Appendix D. Theory of Lattice Representability Proofs.

In the following, we develop the results needed to prove the completeness theorem for $\langle L,R\rangle$-derivability (see 2.2 and 2.4).

D1. Definition and Properties. Let R be a ring, M an R-module, and L a lattice. An *MH-system* is a subset U of $M\times L$ such that U[v] (that is, $\{y \in L: \langle v,y\rangle \in U\}$) is either empty or is a dual ideal of L (possibly equal to L) for each v in M, and $U^{\#}[x]$ (that is, $\{u \in M: \langle u,x\rangle \in U\}$) is in Su(M) for each x in L. A triple $\langle v,x,y\rangle$ in $M\times L\times L$ is called a *defect* for U if $\langle v,x\vee y\rangle$ is in U but there exists no u in M such that $\langle u,x\rangle$ and $\langle v-u,y\rangle$ are in U. For any subset X of $M\times L$, let $\mathcal{S}(X)$ denote the intersection of all MH-systems which contain X and are contained in $M\times L$.

D1a. There is a one-one correspondence between MH-systems U on $M\times L$ and meet homomorphisms $\lambda: L \longrightarrow Su(M)$. (Define λ from U by $\lambda(x) = U^{\#}[x]$ for x in L; define U from λ by $U = \{\langle v,x\rangle \in M\times L: v \in \lambda(x)\}$ for the reciprocal operation.)

D1b. Suppose U and λ correspond as above. Then λ is a lattice homomorphism iff U has no defects. Also, λ is one-one iff for each pair $x < y$ in L, there exists v in M such that $\langle v,y\rangle$ is in U but $\langle v,x\rangle$ is not. In particular, λ is one-one if for each x in L there exists v_x in M such that $U[v_x]$ is the principal dual ideal $\{y: y \geq x\}$ of L.

D1c. The intersection of a set of MH-systems on $M\times L$ is again an MH-system, and $M\times L$ is itself an MH-system. So, for each $X \subseteq M\times L$, $\mathcal{S}(X)$ is the smallest MH-system containing X. We say $\mathcal{S}(X)$ is *generated* by X. So, S is a *closure* operator. That is, $X \subseteq \mathcal{S}(X)$, $\mathcal{S}(\mathcal{S}(X)) = \mathcal{S}(X)$ and $Y \subseteq X$ implies that $\mathcal{S}(Y) \subseteq \mathcal{S}(X)$, for subsets X and Y of $M\times L$. Furthermore, S is algebraic: whenever $\langle u,x\rangle$ is in $\mathcal{S}(X)$, there is a finite $Y \subseteq X$ such that $\langle u,x\rangle \in \mathcal{S}(Y)$. (We can verify that

$$\bigcup \{\mathcal{S}(Y): Y \text{ is a finite subset of } X\}$$

is an MH-system between X and $\mathcal{S}(X)$, and so is equal to $\mathcal{S}(X)$.)

D2. Definitions and Properties. Let R be a ring and L a lattice. Let $R^{(A)}$ be the free R-module on a set of generators A of cardinality α, where α is infinite and is not smaller than the cardinality of R or of L, that is, $\alpha = |A| \geq \aleph_0 + |R| + |L|$. Suppose that A consists of pairwise distinct elements $\mathbf{a}_x$ for each x in L and a_n for each ordinal number ν, $\nu < \alpha$. Since $E = R^{(A)} \times L \times L$ has cardinality at most α, we choose a well-ordering $\langle E,<\rangle$ of E such that each element of E has fewer than α predecessors. For $\nu < \alpha$, define:

$$A_\nu = A - \{a_\mu:\ \nu \leq \mu < \alpha\}$$

If v is in the submodule of $R^{(A)}$ generated by A_ν, say that v has support on A_ν. (Every element of $R^{(A)}$ has support on A_ν for some sufficiently large ν.) Define transfinite sequences X_ν ($\nu < \alpha$) and Y_ν ($\nu < \alpha$) of subsets of $R^{(A)} \times L$ as follows:

$X_0 = \{\langle \mathbf{a}_x, x\rangle:\ x \in L\}$,

$X_\nu = X_0 \cup \bigcup_{\mu<\nu} Y_\mu$ for $0 < \nu \leq \alpha$, and

$Y_\nu = \{\langle a_\nu, x_\nu\rangle, \langle v_\nu - a_\nu, y_\nu\rangle\}$ if $\nu < \alpha$, where $\langle v_\nu, x_\nu, y_\nu\rangle$ is the smallest element of $\langle E,<\rangle$ which is a defect for $\mathcal{S}(X_\nu)$ and such that v_ν has support on A_ν, or

$Y_\nu = \emptyset$ if there is no such element of E.

