perp + p_1^\perp p_2^\perp$. Secondly, we have the Fano plane with $p^\perp = d$. Define $\alpha v = v$, $\alpha v^\perp = v^\perp$, $\alpha p = p$, $\alpha d = p^\perp$, $\alpha a = (v+p)(p^\perp + v^\perp) = wp_1^\perp$, $\alpha b = (v^\perp + p)(p^\perp + v) = u^\perp p_2^\perp$, $\alpha c = (v + v^\perp)(p + p^\perp)$. Then the collinearity $\alpha a, \alpha b, \alpha c$ follows from the Fano axiom, the others from modularity and the join relations of Lemmas 50 and 51. Thus, Corollary 8 of 2.4 applies. Finally, $(\alpha a)^\perp \approx w^\perp + p_1$, $(\alpha b)^\perp \approx u + p_2$, and $u + w^\perp \leq \alpha c$ whence $(\alpha c)^{\perp\perp} = p^\perp + \alpha c \simeq p^\perp + p$ and $(\alpha c)^\perp = q$.

No. 12: $v, w|v^\perp|p^\perp$; $p_1(p^\perp + v^\perp)$, $v^\perp(p^\perp + p_1)$, $p^\perp(v^\perp + p_1)$; $y_1 \cong_0 p_1^\perp + p_1$, $y_2 \cong_0 v + pp_1^\perp$, $y_3 \cong_0 p + vp_1^\perp$. Secondly, $v, u^\perp|p^\perp$; $v(p^\perp + p)$, $p(p^\perp + v)$, $p^\perp(v + p)$; $y_1 \cong_0 v^\perp + q$, $y_2 \cong_0 p^\perp + p_2$, $y_3 \cong_0 p + p^\perp v^\perp$.

No. 14: Define $\alpha v = vq^\perp p_1^\perp$ whence $(\alpha v)^\perp \cong_0 v^\perp + q + p_1$. Define $\alpha a = p^\perp(pp_1^\perp + v)$ whence $(\alpha a)^\perp \cong (p + (pp_1^\perp + v)^\perp)^{\perp\perp}(p + (p(p^\perp + v^\perp))^\perp v^\perp)^{\perp\perp} \cong_0 (p + (p^\perp + p_1)v^\perp)^{\perp\perp} \simeq (u^\perp(w + (p^\perp + p_1)v^\perp)^{\perp\perp} = u^\perp(w + (p^\perp + p_1)v^\perp)$ since $p^\perp \geq w^\perp$. Define $\alpha p = pp_1^\perp p_2^\perp$ whence $(\alpha p)^\perp \approx_0 p^\perp + p_1 + p_2$. Define $\alpha c = (p^\perp w + p^\perp u^\perp)(v + v^\perp)$. $(\alpha c)^\perp \geq p$. Assuming $(\alpha c)^\perp \geq u$ leads to $\alpha c \leq u^\perp$ and a contradiction in no. 14. Similarly for $(\alpha c)^\perp \geq w^\perp$ and $(\alpha c)^\perp \geq p^\perp$. Since $(\alpha c)^\perp$ is closed we have $(\alpha c)^\perp \leq p$ by Lemma 50. Define $\alpha d = (v + p^\perp u^\perp)(v^\perp + p^\perp w) \geq u + w^\perp$, whence $(\alpha d)^{\perp\perp} \geq p^\perp$, $(\alpha d)^{\perp\perp} = p^\perp + \alpha d \simeq (v + p^\perp w)(v^\perp + p^\perp u^\perp) + p^\perp \simeq p(v + p^\perp)(v^\perp + u^\perp) + p^\perp \simeq p_1^\perp p_2^\perp q^\perp$. It follows $(\alpha d)^\perp \approx_0 p_1 + p_2 + q$. Symmetrically,

$\alpha v^\perp = v^\perp q^\perp p_2^\perp$ and $\alpha b = p^\perp(pp_2^\perp + v^\perp)$. That α is join preserving is most easily verified working through the factors of $\mathcal{M}$ (no. 10-12 suffice). Thus, Corollary 8 applies.

No. 18: 11001, $vp^\perp$, $y \approx_0 v^\perp + p$.

THEOREM 52 (CLASSIFICATION). *Table 3 gives a complete list of invariants for the isometry class of a reduced triple (E, F, C) in case 1. By adding the invariants of the summands defined in the reduction steps (with the exception of 4 and 5) one obtains a complete classification of a pair (E, F) in case 1.*

COROLLARY 53. *If the dimension of E is finite, then a pair (E, F) is classified by*

(a) *The invariants associated to the summands 1, 2, 3, 6, 7, 8 of the reduction steps and*

(b) *The invariants labelled by the numbers 0, 1, 2, 3, 4, 5, 6, 18, 19, 20 in Table 3.*

Remarks. 1. In Table 3 read $U = u$, $V = v$, $W = w$, $P = p$, $P_i = p_i$, $R = q$. The summand no. 20 does not occur in the classification of a reduced triple, since it does not satisfy $E^{*\perp} \cap F^\perp = (0)$.

2. The presentation of the problem answered by the above corollary is already contained in [GLS85] and [Mor86]. A complete solution was given in [GW89] through the computation of the quadratic lattice $\mathcal{V}(E, F, E^*)$. The corollary gives a slightly different version and shows incidentally that there is not a 1-1-correspondence between the subdirectly irreducible factors of the lattices $\mathcal{V}(H, W, V, U)$ and $\mathcal{V}(E, F, E^*)$ (more precisely, the correspondence fails for the factors 2, 3 and 5).

3. The factors of $\mathcal{V}(H, W, V, U)$ with $W = U$ may equally be replaced by the corresponding quadratic lattice $\mathcal{V}(E = E^*, F)$: they classify an orthogonally closed subspace of a trace-valued space.

4. The classification of the quadruple (H, W, V, U) is given by the indexed quadratic lattice $\mathcal{V}(H, W, V, U)$ (which is a homomorphic image of $\mathcal{V}_I$ above) and the isometry type of $P = W \cap U^\perp$. A particular discussion is required for the factors No. 11 and 14 of Table 3, because they are the only ones which are not acyclic. In both lattices the crucial interval is that isomorphic to $\mathbb{P}(\mathbb{F}_2^3)$. We know already that linear representations are unique because all admissible fields have characteristic 2 (cf Lemma 5 of 2.2). In order to induce metric isomorphisms it suffices to use (37) and $\hat{P}$ (observe that the crucial interval is governed by $V/U + P_1$ and $P/P_1 + P_2$ whereas $P_1 + P_2$ is totally isotropic). The remaining factors are acyclic (yet, condition Δ may fail). Since H is an alternate space, one can adapt the methods of section 3 and the proof of the Congruence Theorem 47 to obtain a mapping Theorem for homomorphic images of $\mathcal{V}_I$. A Decomposition Theorem is then derived as in 4.5 using the elementary examples.

