$L^p$-Regularity Theory for Linear Elliptic and Parabolic Equations

Dem Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung der Venia Legendi
vorgelegte

Habilitationsschrift

von
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Part I.

Summary
1. Introduction

One of the central topics in the theory of partial differential equations, as soon as existence and uniqueness of solutions are known, is the regularity of these solutions. For linear problems such investigations are of special importance, since many methods for solving non-linear equations are based on suitable regularity assumptions for an associated linear problem. In my research I consider several questions in this context, treating abstract operator theoretical topics as well as concrete classes of differential equations. While the results obtained in my PhD thesis [Hal04] are mainly of the first kind (cf. [HHN02, HHH03, Hal05]) with one application to parabolic systems with VMO-coefficients [HHH06], the present collection mainly contains results on various linear parabolic and elliptic partial differential equations, that I achieved after my PhD. An exception is Article 8 that contains operator theoretical results not contained in my thesis.

The regularity theory of solutions to linear elliptic and parabolic partial differential equations with bounded smooth coefficients on bounded smooth domains is nowadays classical, see e.g. the monographs [Agm65, LSU67, Lun95, Ama95, GT01, ABHN01]. Unfortunately (or happily for mathematical research), modelling real world problems one often is confronted with phenomena that do not fit into this framework.

The papers presented here treat three different situations. In a first part (Articles 1 to 4) I gathered works on elliptic operators in divergence form with real $L^\infty$-coefficients on bounded Lipschitz domains, complemented with mixed boundary conditions. Operators of generalised Ornstein-Uhlenbeck type, i.e. with an unbounded coefficient of the first order term, are investigated in Articles 5 and 6. Thirdly, some result on spectral theory for the weighted Laplacian on a star-shaped network is contained in Article 7.

Finally, Article 8 contains a result on the bounded $H^\infty$-calculus for products of sectorial operators.

In the first article [HMR08] we treat the classical problem of Hölder continuity of the solution to the elliptic problem

$$-\nabla \cdot \mu \nabla u + u = f + \sum_{j=1}^d \frac{\partial f_j}{\partial x_j},$$

where $f \in L^{q/2}(\Omega)$ and $f_j \in L^q(\Omega)$ for some $q > d$ (cf. [GT01, Chapter 8]) in the setting described above, i.e. $\mu \in L^\infty(\Omega; \mathbb{R}^{d \times d})$, $\Omega \subseteq \mathbb{R}^d$ is a Lipschitz domain and mixed boundary conditions are incorporated. In the case of pure Dirichlet boundary conditions this regularity result is well-known by the pioneering work of De Giorgi [DeG57] (see also [Nas58] and [Mos60]), on which also our considerations eventually are based.

It turns out that the solution indeed is Hölder continuous for $d \leq 4$ and if a rather weak compatibility condition for the Dirichlet and the Neumann part of the boundary is fulfilled. This generalises a result of Stampacchia [Sta60]. Finally, the article contains some applications of this result to optimal control theory.

Parabolic linear and quasilinear problems involving $A_\Gamma = -\nabla \cdot \mu \nabla$, where $\Gamma$ is the part of $\partial \Omega$, where Neumann (or Robin) boundary conditions are imposed, are examined in [HR08] (Article 2). In this comprehensive paper we determine the domain of the square root of $A_\Gamma$ in some cases and use this to obtain maximal $L^p$-regularity results for the realisation of $A_\Gamma$ in the negative Sobolev space $W^{-1,q}_\Gamma(\Omega)$ for suitable $q$. Such a maximal regularity result
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was already known in $L^q(\Omega)$ by [HR06]. In the special case of the Laplacian with Dirichlet boundary condition on Lipschitz domains it also follows from the results of Jerison/Kenig [JK95].

In a second step, developing further the ideas of [HR06], the maximal regularity of $A_\Gamma$ allows to treat quasilinear problems with mixed non-linear and non-homogeneous boundary conditions and distributional right hand sides, e.g. surface densities, by the method of Clément/Li [CL94], see also [Prü02].

Two main ingredients of this work are heat kernel estimates known for $A_\Gamma$ on $L^2(\Omega)$ by the works of Arendt/terElst [AtE97] and Ouhabaz [Ouh05] and regularity results on the square root of these operators in the pure Dirichlet case proved by Auscher/Tchamitchian [AT01].

A characteristic of such ‘non-smooth’ elliptic operators is that $A_\Gamma + 1$ in general does not even provide an isomorphism between $W^{1,q}_\Gamma(\Omega)$ and $W^{-1,q}_\Gamma(\Omega)$ for $q \neq 2$. However, knowing this optimal elliptic regularity for $q$ bigger than the space dimension is of great help in many circumstances, see e.g. the section on quasilinear problems in [HR08] (Article 2). So it is worthwhile to spend some work on identifying classes of coefficient functions and geometric constellations that still guarantee this isomorphism property.

In [Dau92] Dauge proved this for some $q > 3$ for the Laplacian on a convex polyhedron in $\mathbb{R}^3$, if the border between the Dirichlet and the Neumann boundary part consists of a finite number of line segments, by estimating the appearing edge and vertex singularities (see also [MR03]).

Article 3 ([HKR08]) contains a generalisation of this result to prototypical situations where mixed boundary conditions and jumps in the coefficient function $\mu$ appear simultaneously. The proof heavily rests on a deep result of Maz’ya [MERS04] that allows to conclude by only considering the edge singularities of some equivalent problem with pure Dirichlet boundary conditions.

Finally, Article 4 ([HHR08]) contains a result on irreducibility of the semigroup generated by $bA_\Gamma$ on $L^q(\Omega)$, where $b$ is a strictly positive $L^\infty$-function.

A further focus of my research lies on generalised Ornstein-Uhlenbeck operators. The problematic part of these operators of second order is the coefficient of the first order term that grows at infinity. These operators appear naturally in stochastic analysis, describing a deterministic drift, perturbed by a Brownian motion. In this context they are quite well understood in spaces of continuous functions (see [DPL95, Lun98]), on $L^p(\mathbb{R}^d)$ (see [LV96, Met01, MPV05, PRS06]) and on $L^p(\mathbb{R}^d, \mu)$, where $\mu$ is the invariant measure of the corresponding stochastic process (see [Lun97, Gol99, MPP02, MPRS02, FL06]). An overview on these topics may be found in the monograph [LB07].

On the other hand, for unbounded domains $\Omega \subseteq \mathbb{R}^d$ very few was known. Using results for the full space from [Met01] and for bounded domains (where the operator may be seen as a perturbation of the elliptic second order part), in [HW05] (Article 5) we showed that Ornstein-Uhlenbeck operators with Dirichlet boundary conditions are generators of consistent, positive and quasi-contractive $C_0$-semigroups on $L^p(\Omega)$ for any $1 \leq p < \infty$ and arbitrary domains $\Omega \subseteq \mathbb{R}^d$. In the special case of an exterior domain a result on the spectrum is given that shows that these semigroups fail to be analytic. Generalisations of some of these considerations to more general coefficients that even grow a little bit faster than linearly are contained in [HW06] (Article 6). A detailed presentation of these and other results concerning (generalised) Ornstein-Uhlenbeck operators on domains is given in [Wie07].

An entirely different problem is at the heart of [AHR08] (Article 7). Here, the underlying geometry is given by $n$ intervals $[0, \infty)$ that are glued together at the origins, thus forming a sort of star that even fails to be a manifold in the usual sense. Nevertheless, it has a natural differential structure and can be seen as a model domain for network problems. Following the approach of [AM96] that treats the case $n = 2$, we explicitly construct by spectral theory a Fourier type transformation that diagonalises the weighted Laplace operator on the $n$-star.
An approach often used to obtain maximal regularity for a given sectorial operator is to show that it admits a bounded $H^\infty$-calculus. In order to treat complicated operators in this way, it is of natural interest to know under which circumstances sectoriality and the bounded $H^\infty$-calculus are inherited by sums and products of operators having these properties.

In the case of sums of commuting operators, based on the fundamental paper of Da Prato/Grisvard [DPG75], this is the topic of the famous Dore/Venni [DV87] and Kalton/Weis [KW01] results on operator sums. Generalisations of these to non-commuting operators were given in [MP97] and [PS07]. While a corresponding result for products of commuting operators still follows from the works on joint functional calculi in [KW01], it was the aim of [HH05] (Article 8) to treat the remaining case and to give conditions, under which products of non-commuting operators are sectorial and admit a bounded $H^\infty$-calculus, thus extending earlier works of Weber [Web98] and Štrkalj [Str01].
2. Divergence form operators with mixed boundary conditions

During my stay at the Weierstrass Institute in Berlin, I started to work on elliptic operators $-\nabla \cdot \mu \nabla$ of second order in divergence form with non-smooth $L^\infty(\Omega; \mathbb{R}^{d \times d})$-coefficients $\mu$ on bounded Lipschitz domains complemented with mixed Dirichlet-Neumann or even Dirichlet-Robin boundary conditions. Such operators appear for example in the simulation of semiconductor devices, where domains with edges and corners, heterogeneous materials, causing jumps in the coefficient functions, and mixed boundary conditions are common phenomena.

When investigating their $L^p$-regularity theory, due to the very poor regularity of the data, we have to forget several reflexes from the smooth theory. In particular, there is no hope of explicitly determining the domain of the operator even in $L^2$ and, naturally, spaces with differentiability index two are completely out of range. But even if we restrict our attention to spaces with differentiability index between $-1$ and $1$, which is the natural setting for these operators, their behaviour is sometimes quite nasty. In fact, any of the three irregular ingredients ($L^\infty$-coefficients, Lipschitz domain, mixed boundary conditions) alone can already destroy optimal elliptic regularity for $q > 3$, see [EKRS07, JK95, Sha68] for the corresponding counterexamples. For instance, the striking result of Elschner (cf. [EKRS07] and [ERS07]) shows that even in the case of pure Dirichlet boundary conditions and smooth domains $\Omega$, operators with $L^\infty$-coefficients do not provide an isomorphism between $W^1_0,q(\Omega)$ and $W^{-1},q(\Omega)$ in general, whenever $q > 2$.

Thus, in many situations we have to conclude without knowing the domains of our operators, which makes regularity theory a delicate matter. Nonetheless, we are able to prove Hölder continuity of solutions to elliptic, parabolic and even quasilinear problems involving these operators in a very general framework.

Before summing up in detail the results of the Articles 1–4, we will shortly introduce some common notation and concepts of these papers.

All of our analysis is done in bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$. Unfortunately, this terminology is used for two different notions in the literature. Following [Gri85], we demand the domain to be locally homeomorphic to the lower half of the unit cube by bi-Lipschitz mappings with the boundary going to the upper plate. Such domains sometimes are also called ‘bi-Lipschitz domains’, cf. [VV06]. This notion is more general than that of a ‘strongly Lipschitz domain’, where the domain locally has to lie under a Lipschitz graph, and that is also often referred to as ‘Lipschitz domain’, for instance in the works of Jerison/Kenig. An example of a bi-Lipschitz domain that is not a strongly Lipschitz domain is the double beam treated in Section 2.7.3 in Article 2.

We start with the mathematical description of mixed boundary conditions. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain in the above sense and let $\Gamma$ be an open part of $\partial \Omega$. Then for $1 < q < \infty$ and $0 < s \leq 1$ we define the space $W^{s,q}_\Gamma(\Omega)$ as the closure of

$$C^{\infty}_\Gamma(\Omega) := \{ v|_\Omega : v \in C^{\infty}(\mathbb{R}^d), \text{supp}(v) \cap (\partial \Omega \setminus \Gamma) = \emptyset \}$$

in $W^{s,q}(\Omega)$ and $W^{-s,q}_\Gamma(\Omega) = (W^{s,q}_\Gamma(\Omega))'$, if $1/q + 1/q' = 1$. 

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2. Divergence form operators with mixed boundary conditions

Given a coefficient function $\mu \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ that is symmetric and satisfies the usual ellipticity condition, we may define the operator $A_\Gamma = -\nabla \cdot \mu \nabla$ on $L^2(\Omega)$ via the form

$$a(u, v) := \int_\Omega \mu \nabla u \cdot \nabla v \, dx, \quad u, v \in D(a) := W^{1,2}_\Gamma(\Omega). \quad (2.0.1)$$

This corresponds to the realisation of $-\nabla \cdot \mu \nabla$ with Neumann boundary conditions on $\Gamma$ and Dirichlet boundary conditions on $\partial\Omega \setminus \Gamma$ in $L^2(\Omega)$.

As for admissible Neumann boundary parts $\Gamma$ of $\partial\Omega$ we mainly employ the concept of regular sets introduced by Gröger in [Grö89]. A Lipschitz domain $\Omega$ together with some open part $\Gamma \subseteq \partial\Omega$ is called regular, if for every point $x \in \partial\Omega$ there is a neighbourhood $U_x$ of $x$, such that $(\Omega \cup \Gamma) \cap U_x$ may be mapped by a bi-Lipschitz function $\phi_x$ to one of the three model sets $K_-, K_- \cup \Sigma$ or $K_- \cup \Sigma_0$ (cf. Figure 2.1) with $\phi_x(x) = 0$. An equivalent condition for this in two and three space dimensions that can be checked by appearance is given in [HMR08] (Article 1).

The notion of regular sets has turned out to fit very well to the treatment of mixed boundary value problems. Especially, since it provides a framework that allows to establish interpolation results for function spaces with a trace zero condition on part of the boundary, see [GGKR02], which we exploit frequently in Article 2. Finally, it has the advantage to allow for general Lipschitz domains $\Omega$.

One final remark on the expression $A_\Gamma$ is in order. As defined above it describes the self-adjoint operator associated to the form (2.0.1) in $L^2(\Omega)$. For ease of notation we use the same symbol for the same operator, when it acts in other Lebesgue and Sobolev spaces. While it is defined again by the form as an operator on the space $W^{1,2}_\Gamma(\Omega)$ with domain $W^{1,2}_\Gamma(\Omega)$, we define it as the maximal restrictions of these two operators to $L^2(\Omega)$, $q \geq 2$, and to $W^{1,q}_\Gamma(\Omega)$, $q \geq 2$, respectively. Finally, for $1 < q < 2$ the operator is understood as the closure of the one in $L^2(\Omega)$ or $W^{1,2}_\Gamma(\Omega)$. Naturally, when doing so one has to pay attention to the underlying spaces and domains very carefully.

2.1. Hölder continuity for the elliptic problem

We consider the elliptic problem

$$-\nabla \cdot \mu \nabla u + u = f + \sum_{j=1}^d \frac{\partial f_j}{\partial x_j}$$

with $f \in L^{q/2}(\Omega)$ and $f_j \in L^q(\Omega)$ for some $q > d$, equipped with mixed boundary conditions as described above. As already mentioned earlier, in the situation of $L^\infty$-coefficients, mixed boundary conditions and/or Lipschitz domains we do not have in general that the domain of $A_\Gamma = -\nabla \cdot \mu \nabla$, seen as an operator on $W^{-1,q}_\Gamma(\Omega)$, is contained in $W^{1,q}_\Gamma(\Omega)$. So Hölder continuity of the variational solution cannot be deduced by Sobolev embedding and gets an
2.1. Hölder continuity for the elliptic problem

interesting issue in this situation. In the case of $\Gamma = \emptyset$, i.e. pure Dirichlet boundary conditions, Hölder continuity follows from the classical De Giorgi result [DeG57], see also [Nas58] and [Mos60].

In the case of space dimension 2, 3 or 4 we show in Article 1 that the Hölder continuity of the solution to the above equation remains true for mixed boundary conditions, when only $\Omega$ and $\Gamma$ are regular in the sense introduced above. This generalises a result of Stampacchia [Sta60], see also [Lie86] and [Lie89].