The limit MH-system $\mathcal{S}(X_\alpha)$ is called *an* $\langle L,R\rangle$*-standard MH-system*, and the $\lambda: L \longrightarrow Su(R^{(A)})$ corresponding to it as in D1a is called *an* $\langle L,R\rangle$*-standard homomorphism*.

D2a. Any $\langle L,R\rangle$-standard MH-system U has no defects. (A defect $\langle v,x,y\rangle$ for $U = \mathcal{S}(X_\alpha)$ would be a defect for $\mathcal{S}(X_\nu)$ for some $\nu < \alpha$ by D1c, and we can choose ν sufficiently large so that v has support on A_ν. Now, each step of the transfinite induction repairs the smallest suitable defect of E under <, but none repairs $\langle v,x,y\rangle$. So, there is a one-one function from $\{\mu:\ \nu < \mu < \alpha\}$ into the set of predecessors of $\langle v,x,y\rangle$ in E, contradicting the set cardinalities.) By D1b, an $\langle L,R\rangle$-standard homomorphism is a lattice homomorphism.

D3. Definitions. Let R be a ring and L a lattice. As in 2.2, let B contain distinct variables $\mathbf{b}_x$ for each x in L and b_i for each $i \geq 1$. An R-linear map $f:R^{(B)} \longrightarrow R^{(A)}$ is called an *insertion* if $f(\mathbf{b}_x) = \mathbf{a}_x$ for x in L, $f(b_i)$ is in $\{a_\nu: \nu < \alpha\}$ for $i \geq 1$, and f is one-one. A finite subset J of $R^{(A)} \times L$ is said to be $\langle L,R \rangle$-*coverable* via an insertion f and an $\langle L,R \rangle$-sequence $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ if:

$$J \subseteq \{\langle f(u_i), x_i \rangle : i \leq n\},$$

where $w_i = \langle u_i, x_i \rangle$ for $i \leq n$.

D4. Proposition. Suppose R is a ring, L is a lattice, and J and K are finite subsets of $R^{(A)} \times L$ such that $J \subseteq \mathcal{S}(K)$. If K is $\langle L,R \rangle$-coverable, then so is J.

Proof: Assume the hypotheses, so for some insertion $f:R^{(B)} \longrightarrow R^{(A)}$ and some $\langle L,R \rangle$-sequence $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$, with $w_i = \langle u_i, x_i \rangle$ for $i \leq n$:

$$K \subseteq \{\langle f(u_i), x_i \rangle : i \leq n\}.$$

We will call an $\langle L,R \rangle$-sequence $\mathbf{w}'$ of n or more terms an n-*restricted extension* of $\mathbf{w}$ if its first n terms are the same as the corresponding terms of $\mathbf{w}$ and there are no uses of rule 2.2e in forming $\mathbf{w}'$ except those used in forming $\mathbf{w}$. (Equivalently, each variable b_j with $j \geq n$ has coefficient 0 in every R-module term of $\mathbf{w}'$.)

Let H be the set of *reachable* pairs of $R^{(A)} \times L$; $\langle v,y \rangle$ is reachable if $\{\langle v,y \rangle\}$ is $\langle L,R \rangle$-coverable via f and some n-restricted extension of $\mathbf{w}$. A finite set of reachable pairs is $\langle L,R \rangle$-coverable, by splicing together finitely many n-restricted extensions of $\mathbf{w}$. Two n-restricted extensions $\mathbf{w}'$ and $\mathbf{w}''$ of $\mathbf{w}$ can be spliced by concatenating first the terms of $\mathbf{w}'$ and then the terms of $\mathbf{w}'$ with the first n omitted; the result is again an n-restricted extension of $\mathbf{w}$.)

Now, H can be showed to be an MH-system as follows: Given v in $R^{(A)}$, H[v] contains $x \wedge y$ if it contains x and y by rule 2.2b. If $z \geq y$ in L and $y \in$ H[v], then $z \in$ H[v] by 2.3. So, H[v] is empty or is a dual ideal of L (possibly equal to L). For x in L, $H^{\#}[x]$ contains $\mathbf{a}_x$ by 2.2a. For u, v in $H^{\#}[x]$ and r in R, $ru \in H^{\#}[x]$ by rule 2.2c and $u+v \in H^{\#}[x]$ by rule 2.2d. So, $H^{\#}[x] \in Su(R^{(A)})$.

Now $K \subseteq H$ by the construction of $\mathbf{w}$, so $J \subseteq \mathcal{S}(K) \subseteq H$ because H is an MH-system. So, J is $\langle L,R\rangle$-coverable. ∎

D5. Proposition. Suppose R is a ring, L is a lattice, and J is a finite subset of an $\langle L,R\rangle$-standard MH-system $\mathcal{S}(X_\alpha)$. Then J is $\langle L,R\rangle$-coverable.