5. In order to classify a reduced triple (E, F, C) we proceed as follows: Consider first the interval $J := W/U$ as a sublattice of $\mathcal{V}(H, W, V, U)$. Through the embedding $\mathcal{V}(H, W, V, U) \subset \mathcal{V}(E, H, F, E^*)$ there is a lattice monomorphism

$$\sigma : J \longrightarrow \mathcal{L}(E^{*\perp}) \text{ defined by } \sigma(x) := E^{*\perp} \cap (X + F).$$

Because of the rigidity of $E^{*\perp}$ in E, every element of $\sigma(J)$ is an invariant of the congruence class of F in E.

Using the total isotropy of $E^{*\perp *}$ and the anisotropy of any supplement of $E^{*\perp *}$ in $E^{*\perp}$, it is possible to find an orthogonal decomposition of $E^{*\perp} + C$ into 22 summands.

How is it possible to recover F_i in a summand $i \in \{0, \dots, 21\}$? If $i \notin \{5, 6, 18, 19\}$ then there is no difficulty because (46) holds. If $i = 5$, then one requires $\widehat{F^*}$. In case $i = 6$, $\hat{E}$ is needed together with some matrix in the factor set $GL_{\gamma_6}(k)/O(H)$, whereas in the other two cases only $\hat{E}$ is needed.

5.4 The classification in case II

Let $\mathcal{V}_{II}$ be the free (modular) Galois lattice generated by a 2-chain $u \le w$ under the relation

$$\delta(w/u) < \aleph_0 . \tag{48}$$

We define

$$p = w^{\perp\perp}u^{\perp}, \;\; p_0 = wu^{\perp}, \;\; q = pp^{\perp}, \text{ and } q_0 = pp_0^{\perp} .$$

Theorem 54. *There are 22 subdirectly irreducible factors of $\mathcal{V}_{II}$. The diagrams and generators thereof are given in Figure 15 and Table 4. The index functions on factor no. i are in 1-1-correspondence with the index ε_i $(i = 0, \dots, 21)$ of a prime quotient in the minimal congruence. ε_i is finite if $i \le 19$. All factors are implemented by hermitean spaces.*

Remarks. 1. We use the same conventions as in 5.3 except that the critical index is denoted by ε. In Table 4 read $R = q$ and $R_0 = q_0$.

2. The epimorphism are no. 0,1 onto no. 2, no. 4,6-10,12,15 and 16 onto no. 20, no. 17 onto no. 21, and no. 11,14,18 and 19 onto both no. 20,21. In particular, the factors no. 20,21 are redundant in an image $\mathcal{V}$ of $\mathcal{V}_{II}$ if $\delta(a/0) = \aleph_0$ and $\delta(b^{\perp}/0) = \aleph_0$, respectively.

3. $\mathcal{V}$ is distributive iff $\varepsilon_8 + \varepsilon_{10} + \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{14} + \varepsilon_{15} = 0$, iff $q_0 = p_0u^{\perp\perp} + ww^{\perp}$ and $p_0u^{\perp\perp} = uu^{\perp} + u^{\perp\perp}ww^{\perp}$ and $pw^{\perp} = b^{\perp}(p_0 + u^{\perp\perp})$ and $pp_0^{\perp} = p_0p_0^{\perp} + u^{\perp\perp}u^{\perp} + pw^{\perp}$. $\mathcal{V}$ is of finite type iff $\delta(a) + \delta(b^{\perp}) < \aleph_0$. This occurs, necessarily, if only no. 3,5 are involved, possibly if also no. 2,20,21 are allowed, and is impossible, otherwise.

Lemma 55. *Let $\mathcal{L}$ be any model of $\mathcal{V}_{II}$ having no subdirect factor no. 0-2. Then $\mathcal{L}$ satisfies the relations*

$$w^{\perp\perp} = w + u^{\perp\perp} \;\; \textit{and} \;\; uw^{\perp} = 0$$

No.	Geometric invariants	Metric $V = W$	invariants $V = U$	Arithmetic invariants
0	$\mathcal{V}_{11},\ u = a, w = a^{\perp}$	$\hat{E}, \hat{C}$	$\hat{E}$	
1	$\mathcal{V}_{11},\ u = w = a$	$\hat{E}, \hat{C}$	$\hat{E}$	
2	$\mathcal{V}_{2},\ u = w = a$	$\hat{E}$	$= \hat{H}$	
3	$\mathcal{V}_{2},\ u = a, w = 1$	$\hat{E}, \hat{C}$	$\hat{E}$	
4	$\mathcal{V}_{12},\ u = a, w = a^{\perp\perp}$	$\hat{E}, \hat{C}$	$\hat{E}$	
5	$\mathcal{V}_{2},\ u = 0, w = a$	$\hat{E}, \hat{C}$	$\hat{E}$	
6	$\mathcal{V}_{12},\ u = w = a$	$\hat{E}, \hat{C}$	$\hat{E}$	
7	$\mathcal{V}_{12},\ u = a, w = 1$	$\hat{E}, \hat{C}$	$\hat{E}$	$\sigma(U^{\perp\perp})$
8	$\mathcal{V}_{17}$	$\hat{E}, \hat{C}$	$\hat{E}$	$\sigma(U + R_0)$
9	$\mathcal{V}_{13}$	$\hat{E}, \hat{C}$	$\hat{E}$	
10	$\mathcal{V}_{15}$	$\hat{E}, \hat{C}$	$\hat{E}$	
11	$\mathcal{V}_{5},\ u = a, w = c$	$\hat{E}, \hat{C}$	$\hat{E}$	$\sigma(U + R)$
12	$\mathcal{V}_{16}$	$\hat{E}, \hat{C}$	$\hat{E}$	
13	$\mathcal{V}_{1},\ u = 0, w = 1$	$\widehat{F^*}, \hat{C}$	$\hat{E}$	$\mu \in GL_{\varepsilon_{13}}(k)/\mathcal{O}(H)$ if $V = U$
14	$\mathcal{V}_{18}$	$\hat{E}, \hat{C}$	$\hat{E}$	$\sigma(W \cap R^{\perp})$
15	$\mathcal{V}_{14}$	$\hat{E}, \hat{C}, \widehat{U^{\perp}}$	$\hat{E}, \widehat{U^{\perp}}$	
16	$\mathcal{V}_{10},\ u = a, w = 1$	$\hat{E}, \hat{C}$	$\hat{E}$	
17	$\mathcal{V}_{10},\ u = w = a$	$\hat{E}$	$= \hat{H}$	
18	$\mathcal{V}_{4},\ u = a, w = b$	$\hat{E}, \hat{C}$	$\hat{E}$	
19	$\mathcal{V}_{3},\ u = w = a$	$\hat{E}$	$= \hat{H}$	
20	$\mathcal{V}_{1},\ u = w = 1$	$\hat{E}$	$= \hat{H}$	
21	$\mathcal{V}_{1},\ u = w = 0$	$\hat{E}$	$= \hat{H}$	