The regularity assumption on $\Omega$ and $\Gamma$ is given in local coordinates, thus providing a good starting point for analysis. However, it is rather difficult to decide by appearance if a given constellation of $\Omega$ and $\Gamma$ satisfies the conditions to be regular or not. At least for $d = 2$ and $d = 3$, we were able to give an alternative characterisation of regular sets $\Omega \cup \Gamma$ that decouples the conditions on $\Omega$ and $\Gamma$, demanding for $\Omega$ to be a Lipschitz domain and requiring a condition for $\Gamma$ that is purely topological in nature. This is the main result of the second part of Article 1.

The article contains a third part that is devoted to an application in control theory. Here, continuity of the solution is a crucial property in many situations. For instance, it plays an important role in the treatment of semilinear optimal control problems that include pointwise inequality constraints on the state. In this context our continuity result allows to generalise results of Casas, de los Reyes and Tröltzsch [CdlRT] to mixed boundary conditions, giving a second-order sufficient optimality condition for such problems.

2.1.1. Results

We consider the operator $A_\Gamma = -\nabla \cdot \mu \nabla$ with mixed boundary conditions that is defined by (2.0.1) as an operator in $W^{-1,2}_\Gamma(\Omega)$ with domain $W^{1,2}_\Gamma(\Omega)$ and its realisation in $W^{-1,q}_\Gamma(\Omega)$ for $q > 2$, that is given by its maximal restriction to this space. Then the first main result of Article 1 ([HMR08]), concerning the Hölder continuity of solutions to the elliptic problem, reads as follows.

**Theorem 2.1.1.** Suppose $d \in \{2, 3, 4\}$ and $q > d$. If $\Omega \cup \Gamma$ is regular, then there is an $\alpha > 0$, such that $(A_\Gamma + 1)^{-1}$ maps $W^{-1,q}_\Gamma(\Omega)$ continuously into $C^\alpha(\Omega)$.

Having this theorem at hand, there are two straightforward generalisations by perturbation theory, see Corollary 1.3.5 and Corollary 1.3.6 in Article 1. Firstly, we can replace the mixed Dirichlet-Neumann boundary conditions by mixed Dirichlet-Robin boundary conditions, whenever the representing function of the Robin boundary condition on $\Gamma$ is in $L^\infty(\Gamma)$.

Secondly, the result also applies to problems of the form

$$-\nabla \cdot \mu \nabla u + Vu = f, \quad f \in W^{-1,q}_\Gamma(\Omega),$$

where $V \in L^\infty(\Omega)$ is non-negative (and not identically zero in the pure Neumann case).

The second contribution of this paper is a characterisation for a set $\Omega \cup \Gamma$ to be regular in the two- and three-dimensional case that mainly is based on topological conditions for the boundary part $\Gamma$, see Theorems 1.5.2 and 1.5.4 in Article 1.

**Theorem 2.1.2.** Let $d = 2$ or $d = 3$, let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain and let $\Gamma \subseteq \partial \Omega$ be relatively open. Then $\Omega \cup \Gamma$ is regular, if and only if

$d = 2$: the set $\Gamma \cap (\partial \Omega \setminus \Gamma)$ is finite and no connected component of $\partial \Omega \setminus \Gamma$ consists of a single point.

$d = 3$: $\partial \Omega \setminus \Gamma$ is the closure of its interior within $\partial \Omega$ and for any point $x \in \overline{\Gamma \cap (\partial \Omega \setminus \Gamma)}$ exists an open neighbourhood $U_x$ of $x$ and a bi-Lipschitz mapping $\kappa_x : U_x \cap \overline{\Gamma \cap (\partial \Omega \setminus \Gamma)} \to (-1, 1)$.

Here the set $\overline{\Gamma \cap (\partial \Omega \setminus \Gamma)}$ is just the interface between the Dirichlet and the Neumann part of the boundary.
2.1.2. Methods

The proof of the regularity result in Theorem 2.1.1 eventually stems on the following well known result of De Giorgi, cf. [DeG57], see also [KS80, Theorem C.2]: If \( \Omega \) is a ball or a cuboid and \( q > d \), then \( A_{\Gamma}^{-1} : W^{-1,q}(\Omega) \rightarrow C^\alpha(\Omega) \) is continuous for some \( \alpha > 0 \), i.e. our result is true for the pure Dirichlet case on balls and cuboids. Having this result at hand, the strategy is the following: We localise the elliptic problem to the three model constellations in Figure 2.1 and show that the assertion is true for them. This is immediate for the first one and can be obtained by even reflection on the Neumann boundary for the second one. Finally, the third constellation is reduced to the second one by some bi-Lipschitz transformation. While doing so, the main analytic problem is to handle the additional terms in the right hand side, coming from the cut-off in the localisation procedure. During these considerations our method gets restricted to space dimension at most four due to the lack of integrability of the gradient of the solution and the required Sobolev embeddings.

While the proof of Theorem 2.1.2 is straightforward in the case \( d = 2 \), it is rather involved for \( d = 3 \). The difficulty here is to handle the third model set, i.e. to show that, under the given hypotheses, for every \( x \in \Gamma \cap (\partial \Omega \setminus \Gamma) \), there is a bi-Lipschitz mapping from a neighbourhood \( U \) of \( x \) to \( \mathbb{R}^3 \), that maps \( U \cap (\Omega \cup \Gamma) \) to \( K \cup \Sigma_0 \) (cf. Figure 2.1). The main ingredient here is a deep lying theorem of Tukia, see [Tuk80], that allows to extend a bi-Lipschitz map defined on a line segment in \( \mathbb{R}^2 \) to a bi-Lipschitz map from the whole of \( \mathbb{R}^2 \) into itself. Using this, the ‘one-dimensional’ information given in the hypotheses provides us with a two- and later three-dimensional bi-Lipschitz map that fulfils the required mapping properties.

Both results were developed together with Joachim Rehberg during my stay at Berlin. The application to problems of optimal control in Section 6 of the paper was contributed by Christian Meyer.

2.2. Maximal regularity for the parabolic problem

The main project of my research stay at the Weierstrass Institute in Berlin was to treat quasilinear problems of the formal type

\[
\begin{aligned}
\frac{d}{dt} (F(u))' - \nabla \cdot G(u) \mu \nabla u &= R(t,u), \\
u(T_0) &= u_0,
\end{aligned}
\tag{2.2.1}
\]

combined with mixed, non-linear boundary conditions

\[
\nu \cdot G(u) \mu \nabla u + b(u) = g \text{ on } \Gamma \quad \text{and} \quad u = h \text{ on } \partial \Omega \setminus \Gamma \tag{2.2.2}
\]

in the before described general non-smooth situation, in order to provide a very general setting as it is needed in applications, e.g. in the modelling of semiconductor devices, see for example [Sel84]. Having these applications in mind, it is desirable to include also distributional right hand sides (e.g. surface densities), as well as inhomogeneous Neumann boundary conditions. This cannot be done in the usual way, using maximal \( L^p \)-regularity of the linearised equation in Lebesgue spaces (cf. [HR06]). The way out was to achieve maximal \( L^p \)-regularity results for the operator \( A_{\Gamma} = -\nabla \cdot \mu \nabla \) in Sobolev spaces \( W^{-1,q}_{\Gamma}(\Omega) \) for large enough values of \( q \).

The maximal regularity result for these operators in \( W^{-1,q}_{\Gamma}(\Omega) \) is interesting in itself, since, together with the already known maximal regularity on \( L^p \)-spaces and interpolation, one obtains this property on a huge variety of spaces in between, which allows to choose appropriate solution spaces for each given quasilinear problem.
While the application of the maximal regularity result to the quasilinear equation, however involved, follows well known procedures, cf. [CL94], [Pr"u02] and [HR06], showing maximal regularity for \( A_\Gamma \) on \( W^{-1,q}_\Gamma(\Omega) \) required new ideas. The basic idea is to prove that the square root of this operator provides a topological isomorphism between \( L^q(\Omega) \) and \( W^{-1,q}_\Gamma(\Omega) \) and to carry over the maximal regularity property from \( L^q(\Omega) \) to \( W^{-1,q}_\Gamma(\Omega) \) via this isomorphism. While doing so, we generalised results of Auscher and Tchamitchian on the square root of divergence form operators, cf. [AT01], to some special situations with mixed boundary conditions. All these results, together with some examples to which they apply, are contained in [HR08] (Article 2).

### 2.2. Maximal regularity for the parabolic problem

#### 2.2.1. Results

The paper under consideration contains three major results. The first concerns regularity of the square root of \( A_\Gamma \) in one local constituent of our Lipschitz domain, cf. Theorems 2.4.3 and 2.4.4 of Article 2.

**Theorem 2.2.1.** If \( \Omega \cup \Gamma \) as a whole is the image under a multiple of a volume-preserving bi-Lipschitz map of one of the three model sets \( K_-, K_- \cup \Sigma \) or \( K_- \cup \Sigma_0 \), cf. Figure 2.1, and \( q \in [2, \infty) \), then \( A^{1/2}_\Gamma : L^q(\Omega) \rightarrow W^{-1,q}_\Gamma(\Omega) \) is a topological isomorphism.

In order to formulate our maximal regularity result (Theorem 2.5.4 in Article 2), we define

\[
q_{iso} := \sup \{ q \geq 2 : A_\Gamma : W^{1,q}_\Gamma(\Omega) \rightarrow W^{-1,q}_\Gamma(\Omega) \text{ is a topological isomorphism} \}
\]

and set \( q_{iso}^* \) to be the Sobolev conjugated index, i.e. \( 1/q_{iso}^* = 1/q_{iso} - 1/d \) for \( q_{iso} < d \) and \( q_{iso}^* = \infty \) else.

**Theorem 2.2.2.** Let \( \Omega \cup \Gamma \) be regular with the additional requirement that for any \( x \in \partial \Omega \) the bi-Lipschitz maps \( \phi_x \) in the definition of regular sets are multiples of volume-preserving maps, i.e. \( |\det(D\phi_x)| \) is constant, where \( D\phi_x \) is the Jacobian of \( \phi_x \). Then \( A_\Gamma \) admits maximal regularity in \( W^{-1,q}_\Gamma(\Omega) \) for all \( 2 \leq q < q_{iso}^* \).

Similarly to Theorem 2.1.1, by perturbation theory this result can be generalised to the operator \( -\nabla \cdot \mu \nabla \) with mixed Dirichlet-Robin boundary conditions, see Theorem 2.5.16 of Article 2.

Another easy but powerful consequence of this result is maximal regularity for the same operator in all real and complex interpolation spaces

\[
[L^p(\Omega), W^{-1,q}_\Gamma(\Omega)]_\theta \quad \text{and} \quad (L^p(\Omega), W^{-1,q}_\Gamma(\Omega))_{\theta,s}
\]

for all choices of \( p \in [2, \infty) \), \( q \in [2, q_{iso}^*] \), \( \theta \in (0,1) \), and \( s \in (1, \infty) \), which is also contained in Theorem 2.5.16 of Article 2.

In our third result, see Theorem 2.6.14 and Corollary 2.6.16 of Article 2, we apply this to the quasilinear problem \((2.2.1)/(2.2.2)\) and get the following local existence and uniqueness result.

**Theorem 2.2.3.** Let \( \Omega \cup \Gamma \) be as in Theorem 2.2.2 and \( q_{iso} > d \). Take \( q \in (d, q_{iso}) \), \( \varsigma \in (\max\{1 - 1/q, d/q\}, 1) \), \( s > 2/(1 - \varsigma) \) and set \( X := W^{-\varsigma,q}_\Gamma(\Omega) \). If

- there exists \( f \in C^2(\mathbb{R}) \) positive, with strictly positive derivative, such that \( F \) is the superposition operator induced by \( f \),
- the mapping \( G : W^{1,q}(\Omega) \rightarrow W^{1,q}(\Omega) \) is locally Lipschitz continuous,
• for any ball in $W^{1,q}(\Omega)$ there exists $\delta > 0$, such that $G(u) \geq \delta$ for all u from this ball,

• the function $\mathcal{R} : J \times W^{1,q}(\Omega) \rightarrow X$ is of Carathéodory type, i.e. $\mathcal{R}(\cdot, u)$ is measurable for all $u \in W^{1,q}(\Omega)$ and $\mathcal{R}(t, \cdot)$ is continuous for almost all $t \in J$,

• $\mathcal{R}(\cdot, 0) \in L^s(J; X)$ and for $M > 0$ there exists $h_M \in L^s(J)$, such that

$$\|\mathcal{R}(t, u) - \mathcal{R}(t, \tilde{u})\|_X \leq h_M(t)\|u - \tilde{u}\|_{W^{1,q}(\Omega)}, \quad t \in J,$$

provided $\max\{\|u\|_{W^{1,q}(\Omega)}, \|\tilde{u}\|_{W^{1,q}(\Omega)}\} \leq M$,

• $b$ is an operator of the form $b(u) = Q(b_0(u))$, where $b_0$ is a possibly non-linear, locally Lipschitzian operator from $C(\overline{\Omega})$ into itself (cf. Lemma 2.5.15 of Article 2),

• $q \in L^q(\Gamma)$ and $u_0 \in (X, \text{dom}_X(A_{\Gamma}))_{1-q}$,

then (2.2.1)/(2.2.2) has a local in time, unique solution in $W^{1,q}(J; X) \cap L^q(J; \text{dom}_X(A_{\Gamma}))$ and this solution lies in $C^\alpha(J; C^\beta(\Omega))$ for some $\alpha, \beta > 0$.

The most severe restriction in this theorem is the condition $q_{\text{fin}} > d$. Thus, it is worthwhile to note that this hypotheses is not necessary in the case of semilinear equations, see Theorem 2.6.17 in Article 2.

### 2.2.2. Methods

It follows from results of Arendt [Are97], see also [HR06], that $A_{\Gamma}$ as an operator on $L^q(\Omega)$ admits maximal regularity for all $1 < q < \infty$. The main idea for the proof of maximal regularity on $W^{-1,q}_\Gamma(\Omega)$ was to use on one hand that a sectorial operator $A$ on some Banach space $X$ admits maximal regularity, if and only if the operator $\mathcal{L}$ with

$$(\mathcal{L}f)(t) = A \int_0^t e^{(s-t)A}f(s) \, ds, \quad t > 0,$$

can be extended to a continuous operator from $L^p(J; X)$ into itself for some (equivalently all) $p \in (1, \infty)$. On the other hand, we headed for the property that $(A_{\Gamma} + 1)^{1/2} : L^q(\Omega) \rightarrow W^{-1,q}_\Gamma(\Omega)$ is a topological isomorphism. These two properties then allow to write $\mathcal{L}$, seen as an operator in $L^p(J; W^{-1,q}_\Gamma(\Omega))$, as the same operator in $L^p(J; L^q(\Omega))$ conjugated with $(A_{\Gamma} + 1)^{1/2}$, since $A_{\Gamma}$ commutes with the corresponding operator $\mathcal{L}$ and to carry over the continuity of $\mathcal{L}$ from $L^q(\Omega)$ to $W^{-1,q}_\Gamma(\Omega)$.

It turned out that such a global regularity result for the square root is a touchy business (cf. the monograph [Aus07]) and eventually we succeeded in proving the local result in Theorem 2.2.1. In order to show this we put in four main ingredients. Firstly, we heavily rely on the fact that the semigroup generated by $A_{\Gamma}$ on $L^2(\Omega)$ satisfies Gaussian estimates even in the case of mixed boundary conditions. This was shown by Arendt and ter Elst in [AtE97], see also [Ouh05]. The second cornerstone is a lemma, showing that the isomorphism property we are interested in remains invariant under bi-Lipschitz transformations of the domain. This allows to reduce the investigation to the three model sets in Figure 2.1. Thirdly, we reduced the problems on the model sets to problems with a pure Dirichlet boundary condition by bi-Lipschitz transforms and reflection. The isomorphism property for the Dirichlet problems finally is contained in the work of Auscher and Tchamitchian [AT01].