Proof: Assuming the hypotheses, we prove by transfinite induction:

(G_ν) If $J \subseteq \mathcal{S}(X_\nu)$, then J is $\langle L,R\rangle$-coverable.

Suppose $J \subseteq \mathcal{S}(X_0)$, so $J \subseteq \mathcal{S}(J_0)$ for some finite $J_0 \subseteq X_0$ by D1c. But J_0 is $\langle L,R\rangle$-coverable by any insertion map and an appropriate $\langle L,R\rangle$-sequence consisting of instances of axiom scheme 2.2a. Then G_0 follows by D4.

Assume $0 < \nu < \alpha$ and G_μ holds for all $\mu < \nu$. If ν is a limit ordinal, then G_ν holds by D1c. Suppose ν is a successor ordinal, say $\nu = \mu + 1$. Now $J \subseteq \mathcal{S}(Z)$ for some finite subset Z of X_ν by D1c. If Y_μ is empty, then Z is $\langle L,R\rangle$-coverable by G_μ, hence so is J. So, assume Y_μ contains $\langle a_\mu, x\rangle$ and $\langle v - a_\mu, y\rangle$ where v has support on A_μ and $\langle v, x \vee y\rangle$ is in $\mathcal{S}(X_\mu)$. So, by G_μ:

$$Z' = (Z - Y_\mu) \cup \{\langle v, x \vee y\rangle\} \text{ is } \langle L,R\rangle\text{-coverable,}$$

via some insertion f and $\langle L,R\rangle$-sequence $\mathbf{w} = \langle w_1, \ldots, w_n\rangle$. Now extend $\mathbf{w}$ to w′ by rule 2.2e applied to a term $\langle u_j, x_j\rangle$ such that $f(u_j) = v$ and $x_j = x \vee y$, so $w_{n+1} = \langle b_n, x\rangle$ and $w_{n+2} = \langle u_j - b_n, y\rangle$. Define an insertion map $g: R^{(B)} \longrightarrow R^{(A)}$ by $g(\mathbf{b}_x) = \mathbf{a}_x$, $g(b_n) = a_\mu$, and $g(b_i) = f(b_i)$ for $i \neq n$, $i \geq 1$, except that $g(b_k) = f(b_n)$ if there exists $k \neq n$ such that $f(b_k) = a_\mu$. We observe that $g(u) = f(u)$ if $\langle f(u), x\rangle$ is in Z′ since then u has support on A_μ, which doesn't contain a_μ. Then Z is $\langle L,R\rangle$-coverable, and so J is also, by D4. This completes the transfinite induction, and G_α is the desired result. ∎

We can now prove an extension of Theorem 2.4, verifying that our logic of $\langle L,R\rangle$-derivability is complete.

D6. Completeness Theorem. Suppose R is a ring and L is a lattice. For x,y in L, the following are equivalent:

D6a. $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable.

D6b. If $h: L \longrightarrow Su(M)$ is a lattice homomorphism for some R-module M, then $h(x) \le h(y)$.

D6c. If $\lambda: L \longrightarrow Su(R^{(A)})$ is an $\langle L,R\rangle$-standard homomorphism, then $\lambda(x) \le \lambda(y)$.

Proof: We have proved D6a $\Rightarrow$ D6b (see 2.4), and D6b $\Rightarrow$ D6c follows by D2a. Assume D6c, and let U be the $\langle L,R\rangle$-standard MH-system corresponding to λ. Now $\langle \mathbf{a}_x, x\rangle \in X_0 \subseteq U$, so $\mathbf{a}_x \in \lambda(x) \leqq \lambda(y)$, and so $\langle \mathbf{a}_x, y\rangle \in U$, using D2a. By D5, there is an insertion map $f: R^{(B)} \longrightarrow R^{(A)}$ and $\langle L,R\rangle$-sequence $\langle \mathbf{w}_1, \dots, \mathbf{w}_n\rangle$, $\mathbf{w}_i = \langle u_i, x_i\rangle$ for $i \le n$, such that $\langle \mathbf{a}_x, y\rangle = \langle f(u_k), x_k\rangle$ for some $k \le n$. Since f is an insertion, $u_k = \mathbf{b}_x$. So, $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable via $\langle \mathbf{w}_1, \dots, \mathbf{w}_k\rangle$. ■

By D6, the kernel congruence $\theta_R = \{\langle x,y\rangle : \lambda(x) = \lambda(y)\}$ of L associated with any $\langle L,R\rangle$-standard homomorphism $\lambda: L \longrightarrow Su(R^{(A)})$ is the unique smallest congruence such that L/θ_R is representable by an R-module, with $\langle x,y\rangle$ in θ_R iff both $\langle \mathbf{b}_x, y\rangle$ and $\langle \mathbf{b}_y, x\rangle$ are $\langle L,R\rangle$-derivable. In fact, there is a functor from the category $\mathcal{L}$ of all lattices and τ_L-homomorphisms into the subcategory $\mathcal{L}(R)$ which maps each L into L/θ_R. This functor, which is a left adjoint to the inclusion functor from $\mathcal{L}(R)$ into $\mathcal{L}$, is called a *reflection* by some authors and a *coreflection* by others.