Table 4: Factors of $\mathcal{V}_{II}$ and classification of a reduced triple (E, F, C) in case II

and is a union of intervals and so is its set of closed elements

$$\mathcal{L} = [0, p_0] \cup [u, w] \cup [u^{\perp\perp} w^{\perp}, p] \cup [u + u^{\perp\perp} w^{\perp}, w^{\perp\perp}] \cup [w^{\perp}, u^{\perp}] \cup [u + w^{\perp}, (u^{\perp\perp} w^{\perp})^{\perp}] \cup [p_0^{\perp}, 1]$$

$$\mathcal{L}^{\perp\perp} = [0, p_0]_{\mathcal{L}} \cup [u^{\perp\perp} w^{\perp}, p]_{\mathcal{L}} \cup [u^{\perp\perp}, w^{\perp\perp}]_{\mathcal{L}} \cup [w^{\perp}, u^{\perp}]_{\mathcal{L}} \cup [p^{\perp}, (u^{\perp\perp} w^{\perp})^{\perp}]_{\mathcal{L}} \cup [p_0^{\perp}, 1]_{\mathcal{L}}.$$

Proof. Observe that $\delta(w + u^{\perp\perp}/u^{\perp\perp}) \leq \delta(w/u) < \aleph_0$ whence $w + u^{\perp\perp}$ is closed in view of Lemma 19 in 2.7 and so $w^{\perp\perp} = w + u^{\perp\perp}$. Also, $(uw^{\perp})^{\perp} \geq u^{\perp} + w^{\perp\perp}$ whence $\mathcal{L} = \{0, 1\} \cup [uw^{\perp}, (uw^{\perp})^{\perp}]$. Again, we use the Split Lemma.

No. 0: $\alpha a = uw^{\perp}$, $\alpha(a^{\perp}) = w(uw^{\perp})^{\perp\perp}$, $\alpha 1 = 1$, $\pi 0 = 0$, $\pi a = u$, $\pi(a^{\perp}) = (uw^{\perp})^{\perp}$. Ad (6): $(\alpha(a^{\perp}))^{\perp} \geq (uw^{\perp})^{\perp\perp\perp} = (\alpha(a^{\perp})\pi a)^{\perp}$. It follows $w(uw^{\perp})^{\perp\perp} \leq u$.

No. 1: $\alpha a = uw^\perp$, $\alpha(a^\perp) = (uw^\perp)^{\perp\perp}$, $\alpha 1 = 1$, $\pi 0 = 0$, $\pi a = w$, $\pi(a^\perp) = (uw^\perp)^\perp$. It follows $(uw^\perp)^{\perp\perp} \leq w$ and, with no. 0, that $uw^\perp$ is closed.

No. 2: $\alpha = uw^\perp$, $\alpha 1 = 1$, $\pi 0 = 0$, $\pi a = (u + w^\perp)^\perp$. Ad (7): For $1/a$ we have $z = a$ and $\alpha z = uw^\perp = (uw^\perp)^{\perp\perp} = (\pi a)^\perp = (\alpha 1 \cap \pi a)^\perp$ by the preceeding. It follows $uw^\perp = 0$.

Clearly, $\mathcal{L}$ splits into intervals, as indicated, since the union of these intervals is a subalgebra containing the generators. The quotients $u^\perp/w^\perp$ and $w^{\perp\perp}/u^{\perp\perp}$ have finite index by Lemma 18 of 2.7. The quotient $p_0/0$ is composed from $wu^\perp \geq uu^\perp \geq uw^\perp = 0$ hence it has finite index, too, and so does $1/p_0^\perp$. Simliarily, for $p = w^{\perp\perp}u^\perp \geq w^{\perp\perp}w^\perp \geq u^{\perp\perp}w^\perp$ and $(u^{\perp\perp}w^\perp)^\perp/p$. By Lemma 19 of 2.7 all these intervals consist of closed elements. Finally, observe that $(u + u^{\perp\perp}w^\perp)^{\perp\perp} = u^{\perp\perp}$ and $(u + w^\perp)^{\perp\perp} = p^\perp$, so we conclude that all closed elements are contained in those intervals. It follows $w^{\perp\perp} = w + u^\perp$.

Lemma 56. *Let $\mathcal{L}$ be as in Lemma 55 with no subdirect factor no. 3-7,9,13,16,17,19. Then every element in $[p^\perp, 1]$ is closed and $\mathcal{L}$ satisfies the relations*

$$uu^\perp = ww^\perp = w^\perp u^{\perp\perp} = p_0 u^{\perp\perp} = 0, \;\; p_0 \leq p_0^\perp, \;\; wu^{\perp\perp} \leq u + w^\perp$$

$$w + u^\perp = w + p_0^\perp = u^\perp + p_0^\perp = 1, \;\; p^\perp + w = (w^{\perp\perp}w^\perp)^\perp, \; p^\perp + u = (u^{\perp\perp}u^\perp)^\perp.$$

Proof. We use the Splitting Method. No. 3: $\alpha a = uu^\perp$, $\alpha 1 = 1$, $\pi 0 = 0$, $\pi a = (uu^\perp)^\perp$. Ad (7): For $1/a$ we have $z = a$ and $\alpha z = uu^\perp = (uu^\perp)^{\perp\perp} = (\pi a)^\perp = (\alpha 1 \cap \pi a)^\perp$. It follows $uu^\perp = 0$.

No. 5: $\alpha a = ww^\perp$, $\alpha 1 = u^\perp$, $\pi 0 = u^{\perp\perp}$, $\pi a = (ww^\perp)^\perp$. Ad (7): For $a/0$ we have $z = 1$ and $\alpha 1 + \pi(a^\perp) = u^\perp + (ww^\perp)^\perp \approx_0 (u^{\perp\perp}ww^\perp)^\perp = (\alpha a \cap \pi 0)^\perp$. For $1/a$ we have $z = a$ and $\alpha a + \pi 0 = ww^\perp + u^{\perp\perp} \approx_0 (u^\perp(ww^\perp)^\perp)^\perp = (\alpha 1 \cap \pi a)^\perp$. It follows $ww^\perp \leq u^{\perp\perp}$.

No. 13: $\alpha 1 = p_0$, $\pi 0 = p_0^\perp$. Ad (7): $p_0 + p_0^\perp \approx_0 (p_0 p_0^\perp)^\perp$. It follows $p_0 \leq p_0^\perp$.

No. 16: $\alpha a = u$, $\alpha 1 = wu^{\perp\perp}$, $\pi 0 = (wu^{\perp\perp})^\perp$, $\pi a = u + w^\perp$. Ad (7): $(wu^{\perp\perp})^\perp \geq w^\perp = w^\perp + uu^\perp \simeq (u + w^\perp)u^\perp = (u + w^\perp)(u + uw^\perp)^\perp \simeq (u + w^\perp)(wu^{\perp\perp}(u + w^\perp))^\perp$ using no. 3 and 5. It follows $wu^{\perp\perp} \leq u + w^\perp$.