How to globalise this local result for the square root is not at all obvious, since $A_{\Gamma}^{1/2}$ is not a local operator. Thus, in order to obtain maximal regularity in the global setting, we localised the parabolic problem instead and used the result on the square root in every local constituent. During this procedure we pick up the restriction to $q < q_{\text{fino}}$. 

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2.3. Optimal elliptic regularity in model situations

Once maximal regularity of $A_\Gamma$ is known, quasilinear and semilinear problems of the form (2.2.1)/(2.2.2) can be treated by the well known local solvability results of Amann [Ama93] or Clément and Li [CL94], see also [Pru02]. The crucial point here is to choose the right space, in which the problem is considered. The natural choice $W^{1,2}_\Gamma(\Omega)$ is not ideal, since then quadratic gradient terms of the solution in the right hand side cannot be treated, in contrast to the classical theory. It turns out that these can be included by taking $W^{s-\varsigma,q}_\Gamma(\Omega)$ for some $\varsigma < 1$ arbitrarily close to 1. On this space we also have maximal regularity for $A_\Gamma$ by the interpolation result mentioned above.

During all our considerations trace and interpolation results on the scale $W^{s,q}_\Gamma$, $-1\leq s\leq 1$, are of great importance. For these we mainly profited from earlier works of Griepentrog, Gröger, Kaiser and Rehberg in [GGKR02].

The contents of this article is the outcome of a one-year intense collaboration. In this process the regularity result of the square root was under my responsibility.

2.3. Optimal elliptic regularity in model situations

In many situations it is important to have information on the number $q_{iso}$ that describes up to what integrability index $q$ the operator $A_\Gamma$ has optimal elliptic regularity, i.e. it provides a topological isomorphism from $W^{1,q}_\Gamma(\Omega)$ onto $W^{1,q}_\Gamma(\Omega)$. Especially, one is interested in $q_{iso} > d$, for instance in the hypotheses of Theorem 2.2.3. While for $d = 2$ this is always true by a result of Gröger [Grö89] and for $d \geq 4$ there are several generic counterexamples, the case $d = 3$ is of particular interest, not only due to the importance in view of applications.

The counterexample of Elschner (cf. [EKR07] and [ERS07]) shows that $q_{iso}$ exceeds 2 only arbitrarily little in general, however, there are several constellations of domains $\Omega \subseteq \mathbb{R}^3$, boundary parts $\Gamma$ and coefficient functions $\mu$ that allow for a $q > 3$ see [Dau92, EKRS07, ERS07, HKR] and Article 3 that has been published in [HKR08]. Here we show $q_{iso} > 3$ for the geometric constellations in Figure 2.2. In both cases $\Omega$ is a prismatic domain and half a side or one or two complete sides form the Neumann boundary part (hatched area $\Sigma$ in Figure 2.2), whereas we impose Dirichlet boundary conditions on the rest of $\partial \Omega$. As for the coefficient $\mu$ we suppose it to be constant up to one jump across a plane that runs through the prism ($\Xi$ in Figure 2.2).

![Figure 2.2.](image)

Figure 2.2.: The Lipschitz domains from Article 3 with Neumann boundary part $\Sigma$ and interface $\Xi$.

At a first glance, the setting treated here seems to be very special, but it should be taken into account that the isomorphism property under consideration remains invariant under bi-Lipschitz transformations and is not destroyed by the usual localisation procedures. Thus,
combining localisation with perturbation techniques, the setting has to be seen as a model constellation for rather complex global situations. This has exemplarily been worked out in [ERS07].

2.3.1. Results

Precisely, the result for our three model situations can be described as follows.

**Theorem 2.3.1.** Let \( \Lambda \subseteq \mathbb{R}^2 \) be an open triangle and let \( P \) be the centre of one of its sides. Define \( \Omega := \Lambda \times (-1, 1) \subseteq \mathbb{R}^3 \) and \( \Gamma := \Upsilon \times (-1, 1) \), where

a) \( \Upsilon \) is the open leg between \( P \) and one of its neighbouring vertices or

b) \( \Upsilon \) is one of the open sides of \( \Lambda \) or

c) \( \partial \Lambda \setminus \Upsilon \) is one of the closed sides of \( \Lambda \).

Suppose \( \Xi \) to be a plane within \( \mathbb{R}^3 \) that intersects the boundary of \( \Gamma \) in a finite number of points and assume that the elliptic coefficient function \( \mu \) takes its values in the set of real, symmetric, positive definite \( 3 \times 3 \) matrices and is constant on both components of \( \Omega \setminus \Xi \). Then there is a \( q > 3 \) such that

\[
A_{\Gamma} : W^{1,q}_{\Gamma}(\Omega) \to W^{-1,q}_{\Gamma}(\Omega)
\]

is a topological isomorphism.

2.3.2. Methods

When treating mixed boundary value problems with discontinuous coefficients on a polyhedron, one usually has to examine all the edge and vertex singularities that occur, see e.g. [Dau92]. Applying the Mellin transform, it emerges that for the case of vertices one has to control the spectrum of some associated Laplace-Beltrami operator and for a point from an edge one has to consider some generalised Sturm-Liouville operator.

As one might imagine this is a difficult and technically involved matter, so it is of great help that by a deep lying theorem of Maz’ya [MERS04] in the case of a pure Dirichlet problem the consideration of the vertex singularities can be omitted, as long as one is interested in the asserted isomorphism property for a \( q > 3 \).

Thus, the strategy of proof is in two steps. First, we transform our problem to an equivalent one with purely Dirichlet boundary conditions. In order to do so, we start by deforming it with a bi-Lipschitz (in fact even piecewise linear) mapping to another prism, where the Neumann boundary part is a whole side of the prism. Here we use again that the isomorphism property we are interested in is stable under bi-Lipschitz transformations. An even reflection of the whole setting across this Neumann boundary, then provides us a suitable problem with Dirichlet boundary conditions.

Naturally, during this procedure we induce quite a lot of additional edges in \( \Omega \) and on its boundary, which then, together with the original edges, have to be treated in the second step. Here, based on explicit formulae, given in [CDL01], see also [MERS04], for the elementary solutions to the Sturm-Liouville problems, which have to be considered, we obtain some transcendental equations for which we have to show that there are no solutions with real part in \((0, 1/3 + \varepsilon)\) for some \( \varepsilon > 0 \).
2.4. Irreducibility of the semigroup

In the paper [HHR08] (Article 4) we consider multiplicative perturbations of $A_{Γ}$ by $L^∞$-functions $b$, bounded away from 0, in the spaces $L^q(Ω)$, $1 < q < ∞$. Since the positive, analytic semigroups generated by $A_{Γ}$ on $L^q(Ω)$ admit Gaussian estimates, the operators $bA_{Γ}$ under consideration also generate analytic semigroups on $L^q(Ω)$, admitting a similar bound by a result of Duong and Ouhabaz [DO99].

The aim of Article 4 is to show that this semigroup is positive, too, and that the semigroups generated by $A_{Γ}$ and $bA_{Γ}$ on $L^q(Ω)$ both are irreducible for $1 < q < ∞$. By irreducibility we mean here that there is no ideal of $L^q(Ω)$ except for $\{0\}$ and the whole space that is invariant under the semigroup. This notion has several applications to evolutionary problems (see e.g. [EN00], [ABHN01]) and is linked to many other characteristics of the operators, for instance degeneracy of the ground state, a positivity improving resolvent or the behaviour of the kernel of the semigroup and the resolvent, cf. [Nag86].

Meanwhile, Arendt pointed out to us that the same result can be deduced more easily by combining techniques of Ouhabaz [Ouh05] and himself [Are04].

2.4.1. Results

Let $A_{Γ,2}$ be the realisation of $−∇ · µ∇$ with mixed boundary conditions in $L^2(Ω)$ in the sense of (2.0.1). Furthermore, for $b ∈ L^∞(Ω)$ with $b ≥ δ > 0$, we denote by $M_b$ the bounded multiplication operator on $L^q(Ω)$. Since the semigroup generated by $M_bA_{Γ,2}$ admits Gaussian estimates, this semigroup can be extended to consistent $C_0$-semigroups on $L^q(Ω)$ for all $1 ≤ q < ∞$. Let $M_bA_{Γ,q}$ denote the generator of this semigroup on $L^q(Ω)$. Then we have the following result.

**Theorem 2.4.1.** Let $1 < q < ∞$ and let $b ∈ L^∞(Ω; ℝ)$ such that $b ≥ δ$ for some $δ > 0$. Then the semigroup $(e^{tM_bA_{Γ,q}})_{t ≥ 0}$ is positive, irreducible and compact on $L^q(Ω)$.

2.4.2. Methods

The starting point for the proof of Theorem 2.4.1 are the Gaussian estimates satisfied by $e^{tM_bA_{Γ,2}}$ already mentioned above. These estimates provide the additional precious information that the semigroup generated by $M_bA_{Γ,q}$ and the resolvent of this operator are given by integral kernels. Thus, we can use that in our case irreducibility is equivalent to the kernel of the semigroup being strictly positive, cf. [Nag86, Example C-III.3.4]. This in turn follows when the resolvent is a positivity improving operator. Reducing the problem via $R(λ, M_bA_{Γ,q}) = (λM_b−1 − A)^{−1}M_b−1$ to additive perturbations and thus to the theory of self-adjoint operators, we can finally conclude by showing that the lowest eigenvalue is simple and that there is a strictly positive eigenfunction to this eigenvalue, using classical results, see e.g. [RS78, Theorem XIII.44].

The idea for the paper and a very rough outline of the proof were born in one afternoon during a discussion among the authors. Working this out and filling the gaps in the argumentation that then showed up, was to a large extent my work.
3. Operators with unbounded drift coefficients

In the classical approach to second order differential operators by coercive forms in \( L^2 \) the boundedness of the coefficients is one of the central hypotheses. If this is no longer fulfilled the properties of the forms and the corresponding operators may change heavily, see [Vog01] and also [SV02] and [LSV02] for a systematic theoretical treatment.

A special, widely studied class of such operators are Ornstein-Uhlenbeck operators, which are given as

\[
A u(x) = \sum_{j,k=1}^{d} q_{jk} D_j D_k u(x) + B x \cdot \nabla u(x),
\]

where \( Q = (q_{jk})_{j,k=1}^{d} \in \mathbb{R}^{d \times d} \) is a positive definite matrix and \( B \in \mathbb{R}^{d \times d} \) is an arbitrary matrix. They are of high interest in stochastic analysis, describing a drift, perturbed by a Brownian motion. In this context they are well studied in \( C_b(\mathbb{R}^d) \) and \( L^p(\mathbb{R}^d, \mu) \), where \( \mu \) is the invariant measure induced by the corresponding stochastic process, see e.g. the monograph [LB07] and references therein.

During the last decade it became apparent that operators of this form also naturally emerge in the treatment of the Stokes and the Navier-Stokes equations in the exterior of a rotating obstacle, cf. [His99], [GHH06], [FHM04]. This served as a motivation to look at these operators in \( L^p \)-spaces relative to the Lebesgue measure on domains \( \Omega \subseteq \mathbb{R}^d \) subjected to Dirichlet boundary conditions. When replacing the invariant measure by the Lebesgue measure many properties of the Ornstein-Uhlenbeck operator change considerably, for example the semigroup generated by \( A \) on \( L^p(\mathbb{R}^d) \) is no longer analytic, its spectrum even contains a vertical line. Furthermore, the spectrum turns out to be \( p \)-dependent, whenever \( \text{tr}(B) \neq 0 \).

For these and other results concerning \( A \) on \( L^p(\mathbb{R}^d) \), see [Met01]. While there are still some results on these operators in \( L^p(\mathbb{R}^d) \), equipped with the Lebesgue measure, the situation for other unbounded domains \( \Omega \subseteq \mathbb{R}^d \), as needed for the treatment of the Navier-Stokes equations, is much less satisfactory. In the case of an exterior domain \( \Omega \) with \( C^{1,1} \)-boundary a generation result in \( L^p(\Omega) \) is contained in [GHHW05]. For results in spaces of continuous functions or in \( L^2 \)-spaces equipped with infinitesimally invariant measures see [DPL04], [FMP04] and [LMP05].

3.1. Kolmogorov kernel estimates for Ornstein-Uhlenbeck operators

It is well known that the semigroups generated by second order divergence form operators with bounded coefficients admit Gaussian estimates, as long as the coefficients are real valued and some suitable boundary conditions are imposed, cf. [Ouh05]. Since Gaussian estimates imply \( p \)-independence of the spectrum, such a result cannot be expected for Ornstein-Uhlenbeck operators. Still, it turns out that the domination results for these operators on domains remain true, if one replaces the Gaussian semigroup by the Kolmogorov semigroup, i.e. the semigroup generated by the same Ornstein-Uhlenbeck operator on the whole of \( \mathbb{R}^d \).
3. Operators with unbounded drift coefficients

Combining these ‘Kolmogorov kernel estimates’ and a generation result on \( L^2(\Omega) \), we could show in [HW05] (Article 5) that Ornstein-Uhlenbeck operators are generators of consistent, positive, quasi-contractive \( C_0 \)-semigroups on \( L^p(\Omega) \) for every \( 1 < p < \infty \) and for arbitrary domains \( \Omega \subseteq \mathbb{R}^d \). Furthermore, in the special case of an exterior domain with \( C^{1,1} \)-boundary we explicitly determined the boundary spectrum of the operator.

3.1.1. Results

Let \( \Omega \subseteq \mathbb{R}^d \) be an open domain and let \( Q, B \in \mathbb{R}^{d \times d} \) with \( Q \) positive definite. Then we consider the operator

\[
A_{\Omega,2} u(x) := A u(x) \quad \text{with} \quad D(A_{\Omega,2}) = H^1_0(\Omega) \cap \{ v \in H^2_{\text{loc}}(\Omega) : A v \in L^2(\Omega) \}
\]

in \( L^2(\Omega) \). For this we get the following two theorems.