To illustrate our specialized logic, we demonstrate a particular result of the form $R \vDash \Psi$ implies $\mathcal{L}(R) \vDash \Gamma$, using a less formal approach than that of 2.9. Our proof adapts that of Czédli and Hutchinson [2E], which uses series-parallel graphs by the method of Czédli [2G].

D7. Example. For the set of variables $Y = \{y_1, y_2, y_3, y_4\}$, define the lattice polynomials:

$$z_{12} = (y_1 \vee y_2) \wedge (y_3 \vee y_4), \quad z_{13} = (y_1 \vee y_3) \wedge (y_2 \vee y_4),$$

$$z_{14} = (y_1 \vee y_4) \wedge (y_2 \vee y_3), \quad w_1 = (y_1 \vee y_4) \wedge (z_{12} \vee z_{13}), \text{ and}$$

$$w_2 = (y_2 \vee y_4) \wedge (z_{12} \vee z_{14}).$$

Let Γ denote the formula

$$(\forall y_1, y_2, y_3, y_4)(p_1 \le q_1 \Rightarrow p_2 \le q_2),$$

where $p_1 = y_2 \wedge (y_3 \vee w_1)$, $q_1 = y_1 \vee y_3 \vee w_2$,

$p_2 = z_{12}$ and $q_2 = y_1 \vee (y_2 \wedge (y_3 \vee w_1))$.

We first observe that $\mathcal{L}(S) \models \Gamma$ fails if there is an S-module N which is (additively) a free Z(4)-module. For ${}_SM = N \oplus N \oplus N$, it is convenient to denote certain submodules of ${}_SM$ by bracketed triples $[e_1, e_2, e_3]$, where variables occurring in the expressions e_1, e_2, e_3 may be assigned arbitrary elements of N. For example, $[u, 0, u+2v]$ denotes

$$\{\langle a,b,c\rangle \in M: \ u,v \text{ in } N \text{ such that } a = u,\ b = 0,\ c = u+2v\}.$$

Let $y_1 = [u,0,0]$, $y_2 = [0,u,0]$, $y_3 = [0,0,u]$ and $y_4 = [u,u,u]$ in $\mathrm{Su}({}_SM)$. Computation shows that $z_{12} = [u,u,0]$, $z_{13} = [u,0,u]$, $z_{14} = [0,u,u]$, $w_1 = [2u,u,u]$ and $w_2 = [u,2u,u]$. Furthermore, we have:

$$p_1 = \{\langle 0,v,0\rangle: \ v \in N \text{ and } 2v = 0\},$$

$q_1 = [u,2v,w]$, $p_2 = [u,u,0]$ and $q_2 = [u,0,0] \vee p_1$. Since N is additively a direct sum of cyclic groups of order four, we have $p_1 = [0,2u,0]$, so $p_1 \le q_1$ but $p_2 \le q_2$. Therefore, Γ fails in $\mathrm{Su}({}_SM)$.

Now let Ψ denote the system of ring equations formula

$$(\exists t)(4 = 0 \ \&\ 2t = 0 \ \&\ t^2 - 2 = 0),$$

and suppose $R \models \Psi$. That is, R is a ring with characteristic dividing four which contains an element t such that $2t = 0$ and $t^2 = 2$. (If R has characteristic four, then it contains an eight element subring isomorphic to $Z(4)[t]/K$, where K is the ideal generated by $\{2t, t^2 - 2\}$. These eight elements may be denoted by 0,1,2,3,t,t+1,t+2,t+3. If R has characteristic two or is trivial, then $R \models \Psi$ by taking $t = 0$.) Any R-module is a Z(4)-module, but a nontrivial free Z(4)-module can never be given an R-module structure. (If there was such an ${}_RM$, then $\mathrm{Ker}\, f = \mathrm{Im}\, f$ for $f = 2\cdot 1_M = 1_M + 1_M$. Choosing v in M with $2v \ne 0$, we would have $f(tv) = 2tv = 0$, so $2u = f(u) = tv$ for some u in M. But then $2v = t^2 v = t2u = 0$, a contradiction.)

To prove that $R \models \Psi$ implies $\mathcal{L}(R) \models \Gamma$, we assume that $L \in \mathcal{L}(R)$ and $h:P(Y,\tau_L) \longrightarrow L$ is a τ_L-homomorphism satisfying $h(p_1) \subseteq h(q_1)$. Let $z = h(p_2) = h(z_{12})$, so that our objective is to show that $\langle \mathbf{b}_z,\ h(q_2)\rangle$ is $\langle L,R\rangle$-derivable, hence $h(p_2) \le h(q_2)$ by 2.5.