No. 17: $\alpha a = u$, $\alpha 1 = u^{\perp\perp}$, $\pi 0 = u^\perp$, $\pi a = w + u^\perp$. Ad (7): $u^\perp \geq (u^{\perp\perp}(w + u^\perp))^\perp$ since $u \leq u^{\perp\perp}(w + u^\perp)$. It follows $u^{\perp\perp} \leq w + u^\perp$.

No. 19: $\alpha a = u$, $\alpha a^\perp = w^\perp$, $\alpha 1 = p^\perp$, $\pi a = w^{\perp\perp}$, $\pi a^\perp = u^\perp$, $\pi(a + a^\perp) = w^{\perp\perp} + u^\perp$. Observe $\pi 0 = p$. Ad (6): The critical quotient is $1/(a + a^\perp)$: here $(\alpha 1)^\perp = p = \pi(1^\perp)$. (7) holds trivially. It follows $p^\perp \leq w^{\perp\perp} + u^\perp$.

No. 4: $\alpha a = u$, $\alpha a^\perp = ww^\perp u^{\perp\perp}$, $\alpha 1 = 1$, $\pi a = u$, $\pi a^\perp = u^\perp$, $\pi(a^{\perp\perp}) = (ww^\perp u^{\perp\perp})^\perp \geq u^{\perp\perp}$. Observe that $\pi 0 = uu^\perp = 0$ by no. 3. Ad (7): For $1/a^{\perp\perp}$ we use that $\alpha(a^\perp) = ww^\perp u^{\perp\perp} = (ww^\perp u^\perp)^{\perp\perp} = (\alpha 1 \cap \pi a^{\perp\perp})^\perp$. It follows $ww^\perp u^{\perp\perp} = 0$.

No. 6: $\alpha a = u$, $\alpha a^\perp = w^\perp u^{\perp\perp}$, $\alpha 1 = p_0^\perp$, $\pi a = w$, $\pi a^\perp = u^\perp$, $\pi(a^{\perp\perp}) = (w^\perp u^{\perp\perp})^\perp \geq u^{\perp\perp}$ since $w^\perp \leq u^\perp$. Observe that $\pi 0 = p_0$. Ad (7): For $a^\perp/0$ we use $p_0^\perp + (w^\perp u^{\perp\perp})^\perp \approx_0 (w^\perp u^{\perp\perp} p_0)^\perp$. For $1/a^{\perp\perp}$ we have $w^\perp u^{\perp\perp} + p_0 \approx_0 (p_0^\perp(w^\perp u^{\perp\perp})^\perp)^\perp$. It follows $w^\perp u^{\perp\perp} \leq p_0$.

No. 7: $\alpha a = u, \alpha a^\perp = p_0 u^{\perp\perp}, \alpha 1 = w, \pi a = u + w^\perp, \pi a^\perp = u^\perp, \pi(a^{\perp\perp}) = (p_0 u^{\perp\perp})^\perp \geq u^{\perp\perp}$. Observe that $\pi 0 = (u + w^\perp)u^\perp \simeq w^\perp + uu^\perp = w^\perp$ by no. 3. Ad (7): For $a^\perp/0$ we use $w + (p_0 u^{\perp\perp})^\perp \approx_0 (p_0 u^{\perp\perp} w^\perp)^\perp$. For $1/a^{\perp\perp}$ we have to show $x := p_0 u^{\perp\perp} + w^\perp \geq y := (w(p_0 u^{\perp\perp})^\perp)^\perp$. Since x and y are closed, this amounts to $x^\perp \leq y^\perp$. Now, indeed, $x^\perp = (p_0 u^{\perp\perp})^\perp w^{\perp\perp} = (p_0 u^{\perp\perp})^\perp (w + u^{\perp\perp}) \simeq w(p_0 u^{\perp\perp})^\perp + u^{\perp\perp} = (w(p_0 u^{\perp\perp})^\perp)^{\perp\perp} = y^\perp$. It follows $p_0 u^{\perp\perp} \leq w^\perp$.

We derive $uw^\perp = ww^\perp u^{\perp\perp} = 0$ with no. 5 and 4; $w^\perp u^{\perp\perp} \leq ww^\perp = 0$ with no. 6; $p_0 u^{\perp\perp} = w^\perp u^{\perp\perp} = 0$ with no. 7; $(w^\perp u^{\perp\perp})^\perp = 1$ and $x = x^{\perp\perp}$ for all $x \geq p^\perp$; also $p^\perp \leq u^\perp + w^{\perp\perp} \approx_0 (u^{\perp\perp} w^\perp)^\perp = 0^\perp = 1$ with no. 19; and $w + u^\perp = w + u^{\perp\perp} + u^\perp = w^{\perp\perp} + u^\perp = 1$ with no. 17. Now, $w + p_0^\perp \approx_0 (w^\perp p_0)^\perp = 1$ and $u^\perp + p_0^\perp \approx_0 (u^{\perp\perp} p_0)^\perp = 1$. Also, $p^\perp + w \approx_0 (pw^\perp)^\perp = (w^{\perp\perp} w^\perp)^\perp$ and $p^\perp + u^\perp \approx_0 (pu^{\perp\perp})^\perp = (u^{\perp\perp} u)^\perp$.

Proof. (Theorem 54.) Let $\mathcal{F}$ be the free object with presentation $\mathcal{V}_{II}$ and the relations of Lemma 55 and 56 and $\mathcal{M}$ the 0-1-sublattice of $\mathcal{F}$ generated by $w, u^{\perp\perp}, u, u^\perp, w^\perp, p_0^\perp, p^\perp$. Observe that $p, p_0, w^{\perp\perp} \in \mathcal{M}$. Then $\mathcal{M}$ is a modular lattice generated by a poset $w \geq u \leq u^{\perp\perp} \leq p^\perp \leq p_0^\perp, w^\perp \leq u^{\perp\perp}, w^\perp \leq u^\perp$ whence, according to Lemma 3 of 2.2, a subdirect product of lattices $\mathcal{D}_2$ and $\mathcal{M}_3$. The relations admit all 8 $\mathcal{M}_3$-models generated by the poset and 9 out of the 27 $\mathcal{D}_2$-models. These are exactly the subdirectly irreducible factors of the lattices no. 8-12,14,15,18,20,21. We proceed as in 5.3 Observe that $u \not\leq u^\perp$, $u^{\perp\perp} w^\perp \not\leq w$, and $w^\perp \not\leq w^{\perp\perp}$ in view of the examples no. 8 and 11. Thus, there is no overlap between the intervals - if we don't count $[p_0^\perp, 1]$. Recall that $w^{\perp\perp} = w + u^{\perp\perp}$.