**Theorem 3.1.1.**

a) **Generation on \( L^2(\Omega) \):** The operator \( A_{\Omega,2} \) generates a positive \( C_0 \)-semigroup \( (T_{\Omega,2}(t))_{t \geq 0} \) on \( L^2(\Omega) \).

b) **Generation on \( L^p(\Omega) \):** For all \( 1 \leq p < \infty \) there are positive \( C_0 \)-semigroups \( (T_{\Omega,p}(t))_{t \geq 0} \) on \( L^p(\Omega) \) that are consistent with \( (T_{\Omega,2}(t))_{t \geq 0} \).

c) **Growth bounds of the semigroups:** \( \| T_{\Omega,p}(t) \|_{L(L^p(\Omega))} \leq e^{-t \text{tr}(B)/p} \) for all \( 1 \leq p < \infty \) and all \( t \geq 0 \).

d) **Consistency of the generators:** For the generator \( A_{\Omega,p} \) of \( (T_{\Omega,p}(t))_{t \geq 0} \) we have \( D := \{ f \in D(A_{\Omega,2}) \cap L^p(\Omega) : A_{\Omega,2} f \in L^2(\Omega) \} \subseteq D(A_{\Omega,p}) \) and \( A_{\Omega,2} f = A f \) for all \( f \in D \).

e) **Consistency of the resolvents:** \( R(\lambda, A_{\Omega,p}) f = R(\lambda, A_{\Omega,q}) f \) for all \( f \in L^p(\Omega) \cap L^q(\Omega) \), all \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > \max \{-\text{tr}(B)/p, -\text{tr}(B)/q\} \) and all \( 1 \leq p, q < \infty \).

f) **Domination of the resolvents:** For all \( 1 \leq p < \infty \) we have

\[
| R(\lambda, A_{\Omega,p}) f | \leq R(\lambda, A_{\Omega,q}) | f |, \quad f \in L^p(\Omega), \; \lambda \geq -\text{tr}(B)/p.
\]

g) **Domination of the semigroups – Kolmogorov kernel estimates:** For all \( 1 \leq p < \infty \) it holds

\[
| T_{\Omega,p}(t) f | \leq T_{\Omega,q,p}(t) | f | = k_t * | f |(e^{tB}) \quad f \in L^p(\Omega), \; t > 0,
\]

where the Kolmogorov kernel \( k_t \) is given by

\[
k_t(x) = \frac{1}{(4\pi)^{n/2} \det(Q_t)^{1/2}} \exp \left( -\frac{1}{4} Q_t^{-1} x \cdot x \right) \quad \text{and} \quad Q_t = \int_0^t e^{sB} Q e^{sB^*} \, ds.
\]

In particular, \( T_{\Omega,p}(t) \) is given by an integral kernel.

h) **\( H^\infty \)-calculus:** \( A_{\Omega,p} - |\text{tr}(B)| \) has a bounded \( H^\infty \)-calculus for any \( 1 < p < \infty \).

**Theorem 3.1.2.**

Let \( K \subseteq \mathbb{R}^d \) be compact with a \( C^{1,1} \)-boundary and \( \Omega = \mathbb{R}^d \setminus K \). Then for any \( 1 < p < \infty \) the domain of \( A_{\Omega,p} \) is given by

\[
D(A_{\Omega,p}) = W^1_0(\Omega) \cap W^2,p(\Omega) \cap \{ f \in L^p(\Omega) : B x \cdot \nabla f \in L^p(\Omega) \}
\]

and the spectral bound is \( s(A_{\Omega,p}) = -\text{tr}(B)/p \). Furthermore, \( -\text{tr}(B)/p + \mathbb{i} \mathbb{R} \subseteq \sigma(A_{\Omega,p}) \), if \( \text{tr}(B) \neq 0 \) and in the case \( \text{tr}(B) = 0 \) this inclusion is at least true for some non-trivial additive subgroup of \( \mathbb{i} \mathbb{R} \).
3.2. Generalisation to superlinearly growing drift coefficients

3.1.2. Methods

All the results in Theorem 3.1.1 follow in a standard and straightforward manner as soon as we have the generation result a) and the Kolmogorov kernel estimates in g) for \( p = 2 \).

Since \( A_{\Omega,2} + \text{tr}(B)/2 \) is dissipative in \( L^2(\Omega) \), in view of the Lumer-Philipps Theorem, one has to show surjectivity of the corresponding elliptic problem for \( \lambda > 0 \). This follows by approximating the domain from the interior by bounded domains with smooth boundary.

Here it is well known from the classical theory that the elliptic problem has a unique solution with good norm estimates, see e.g. [GT01]. Since these estimates turn out to be independent of the approximating domain, this allows to pass to the limit and to show that the limit is a suitable solution in the whole of \( \Omega \).

As for the Kolmogorov kernel estimates the corresponding proof for the Dirichlet Laplacian from [AB99] can be adapted. This, in fact, even shows that the solution of the homogeneous parabolic problem with given initial state in some domain is dominated pointwise by the solution of the same problem in every larger domain, i.e. the semigroup behaves monotonously when the domain is enlarged.

The idea behind Theorem 3.1.2 is to adapt a technique developed by Davies and Simon, see [DS86], and to shrink down the set \( K \) to \( \{0\} \) without changing the spectral behaviour, by a scaling of our operator. This allows to relate its boundary spectrum to the corresponding spectrum in the situation of the whole space. Here the spectrum is known by [Met01].

This article is a part of the PhD thesis of Julian Wiedl that I advised mathematically. Thus, my contributions were mainly conceptual. Still, I proved the domination result in Section 5.4 of Article 5.

3.2. Generalisation to superlinearly growing drift coefficients

A natural generalisation of Article 5 is to consider variable coefficient functions instead of the matrices \( Q \) and \( B \) in the Ornstein-Uhlenbeck operators. This is motivated also by applications to the Stokes and Navier-Stokes system, when these are considered with a linearly or even superlinearly growing initial value, cf. [HRS07] and [Wie07]. Article 6 (cf. [HW06]) contains generalisations of the results in Theorem 3.1.1 to this larger class of operators.

Similar results, but from the point of view of Schrödinger operators with unbounded drift coefficients are included in [AMP06].

3.2.1. Results

We consider the formal differential operator

\[
A u(x) = \text{div} (a(x) \nabla u(x)) + b(x) \cdot \nabla u(x), \quad x \in \Omega,
\]

with \( a \in C^1_\text{loc}(\Omega; \mathbb{R}^{d \times d}) \) and \( b \in W^{1,\infty}_\text{loc}(\Omega; \mathbb{R}^d) \) on some open set \( \Omega \subseteq \mathbb{R}^d \). Additionally, we require the following:

- \( \xi^T a(x) \xi \geq \alpha |\xi|^2 \) for all \( \xi \in \mathbb{R}^d \) and almost all \( x \in \Omega \) (ellipticity),
- \( \text{div} b \) is bounded from below (dissipativity),
- \( |b(x)| \leq c (1 \vee |x| \log(|x|)) \) for almost all \( x \in \Omega \) (superlinear growth condition for \( b \)).

Defining again the realisation of \( A \) in \( L^2(\Omega) \) as

\[
D(A_{\Omega,2}) = \bigcap \{ u \in H^1_\text{loc}(\Omega) : A u \in L^2(\Omega) \}, \quad A_{\Omega,2} u = A u,
\]

and setting \( \omega_p := \text{essinf}_{x \in \Omega} \frac{\text{div} b(x)}{p} \), we get analogously to Theorem 3.1.1 the following result.
Theorem 3.2.1. The operator $A_{\Omega,2}$ generates a positive $C_0$-semigroup $(T_{\Omega,2}(t))_{t \geq 0}$ on $L^2(\Omega)$ with $\|T_{\Omega,2}(t)\| \leq e^{-\omega_2 t}$ and for $1 < p < \infty$ there exists a family of consistent positive $C_0$-semigroups $(T_{\Omega,p}(t))_{t \geq 0}$ on $L^p(\Omega)$ with $\|T_{\Omega,p}(t)\| \leq e^{-\omega_p t}$. Furthermore, for every $\lambda > -\omega_p$ and every $t \geq 0$ we have the domination properties

$$|T_{\Omega,p}(t)f| \leq T_{\mathbb{R}^d,p}(t)|\tilde{f}|, \quad f \in L^p(\Omega),$$

$$|R(\lambda, A_{\Omega,p})f| \leq R(\lambda, A_{\mathbb{R}^d,p})|\tilde{f}|, \quad f \in L^p(\Omega),$$

where $\tilde{f}$ denotes the extension of $f$ by 0 and $A_{\Omega,p}$ is the generator of $(T_{\Omega,p}(t))_{t \geq 0}$.

3.2.2. Methods

The proof of the above theorem is in large parts along the lines of the proof of Theorem 3.1.1. However, there is one important difference. Having established the domination of the $L^2$-semigroup on a domain by the semigroup on $L^2(\mathbb{R}^d)$, in the case of Ornstein-Uhlenbeck operators we could argue by the explicitly known kernel of this semigroup in order to extend the $L^2$-semigroup consistently to $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$. Since in this more general case such a kernel was not known to exist (in fact it is now, cf. [Wie07]), we had to show the existence of a consistent family of positive $C_0$-semigroups on $L^p(\mathbb{R}^d)$ for these operators by hand.

This was again done by showing dissipativity in $L^p(\mathbb{R}^d)$ and solving the elliptic problem by approximating $\Omega$ from the interior with bounded open domains with smooth boundary.

The contributions of the two authors were comparable to the case of Article 5.
4. The weighted Laplacian on star-shaped networks

Take $n$ copies of the interval $[0, \infty)$ and glue them together at 0. The resulting object, the $n$-star, is a very simple network and may serve as a model for investigating evolution phenomena locally in time and space near the ramification points of more complicated networks.

In [AHR08] (Article 7) we were interested in spectral theory for an easy, but non-trivial differential operator $A$ on this structure, the second derivative on each branch weighted with constant factors that might be different on different branches. The special nature of the point 0 has to be taken into account, of course, so we impose Kirchhoff conditions as natural transmission conditions in this point.

Starting from an explicitly given total system of generalised eigenfunctions of this operator, the analytical core of the paper is a representation formula for the kernel of the resolvent of the operator in terms of these generalised eigenfunctions. This then serves to construct an explicit Fourier type transformation $V$ that diagonalises $A$ together with a left inverse $Z$, both in terms of the generalised eigenfunctions, for which a Plancherel type formula is proved. Finally, this leads to a functional calculus for our operator that in a next step should allow to represent solutions of evolution equations on the $n$-star containing $A$ and to obtain qualitative information like decay properties in time.

There remains an unsatisfactory point in this paper: the Fourier type transform $V$ we found fails to be a spectral representation of $A$ in the classical sense, since the natural norm on the range of $V$ that makes $V$ an isometry is not just a weighted $L^2$-norm as it is for instance in the Plancherel Theorem. Nevertheless, $V$ diagonalises the operator. Very recently we proved that in fact for $n \geq 3$ it is not possible both to represent $V$ and $Z$ in terms of any generalised eigenfunctions of $A$ and to get a weighted $L^2$-norm in the range of $V$ at the same time. Thus, there seems to be a principal obstruction and thus an interesting problem behind it. We will discuss this in a forthcoming paper.

4.0.1. Results

We denote the $j$-th branch $(0, \infty)$ of the $n$-star by $N_j$ and the whole network by $N := \bigcup_{j=1}^{n} N_j$. On $N$ we define the weighted Laplacian by

$$D(A) := \left\{ u = (u_j)_{j=1}^{n} \in \prod_{j=1}^{n} H^2(N_j) : \text{u satisfies (T_0) and (T_1)} \right\}, \quad Au := (-c_j^2 u_j'')_{j=1}^{n},$$

where $c_j > 0$ are constants for all $j \in \{1, \ldots, n\}$ and (T_0) and (T_1) are the Kirchhoff conditions

$$(T_0): \quad u_j(0) = u_k(0) \text{ for all } j, k \in \{1, \ldots, n\},$$

$$(T_1): \quad \sum_{j=1}^{n} c_j^2 u_j'(0^+) = 0.$$  

We use the two total sets of generalised eigenfunctions defined by

$$F_{\lambda}^{\pm, j}(x) = \begin{cases} \frac{1+s_j}{2} \exp(\pm ic_j^{-1}\sqrt{\lambda}x) + \frac{1-s_j}{2} \exp(\mp ic_j^{-1}\sqrt{\lambda}x), & x \in N_j, \\ \exp(\pm ic_k^{-1}\sqrt{\lambda}x), & x \in N_k, k \neq j, \end{cases}$$
4. The weighted Laplacian on star-shaped networks

for \( j = 1, \ldots, n \) and \( \lambda \in \mathbb{C} \), where \( s_j := -c_j^{-1} \sum_{k \neq j} c_k \). Using these functions the representation formula for the kernel of the resolvent of \( A \) reads as follows.

**Theorem 4.0.1.** For all \( f \in H := \prod_{j=1}^{n} L^2(N_j) \) and every \( \lambda \in \mathfrak{a}(A) \) it holds

\[
[R(\lambda, A)f](x) = \int_{N} K(x, x', \lambda)f(x') \, dx', \quad x \in N,
\]

where the kernel \( K \) is given for \( x \in N_j \), \( j = 1, \ldots, n \), by

\[
K(x, x', \lambda) = \begin{cases} 
\frac{1}{\sqrt{x}} F_\lambda^x (x) F_\lambda^{x'} (x'), & \text{for } x' \in N_j \text{ with } x' > x, \\
\frac{1}{\sqrt{x'}} F_\lambda^{x'} (x) F_\lambda^x (x'), & \text{for } x' \in N_j \text{ with } x' < x \text{ or } x' \in N_k, \text{ } k \neq j,
\end{cases}
\]

with \( C := \sum_{j=1}^{n} c_j \). In the whole formula the sign \( + \) is chosen if \( \text{Im}(\lambda) > 0 \) and \( - \) for \( \text{Im}(\lambda) \leq 0 \) and the index \( j \) is to be understood modulo \( n \), i.e. if \( j = n \), then \( j + 1 = 1 \).

The Fourier type transform \( V \) and its left inverse \( Z \) are defined as follows. For \( f \in L^1(N) \) we define the function \( Vf = (V_j f)^n_{j=1} : [0, \infty) \to \mathbb{C}^n \) by

\[
(V_j f)(\lambda) := \int_{N} f(x) F_{\lambda}^{-j}(x) \, dx, \quad \lambda \in [0, \infty), \quad j = 1, \ldots, n.
\]

Furthermore, for all \( K = (K_j)^n_{j=1} \in C^{\infty}((0, \infty); \mathbb{C})^n \), such that every component is rapidly decreasing at \( \infty \), we define \( Z(K) : N \to \mathbb{R} \) by

\[
[Z(K)](x) := \frac{1}{\pi} \text{Re} \left( \int_{0}^{\infty} \frac{1}{C \sqrt{\lambda}} \sum_{j=1}^{n} 1_{N_j}(x) K_j(\lambda) F_{\lambda}^{\infty-j+1}(x) \, d\lambda \right), \quad x \in N.
\]

With these two operators we proved a Plancherel type formula:

**Theorem 4.0.2.** Let \( f \in \prod_{j=1}^{n} D(N_j) \) and \( G \in C^{\infty}((0, \infty); \mathbb{C})^n \), such that every component is rapidly decreasing at \( \infty \). Defining

\[
\langle Vf, G \rangle_V := \frac{1}{\pi} \text{Re} \left( \sum_{j=1}^{n} \int_{0}^{\infty} \frac{1}{C \sqrt{\lambda}} [V_{j+1}(1_{N_j} f)](\lambda) G_{j}(\lambda) \, d\lambda \right),
\]

it holds \( \langle Vf, G \rangle_V = \langle f, Z(G) \rangle_H \).

If we denote the completion of \( V(\prod_{j=1}^{n} D(N_j)) \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_V \) by \( L^2_V \), it turns out that \( V \) defined on \( \prod_{j=1}^{n} D(N_j) \) extends to a surjective isometry \( \tilde{V} : H \to L^2_V \) and \( Z \) extends to a surjective isometry \( \tilde{Z} : L^2_V \to H \).

The final result of our paper is the following functional calculus for \( A \).

**Theorem 4.0.3.** Let \( h \in C(\mathbb{R}) \) and \( f \in H \), such that \( \lambda \mapsto h(\lambda)\lambda^{1/2} (\tilde{V} f)(\lambda) \) is absolutely integrable on \([0, \infty)\). Then for every \( x \in N \)

\[
(h(A)f)(x) = \frac{1}{\pi} \text{Re} \left( \int_{0}^{\infty} \frac{h(\lambda)}{C \sqrt{\lambda}} \sum_{j=1}^{n} 1_{N_j}(x) (V_j f)(\lambda) F_{\lambda}^{\infty-j+1}(x) \, d\lambda \right),
\]

i.e. we have formally \( h(A)f = \tilde{Z} M_h \tilde{V} f \), where \( M_h \) denotes the operator of multiplication with \( h \).
4.0.2. Methods

The initial idea for this paper was to generalise the results of Felix Ali Mehmeti for the case 
\( n = 2 \) in [AM96]. While doing so, we encountered several difficulties that finally led to the 
above mentioned defect that our transformation is no spectral representation in the classical 
sense. Nevertheless, we proceed very much along the lines of the case \( n = 2 \). In the first place, 
this gives the right idea to find the generalised eigenfunctions and to prove the representation 
formula for the resolvent via an adapted Green’s formula for the \( n \)-star. 