The $\langle L,R\rangle$-sequence below is not the shortest possible, since we use 2.3 frequently for convenience, even when it leads to unnecessary terms. We begin:

1. $\langle \mathbf{b}_z,\ h(z_{12})\rangle$ by 2.2a.

5. $\langle \mathbf{b}_z,\ h(y_1) \vee h(y_2)\rangle$ by 2.3 for 1 (terms 2-4 omitted).

9. $\langle \mathbf{b}_z,\ h(y_3) \vee h(y_4)\rangle$ by 2.3 for 1 (terms 6-8 omitted).

10. $\langle b_9,\ h(y_1)\rangle$ by 2.2e for 5.

11. $\langle \mathbf{b}_z - b_9,\ h(y_2)\rangle$ by 2.2e continued.

12. $\langle b_{11},\ h(y_3)\rangle$ by 2.2e for 9.

13. $\langle \mathbf{b}_z - b_{11},\ h(y_4)\rangle$ by 2.2e continued.

14. $\langle t\mathbf{b}_z + tb_9,\ h(y_2)\rangle$ by 2.2c for 11, since $t = -t$.

20. $\langle b_9 - b_{11},\ h(y_1) \vee h(y_3)\rangle$ by 2.3 for 10 and 12.

26. $\langle b_9 - b_{11},\ h(y_2) \vee h(y_4)\rangle$ by 2.3 using 11 and 13.

27. $\langle b_9 - b_{11},\ h(z_{13})\rangle$ by 2.2b for 20 and 26.

28. $\langle b_9 + \mathbf{b}_z - b_{11},\ h(y_1) \vee h(y_4)\rangle$ by 2.2d for 10 and 13.

29. $\langle \mathbf{b}_z + b_9 - b_{11},\ h(z_{12}) \vee h(z_{13})\rangle$ by 2.2d for 1 and 27.

30. $\langle \mathbf{b}_z + b_9 - b_{11},\ h(w_1)\rangle$ by 2.2b for 28 and 29.

36. $\langle t\mathbf{b}_z + tb_9,\ h(y_3) \vee h(w_1)\rangle$ by 2.3 for 12 and 30.

37. $\langle t\mathbf{b}_z + tb_9,\ h(y_2) \wedge (h(y_3) \vee h(w_1))\rangle$ by 2.2b for 14 and 36.

By hypothesis,

$$h(y_2) \wedge (h(y_3) \vee h(w_1)) = h(p_1) \le h(q_1) = h(y_1) \vee h(y_3) \vee h(w_2),$$

so we continue:

41. $\langle t\mathbf{b}_z + tb_9,\ h(y_1) \vee h(y_3) \vee h(w_2)\rangle$ by 2.3 for 37.

42. $\langle b_{41},\ h(y_1)\rangle$ by 2.2e for 41.

43. $\langle t\mathbf{b}_z + tb_9 - b_{41},\ h(y_3) \vee h(w_2)\rangle$ by 2.2e continued.

44. $\langle b_{43},\ h(y_3)\rangle$ by 2.2e for 43.

45. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43},\ h(w_2)\rangle$ by 2.2e continued.

49. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43},\ h(y_2) \vee h(y_4)\rangle$ by 2.3 for 45.

53. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43},\ h(z_{12}) \vee h(z_{14})\rangle$ by 2.3 for 45.

54. $\langle b_{53},\ h(y_2)\rangle$ by 2.2e for 49.

55. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43} - b_{53},\ h(y_4)\rangle$ by 2.2e continued.

56. $\langle b_{55},\ h(z_{12})\rangle$ by 2.2e for 53.

57. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43} - b_{55},\ h(z_{14})\rangle$ by 2.2e continued.

61. $\langle b_{55},\ h(y_1) \vee h(y_2)\rangle$ by 2.3 for 56.

65. $\langle b_{55},\ h(y_3) \vee h(y_4)\rangle$ by 2.3 for 56.

66. $\langle b_{65},\ h(y_1)\rangle$ by 2.2e for 61.

67. $\langle b_{55} - b_{65},\ h(y_2)\rangle$ by 2.2e continued.

68. $\langle b_{67},\ h(y_3)\rangle$ by 2.2e for 65.

69. $\langle b_{55} - b_{67},\ h(y_4)\rangle$ by 2.2e continued.

73. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43} - b_{55},\ h(y_1) \vee h(y_4)\rangle$ by 2.3 for 57.

77. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43} - b_{55},\ h(y_2) \vee h(y_3)\rangle$ by 2.3 for 57.