No. 8: $w|p_0^\perp|u^\perp$; $w, p_0^\perp, u^\perp$, $y_1 = w^\perp$, $y_2 = p_0$, $y_3 = p$. Secondly, $w|u^{\perp\perp}|w^\perp$; $w(u^{\perp\perp}u^\perp + w^\perp)$, $u^{\perp\perp}(p_0 p^\perp + w^\perp)$, $w^\perp(p_0 p^\perp + u^{\perp\perp})$; $x_1 \simeq p_0(u^{\perp\perp} + w^\perp) \simeq p_0(u^{\perp\perp} + w^{\perp\perp} w^\perp)$ whence $y_1 \cong_0 p_0^\perp + u^\perp(p^\perp + w)$, $p \geq x_2 \simeq pu^{\perp\perp}p^\perp(p_0 + w^\perp) = pu^{\perp\perp}(p_0 + w^\perp)$ whence $y_2 \cong_0 p^\perp + u^\perp + p_0^\perp w^{\perp\perp}$, $p \geq x_3 \simeq pw^\perp p^\perp(p_0 + u^{\perp\perp}) = pw^\perp(p_0 + u^{\perp\perp})$ whence $y_3 \cong_0 p^\perp + w^{\perp\perp} + p_0^\perp + u^\perp$.

No. 9: 0101011, $u^\perp u^{\perp\perp}$, $y = p^\perp + u$. Secondly, 1000010, $wp_0^\perp$, $y \approx_0 w^\perp + p_0$.

No. 10: 0001010, $u^\perp p_0^\perp$, $y \approx_0 u^{\perp\perp} + p_0$. Secondly, $w|u^{\perp\perp}|u^\perp, w^\perp$; $w(u^{\perp\perp} + u^\perp)$, $u^{\perp\perp}(w + w^\perp)$, $w^\perp(w + u^{\perp\perp}) = w^\perp w^{\perp\perp}$; $z_1 = x_1 + u^{\perp\perp} \simeq (w + u^{\perp\perp})(w^\perp + u^{\perp\perp}) \simeq w^{\perp\perp} w^\perp + u^{\perp\perp}$ whence $y_1 = u^\perp(p^\perp + w)$, $z_2 = u^{\perp\perp}$, $y_3 \approx_0 p^\perp + w$.

No. 11: 1001011, $p_0 p^\perp$, $y \approx_0 p_0^\perp + p$. Secondly, $w|p^\perp, p_0^\perp|u^\perp$; $w(u^\perp + p^\perp)$, $u^\perp(w + p^\perp)$, $p^\perp$; $z_1 = x_1 + u^{\perp\perp} \simeq w^{\perp\perp}(u^\perp + p^\perp)$ whence $y_1 \cong_0 w^\perp + u^{\perp\perp}p$, $z_2 = x_2 + w^\perp \simeq u^\perp(w^\perp + w + p^\perp)$ whence $y_2 \cong_0 u^{\perp\perp} + w^{\perp\perp} w^\perp p$, $y_3 = p$.

No. 12: 1001010, $p_0 p_0^\perp$. $y \approx_0 p^\perp + 0 + p_0$. $u^{\perp\perp}, p^\perp, p_0^\perp|w|u^\perp$; $u^{\perp\perp}$, $w(u^{\perp\perp} + u^\perp)$, $u^\perp(u^{\perp\perp} + w) = p$; $y_1 = u^\perp$, $z_2 = x_2 + u^{\perp\perp} \simeq (w + u^{\perp\perp})(u^{\perp\perp} + u^\perp) \simeq u^{\perp\perp} + p$ whence $y_2 = u^\perp p^\perp$, $y_3 = p^\perp$.

No. 14. $w|u^{\perp\perp}|u^\perp$; $w(u^{\perp\perp} + u^\perp p^\perp)$, $u^{\perp\perp}(wp^\perp + u^\perp p^\perp)$, $u^\perp(wp^\perp + u^{\perp\perp})$; $z_1 = x_1 + u^{\perp\perp} \simeq w^{\perp\perp}((u^{\perp\perp} + u^\perp p^\perp) \simeq u^{\perp\perp} p^\perp p$ whence $y_1 \approx_0 (p + p^\perp)u^\perp$, $z_2 = u^{\perp\perp}$, $x_3 \simeq u^\perp p^\perp(w + u^{\perp\perp}) = pp^\perp$ whence $y_3 = q^\perp$. Secondly, $w|p^\perp|u^\perp$; $w(p^\perp + u^\perp p_0^\perp)$, $p^\perp(wp_0^\perp + u^\perp p_0^\perp)$, $u^\perp(p^\perp + wp_0^\perp)$; $z_1 = x_1 + u^{\perp\perp} \simeq w^{\perp\perp} p_0^\perp(p^\perp + u^\perp)$ whence $y_1 \approx_0 w^\perp + p_0 + u^\perp u^{\perp\perp}$, $z_2 = p^\perp$, $z_3 = x_3 + w^\perp \simeq u^\perp p_0^\perp(p\perp + w)$ whence $y_3 \approx_0 u^{\perp\perp} + p_0 + w^\perp w^{\perp\perp}$.

No. 15: $w|p^\perp, u^{\perp\perp}|u^\perp$; $w(u^{\perp\perp}+u^\perp p_0^\perp)$, $u^{\perp\perp}(wp_0^\perp+u^\perp p_0^\perp)$, $u^\perp(u^{\perp\perp}+wp_0^\perp)$; $z_1 = x_1 + u^{\perp\perp} \simeq w^{\perp\perp}(u^{\perp\perp}+u^\perp p_0^\perp) \simeq p_0^\perp p + u^{\perp\perp}$ whence $y_1 = (p_0+p^\perp)u^\perp$, $z_2 = u^{\perp\perp}$, $x_3 \simeq u^\perp p_0^\perp w^{\perp\perp} = pp_0^\perp$ whence $y_3 = p_0 + p^\perp$.

No. 18: 1000011, $wp^\perp$, $y \approx_0 w^\perp + p$. Secondly, 0001011, $u^\perp p^\perp$, $y \approx_0 u^{\perp\perp} + p$.

No. 20: 1110011, u, $y = u^\perp$. No. 21: 0001111, $w^\perp$, $y = w^{\perp\perp}$.

Theorem 57 (Classification). *Table 4 gives a complete list of invariants for the isometry class of a reduced triple (E, F, C) in case II. By adding the invariants of the summands defined in the reduction steps one obtains a complete classification of a pair (E, F) in case II.*

Corollary 58. *If the division ring $(k, \varepsilon, -)$ is perfect in the sense that $\dim S/T \leq 1$, then a pair (E, F) in case II is classified by*

(a) The invariants associated to the summands 1, ..., 8 of the reduction steps and

(b) The invariants labelled by the numbers 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 15, 16, 17, 18, 19, 20, 21.

