GALOIS LATTICES

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1 Introduction

Galois lattices, i.e. lattices equipped with a Galois self-correspondence $\perp$, have been used by H.Gross and his coworkers in the theory of infinite dimensional forms, extensively - see [4]. In particular, the structure has been determined for quite a few finitely presented modular Galois lattices. In the present note we show that the special case of distributive Galois lattices can be adequately captured by the concept of Galois semilattices. This variety is quite similar to the variety of all lattices : On one hand it allows solutions of the word problem in the styles of McKinsey [6] and Dean [1]; on the other, there are infinite 2-generated Galois semilattices, even with rather strong relations, and a finitely based subvariety with 2-variable equations undecidable (for lattices the same occurs with 4 generators).

2 Free distributive extension

A *Galois semilattice* is a meet semilattice S (meets written as products) with a unary operation $\perp$ such that

$$x \leq y \Rightarrow y^\perp \leq x^\perp \text{ and } y \leq x^\perp \rightarrow x \leq y^\perp.$$

Of course, these axioms can be replaced by equations, the second one also by $x \leq x^{\perp\perp}$. An element x is *closed* if $x = x^{\perp\perp}$. All elements of the form $x^\perp$ are closed. If S is a lattice, with joins written as sums, then one speaks of a *Galois lattice* and derives $(x+y)^\perp = x^\perp y^\perp$. This happens if all elements are closed: define $x + y = (x^\perp y^\perp)^\perp$ and call L a *polarity lattice*. Our results are easily modified to the situation where one adds bounds $0, 1$ as constants and the identity $1^\perp = 0$.

Let $\mathcal{I}S$ the lattice of all finitely generated order ideals of S and denote by $\downarrow X$ the order ideal generated by X. Identify x with $\downarrow x$ to have S a Galois subsemilattice of $\mathcal{I}S$ and endow $\mathcal{I}S$ with the unary operation

$$I^\perp = \downarrow \Pi\{x^\perp | x \text{ maximal in } I\}.$$

Lemma 1 *$\mathcal{I}S$ is the free distributive Galois lattice generated by the Galois semilattice S. S consists just of the join irreducibles of $\mathcal{I}S$ and contains all closed elements.*

Proof. $\downarrow X \cap \downarrow Y = \downarrow \{xy | x \in X, y \in Y\}$, thus $\mathcal{I}S$ is a distributive lattice. Every ideal $I^\perp$ is principal, $I^\perp = \downarrow a$, and $I^{\perp\perp} = \downarrow a^\perp$. Then for $y \in I$ we have $y \leq x$ for some maximal $x \in I$, whence $y^\perp \leq x^\perp \leq a$ and $y \leq y^{\perp\perp} \leq a^\perp$. From this one sees that $\perp$ is a Galois map. Given a Galois semilattice homomorphism ϕ of S into a distributive Galois lattice the extension to $L(S)$ can be defined by $\overline{\phi}I = \sum \phi(X)$, X any finite generating set of I .

3 Word problem.

A *partial Galois semilattice* A is a partially ordered set together with two partial operations, a binary written as xy which is $inf(x,y)$ where defined and a unary $x^\perp$ satisfying the following axioms as far as they can be evaluated

$$x < y \to y^\perp \leq x^\perp \ , \ x \leq x^{\perp\perp} \ , \ y < x \to x \leq y^\perp .$$

In particular, $x^{\perp\perp\perp} = x^\perp$ if defined.

If A is weakly embedded into a partial Galois semilattice B generated by A such that every weak homomorphism of A into a Galois semilattice can be extended to B, then we call B a *free partial extension* of A.

A *filter* of A is an order filter closed under all defined meets. Let $\uparrow X$ denote the filter generated by X and $X^\perp = \uparrow \{x^\perp | x \in X\}$. Ordered by the dual of set inclusion, the filters form a meet semilattice $\mathcal{F}A$ extending A. Identify x with $\uparrow x$. Equipped with the partial operation $\perp$ from A $\mathcal{F}A$ becomes a free partial extension of A.

For r in A such that $r^\perp$ is not defined let $r^\perp$ a new element and define $p \leq_r q$ if an only if $p = q$ or $p, q \in A$ and $p \leq q$ or one of the following

$$p = r^\perp, q \in \uparrow X^\perp, X \subseteq \downarrow r \text{ for some finite } X \subseteq A$$

$$q = r^\perp, r = y^\perp, p \leq y \text{ for some } y \in A$$

$$q = r^\perp, p \leq y^\perp, r \leq y \text{ for some } y \in A.$$

Let $A_r = A \cup \{r^\perp\}$ identifying $r^\perp$ with $a \in A$ if $a \leq_r r^\perp \leq_r a$. Endow A_r with the obvious relation and partial operations coming from A with just $r \mapsto r^\perp$ added.