78. $\langle b_{77},\ h(y_1)\rangle$ by 2.2e for 73.

79. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43} - b_{55} - b_{77},\ h(y_4)\rangle$ by 2.2e continued.

80. $\langle b_{79},\ h(y_2)\rangle$ by 2.2e for 77.

81. $\langle t\mathbf{b}_z + tb_9 - b_{41} - b_{43} - b_{55} - b_{79},\ h(y_3)\rangle$ by 2.2e continued.

At this point, we have completed the process of polynomial decomposition (using 2.3 twice for each meet and 2.2e for each join). It remains to derive $\langle \mathbf{b}_z,\ h(q_2)\rangle$ by applying 2.2b, 2.2d and 2.3 a number of times, which is done as follows:

87. $\langle b_{77} - b_{79},\ h(y_1) \vee h(y_2)\rangle$ by 2.3 using 78 and 80.

93. $\langle b_{77} - b_{79},\ h(y_3) \vee h(y_4)\rangle$ by 2.3 using 79 and 81.

94. $\langle b_{77} - b_{79},\ h(z_{12})\rangle$ by 2.2b using 87 and 93.

104. $\langle 2\mathbf{b}_z + 2b_9 + tb_{55} + tb_{77} + tb_{79},\ h(y_1) \vee h(y_3)\rangle$ by 2.3 using 42, 44, 78 and 81, since $t^2 = 2$ and $t = -t$.

110. $\langle b_{53} - b_{55} - b_{77},\ h(y_4)\rangle$ by 2.3 using 55 and 79.

120. $\langle 2\mathbf{b}_z + 2b_9 + tb_{55} + tb_{77} + tb_{79},\ h(y_2) \vee h(y_4)\rangle$ by 2.3 using 11, 54, 80 and 110, since $2 = -2$.

121. $\langle 2\mathbf{b}_z + 2b_9 + tb_{55} + tb_{77} + tb_{79},\ h(z_{13})\rangle$ by 2.2b using 104 and 120.

131. $\langle 2\mathbf{b}_z + 2b_{11},\ h(z_{12}) \vee h(z_{13})\rangle$ by 2.3 using 27, 56, 94 and 121.

135. $\langle 2\mathbf{b}_z + 2b_{11},\ h(y_1) \vee h(y_4)\rangle$ by 2.3 using 13.

136. $\langle 2\mathbf{b}_z + 2b_{11},\ h(w_1)\rangle$ by 2.2b using 131 and 135.

137. $\langle -\mathbf{b}_z + b_9 + b_{11},\ h(w_1)\rangle$ by 2.2d using 30 and 136, since $3 = -1$.

143. $\langle \mathbf{b}_z - b_9,\ h(y_3) \vee h(w_1)\rangle$ by 2.3 using 12 and 137.

144. $\langle \mathbf{b}_z - b_9,\ h(y_2) \wedge (h(y_3) \vee h(w_1))\rangle$ by 2.2b using 11 and 143.

145. $\langle \mathbf{b}_z,\ h(y_1) \vee (h(y_2) \wedge (h(y_3) \vee h(w_1)))\rangle$ by 2.2d using 10 and 144.

This completes the proof that $\langle \mathbf{b}_z,\ h(q_2)\rangle$ is $\langle L,R\rangle$-derivable.

BIBLIO.V2 4-13-1997 6:09pm

Bibliograph of Frequently Used Papers

@ F. W. Anderson and K. R. Fuller, {Rings and Categories of Modules}, Springer Verlag, Berlin, 1973.

@ E. Artin, {Geometric algebra}, Interscience, New York and London, 1957. MR18#553.

@ R. Baer, {Linear algebra and projective geometry}, Academic Press, New York, 1952. MR14#675.

@ G.Birkhoff, {Lattice theory}, 3rd ed., Amer. Math. Soc. Colloquium Pubs. 25, Providence, RI, 1967. MR37#2638.

@ H.-B. Brinkmann, {Addition von Korrespondenzen in abelschen Kategorien}, Math Z. 113 (1970), 344-352.

@ H.-B. Brinkmann and D. Puppe, {Abelsche und exakte Kategorien, Korrespondenzen}, Lecture Notes in Mathematics 96, Springer-Verlag, Berlin, Heidelberg and New York, 1969.

@ S. Brenner, {On four subspaces of a vector space}, J. of Algebra 29 (1974), 587-599.

@ D. Buchsbaum, {Exact Categories}, in H. Cartan and S. Eilenberg, {Homological Algebra}, Princeton Univ. Press, Princeton, NJ, 1956, 379-386.

@ H. Cartan and S. Eilenberg, {Homological Algebra}, Princeton Univ. Press, Princeton, NJ, 1956.

@ W. Craig, {On axiomatizability within a system}, J. Symbolic Logic 18 (1953), 30-32. MR14#1051.