Remarks. 1. The above corollary is a slightly different version of the results contained in [Gro87, ex. 2, p.239-252]. We remember that only the factors 13 and 15 when $V = W$ are not implemented over $\mathbb{F}_2$. In order to implement factor 15, 18, 21 it is required a division ring with $\dim S/T \geq 2$.

2. Only the summands associated to the factors 13, 20, 21 do not satisfy (46). In no. 17 and 19 the isometry type of $\hat{E} = \hat{H}$ is uniquely determined in view of (37).

3. It is evident that for a reduced triple (E, F, C) $E^* \cap E^{*\perp} = (0) \iff U = V$ and $E^{*\perp} \subset E^* \iff V = W$. Thus, the geometric invariant to be dealt with is $\mathcal{V}(H, W, U)$.

5.5 The classification in Case III

Let k be a perfect field and assume that the symmetric bilinear space (E, Φ) over k possesses an orthonormal basis. Then we have in particular $E^{*\perp\perp} = E$. The classification of a pair (E, F) is in this case very complicated and requires many invariants. It was studied in [Wid89] and also presented in [GW89]. Here we just state the final result.

Consider the following sets of subspaces of E, ordered by inclusion:

$$\mathcal{N}(E, F) := \mathcal{L}(E, F) \cup \{A^{*\perp} | A \in \mathcal{L}(E, F)\} \cup \{E^*\}$$
$$\dim ||E|| \leq [k : k^2] = 1 \quad \text{implies} \quad |N(E, F)| \leq 19\,.$$

We set $1 := F^{*\perp}$, $2 := (F + F^\perp)^{*\perp}$, $3 := (F \cap F^\perp)^{\perp *\perp}$ and $4 := F^{\perp *\perp}$. Then $\mathcal{N}(E, F)$ looks as follows as in Fig. 16(a).

$\mathcal{M}(E,F) := \{F^\perp \cap F^{\perp\perp}, 2, 5, 6, 7\}$, where $5 := F^{\perp\perp} \cap F^{*\perp}$, $6 := F^\perp \cap F^{\perp*\perp}$ and $7 := F^{*\perp} \cap F^{\perp*\perp}$. $\mathcal{M}(E,F)$ looks as follows as in Fig. 16(b).

If (E,F) and (E',F') are isometric, then there exists an index preserving isomorphism

$$\eta : \mathcal{N}(E,F) \longrightarrow \mathcal{N}(E',F')$$

such that η induces an ortho-isomorphism of the lattices $\mathcal{L}(E,F)$ and $\mathcal{L}(E',F')$ which maps F onto F' and an isomorphism

$$\psi : \mathcal{M}(E,F) \longrightarrow \mathcal{M}(E',F')$$

which maps $F^\perp \cap F^{\perp\perp}$ onto $F'^\perp \cap F'^{\perp\perp}$, i onto i' and for which $i \subset E^*$ is equivalent to $i' \subset E'^*$ $(i = 2, 5, 6, 7)$.

Provided there exist such η and ψ we say that (E,F) and (E',F') have the same geometric invariants $\mathcal{N}$ and $\mathcal{M}$. There are also thirteen arithmetic invariants $\gamma_i = \gamma_i(E,F)$, $i = 1, \ldots, 13$, which prove to be important for the classification. Each γ_i is linked with a one-dimensional quotient space $H_i := A_i/B_i$ and is defined through a representative $h_i \in A_i \backslash B_i$ as follows:

$H_1 := F/F^*$, $\quad H_2 := F^\perp/F^{\perp*}$, $\quad H_3 := F^{\perp*\perp}/F^{\perp\perp}$, $\quad H_4 := F^{*\perp}/F^{*\perp*}$,
$H_5 := F^{\perp\perp} \cap F^{*\perp}/F^\perp \cap F^{\perp\perp}$, $\quad H_6 := (F + F^\perp)^{*\perp}/F^\perp \cap F^{\perp\perp}$,
$H_7 := F^{\perp*\perp} \cap F^{*\perp}/F^\perp \cap F^{\perp\perp}$, $\quad H_8 := F^{\perp*\perp}/F^{\perp*}$, $\quad H_9 := F \cap F^{*\perp}/F \cap F^\perp$,
$H_{10} := F^{\perp*\perp*}/F^{\perp\perp*}$.

$$\gamma_1 := \frac{<h_2,h_3>^2}{||h_2|| ||h_3||},\ \gamma_2 := \frac{<h_1,h_4>^2}{||h_1|| ||h_4||},\ \gamma_3 := \frac{<h_1,h_5>^2}{||h_1|| ||h_5||},\ \gamma_4 := \frac{<h_1,h_6>^2}{||h_1|| ||h_6||},$$

$$\gamma_5 := \frac{<h_7,h_8>^2\ ||h_1|| ||h_2||}{<h_1,h_8>^2 <h_2,h_7>^2},\ \gamma_6 := \frac{<h_7,h_8>^2\ ||h_1||}{<h_1,h_8>^2\ ||h_7||},$$

$$\gamma_7 := \frac{<h_9,h_{10}>^2\ ||h_1|| ||h_2||}{<h_2,h_{10}>^2 <h_1,h_9>^2}\ \gamma_8 := \frac{<h_2,h_{10}>^2\ ||h_9||}{<h_9,h_{10}>^2\ ||h_2||},$$

$$\gamma_9 := \frac{<h_5,h_{10}>^2\ ||h_1|| ||h_2||}{<h_1,h_5>^2 <h_2,h_{10}>^2},\ \gamma_{10} := \frac{<h_{10},h_8>^2\ ||h_1|| ||h_2||}{<h_2,h_{10}>^2 <h_1,h_8>^2},$$

$$\gamma_{11} := \frac{<h_2,h_7>^2\ ||h_1||}{<h_1,h_7>^2\ ||h_2||},\ \gamma_{12} := \frac{<h_2,h_7>^2}{||h_2|| ||h_7||},\ \gamma_{13} := \frac{<h_1,h_7>^2}{||h_1|| ||h_7||}.$$

It is worth noticing that the set $\Gamma(E,F) \subset k$ of the arithmetic invariants has always a cardinality less or equal two. We can now state the

Classification Theorem 59. *The geometric and arithmetic invariants determine the orbit of (E,F). Each pair is a sum of a trace-valued pair and a subdirectly indecomposable pair of case III. The number of the latter ones is finite if and only if k is finite. It is $33 + 5|k|$ if F or $F^\perp$ is trace-valued, and $\le 55 + 15|k| + 2|k|^2$ otherwise.*

The pairs of ordered sets $(\mathcal{N}, \mathcal{M})$ split into 119 isomorphism classes. In 95 of these classes the arithmetic invariants do not play any rôle. For 58 classes the information delivered by $\mathcal{M}$ is redundant.

Remark. All instances of subdirectly irreducible pairs (E, F) are realizable over $\mathbb{F}_2(\Gamma(E, F))$.