By a standard procedure, cf. [AM96], proving a limiting absorption principle for the resolvent, 
we may insert \( A \) into Stone’s formula in a next step. This provides an explicit formula for 
the resolution of the identity of \( A \) (cf. Proposition 7.4.5 in Article 7):

\[
[E(a, b)f](x) = \text{Re} \left[ \frac{1}{\pi} \int_a^b \sum_{j=1}^{n} \frac{1}{CN_j} 1_{N_j}(x) F_{\lambda}^{-j+1}(x) \left( \int_N f(x') F_{\lambda}^{-j}(x) \, dx' \right) \, d\lambda \right].
\]

The most important feature of this formula is that the two different expressions of the kernel of 
the resolvent that cause a cut in the integral over \( N \) in the beginning can be recombined to one 
single expression allowing to simply write an integral over all of \( N \) in the inner integral. This 
eliminates artificial singularities that would appear in \( Vf \) and that would make it practically 
impossible to say anything about the regularity of \( Vf \) even for very good \( f \). 

The formula for the resolution of the identity also allows to read off the right definitions of 
the operators \( V \) and \( Z \) and is the major analytic tool in the proof of the above mentioned 
results. On the other hand it shows the origin of our problems: the characteristic function 
\( 1_{N_j} \), which may be eliminated by rewriting the formula in the case \( n = 2 \).
5. \( H^\infty \)-calculus for products of non-commuting operators

This final article of the present collection takes a more abstract point of view and deals with sectorial operators in an arbitrary Banach space. Here sectoriality means, that the operator is closed, densely defined with dense range, a sector in the complex plane \( \Sigma_\phi := \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \phi \} \) belongs to the resolvent set and that we have the usual resolvent estimate on this sector. Our aim is to see under which conditions good properties of sectorial operators, and first of all sectoriality itself, carry over from sectorial operators to the product of two of them.

The corresponding problem for sums of operators has been treated in several ways, cf. the articles [DPG75], [DV87], [KW01] and [KW04] for commuting operators and [MP97], as well as [PS07] in the non-commutative case.

Since such results for the product of two commuting operators follow from the works of Kalton and Weis [KW01], cf. Corollary 2.2 in [HH05] (Article 8), it remains to consider the case of non-commuting operators. A first result in this direction was proved by Weber in [Web98]. He showed sectoriality of the product in UMD spaces under conditions of the Dore-Venni type and some commutator estimates. This result has been modified by Štrkalj in [Štr01], imposing conditions of Kalton-Weis type, for B-convex Banach lattices.

Our paper [HH05] (Article 8) now adds the following two results. Firstly, the result of Štrkalj is generalised to arbitrary Banach spaces. In a second step, a modification of our proof also provides a criterion, when a bounded \( H^\infty \)-calculus is inherited by the product of two non-commuting sectorial operators having this property.

5.0.1. Results

Formulated in detail, in Article 8 we prove the following two results.

Theorem 5.0.1. Let \( A \) and \( B \) be sectorial operators on some Banach space \( X \), let \( A \) have a bounded \( H^\infty \)-calculus and let \( B \) be \( \mathcal{R} \)-sectorial. Furthermore, let the following conditions be satisfied:

- \( 0 \in \varrho(A) \) and \( (\mu - B)^{-1}(D(A)) \subseteq D(A) \) for some \( \mu \in \varrho(B) \).
- There are angles \( \theta_A \) superior to the \( H^\infty \)-angle of \( A \) and \( \theta_B \) superior to the angle of \( \mathcal{R} \)-sectoriality of \( B \), such that \( \theta_A + \theta_B < \pi \).
- There are numbers \( c, \alpha \geq 0 \) and \( \beta > 0 \) with \( \alpha + \beta < 1 \) and

\[
\| (A(\mu + B)^{-1} - (\mu + B)^{-1}A)(\lambda + A)^{-1} \| \leq \frac{c}{(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}}
\]

for all \( \lambda \in \Sigma_{\pi - \theta_A} \) and \( \mu \in \Sigma_{\pi - \theta_B} \).

Then there exists some \( \nu \geq 0 \), such that the operator \( \nu + AB \) equipped with the natural domain \( D(AB) = \{ u \in D(B) : Bu \in D(A) \} \) is again sectorial and the angle of sectoriality is at most \( \theta_A + \theta_B \).
5. $H^\infty$-calculus for products of non-commuting operators

If $B$ even admits an $\mathcal{R}$-bounded $H^\infty$-calculus, then, keeping the same conditions on $A$ and $B$, the product also inherits the bounded $H^\infty$-calculus.

**Theorem 5.0.2.** Let $A$ and $B$ be sectorial operators on some Banach space $X$, let $A$ have a bounded $H^\infty$-calculus and $B$ allow for an $\mathcal{R}$-bounded $H^\infty$-calculus. Furthermore, let the following conditions be satisfied:

- $0 \in \rho(A)$ and $(\mu - B)^{-1}(D(A)) \subseteq D(A)$ for some $\mu \in \rho(B)$.
- There are angles $\theta_A$ superior to the $H^\infty$-angle of $A$ and $\theta_B$ superior to the $\mathcal{R}$-$H^\infty$-angle of $B$ with $\theta_A + \theta_B < \pi$.
- There are numbers $c, \alpha \geq 0$ and $\beta > 0$ with $\alpha + \beta < 1$ and
  \[
  \| (A(\mu + B)^{-1} - (\mu + B)^{-1}A)(\lambda + A)^{-1} \| \leq \frac{c}{(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}}
  \]
  for all $\lambda \in \Sigma_{\pi-\theta_A}$ and $\mu \in \Sigma_{\pi-\theta_B}$.

Then there exists $\nu \geq 0$, such that the operator $\nu + AB$ with its natural domain $D(AB)$ has a bounded $H^\infty$-calculus with $H^\infty$-angle at most $\theta_A + \theta_B$.

5.0.2. Methods

The key to the proof of both theorems is the construction of two operators $S_\mu$ and $T_\mu$ that, up to a small remainder treated by the Neumann series, act as a left or right inverse of $\mu + AB$, respectively. It is in this definition, where the bounded $H^\infty$-calculus of $A$ plays a fundamental role.

Having these operators at hand that are in the beginning only defined on $D(A)$, it is the most delicate part of the proof to show that they may really be extended to the whole of $X$ as continuous operators, see Lemma 8.4.2 in Article 8. At this point the commutator estimates and the $\mathcal{R}$-sectoriality or the $\mathcal{R}$-bounded $H^\infty$-calculus of $B$, respectively, enter the analysis. It is mainly this property of $B$ that allows to estimate the singular parts of the path integrals defining $S_\mu$ and $T_\mu$. In the end the problem is reduced to the application of an abstract lemma about the bounded $H^\infty$-calculus due to Kalton and Weis ([KW01, Lemma 4.1]).

The mathematical content of this article was to a large extent worked out by me. Important hints for the adequate definition of $S_\mu$ and $T_\mu$ came from [Web98] and [PS07].
References (Summary)

Own artikels


[HR08] R. Haller-Dintelmann and J. Rehberg: Maximal parabolic regularity for divergence operators including mixed boundary conditions, WIAS-Preprint no. 1288, 2008.

References (Summary)


Other references


References (Summary)


References (Summary)


References (Summary)


Part II.

Selected Articles
1. Hölder Continuity for Second Order Elliptic Problems With Nonsmooth Data

by Robert Haller-Dintelmann, Christian Meyer and Joachim Rehberg

Abstract

The well known De Giorgi result on Hölder continuity for solutions of the Dirichlet problem is re-established for mixed boundary value problems, provided that the underlying domain is a Lipschitz domain and the border between the Dirichlet and the Neumann boundary part satisfies a very general geometric condition. Implications of this result for optimal control theory are presented.

1.1. Introduction

In the last decades it has been anticipated in applied analysis that many elliptic problems originating from science, engineering, and technology possess nonsmooth data. This means that they often live on nonsmooth domains, the coefficients are nonsmooth and, thirdly, they often exhibit mixed boundary conditions, see [Ama93], [MM07] and the references cited therein, see also [Sel84] and [Gaj93]. In this paper we prove the Hölder continuity for the solution \( u \) of

\[
-\nabla \cdot \rho \nabla u + u = f + \sum_{j=1}^{d} \frac{\partial f_j}{\partial x_j}, \quad f \in L^{q/2}, f_j \in L^q, \tag{1.1.1}
\]

where \( q \) is larger than the space dimension, and mixed boundary conditions are incorporated, see Theorem 1.3.3. The result is to be seen in the tradition of Stampacchia’s paper [Sta60] (see also [Lie86, Lie89]), where Hölder continuity already was achieved for mixed boundary value problems, but under rather technical conditions – difficult to verify in applications. Here, we generalize the Stampacchia result in space dimensions \( d = 2, 3 \) and \( 4 \) to Lipschitz domains, provided that the Dirichlet boundary part satisfies a very general compatibility condition – purely topological in nature and easy to check at least for \( d = 2, 3 \), see Theorem 1.5.2 and Theorem 1.5.4 below.

Note that the admissible distributional right hand sides in (1.1.1) allow for jumps in the conormal derivative of solutions across internal interfaces. This means, e.g. in electrostatics, that the jump in the normal component of the displacement \( \nu_+ \cdot \varepsilon \nabla \varphi - \nu_- \cdot \varepsilon \nabla \varphi \) across a prescribed interface equals the surface charge density on the interface, and this surface charge density is represented by a distribution on the underlying domain \( \Omega \).

Divergence type operators as in (1.1.1) are of fundamental significance in many application areas. This is the case not only in mechanics (see [LS87, Ch. IV/V]), thermodynamics (see [Som56]) and electrodynamics (see [Som52]) of heterogeneous media, but also
in mining, multiphase flow, mathematical biology (see [FGR90, BHR05]) and semiconductor device simulation (see [Sel84, Gaj93, GH97]), in particular quantum electronics (see [WV91, BKKR03, KKF06, WHLL03, LTVLS01]). The non-homogeneous coefficient function $\rho$ represents varying material properties as the context requires. It may be thermal conductivity in a heat equation (see [Som56, §21]) or dielectric permittivity in a Poisson equation, or diffusivity in a transport equation (see for instance [Sel84, §2.2] for carrier continuity equations) or effective electron mass in a Schrödinger equation (see [KKF06]).

Continuity of solutions to (1.1.1) plays an important role for the discussion of state-constrained optimal control problems (see for instance [Cas93]). Hence it is rather natural to use the above mentioned result for the discussion of semilinear elliptic control problems with pointwise inequality constraints on the state, which is done here in a very general setting. Such problems have been discussed by numerous authors before (see for instance [Cas93, AR97, CTU00] and the references therein). Concerning second-order sufficient optimality conditions, some progress has recently been made in a contribution of Casas et al. [CdIRT]. Here we show that, based on the regularity results of Theorem 1.3.3, the analysis, developed in [CdIRT], is also applicable to problems with mixed boundary conditions, which are not considered in [CdIRT]. Thus the consideration of mixed boundary conditions for semilinear elliptic state-constrained optimal control problems represents the genuine contribution of this paper from the viewpoint of optimal control theory.

The outline of the paper is as follows: first we introduce some notation. In Section 1.3 we formulate our regularity result, which is proved in Section 1.4. In Section 1.5 we give an alternative characterization for Gröger’s regular sets, which represent the geometric setting for the domains under consideration and the associated Dirichlet boundary parts, in the 2d and 3d case. Finally, the relevance of the Hölder property for the discussion of semilinear elliptic state-constrained optimal control problems with pointwise state constraints is pointed out in Section 1.6.

### 1.2. Notation

Throughout this paper, $\Omega \subset \mathbb{R}^d$ always denotes a bounded Lipschitz domain (see [Gri85, Ch. 1.2] for the definition) and $\Gamma \subset \partial \Omega$ is an open part of its boundary. In particular, we often use the cube $K := \{x \in \mathbb{R}^d : -1 < x_j < 1 \text{ for } 1 \leq j \leq d\}$, the half cube $K_- := \{x \in K : x_d < 0\}$, its upper plate $\Sigma := \{x \in K : x_d = 0\}$ and, lastly, the half of this, $\Sigma_0 := \{x \in \Sigma : x_{d-1} < 0\}$. The symbol $C^\alpha(\Omega)$ stands for the usual Hölder space on $\Omega$, see [KSS80] or [Tri78]. $W^{1,p}(\Omega)$ denotes the Sobolev space on $\Omega$ consisting of those $L^p(\Omega)$ functions whose first order distributional derivatives also belong to $L^p(\Omega)$ (see [Gri85] or [Maz85]). We use the symbol $W^{1,p}_\Gamma(\Omega)$ for the closure of

$$\{v|_\Omega : v \in C^\infty(\mathbb{R}^d), \text{ supp } v \cap (\partial \Omega \setminus \Gamma) = \emptyset\}$$

in $W^{1,p}(\Omega)$. Note that $\Omega$ enjoys the extension property for $W^{1,p}(\Omega)$ in view of being a bounded Lipschitz domain, see [Gin94, Thm. 3.10] or [Maz85, Ch. 1.1.16]. Thus, in case of $\Gamma = \partial \Omega$ the space $W^{1,p}_\Gamma(\Omega)$ is identical with the usual Sobolev space $W^{1,p}(\Omega)$. If $\Gamma = \emptyset$ we write as usual $W^{1,p}_\emptyset(\Omega)$ instead of $W^{1,p}_\emptyset(\Omega)$. $W^{-1,p'}(\Omega)$ denotes the dual to $W^{1,p}_\Gamma(\Omega)$ and $W^{-1,p'}(\Omega)$ denotes the dual to $W^{1,p}_\Gamma(\Omega)$, when $\frac{1}{p} + \frac{1}{p'} = 1$ holds. If $\Omega$ is understood, then we sometimes abbreviate $W^{1,1}_\Gamma$, $W^{1,p}_\emptyset$ and $W^{-1,p}_\emptyset$, respectively. Please notice that all functional spaces under consideration are regarded as complex ones. By $(\cdot, \cdot)_X$ we indicate the duality between a Banach space $X$ and its dual. Finally, $\gamma$ denotes a generic constant not always of the same numerical value.
1.3. The regularity result

**Definition 1.3.1.** Let \( \Lambda \subset \mathbb{R}^d \) be a bounded domain and \( \Upsilon \) a (relatively) open part of its boundary \( \partial \Lambda \). Then we call \( \Lambda \cup \Upsilon \) regular (in the sense of Gröger [Grö89]), if for every \( x \in \partial \Lambda \) there are two open sets \( U_x, V_x \subset \mathbb{R}^d \) and a bi-Lipschitz transform \( \Psi_x \) from \( U_x \) onto \( V_x \), such that \( x \in U_x \), \( \Psi_x(x) = 0 \) and \( \Psi_x(U_x \cap (\Lambda \cup \Upsilon)) \) either coincides with \( K_- \) or with \( K_- \cup \Sigma \) or with \( K_- \cup \Sigma_0 \).

**Assumption 1.3.2.** Let \( \rho \) be a Lebesgue measurable, essentially bounded function on \( \Omega \), taking its values in the set of real \( d \times d \) matrices, that additionally satisfies the usual (strong) ellipticity condition
\[
y \cdot \rho(x) y \geq \iota |y|^2, \quad y \in \mathbb{R}^d,
\]
for almost all \( x \in \Omega \) and some \( \iota > 0 \).