@ P. M. Cohn, {Universal Algebra}, Harper & Row, New York, 1965.

@ G. Czedli and G. Hutchinson, {An irregular Horn sentence in submodule lattices}, Acta. Sci. Math (Szeged), in press.

@ M. Davis, {Computability and Unsolvability}, McGraw-Hill, New York, Toronto and London, 1958.

@ R. A. Dean, {Component subsets of the free lattice on n generators}, Proc. Amer. Math. Soc. 7 (1956), 220-226.

@ R. Freese, {Planar sublattices of FM(4)}, Algebra Universalis 6 (1976), 69-72.

@ P. Freyd, {Abelian Categories: An Introduction to the Theory of Functors}, Harper and Row, New York, 1964.

@ O. Frink, {Complemented modular lattices and projective spaces of infinite dimension}, Trans. Amer. Math. Soc. 60 (1964), 452-467.

@ G. Frobenius, {Theorie der linearen Formen mit ganzen Coefficienten}, J. reine und angewandte Math. 86 (1979), 146-208.

@ K. Godel, {The consistency of the continuum hypothesis}, Annals of Mathematics Studies No. 3, Princeton Univ. Press, Princeton, NJ, 1940.

@ G. Gratzer, {Universal Algebra}, 2nd ed., Springer-Verlag, Berlin, Heidelberg and New York, 1979.

@ C. Herrmann, {On the equational theory of submodule lattices}, Proc. University of Houston Lattice Theory Conference, Houston, 1973, 105-118.

@. C. Herrmann and A. Huhn, {Zum Wortproblem fur freie Untermodulverbande}, Archiv fur Mathematik 26 (1975), 449-453.

@ C. Herrmann and A. Huhn, {Zum Begriff der Charakteristik modularer Verbande}, Math. Z. 144 (1975), 185-194.

@ C. Herrmann and W. Poguntke, {The class of sublattices of normal normal subgroup lattices is not elementary}, Algebra Universalis 4 (1974), 280-286.

@ G. Higman, B. H. Neumann and H. Neumann, {Embedding theorems for Groups}, J. London Math. Soc. 24 (1949), 247-254.

1A. P. Hilton, {Correspondences and exact squares}, Proc. of the Conference on Categorical Algebra, La Jolla, Springer Verlag, Berlin, 1966, 254-271.

@ P. Hilton and Y.-C. Wu, {On the addition of relations in an abelian category}, Canadian J. Math 22, (1970), 66-74.

@ A. Huhn, {Schwach distributive Verbande I}, Acta Sci. Math. 33 (1972), 297-305.

@ G. Hutchinson, {An embedding theorem for abelian relation categories}, dissertation, Columbia University, New York, 1967.

@ G. Hutchinson, {Modular lattices and abelian categories}, J. Algebra 19 (1971), 156-184. MR43#4880.

@ G. Hutchinson, {Recursively unsolvable word problems of modular lattices and diagram-chasing}, J. of Algebra 26 (1973), 385-399.

@ G. Hutchinson, {The representation of lattices by modules}, Bull. Amer. Math. Soc. 79 (1973), 172-176. MR47#1693.

@ G. Hutchinson, {On the representation of lattices by modules}, Trans. Amer. Math. Soc. 209 (1975), 311-351.

@ G. Hutchinson, {On classes of lattices representable by modules}, Proc. University of Houston Lattice Theory Conference, Houston, 1973, 69-94.

@ G. Hutchinson, {Embedding and unsolvability theorems for modular lattices}, Algebra Universalis 7 (1977), 47-84.

@ G. Hutchinson, {A duality principle for lattices and categories of modules}, J. Pure Appl. Algebra 10 (1977), 115-119.

@ G. Hutchinson, {A complete logic for n-permutable congruences}, Algebra Universalis 13 (1981), 206-224.

@ G. Hutchinson, {Exact embedding functors between categories of modules}, J. Pure Appl. Algebra 25 (1982), 107-111.

@ G. Hutchinson and G. Czedli, {A test for identities satisfied in lattices of submodules}, Algebra Universalis 8 (1978), 269-309.

@ G. Hutchinson, {Representations of additive relation algebras by modules}, J. Pure and App. Algebra, in press.

@ G. Hutchinson, {Representations of additive relations by modules}, J. Pure and Applied Algebra 42 (1986), 63-83.

@ G. Hutchinson, {Addendum to "Exact embedding functors between categories of modules"}, J. Pure and Applied Algebra 47 (1987), 99-100.

@ B. Jonsson, {On the representation of lattices}, Math. Scand. 1 (1953), 193-206. MR15#389.

@ B. Jonsson, {Modular lattices and Desargues' Theorem}, Math. Scand. 2 (1954), 295-314.