6 Classification of subspaces of quadratic spaces

6.1 Mapping Theorems

In all of this section let k be a commutative field such that $[k : k^2]$ is finite and (E, Q) non-degenerate diagonal quadratic space over k of dimension $\aleph_\alpha$, $\alpha < \omega$, with associated α-arfian lattice $\mathcal{L}(E)$. Write $Q(X) = \{Q(x) \mid x \in X\}$. Call a finite subalgebra $\mathcal{V}$ of $\mathcal{L}(E)$ *well behaved* if

(a) the lattice $\mathcal{V}$ belongs to the variety generated by $\mathcal{M}_3$

(b) $\mathcal{V}$ contains a largest totally isotropic member R,

(c) $X \subset R$ for all non-prime join irreducibles X with $\dim X/X_* < \aleph_0$

(d) $X \subset R$ or $X \subset R^\perp$ for all $X \in \mathcal{V}$ with $\dim X/rX < \aleph_0$.

Let X_0 denote the sum of all $X \in \mathcal{V}$ with $\dim X/rX < \aleph_0$.

Theorem 60. *Let (E, Q) and (E', Q') diagonal quadratic spaces of dimension $\aleph_\alpha$, $\alpha < \omega$ and $\mathcal{V}$, $\mathcal{V}'$ well behaved subalgebras of $\mathcal{L}(E)$ and $\mathcal{L}(E')$, respectively. For an index preserving isomorphism $\eta : \mathcal{V} \to \mathcal{V}'$ to be induced by an isometry it is necessary and sufficient that*

(a) $Q(X) = Q'(\eta X)$ for all X in a generating set of the lattice $[\kappa R, R]$

(b) $X \cong \eta X$ for all irreducible $X \subset R^\perp$ with $0 < \dim X/rX < \aleph_0$

(c) If $X_0 \not\subset R^\perp$ then $X \cong \eta X$ for some $X \in [X^\perp \cap X_0, X_0]$.

Corollary 61. *For (E, Q) as above and $F, F' \subset E$ assume that the α-arfian lattice $\mathcal{V}_E(F)$ is finite, its interval $[0, rF^{\perp\perp}]$ in the lattice variety generated by $\mathcal{M}_3$, and $[0, K]$ and $[(rF^{\perp\perp})^\perp, E]$ distributive (where $K = \kappa(rF^{\perp\perp})$). Then F and F' are congruent if and only if there is an index preserving isomorphism η between the α-arfian lattices $\mathcal{V}_E(F)$ and $\mathcal{V}_E(F')$ with $\eta F = F'$ and*

(a) $Q(X) = Q(\eta X)$ for all X in a suitable system of lattice generators of $[K, rF^{\perp\perp}]$

(b) If $\dim F/rF < \infty$ then $F \cong F'$ and if $\dim F^\perp/rF^\perp < \infty$ then $F^\perp \cong F'^\perp$

(c) If $\dim F/rF < \infty$ and $\dim F^\perp/rF^\perp < \infty$ then $K^\perp \cong (\eta K)^\perp$.

These are Thms.14.1,2 in [Wal88]. The proof refines the methods of Section 3 to cope with the additional structure, see section 15-17 of [Wal88].

6.2 The relevant lattices

Denote by $\mathcal{V}_\alpha(a,R)$ the free α-arfian lattice with generator a and relations R - within any variety $\mathbf{V}$ of modular lattices containing $\mathcal{M}_3$. Recall that the subdirectly irreducible factors are obtained from those of the associated side lattice $\mathcal{S}(\mathcal{V}_\alpha(a,R))$ (cf Thm. 24) and the $\mathcal{R}_j$, $\mathcal{R}_{\alpha i}$ (Cor. 29) and that

$$r_i = r\sigma_i a = a^\perp \sigma_i a \text{ and } c = \kappa r_0 .$$

Theorem 62. *As a lattice in* $\mathbf{V}$, $\mathcal{S}(\mathcal{V}_1(a))$ *is freely generated by the poset* P *in Fig. 17 under the relations*

$$cr_2 \leq \sigma_1(c\sigma_1 r_2),\ cr_1 \leq \sigma_1(cr_1),\ c\sigma_0(c\sigma_0 r_2) = c\sigma_1(c\sigma_0 r_2),$$

$$c\sigma_0(c\sigma_0 r_1) = c\sigma_1(c\sigma_0 r_1),\ c\sigma_1(c\sigma_0 r_2)\sigma_1(cr_1) = c\sigma_1(cr_1\sigma_0 r_2).$$

Moreover, the set C_i *of* σ_i*-closed elements is the join subsemilattice generated by the* r_j *and elements of the form* $\sigma_j x \in P,\ j \leq i$.

Corollary 63. *As a lattice in* $\mathbf{V}$, $\mathcal{S}(\mathcal{V}_1(a, ra = a))$ *is freely generated by the poset*

$$c,\ r_2,\ \sigma_0(c\sigma_0 r_2) \leq \sigma_1(c\sigma_0 r_2) \leq \sigma_1(c\sigma_1 r_2)$$

under the relations

$$c \leq r_2 + \sigma_0(c\sigma_0 r),\ cr_2 \leq \sigma_1(c\sigma_0 r_2),\ c\sigma_0(c\sigma_0 r_2) = c\sigma_1(c\sigma_1 r_1).$$

Its subdirectly irreducible factors are $\mathcal{A}_{-12}$, $\mathcal{A}_{02}$, *and* $\mathcal{B}_{-1012}$ *with arbitrary choice of* c *and those in Fig. 19.*

Corollary 64. *As a lattice in* $\mathbf{V}$, $\mathcal{S}(\mathcal{V}_1(a, ra = 0))$ *is freely generated by the poset of Fig. 18 under the relations*

$$cr_1 \leq \sigma_1(cr_1),\ c\sigma_0(c\sigma_0 r_1) = c\sigma_1(c\sigma_0 r_1).$$

Its subdirectly irreducible factors are $\mathcal{A}_{-10}$, $\mathcal{A}_{01}$, *and* $\mathcal{A}_{12}$ *with arbitrary choice of* c *and those in Fig. 20.*

Remarks. 1. In Fig. 17-20 we represented σ_0- and σ_1-closed elements large disks and squares, respectively.
2. The subdirectly irreducible factors of $\mathcal{V}_1(a, ra = a)$ are $\mathcal{R}_1$-$\mathcal{R}_3$, $\mathcal{R}_{1i}$, and the $\hat{\mathcal{S}}$ arising according to Case 2 from a side lattice.

3. The subdirectly irreducible factors of $\mathcal{V}_1(a, ra = 0)$ are $\mathcal{R}_1$,$\mathcal{R}_4$, $\mathcal{R}_5$, and the $\hat{\mathcal{S}}$ arising according to Case 1a from a side lattice.