Given a coefficient function \( \rho \), satisfying this assumption, we define the operator \( -\nabla \cdot \rho \nabla + 1 : W^{-1,2}_1(\Omega) \rightarrow W^{-1,-1}_1(\Omega) \) by
\[
\langle -\nabla \cdot \rho \nabla v + v, w \rangle_{W^{-1,2}_1(\Omega)} := \int_{\Omega} (\rho \nabla v \cdot \nabla w + vw) \, dx, \quad v, w \in W^{-1,2}_1(\Omega).
\]

Then our first main result reads as follows.

**Theorem 1.3.3.** Suppose \( 2 \leq d \leq 4 \) and \( q > d \). Suppose further that \( \Omega \cup \Gamma \) is regular and that Assumption 1.3.2 is satisfied. Then there is an \( \alpha > 0 \), such that \((-\nabla \cdot \rho \nabla + 1)^{-1} \) maps \( W^{-1,-1}_1(\Omega) \) continuously into \( C^\alpha(\Omega) \).

**Remark 1.3.4.** a) The cases \( \Gamma = \emptyset \) (Dirichlet boundary condition) and \( \Gamma = \partial \Omega \) (Neumann boundary condition) are explicitly allowed.

b) It is not hard to see that the right hand side of (1.1.1) defines an element of \( W^{-1,-1}_1(\Omega) \). Conversely, any element from \( W^{-1,-1}_1(\Omega) \) may be represented this way, see [Zie89, Ch. 4.3].

**Corollary 1.3.5.** The result of Theorem 1.3.3 carries over to problems with Robin boundary conditions on \( \Gamma \), if the representing function \( \kappa \) is from \( L^{\infty}(\Gamma, \sigma) \) (\( \sigma \) being the induced boundary measure on \( \Gamma \), cf. [HR08, Section 3]).

**Corollary 1.3.6.** Let a non-negative function \( V \in L^{\infty}(\Omega) \) be given. Moreover, assume that, if \( \text{meas}(\partial \Omega \setminus \Gamma) = 0 \), then there is a subset \( \Omega_+ \) of \( \Omega \) of positive measure, where \( V \) is strictly positive. Then, similarly to Corollary 1.3.5, Theorem 1.3.3 also applies to problems of the form \( -\nabla \cdot \rho \nabla v + Vv = f, \ f \in W^{-1,-1}_1(\Omega) \).

**Corollary 1.3.7.** Let \( D_q \) denote the domain of the maximal restriction of \( -\nabla \cdot \rho \nabla + 1 \) to the space \( W^{-1,-1}_1(\Omega) \). Then, under the suppositions of Theorem 1.3.3, even the complex interpolation space \([D_q, W^{-1,-1}_1(\Omega)], \tau \) continuously embeds into a Hölder space \( C^\beta(\Omega) \), if \( \tau \) and \( \beta \) are sufficiently close to 0.

**Remark 1.3.8.** Corollary 1.3.7 may be of use for the treatment of parabolic equations, see [Sob66, Ama88, Prü02].

1.4. Proof of the regularity result

Let us start by commenting on the philosophy of the proof: the problem will be localized by means of a suitably chosen partition of unity, afterwards transformed by bi-Lipschitz mappings and, if necessary, by reflection. In any case one ends up with a Dirichlet problem
on either a ball, the half cube $K_-$ or the cube $K$. Then a well known regularity result (see Proposition 1.4.3) may be applied.

In order to perform this procedure we first quote two results from the literature and afterwards establish some auxiliary results, which will justify the required technical steps.

### 1.4.1. Known results

**Proposition 1.4.1** ([GR89], see also [Grö89]). Suppose that $\rho$ satisfies Assumption 1.3.2. If $\Omega \cup \Gamma$ is regular, then there is a $q_0 > 2$ such that for all $q \in [2, q_0]$ the operator $-\nabla \cdot \rho \nabla + 1$ provides a topological isomorphism between $W^{1,q}_\Gamma(\Omega)$ and $W^{-1,q}_\Gamma(\Omega)$.

**Remark 1.4.2.** It is clear by Sobolev embedding that in the two dimensional case the assertion of Theorem 1.3.3 already follows from Proposition 1.4.1.

**Proposition 1.4.3** (see [LU68, Ch. III.14], [KS80, Thm. C.2], [CW98, Ch. 4], see also [DeG57]). Let $\Lambda$ be a ball or a cuboid and let the coefficient function $\omega$ (mutatis mutandis) satisfy Assumption 1.3.2. If $q > d$, then there is an $\alpha > 0$, such that $( -\nabla \cdot \omega \nabla )^{-1} : W^{-1,q}(\Lambda) \to C^\alpha(\Lambda)$ (1.4.1) is continuous.

**Remark 1.4.4.** Usually, Proposition 1.4.3 is proved only for real spaces, but it is straightforward to extend this to the complex case: one considers for any element $T$ of the complex Sobolev space $W^{-1,q}(\Lambda)$ the linear forms $T^+ \otimes \partial^+\nabla$ and $T^- \otimes \partial^-\nabla$, where $T^\ast := T^\dagger$. Obviously, both take real values when applied to real functions and satisfy $T^+ \otimes \partial^+\nabla + iT^- \otimes \partial^-\nabla = T$.

### 1.4.2. Auxiliary Results

**Lemma 1.4.5.** Let $\Omega \cup \Gamma$ be regular and let $\mathcal{U} \subset \mathbb{R}^d$ be open, such that $\Omega_\mathcal{U} := \Omega \cap \mathcal{U}$ is also a Lipschitz domain. Furthermore, we put $\Gamma_\mathcal{U} := \Gamma \cap \mathcal{U}$ and fix an arbitrary function $\eta \in C^\infty_0(\mathbb{R}^d)$ with $\text{supp } \eta \subset \mathcal{U}$.

a) If $v \in W^{1,q}_\Gamma(\Omega)$, then $\eta v|_{\Omega_\mathcal{U}} \in W^{1,q}_{\Gamma_\mathcal{U}}(\Omega_\mathcal{U})$.

b) Let for any $v \in L^1(\Omega_\mathcal{U})$ the symbol $\tilde{v}$ indicate the extension of $v$ to $\Omega$ by zero. Then the mapping 

$$ W^{1,q}_{\Gamma_\mathcal{U}}(\Omega_\mathcal{U}) \ni v \mapsto \tilde{v} $$

has its image in $W^{-1,q}_\Gamma(\Omega)$ and is continuous.

**Proof.** For the proof of both points we will employ the following well known set inclusion (cf. [Die71, Ch. 3.8]):

$$ (\partial \Omega \cap \mathcal{U}) \cup (\Omega \cap \partial \mathcal{U}) \subset \partial \Omega_\mathcal{U} \subset (\partial \Omega \cap \mathcal{U}) \cup (\partial \mathcal{U} \cap \Omega). \quad (1.4.2) $$

a) First one observes that the multiplication with $\eta$ and the restriction is a continuous mapping from $W^{1,q}_\Gamma(\Omega)$ into $W^{1,q}_{\Gamma_\mathcal{U}}(\Omega_\mathcal{U})$. Thus, it suffices to show the assertion only for elements of the dense subset

$$ \{ v|_{\Omega_\mathcal{U}} : v \in C^\infty(\mathbb{R}^d), \text{ supp } v \cap (\partial \Omega \setminus \Gamma) = \emptyset \}, $$

what we will do now. One has by (1.4.2)

$$ \text{supp}(\eta v) \cap (\partial \Omega_\mathcal{U} \setminus \Gamma_\mathcal{U}) \subset \text{supp } \eta \cap \text{supp } v \cap \left( (\partial \Omega \cap \mathcal{U}) \cup (\partial \mathcal{U} \cap \Omega) \right) \setminus (\Gamma \cap \mathcal{U}). $$
Since \((\Omega \cap \partial U) \cap (\Gamma \cap U) = \emptyset\), we see
\[
\left( (\partial \Omega \cap U) \cup (\Omega \cap \partial U) \right) \setminus (\Gamma \cap U) = \left( (\partial \Omega \cap U) \setminus (\Gamma \cap U) \right) \cup (\Omega \cap \partial U) \cup (\Omega \cap \partial U) \setminus (\Gamma \cap U) = \left( (\partial \Omega \cap \Gamma) \cap U \right) \cup (\Omega \cap \partial U).
\]
This, together with \(\text{supp} \eta \subset U\) yields
\[
\text{supp}(\eta v) \cap \left( (\partial \Omega \cap \Gamma) \cap U \right) \subset \text{supp} \eta \cap \text{supp} v \cap \left( (\partial \Omega \cap \Gamma) \cap U \right) = \emptyset.
\]

b) Let \(v \in C^\infty_0(\mathbb{R}^d)\) with \(\text{supp} v \cap (\partial \Omega \setminus \Gamma) = \emptyset\). Since by the right hand side of (1.4.2) we have
\[
\partial \Omega \setminus \Gamma \supset (\partial \Omega \cap U) \setminus \Gamma = U \cap (\partial \Omega \setminus \Gamma),
\]
it follows \(\text{supp} v \cap (U \cap (\partial \Omega \setminus \Gamma)) = \emptyset\). Combining this with \(\text{supp} \eta \subset U\), we obtain
\[
\text{supp}(\eta v) \cap (\partial \Omega \setminus \Gamma) = \text{supp}(\eta v) \cap (U \cap (\partial \Omega \setminus \Gamma)) = \emptyset,
\]
so \(\eta v|_\Omega \in W^{1,2}_r(\Omega)\). Furthermore, it is not hard to see that \(\|\eta v\|_{W^{1,2}_r(\Omega)} \leq \gamma\eta\|v\|_{W^{1,2}_r(\Omega)}\), where the constant \(\gamma\) is independent from \(v\). Thus, the assertion follows, since \(\{v|_\Omega : v \in C^\infty_0(\mathbb{R}^d)\text{, supp}(v) \cap (\partial \Omega \setminus \Gamma) = \emptyset\}\) is dense in \(W^{1,2}_r(\Omega)\) and \(W^{1,2}_r(\Omega)\) is closed in \(W^{1,2}_r(\Omega)\). \(\square\)

**Lemma 1.4.6.** Let \(\Omega, \Gamma, U, \eta, \Omega_r\) and \(\Gamma_r\) be as in the foregoing lemma. Denote by \(\rho_r\) the restriction of the coefficient function \(\rho\) to \(\Omega_r\) and let the operator \(-\nabla \cdot \rho_r \nabla : W^{1,2}_r(\Omega_r) \to W^{-1,2}_r(\Omega_r)\) be defined analogously to (1.3.2). Assume \(v \in W^{1,2}_r(\Omega)\) to be the solution of
\[
-\nabla \cdot \rho \nabla v + v = f \in W^{-1,2}_r(\Omega). \tag{1.4.3}
\]
Then the following holds true.

a) For all \(q \in ]1, \infty[\) the linear form
\[
f_r : w \mapsto \langle f, \eta \tilde{w}\rangle_{W^{-1,2}_r(\Omega)},
\]
where \(\eta \tilde{w}\) again means the extension by zero to the whole \(\Omega\), is well defined and continuous on \(W^{1,q}_r(\Omega_r)\), whenever \(f \in W^{-1,q}_r(\Omega)\).

b) If we denote the linear form
\[
W^{1,2}_r(\Omega_r) \ni w \mapsto \int_{\Omega_r} v \rho_r \nabla \eta \cdot \nabla w \, dx,
\]
by \(T_v\), then \(u := \eta v|_{\Omega_r}\) satisfies
\[
-\nabla \cdot \rho_r \nabla u = -\eta v|_{\Omega_r} - \rho_r \nabla v|_{\Omega_r} \cdot \nabla \eta|_{\Omega_r} + T_v + f_r =: f^r. \tag{1.4.4}
\]

c) Assume now \(2 \leq d \leq 4\). If \(f \in W^{-1,q}_r(\Omega)\) for a \(q > d\), then there is a \(p > d\) such that \(f^r \in W^{-1,p}_r(\Omega_r)\). Moreover, the mapping \(W^{-1,q}_r(\Omega) \ni f \mapsto f^r \in W^{-1,p}_r(\Omega_r)\) is continuous.

**Proof.** a) The mapping \(f \mapsto f_r\) is the adjoint to \(v \mapsto \eta \tilde{v}\), which maps by the preceding lemma \(W^{1,q}_r(\Omega_r)\) continuously into \(W^{-1,q}_r(\Omega)\).
Let \( \rho \cdot \nabla u \) and \( \eta w \) be \( 1 \)-radial functions, then we have
\[
\langle -\nabla \cdot (\rho \cdot \nabla u), w \rangle_{W^{1,2}_r(\Omega_\ast)} + \int_{\Omega_\ast} \eta w w \, dx = \int_{\Omega_\ast} \rho \cdot \nabla (\eta w) \cdot \nabla w + \int_{\Omega_\ast} \eta w w \, dx
\]
\[
= \int_{\Omega_\ast} \eta \rho \cdot \nabla \eta \cdot \nabla w \, dx + \int_{\Omega_\ast} \eta \rho \cdot \nabla v \cdot \nabla w \, dx + \int_{\Omega_\ast} \eta w \, dx
\]
\[
= \int_{\Omega_\ast} \rho \cdot \nabla \eta \cdot \nabla w \, dx + \int_{\Omega_\ast} \rho \cdot \nabla v \cdot \nabla (\eta w) \, dx - \int_{\Omega_\ast} \rho \cdot \nabla v \cdot \nabla \eta \, dx + \int_{\Omega_\ast} v \eta \, dx
\]
and by (1.4.3) we see
\[
\int_{\Omega_\ast} \rho \cdot \nabla v \cdot \nabla (\eta w) \, dx + \int_{\Omega} v \eta \, dx = \left( f, \eta w \right)_{W^{-1,2}_r(\Omega)}.
\]

Applying the definition of \( T_v \) and \( f \cdot \rho \cdot \nabla \) and afterwards subtracting \( \int_{\Omega_\ast} \eta w \, dx \) from both sides yields the assertion.

We regard the terms in (1.4.4) from left to right. For the first summand the assertion is obvious.

According to Proposition 1.4.1 there is an \( \epsilon > 0 \) such that the solution of \( -\nabla \cdot (\rho \nabla v) = f \) is from \( W^{1,2+\epsilon}_r(\Omega) \), what implies \( \nabla v \in L^{2+\epsilon}_r(\Omega) \). Furthermore, \( |\nabla \eta| \in L\infty(\Omega_\ast) \) and \( |\rho \cdot \nabla v| \in L^{2+\epsilon}_r(\Omega_\ast) \). Consequently we have \( \rho \cdot \nabla v \cdot \eta \in L^{2+\epsilon}(\Omega_\ast) \). Now, whenever \( 1/p > (d - 2 - \epsilon)/(d(2 + \epsilon)) \) we have the embedding \( L^{2+\epsilon}(\Omega_\ast) \hookrightarrow W^{-1,p}_r(\Omega_\ast) \). Since we restricted the dimension to \( 2 \leq d \leq 4 \), there is always a \( p > 4 \geq d \) satisfying that condition, so the second term is also fine.

On the other hand, we have \( v \in W^{1,2+\epsilon}_r(\Omega) \hookrightarrow L^{4+\delta}(\Omega) \) for a \( \delta = \delta(d) > 0 \). Thus, concerning \( T_v \), we can estimate
\[
|\langle T_v, w \rangle_{W^{-1,2+\epsilon}_r(\Omega_\ast)}| \leq \|v\|_{L^{4+\delta}(\Omega_\ast)} \|\rho\|_{L\infty(\Omega; C^d)} \|\nabla \eta\|_{L\infty(\Omega_\ast)} \|w\|_{W^{1,4+\delta}_r(\Omega_\ast)}.
\]

The claim on \( f \cdot \rho \cdot \nabla \) follows from i), while the proof of the last assertion is implicitly contained in the above considerations.