@ B. Jonsson, {Representation of modular lattices and of relation algebras}, Trans. Amer. Math. Soc. 92 (1959), 449-464.

@ B. Jonsson, {The class of Arguesian lattices is self-dual}, Algebra Universalis 2 (1972), 396.

@ S. R. Kogalovskii, {Structural characteristics of universal classes}, Sibirsk. Mat Z. 4 (1963), 97-119. (Russian) MR26#4909.

@ L. Lipshitz, {The undecidability of the word problems for projective geometries and modular lattices}, Trans. Amer. Math. Soc. 193 (1974), 171-180.

@ J. Los, {Quelques remarques, theoremes et problemes sur les classes definissables d'algebres}, Mathematical Interpretation of Formal Systems, North-Holland, Amsterdam, 1955, 98-113. MR17#700.

@ S. Lubkin, {Imbedding of abelian categories}, Trans. Amer. Math. Soc. 97 (1960), 410-417. MR30#133.

1C S. MacLane, {An algebra of additive relations}, Proc. Nat. Acad. Sci. USA 47 (1961), 1043-1051.

@ S. MacLane, {Categories for the working mathematician}, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

@ S. MacLane and G. Birkhoff, {Algebra}, MacMillan, New York, 1967.

@ M. Makkai and G. McNulty, {Universal Horn axiom systems for lattices of submodules}, Algebra Universalis 7 (1977), 25-31.

@ A. Mal'cev, {On the general theory of algebraic systems}, Mat. Sb. (N.S.) 35 (77) (1954), 3-20. (Russian)

@ Ju. V. Matiyasevic, {Simple examples of unsolvable canonical calculi}, Trudy. Mat. Inst. Steklov. 93 (1973), 56-88. (Russian) MR36#4996.

@ J. McCarthy, {A basis for a mathematical theory of computation}, in Computer Programming and Formal Systems, (Braffort and Hirschberg, Eds.), North-Holland, Amsterdam, 1963, 33-70.

@ J. McCarthy, P. W. Abrahams, D. J. Edwards, T. P. Hart and M. I. Levin, {LISP 1.5 Programmer's Manual}, 2nd ed., M.I.T. Press, Cambridge, MA, 1963.

@ B. Mitchell, {Theory of Categories}, Academic Press, New York, 1965.

@ O. Ore, {On the foundations of abstract algebra, I and II}, Math. Ann. 36 (1935), 406-437; 37 (1936), 265-292.

1B D. Puppe, {Korrespondenzen in Abelschen Kategorien}, Math. Ann. 148 (1962), 1-30.

@ J. B. Rosser, {Logic for Mathematicians}, McGraw-Hill, New York, 1953.

@ J. Rotman, {The Theory of Groups, An Introduction}, 2nd ed., Allyn and Bacon, Boston, 1973.

@ J. Rotman, {Notes on Homological Algebra}, Van Nostrand Reinhold, New York, 1970.

@ D. Sachs, {Identities in finite partition lattices}, Proc. Amer. Math. Soc. 12 (1961), 944-945.

@ B. M. Schein, {Relation algebras and function semigroups}, Semigroup Forum 1 (1970), 1-62.

@ A. Shafaat, {On implicationally defined classes of algebras}, J. London Math. Soc. 44 (1969), 137-140.

@ J. Shoenfield, {Mathematical Logic}, Addison-Wesley, Reading, MA, 1967.

@ O. Schreier and E. Sperner, {Introduction to Modern algebra and Matrix Theory}, Chelsea, New York, 1951.

@ R. Vescan, {On subobjects, quotients, kernels, cokernels in a partially ordered category}, Publ. Math. (Debrecen) 22, (1975), 211-218.

@ R. Vescan, {Categories of relations with standard factorizations}, Publ. Math. (Debrecen) 26 (1979), 155-160.

@ J. von Neumann, {Continuous geometry}, Princeton Math. Series No. 25, Princeton Univ. Press, Princeton, NJ, 1960. MR22#10931.

@ C. Watts, {Intrinsic characterizations of some additive functors}, Proc. Amer. Math. Soc. 11 (1960), 5-8.

@ P. M. Whitman, {Free lattices II}. Annals of Math. 43 (1942), 104-115.

@ P. M. Whitman, {Status of the word problems for lattices}, in Lattice Theory (R. P. Dilworth, Ed.), Proceedings of Symposia in Pure Mathematics, Vol. II, Amer. Math. Soc., Providence, RI, 1961.

@ R. Wille, {Primitive Lange und primitive Weite bei modularen Verbanden}, Math. Z. 108 (1969), 129-136.

@ R. Wille, {Kongruenzklassengeometrien}, Lecture Notes in Mathematics No. 113, Springer-Verlag, Berlin, Heidelberg and New York, 1970.