4. The poset P in Thm. 62 is of tame representation type containing the 4-element anti-chain

$$A = \{c, r_2, \sigma_1(cr_1), \sigma_0(c\sigma_0 r_2)\}.$$

The relations are satisfied in any representation ρ where $\rho c \rho r_1 = 0$, $\rho x = 1$ if $x > y$ for some $y \in A$, and $\rho x = 0$ if $x < y$ for some $y \in A$. Therefore, even in the variety $\mathbf{V}_2$ associated with vector spaces over fields of characteristic 2, $\mathcal{V}_1(a)$ is infinite. This extends to $\mathcal{V}_2(a, ra = 0)$ and $\mathcal{V}_2(a, ra = a)$. The first has homomorphic image $\mathcal{V}_1(a)$ with $\sigma_2 = id$. For the second, a generating poset with relations has been given in [Wal88, p.56] containing the anti-chain c, r_2, $\sigma_1(c\sigma_1 r_2)$, $\sigma_2(c\sigma_0 r_2)$ and one may use representations of defect -1.

5. To prove the Corollaries one just has to derive from the side lattice relations $\sigma_i r_j = r_i$, $i < j$, resp. $r_2 = r_3$ the additional lattice relations between the elements of P. Then Lemma 3 assures that there are only factors $\mathcal{D}_2$ and $\mathcal{M}_3$ and it suffices to verify that those which are models of relations are exactly the ones occurring as subdirect lattice factors of the given side lattices.

6. The proof of the Theorem relies on the obvious Lemma below: choose $\mathcal{K}_i$ as the ideal lattice of the poset $P_i \subset P$ consisting of c, the r_i, and the elements denoted in the form $\sigma_j t$, $j \leq i$, with a side-lattice term t. For such, let $\alpha_i t = t$. Put $\alpha_i t = ct$ and $\alpha_i r_j = cr_j$. Then, define $\alpha_i x = \sum\{\alpha_i y \mid x \geq y \in J(\mathcal{K}_i)\}$. Conversely, in the modular lattice with the presentation, considered, the subsemilattices C_i are indeed sublattices and give rise to a side-lattice structure. For $\alpha \leq 2$ there is an analogous result [Wal88, Thm.86]: the point is that for the congruence generated by r_0/c the quotient $\mathcal{V}_\alpha(a)/\theta$ is distributive and that the projection map has smallest preimages.

Lemma 65. *Let $\mathcal{S}$ be an α-side lattice with generator set E. For each $i \leq \alpha$ let $\mathcal{K}_i$ be a finite lattice and α_i a join homomorphism from $\mathcal{K}_i$ into $\mathcal{S}$ such that $\pi_i x = \sigma_i \alpha_i x$ is meet preserving and $E \cup \{r_0, \ldots, r_{\alpha+1}\} \subset \bigcup_{x \in \mathcal{K}_i} [\alpha_i x, \pi_i x]$. Suppose that $r_i \in \pi_i(\mathcal{K}_i) \subset \pi_i(\mathcal{K}_j)$ for all $j \leq i$. Then, the set C_i of σ_i-closed elements is the join subsemilattice generated by $X_i = \{\sigma_i \alpha_i x \mid x \in J(\mathcal{K}_i)\}$ and, as a lattice, $\mathcal{S}$ is generated by E, the r_i, and $\bigcup_i X_i$.*

6.3 Classification of subspaces

Corollaries 61, 63, and 64 yield the following characterization of congruence in the case $\alpha = 1$ and $rF = F$ or $rF = 0$. These are Thm.19.1 and 20.1 in [Wal88]. We write $K = \kappa(F^{\perp\perp})$, $G = F \cap K$, $H = r\sigma F \cap K$, G', K', H' correspondingly, and $\sigma = \sigma_1$.

Theorem 66. *For (E, Q) as above let $\dim E = \aleph_1$ and F, F' totally isotropic subspaces of E. Then F and F' are congruent if and only if*

(a) $Q(F) = Q(F')$, $Q(G^{\perp\perp}) = Q(G'^{\perp\perp})$, $Q(\sigma G) = Q(\sigma G')$, $Q(\sigma K) = Q(\sigma K')$, $Q(G^{\perp\perp} \cap F) = Q(G'^{\perp\perp} \cap F')$, $Q(\sigma G \cap F) = Q(\sigma G' \cap F)$

(b) If $\dim F^{\perp}/F^{\perp\perp} < \infty$ then $F^{\perp} \cong F'^{\perp}$ and $K^{\perp} \cong K'^{\perp}$
(c) $\dim(F) = \dim(F')$, $\dim(\sigma F/F) = \dim(\sigma F'/F')$, $\dim(F^{\perp\perp}/\sigma F) = \dim(F'^{\perp\perp}/\sigma F')$, $\dim(F^{\perp}/F^{\perp\perp}) = \dim(F'^{\perp}/F'^{\perp\perp})$.

Theorem 67. *For (E, Q) as above let $\dim(E) = \aleph_1$ and F, F' non-degenerate subspaces of E. Then F and F' are congruent if and only if*
(a) $Q(rF^{\perp\perp}) = Q(rF'^{\perp\perp})$, $Q(r\sigma F) = Q(r\sigma F')$, $Q(K^{\perp\perp}) = Q(K'^{\perp\perp})$, $Q(\sigma K) = Q(\sigma K')$, $Q(H^{\perp\perp}) = Q(H'^{\perp\perp})$, $Q(\sigma(H^{\perp\perp} \cap K))) = Q(\sigma(H'^{\perp\perp} \cap K'))$, $Q(\sigma H) = Q(\sigma H')$, $Q(H^{\perp\perp} \cap r\sigma F) = Q(H'^{\perp\perp} \cap r\sigma F')$,
(b) If $\dim(F) < \infty$ then $F \cong F'$; if $\dim(F^{\perp}/rF^{\perp}) < \infty$ then $F^{\perp} \cong F'^{\perp}$
(c) $\dim(F) = \dim(F)'$, $\dim(\sigma F/r\sigma F + F) = \dim(\sigma F'/r\sigma F' + F')$, $\dim(F^{\perp\perp}/rF^{\perp} + \sigma F) = \dim(F'^{\perp\perp}/rF'^{\perp}+\sigma F')$, $\dim(F^{\perp}/rF^{\perp}) = \dim(F'^{\perp}/rF'^{\perp})$, $\dim((rF^{\perp\perp})^{\perp}/F^{\perp\perp}+F^{\perp}) = \dim((rF'^{\perp\perp})^{\perp}/F'^{\perp\perp} + F'^{\perp})$, $\dim(r\sigma F) = \dim(r\sigma F')$, $\dim((r\sigma F)^{\perp\perp}/r\sigma F) = \dim((r\sigma F')^{\perp\perp}/r\sigma F')$, $\dim(rF^{\perp\perp}/(r\sigma F)^{\perp\perp}) = \dim(rF'^{\perp\perp}/(r\sigma F')^{\perp\perp})$.