**Remark 1.4.7.** It is the lack of integrability of the gradient of \( v \) (see the counterexample in [ERS07, Ch. 4]) together with the quality of the needed Sobolev embeddings, which prevents the applicability of this localization procedure to higher dimensions and thus limits our central result to the dimensions up to 4.

The reader may wonder why we start with the operator \( -\nabla \cdot (\rho \cdot \nabla) + 1 \) and consider the operator \( -\nabla \cdot (\rho \cdot \nabla) \) after the localization. The reason for this is the following: in order to include the pure Neumann case one should consider the operator \( -\nabla \cdot (\rho \cdot \nabla) + 1 \). On the other hand, the resulting localized operators \( -\nabla \cdot (\rho \cdot \nabla) \) exhibit in any case a nontrivial Dirichlet boundary part (making them invertible) and the subsequent transformation techniques are technically simpler for the pure operator \( -\nabla \cdot (\rho \cdot \nabla) \).

**Proposition 1.4.8.** Let \( \Lambda \subset \mathbb{R}^d \) be a bounded Lipschitz domain and \( \Upsilon \) be an open subset of its boundary. Assume that \( \phi \) is a mapping from a neighborhood of \( \Lambda \) into \( \mathbb{R}^d \) that is bi-Lipschitz. Let us denote \( \phi(\Lambda) := \Lambda_\Delta \) and \( \phi(\Upsilon) := \Upsilon_\Delta \). Then the following is true.

a) For any \( p \in [1, \infty[ \), the mapping \( \phi \) induces a linear, topological isomorphism
\[
\Phi_p : W^{1,p}_{\Gamma_\delta}(\Lambda_\delta) \to W^{1,p}_r(\Lambda),
\]
which is given by \( (\Phi_p f)(x) = f(\phi(x)) = (f \circ \phi)(\phi(x)) \).

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b) $\Phi^*$ is a linear, topological isomorphism between $W_{-1,p}(\Lambda)$ and $W_{-1,p}(\Lambda_\Delta)$.

c) If $\omega$ is a bounded, measurable function on $\Lambda$, taking its values in the set of $d \times d$ matrices, then

$$\Phi^* \nabla \cdot \omega \nabla \Phi = \nabla \cdot \omega_\Delta \nabla$$

with

$$\omega_\Delta(y) := \frac{1}{|\det(D\phi)(\phi^{-1}(y))|} (D\phi)(\phi^{-1}(y)) \omega(\phi^{-1}(y)) (D\phi)^T(\phi^{-1}(y)) \tag{1.4.5}$$

for almost all $y \in \Lambda_\Delta$. Here, $D\phi$ denotes the Jacobian of $\phi$ and $\det(D\phi)$ the corresponding determinant.

d) If $\omega$ satisfies Assumption 1.3.2, then $\omega_\Delta$ also does.

Proof. The proof of i) is contained in [GGKR02, Thm. 2.10]. Assertion ii) follows from i) by duality, while iii) is well known, see [HKR07] for an explicit verification or [AT98, Ch. 0.8]. Finally, iv) is implied by (1.4.5) and the fact that for a bi-Lipschitz $\phi$ the Jacobian $D\phi$ and its inverse $(D\phi)^{-1}$ are essentially bounded (see [EG92, Ch. 3.1]).

The next lemma makes clear that within the class of bi-Lipschitz transformations one only needs the two local model sets $K_-$ and $K_- \cup \Sigma$, if one dispenses with the condition $\Psi(x) = 0$ (cf. Definition 1.3.1):

Lemma 1.4.9. There is a bi-Lipschitz mapping $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ which maps $K_- \cup \Sigma_0$ onto $K_- \cup \Sigma$.

Proof. Let us first consider the case $d = 2$. We define on the lower halfspace $\{(x, y) : y \leq 0\}$

$$\chi_1(x, y) := \begin{cases} (x - y/2, y/2), & \text{if } x \leq 0, y \geq x, \\ (x/2, -x/2 + y), & \text{if } x \leq 0, y < x, \\ (x/2, x/2 + y), & \text{if } x > 0, y < -x, \\ (x + y/2, y/2), & \text{if } x > 0, y \geq -x. \end{cases} \tag{1.4.6}$$

Observing that $\chi_1$ acts as the identity on the $x$-axis, we may define $\chi_1$ on the upper half space $\{(x, y) : y > 0\}$ also as the identity and thus obtain a globally bi-Lipschitz transformation $\chi_1$ from $\mathbb{R}^2$ onto itself that transforms $K_- \cup \Sigma_0$ onto the triangle shown in Figure 1.1.

![Figure 1.1: $K_- \cup \Sigma_0$ and $\chi_1(K_- \cup \Sigma_0)$](image)
Next, we define the bi-Lipschitz mapping \( \chi_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
\chi_2(x, y) := \begin{cases} 
(x, x + 2y + 1), & \text{if } x \leq 0, \\
(x, -x + 2y + 1), & \text{if } x > 0,
\end{cases}
\tag{1.4.7}
\]
in order to get the geometric constellation in Figure 1.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{\( \chi_2(\rho_1(K_\cup \Sigma_0)) \)}
\end{figure}

If \( \vartheta \) is the (clockwise) rotation by \( \pi/4 \), we thus achieved that \( \chi := \vartheta \chi_2 \chi_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) is bi-Lipschitz and satisfies
\[
\chi(K_\cup \Sigma_0) = \left\{ (x, y) : -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} < y \leq \frac{1}{\sqrt{2}} \right\}.
\]

Let \( \varsigma : \mathbb{R}^2 \to \mathbb{R}^2 \) be the affine mapping \( (x, y) \mapsto (\sqrt{2}x, \frac{1}{\sqrt{2}}y - \frac{1}{2}) \). Then \( \Psi_2 := \varsigma \chi \) maps \( K_\cup \Sigma_0 \) bi-Lipschitzian onto \( K_\cup \Sigma \) in the 2\( d \) case.

If \( d > 2 \), one simply puts \( \Psi(x_1, \ldots, x_d) := (x_1, \ldots, x_{d-2}, \Psi_2(x_{d-1}, x_d)) \).

\begin{proposition}
Let for any \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) the symbol \( x_- \) denote the element \( (x_1, \ldots, x_{d-1}, -x_d) \). Further, for a \( d \times d \) matrix \( \varrho \), we define the matrix \( \varrho^- \) by
\[
\varrho_{j,k}^- := \begin{cases} 
\varrho_{j,k}, & \text{if } j, k < d, \\
-\varrho_{j,k}, & \text{if } j = d \text{ and } k \neq d \text{ or } k = d \text{ and } j \neq d, \\
\varrho_{j,k}, & \text{if } j = k = d.
\end{cases}
\tag{1.4.8}
\]

Let \( \omega \) be a bounded, measurable function on \( K_- \) taking its values in the set of real, symmetric \( d \times d \) matrices. We define the matrix valued function \( \hat{\omega} \) on \( K \) by
\[
\hat{\omega}(x) := \begin{cases} 
\omega(x), & \text{if } x \in K_-,
\omega(x_-)^-, & \text{if } x_- \in K_-,
0, & \text{if } x \in \Sigma.
\end{cases}
\]
Then we have the following assertions for every \( p \in [1, \infty[ \).
a) If \( \psi \in W^{1,p}_\Sigma(K_-) \) satisfies \(-\nabla \cdot \omega \nabla \psi = f \in W^{-1,p}_\Sigma(K_-)\), then \(-\nabla \cdot \omega \nabla \hat{\psi} = \hat{f} \in W^{-1,p}(K)\) holds for \(\hat{\psi}\) with
\[
\hat{\psi}(x) = \begin{cases} 
\psi(x), & \text{if } x \in K_, \\
\psi(x_*), & \text{if } x_* \in K_-
\end{cases}
\]
and \(\hat{f}\) defined by \(\langle \hat{f}, \varphi \rangle_{W^{-1,p}(K)} := \langle f, \varphi|_{K_-} + \varphi_0|_{K_-} \rangle_{W^{-1,p}(K_-)}\). Here, the function \(\varphi_-\) is defined by \(\varphi_-(x) := \varphi(x_*)\).

b) The mapping \(W^{-1,p}_\Sigma(K_-) \ni f \mapsto \hat{f} \in W^{-1,p}(K)\) is continuous.

**Proof.**

a) It is known that \(\hat{\psi}\) belongs to \(W^{1,p}_0(K)\), see [Giu94, Lemma 3.4]. Thus, it remains to show \(-\nabla \cdot \omega \nabla \hat{\psi} = \hat{f}\) as an equation in \(W^{-1,p}(K)\). Since every test function from \(W^{-1,p}_0(K)\) may be split up into a symmetric and an anti-symmetric part, we may look at these two cases separately. For every anti-symmetric \(\varphi \in W^{-1,p}_0(K)\), i.e., \(\varphi(x_-) = -\varphi(x)\), the symmetry of \(\omega\) yields immediately
\[
\langle -\nabla \cdot \omega \nabla \hat{\psi}, \varphi \rangle_{W^{-1,p}(K)} = 0 = \langle \hat{f}, \varphi \rangle_{W^{-1,p}(K)}.
\]
In the case of symmetric functions \(\varphi\) the assertion is obtained by the definitions of \(\hat{\psi}, \hat{f}\), \(-\nabla \cdot \omega \nabla_\Sigma \hat{\psi} = \hat{f}\) and straightforward calculations, based on Proposition 1.4.8 when applied to the transformation \(x \mapsto x_*\).

b) The operator \(f \mapsto \hat{f}\) is the adjoint to \(\varphi \mapsto (\varphi|_{K_-} + \varphi_0|_{K_-})\).

### 1.4.3. Core of the proof

By Definition 1.3.1 and Lemma 1.4.9, for every \(x \in \partial \Omega\) there is an open neighborhood \(U_x\) of \(x\), an open set \(W_x\) and a bi-Lipschitz mapping \(\Psi_x\) from \(U_x\) onto \(W_x\) such that \(\Psi_x((\Omega \cup \Gamma) \cap U_x)\) equals either \(K_-\) or \(K_+ \cup \Sigma\). Take for every point \(x \in \Omega\) a ball \(B_x \subset \Omega\) centered at \(x\). Obviously, the system \(\{U_x\}_{x \in \partial \Omega} \cup \{B_{x_j}\}_{x \in \Omega}\) forms an open covering of \(\Omega\). Take a finite subcovering \(U_{s_1}, \ldots, U_{s_k}, B_{x_{s_1}}, \ldots, B_{x_{s_k}}\) and choose a partition of unity \(\eta_1, \ldots, \eta_k, \zeta_1, \ldots, \zeta_l\) over \(\Omega\), which is subordinated to this subcovering.

Assume now \(f \in W^{-1,q}_\Gamma(\Omega)\) with \(q > d\) and that \(v\) is the solution of \(-\nabla \cdot \rho \nabla v = f\). Then, according to Lemma 1.4.6 with \(\Gamma_\star = \emptyset\) and \(\Omega_\star = B_{x_{s_j}}\), every function \(\zeta_{s_j} v|_{B_{x_{s_j}}}\) satisfies an equation \(-\nabla \cdot \rho \nabla (\zeta_{s_j} v|_{B_{x_{s_j}}} = g_{s_j}\), where \(g_{s_j} \in W^{-1,p}(B_{x_{s_j}})\) with \(p > d\) and, additionally,
\[
\|g_{s_j}\|_{W^{-1,p}(B_{x_{s_j}})} \leq \gamma\|f\|_{W^{-1,q}(\Omega)} \tag{1.4.9}
\]
with \(\gamma\) independent from \(f\). Hence, by Proposition 1.4.3, we have \(\zeta_{s_j} v|_{B_{x_{s_j}}} \in C^\alpha(B_{x_{s_j}})\) for an \(\alpha = \alpha(j) > 0\) and, moreover,
\[
\|\zeta_{s_j} v|_{B_{x_{s_j}}} \|_{C^\alpha(B_{x_{s_j}})} \leq \gamma\|g_{s_j}\|_{W^{-1,p}(B_{x_{s_j}})} \tag{1.4.10}
\]
Clearly, (1.4.10) together with (1.4.9) implies
\[
\|\zeta_{s_j} v\|_{C^\alpha(\Omega)} = \|\zeta_{s_j} v|_{B_{x_{s_j}}} \|_{C^\alpha(B_{x_{s_j}})} \leq \gamma\|g_{s_j}\|_{W^{-1,p}(B_{x_{s_j}})} \leq \gamma\|f\|_{W^{-1,q}(\Omega)} \tag{1.4.11}
\]
Let us now consider the functions \(\eta_{s_j} v\) for fixed \(j\): putting \(\Omega_j := \Omega \cap U_{s_j}\) and \(\Gamma_j := \Gamma \cap U_{s_j}\), we obtain by Lemma 1.4.5 a)) that each \(\eta_{s_j} v|_{\Omega_j}\) belongs to \(W^{1,2}_{\Gamma_j}(\Omega_j)\). Furthermore, Lemma 1.4.6

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shows that $\eta_j v|_{\Omega_j}$ satisfies an equation $-\nabla \cdot \rho \nabla (\eta_j v|_{\Omega_j}) = f_j$, where $f_j \in W^{-1,p}_\Gamma(\Omega_j)$ with $p > d$ and, additionally,

$$\|f_j\|_{W^{-1,p}_\Gamma(\Omega_j)} \leq \gamma \|f\|_{W^{-1,q}(\Omega)}$$

(1.4.12)

with $\gamma$ independent from $f$. Next we consider the ‘transformed’ function (cf. Proposition 1.4.8 with $\phi = \Psi_{x_j}^{-1}$) $\psi_j := \Phi_\rho(\eta_j v|_{\Omega_j}) = (\eta_j v|_{\Omega_j}) \circ \Psi_{x_j}^{-1}$ on $K_-$, from now on distinguishing the cases

$$\Psi_{x_j}((\Omega \cup \Gamma) \cap \mathcal{U}_{x_j}) = K_-$$

(1.4.13)

and

$$\Psi_{x_j}((\Omega \cup \Gamma) \cap \mathcal{U}_{x_j}) = K_- \cup \Sigma.$$  

(1.4.14)

If (1.4.13) is true, then $\Gamma_j := \Gamma \cap \mathcal{U}_{x_j} \subset \partial(\Omega \cup \mathcal{U}_{x_j})$ must be empty, since $K_-$ has only inner points and inner points pass to inner points and boundary points to boundary points under a bi-Lipschitz transformation. In particular, this means $\eta_j v|_{\Omega_j} \in W^{1,2}_0(\Omega_j)$. By Proposition 1.4.8 the function $\psi_j$ belongs to $W^{1,2}_0(K_-)$ and satisfies an equation $-\nabla \cdot \omega \nabla \psi_j = h_j$ with $h_j = (\Phi_\rho)^{-1}f_j \in W^{-1,p}(K_-)$ for the same $p > d$ as above. Thanks to Proposition 1.4.8(d)), the coefficient function $\omega$ again satisfies Assumption 1.3.2. Thus, by Proposition 1.4.3, $\psi_j \in C^{\alpha}(K_-)$, where $\alpha$ depends on $j$, and

$$\|\psi_j\|_{C^{\alpha}(K_-)} \leq \gamma \|h_j\|_{W^{-1,p}(K_-)}$$

with $\alpha$ and $\gamma$ independent from $h_j$. Transforming back, this gives $\eta_j v|_{\Omega_j} \in C^{\alpha}(\Omega_j)$ with

$$\|\eta_j v|_{\Omega_j}\|_{C^{\alpha}(\Omega)} \leq \gamma \|f_j\|_{W^{-1,p}(\Omega_j)} \leq \gamma \|f\|_{W^{-1,q}(\Omega)},$$

(1.4.15)

where the last inequality is just (1.4.12). As the support of $\eta_j v$ has a positive distance to $\Omega \setminus \Omega_j$, the function $\eta_j v$ is from $C^{\alpha}(\Omega)$ with the norm equality $\|\eta_j v\|_{C^{\alpha}(\Omega)} = \|\eta_j v|_{\Omega_j}\|_{C^{\alpha}(\Omega)}$. This, together with (1.4.15) gives the desired estimate, where $\gamma$ is independent from $f$.