Lemma 2 *A_r is a free partial extension of A.*

Proof. First, we have to show transitivity of the relation $\leq_r$. Let $p \leq_r r^\perp \leq_r q$. Then $q \in X^\perp$ and $X \subseteq \downarrow r$ for some X.
Case 1: There is $y \in A$ with $r = y^\perp$ and $p \leq y$. Then $x \leq y^\perp$ and $y \leq x^\perp$ for all $x \in X$ whence $y \leq q$ and $p \leq q$. Also, if $p^\perp$ is defined then $r = y^\perp \leq p^\perp$.
Case 2: $p \leq y^\perp$ and $r \leq y$. Then $x \leq y$, $y^\perp \leq x^\perp$, and $p \leq y^\perp$ for all $x \in X$ whence $p \leq q$. Also, $r \leq y \leq p^\perp$ if $p^\perp$ is defined.
In particular, we have $r^\perp$ identified with at most one $a \in A$ and $r^{\perp\perp}$ defined if and only if $a^\perp$ defined - and then $r \leq a^\perp = r^{\perp\perp}$ by the above. Also, if $r = y^\perp$ then $y \leq_r r^\perp = y^{\perp\perp}$. This leaves us with verifying order reversion. Assume $r^{\perp\perp}$ defined. If $x \leq_r r^\perp$ then $x \leq a$ and $a^\perp \leq x^\perp$. If $r^\perp \leq_r y$ then $a \leq y$ and $y^\perp \leq a^\perp$. The remaining instances are obvious by the definition of $\leq_r$.

Finally, we have to show that products are meets in the enlarged order, too. The only case to be considered is $r^\perp \leq_r q_i$, $i = 1, 2$, and $q = q_1 q_2$. With suitable X_i we have $q \in (X_1 \cup X_2)^\perp$ whence $r^\perp \leq_r q$.

The freeness of the extension is obvious since elements are related in $\leq_r$ only if this is forced by the structure of A and the axioms.

For finite A let $\mathcal{P}A$ the finite partial Galois semilattice with all $x^\perp$ defined which arises from A by several steps according to the Lemma (in any order).

Theorem 1 *For a finite partial Galois semilattice A, the free Galois semilattice generated by A can be constructed as*

$$FGS(A) = \bigcup_{n=0}^{\infty} (\mathcal{P}\mathcal{F})^n A.$$

In particular, every partial Galois semilattice can be embedded into a Galois semilattice.

This is analogous to the decription of the lattice freely generated by a finite partial lattice given in Freese and Nation [3]. Also, this gives rise to the obvious "dynamic" solution of the word problem: Consider the subterms of the presentation and the two terms s, t, the equality of which has to be decided, as a partial algebraic structure in the natural way and produce the smallest quasiorder $\leq$ which, factored by the equivalence $x \approx y \Leftrightarrow x \leq y \leq x$, yields a partial Galois semilattice ; then the relation $s = t$ follows from the presentation if and only if $s \approx t$. Complexity bounds can be given as in Cosmadakis [2].

4 Negative results

Proposition 1 *There is an infinite Galois semilattice with two generators a, b and the relations $a = a^{\perp\perp} \leq b = b^{\perp\perp}$ and $tt^{\perp} \leq a^{\perp}b$ for all t.*

Proof. The free lattice L generated by $a \leq b$, $c \leq d$ is infinite according to Rolf [7]. Let ϕ the homomorphism onto the dual lattice interchanging a and d, b and c. Then $x^{\perp} = \phi x$ turns L into a polarity lattice. Inspection of the diagram yields $tt^{\perp} \leq a^{\perp}b$ for all $t \in L$.

Theorem 2 *There is a complete lattice embedding taking lattice varieties to Galois semilattice varieties.*

Proof. Given a lattice L one can construct a polarity lattice $L^{\bullet}$ choosing a disjoint isomorphic copy M of L and isomorphism ϕ of L onto M and letting $L^{\bullet}$ the linear sum of the dual M^{δ} of M and L, i.e. $L^{\bullet} = M^{\delta} \cup L$ with $x \leq y$ for all $x \in M^{\delta}, y \in L$ endowed with the polarity $x^{\perp} = \phi x$ for $x \in L$ and $x^{\perp} = \phi^{-1}x$ for $x \in M^{\delta}$. Cleary, L is a finitely generated subdirectly irreducible lattice if and only if $L^{\bullet}$ is the obvious manner a subdirect product of the 2-element Boolean algebra B_2 and a finitely generated subdirectly irreducible polarity lattice not isomorphic B_2.

Moreover, if $\mathcal{C}$ is any class of such lattices L and $\mathcal{C}^{\bullet}$ the class of the corresponding $L^{\bullet}$ then

$$(GHSP_u\mathcal{C})^{\bullet} = \{B_2\} \cup GHSP_u(\mathcal{C}^{\bullet})$$

where H, S and P_u denote the operators of forming the homomorphic images, subalgebras, and ultraproducts , respectively, and $G\mathcal{C}$ the subclass of all finitely generated subdirectly irreducibles in $\mathcal{C}$. Thus, in view of Jónsson's Lemma, by mapping $\mathcal{V}$ onto the variety generated by $(G\mathcal{V})^{\bullet}$ one obtains an embedding of the lattice of lattice varieties into the filter generated by B_2 in the lattice of polarity lattice varieties.

Proposition 2 *There is a finitely axiomatized variety of Galois semilattices for which the free algebra in two generators has an unsolvable word problem.*

Proof. The free modular lattice on four generators is turned into a polarity lattice by the homomorphism onto its dual switching the generators, pairwise. Thus the claim follows from Herrmann [5].

Corollary 1 *The above results hold equally in the setting of distributive Galois lattices.*

Proof. First observe that a generating set of a polarity lattice L is also a generating set of the Galois semilattice L. Now, for the first claim, it is obvious that $tt^{\perp} \leq a^{\perp}b$ holds for all $t \in \mathcal{F}L$, too. The two others follow using the fact that L can be recovered in $\mathcal{F}L$ as the set of closed elements.

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