Let us now consider the case (1.4.14). Analogously as before Proposition 1.4.8 yields that $\psi_j$ belongs to $W^{1,2}_0(K_-)$ and satisfies an equation $-\nabla \cdot \omega \nabla \psi_j = h_j$ with $h_j \in W^{-1,p}_\Sigma(K_-)$ and $p > d$, where $\omega$ again satisfies Assumption 1.3.2. Now, we apply the reflection principle from Proposition 1.4.10. This leads to a homogeneous Dirichlet problem $-\nabla \cdot \omega \nabla \hat{\psi}_j = h_j$, where $\hat{h}_j \in W^{-1,p}(K)$. But then Proposition 1.4.3 gives $\hat{\psi}_j \in C^{\alpha}(K)$ and $\|\hat{\psi}_j\|_{C^{\alpha}(K)} \leq \gamma \|\hat{h}_j\|_{W^{-1,p}(K)}$ (alpha depending on $j$). Clearly, this, together with Proposition 1.4.10 b)), implies $\|\psi_j\|_{C^{\alpha}(K_-)} \leq \gamma \|h_j\|_{W^{-1,p}(K_-)}$ and, consequently,

$$\|\eta_j v|_{\Omega_j}\|_{C^{\alpha}(\Omega)} \leq \gamma \|f_j\|_{W^{-1,p}(\Omega_j)} \leq \gamma \|f\|_{W^{-1,q}(\Omega)}.$$  

Thus, we get $\eta_j v \in C^{\alpha}(\Omega)$ and $\|\eta_j v\|_{C^{\alpha}(\Omega)} \leq \gamma \|f\|_{W^{-1,q}(\Omega)}$ with $\gamma$ independent from $f$ as in the previous case. Passing to the minimal $\alpha(j)$, this finishes the proof of Theorem 1.3.3.

Proof of Corollary 1.3.5. As is well known ([Cia79, Ch. 1.2], [GGZ74, Ch II.2]), in case of a Robin boundary condition the operator is defined via the bilinear form

$$W^{1,2}_\Gamma(\Omega) \times W^{1,2}_\Gamma(\Omega) \ni (v,w) \mapsto \int_\Omega \rho \nabla v \cdot \nabla w \, dx + \int_\Omega vw \, dx + \int_\Gamma \kappa vw \, d\sigma$$

for some $\kappa \in L^\infty(\Gamma, \sigma)$. It is easy to see that the linear mapping $T : L^\infty(\Gamma, \sigma) \to W^{-1,p}_\Gamma(\Omega)$, given by

$$\langle T \psi, \varphi \rangle_{W^{-1,p}_\Gamma(\Omega)} = \int_\Gamma \kappa \psi \varphi \, d\sigma,$$
1.5. Alternative characterization for regular sets

Gröger’s concept of regular sets [Grö89] turned out to be a powerful tool for the treatment of mixed boundary value problems. Not only his regularity result [Grö89], based on this, is exploited in some tens of papers; but the regular sets proved also to be an adequate frame for establishing interpolation results for function spaces which include a trace zero condition on part of the boundary, see [GGKR02]. Moreover, it allowed to recover resolvent estimates and thus provided tools for the treatment of parabolic equations, which incorporate mixed boundary conditions, see [GR89], [GKR01]. All of this shows that the concept of regular sets in itself deserves some investigation in order to simplify things.

In this spirit, the aim of this chapter is to prove that for two and three space dimensions the property of a set $\Lambda \cup \Upsilon$ to be regular in the sense of Gröger (see Definition 1.3.1) can be characterized by $\Lambda$ being a Lipschitz domain and a certain topological property of $\Upsilon$ (to be specified in a moment). The point is that the resulting conditions usually can be checked is well defined and continuous. Denoting the domain of $-\nabla \cdot \rho \nabla + 1$ again by $D_q$, we have by Theorem 1.3.3 the compact embedding $D_q \hookrightarrow C^\alpha(\Omega) \hookrightarrow L^\infty(\Gamma, \sigma)$. Hence, the mapping $T$ is relatively compact with respect to $-\nabla \cdot \rho \nabla + 1$ and a classical perturbation theorem [Kat80, Ch. IV.1.3] applies.

Proof of Corollary 1.3.6. We argue analogously to the proof of Corollary 1.3.5. Here, $T : L^\infty(\Omega) \rightarrow W^{-1,q}_\Gamma(\Omega)$ is defined by

$$\langle T\psi, \varphi \rangle_{W^{-1,q}_\Gamma(\Omega)} = \int_\Omega V \psi \varphi \, dx.$$  

Then, due to the compact embedding $D_q \hookrightarrow L^\infty(\Omega)$, the same perturbation argument as above yields that the domain of the maximal restriction of $-\nabla \cdot \rho \nabla + V$ to $W^{-1,q}_\Gamma(\Omega)$ coincides with $D_q$. Together with the coercivity of the associated bilinear form that follows from the assumptions on $V$ stated in Corollary 1.3.6, this gives the assertion.

Proof of Corollary 1.3.7. Applying a well known re-iteration result on complex interpolation (see [Tri78, Ch. 1.9.3]), we obtain for $\tau \in ]0,1[$

$$[D_q, W^{-1,q}_\Gamma(\Omega)]_\tau \hookrightarrow [D_q, [D_q, W^{-1,q}_\Gamma(\Omega)]_\frac{1}{\tau}]_\tau. \quad (1.4.16)$$

But the embedding $W^{-1,q}_\Gamma(\Omega) \hookrightarrow W^{-1,2}_\Gamma(\Omega)$ gives $D_q \hookrightarrow W^{-1,2}_\Gamma(\Omega)$. Together with $D_q \hookrightarrow C^\alpha(\Omega)$, we obtain by (1.4.16)

$$[D_q, W^{-1,q}_\Gamma(\Omega)]_\tau \hookrightarrow [C^\alpha(\Omega), [W^{1,2}_\Gamma(\Omega), W^{-1,2}_\Gamma(\Omega)]_\frac{1}{\tau}]_\tau.$$  

One identifies the interpolation space $[W^{1,2}_\Gamma(\Omega), W^{-1,2}_\Gamma(\Omega)]_\frac{1}{\tau}$ as the space $L^2(\Omega)$ (see [Tri78, Ch. 1.18.10]), what gives

$$[D_q, W^{-1,q}_\Gamma(\Omega)]_\tau \hookrightarrow [C^\alpha(\Omega), L^2(\Omega)]_\tau. \quad (1.4.17)$$

This latter interpolation space is known to embed into another Hölder space $C^\beta(\Omega)$, if $\tau > 0$ is chosen sufficiently small (see [GKR01, Ch. 7], see also [Tri78a]).

Remark 1.4.11. If the coefficient matrices are symmetric, the Hölder continuity of the solution for (1.1.1) may be deduced from the results of [GR01] and [Gri02] by means of suitable (but nontrivial) embedding theorems. Unfortunately, this is not carried out there.
by appearance’ – in contrast to the original definition. Let us explicitly mention that the underlying class of Lipschitz domains is broad enough to contain e.g. the case of two balks, lying on each other with an angle \( \neq \pi \), which together do not form a domain with Lipschitz boundary.

We start with the following observation.

**Theorem 1.5.1.** If \( \Lambda \cup \Upsilon \) is regular, then \( \Lambda \) is a Lipschitz domain. 

*Proof.* Let \( x \in \partial \Lambda \). Then there is, due to the definition, an open neighborhood \( \mathcal{U} \) of \( x \) and a bi-Lipschitz mapping \( \Psi : \mathcal{U} \to \mathbb{R}^d \), such that \( \Psi(x) = 0 \) holds and \( \Psi(\mathcal{U} \cap (\Lambda \cup \Upsilon)) \) equals \( K_- \cup \Sigma \), where \( \Sigma = \emptyset \) or \( \Sigma = \Sigma \) or \( \Sigma = \Sigma_0 \). This means

\[
\Psi(\mathcal{U} \cap (\Lambda \cup \Upsilon)) = \Psi((\mathcal{U} \cap \Lambda) \cup (\mathcal{U} \cap \Upsilon)) = K_- \cup \Sigma
\]  

(1.5.1)

and since inner points pass to inner points and boundary points to boundary points under a bi-Lipschitz transformation, (1.5.1) implies \( \Psi(\mathcal{U} \cap \Lambda) = K_- \) in all three cases. Hence, \( \mathcal{U} \) may serve as the local chart neighborhood required in the definition of a Lipschitz domain, see [Gri85, Def. 1.2.1.2].

We first deal with the easier case of \( d = 2 \).

**Theorem 1.5.2.** Let \( \Lambda \subset \mathbb{R}^2 \) be a bounded Lipschitz domain and \( \Upsilon \subset \partial \Lambda \) be an open part of the boundary. Then \( \Lambda \cup \Upsilon \) is regular in the sense of Gröger, iff the set \( \Upsilon \cap (\partial \Lambda \setminus \Upsilon) \) is finite and no connected component of \( \partial \Lambda \setminus \Upsilon \) consists of a single point.

*Proof.* In view of Theorem 1.5.1 and an application of the definition for regular sets it is clear that the condition is necessary. Sufficiency follows from the Lipschitz domain property and the fact that any point \( x \in \partial \Lambda \) can only lie in \( \Upsilon \), \( \Upsilon \cap (\partial \Lambda \setminus \Upsilon) \) or in the (relative) interior of \( \partial \Lambda \setminus \Upsilon \).

**Remark 1.5.3.** It is not hard to see that the given condition is equivalent to the following: \( \Upsilon \) is a finite union of open arc pieces from \( \partial \Lambda \) and \( \partial \Lambda \setminus \Upsilon \) is a finite union of (nondegenerate) closed arc pieces.

Now we come to an intrinsic characterization of regular sets in \( \mathbb{R}^3 \), which we regard as the second essential result of this work.

**Theorem 1.5.4.** Let \( \Lambda \subset \mathbb{R}^3 \) be a bounded Lipschitz domain. Assume \( \Upsilon \) to be an open subset of \( \partial \Lambda \). Then \( \Lambda \cup \Upsilon \) is regular in the sense of Gröger, iff the following two conditions are satisfied.

a) \( \partial \Lambda \setminus \Upsilon \) is the closure of its interior (within \( \partial \Lambda \)).

b) For any \( x \in \overline{\Upsilon} \cap (\partial \Lambda \setminus \Upsilon) \) there is an open neighborhood \( \mathcal{N} \) of \( x \) and a bi-Lipschitz mapping \( \kappa : \mathcal{N} \cap (\partial \Lambda \setminus \Upsilon) \to [-1, 1] \).

*Proof.* According to the definition of regular sets the conditions are necessary.

In order to prove sufficiency, we have to show that for every \( x \in \partial \Lambda \) there is an open neighborhood \( \mathcal{U} \) of \( x \) and a bi-Lipschitz mapping \( \Psi \) with \( \Psi(x) = 0 \), such that \( \Psi(\mathcal{U} \cap (\Lambda \cup \Upsilon)) \) is either \( K_- \) or \( K_- \cup \Sigma \) or \( K_- \cup \Sigma_0 \).

We first observe that

\[
\partial \Lambda = \Upsilon \cup (\partial \Lambda \setminus \Upsilon)^\circ \cup [(\partial \Lambda \setminus \Upsilon) \cap \overline{\Upsilon}],
\]  

(1.5.2)

where the closure and the interior are again taken with respect to the topology of \( \partial \Lambda \). In the following we will treat these three cases separately.
Let $x \in \Upsilon$. Since $\Upsilon$ was supposed to be open, there is an open set $U_1 \subset \mathbb{R}^d$ that contains $x$ and satisfies $U_1 \cap \partial \Lambda \subset \Upsilon$. Furthermore, as $\Lambda$ is a Lipschitz domain, there is another open neighborhood $U_2 \subset \mathbb{R}^d$ of $x$ and a bi-Lipschitz transform $\Phi$ from $U_2$ onto the (open) cube $K$, such that $\Phi(x) = 0$, $\Phi(\Lambda \cup U_2) = K_-$ and $\Phi(\partial \Lambda \cup U_2) = \Sigma$. Since $\Phi$ is in particular a homeomorphism, the set $\Phi(U_1 \cap U_2)$ is an open neighborhood of $0$ and it is contained in $K$. Thus, it contains a homothety $tK$ of $K$ for some $t > 0$. If we define $U := \Phi^{-1}(tK)$, then $\Phi|_U$ is a bi-Lipschitz mapping from the open neighborhood $U$ of $x$ onto $tK$, such that $\Phi(U \cap (\Lambda \cup \Upsilon))$ is the set $t(K^- \cup \Sigma)$. Combining $\Phi$ with a homothety, we get a bi-Lipschitz mapping $\Psi$ from $U$ onto $K$ that satisfies $\Psi(x) = 0$ and $\Psi(U \cap (\Lambda \cup \Upsilon)) = K_- \cup \Sigma$.

Analogously, one proves for the (relatively) inner points $x \in (\partial \Lambda \setminus \Upsilon)^o$ the existence of a neighborhood $U$ and a bi-Lipschitz mapping $\Psi$ onto the open cube $K$ such that $\Psi(x) = 0$ and $\Psi(U \cap (\Lambda \cup \Upsilon))$ is the set $K_-$.

It remains to consider the points of $(\partial \Lambda \setminus \Upsilon) \cap \Upsilon$. Let $x$ be an element of this set. As $\Lambda$ is a Lipschitz domain, there is an open neighborhood $O$ of $x$ in $\mathbb{R}^3$ and a bi-Lipschitz mapping $\Phi$ from $O$ onto the cube $K \subset \mathbb{R}^3$, such that $\Phi(x) = 0$, $\Phi(\Lambda \cap O) = K_-$ and $\Phi(\partial \Lambda \cup O) = \Sigma$. Exploiting ii), we find another open neighborhood $N$ of $x$ and a bi-Lipschitz mapping $\kappa$, such that $\kappa(\Upsilon \cap (\partial \Lambda \setminus \Upsilon) \cap N) = [-1,1]$. Without loss of generality we may assume $\kappa(x) = 0 \in \mathbb{R}$. Our job is now to combine the good properties of $\Phi$ and $\kappa$. In order to do so, we first define a smaller neighborhood of $x$ that is contained in $O \cap N$. Since $\Phi(O \cap N)$ is an open neighborhood of $0 \in \mathbb{R}^3$, we find a number $t \in ]0,1[$, such that $tK \subset \Phi(O \cap N)$ and we set $\Lambda := \Phi^{-1}(tK)$. Clearly, $\Lambda$ then is an open neighborhood of $x$ that is contained in $O \cap N$. Additionally, one has $\Phi(\Lambda \cap \Lambda) = \Phi(\Lambda) \cap tK = \Phi(\Lambda) \cap \Sigma \cap tK = K_- \cap tK = tK_-$ and $\Phi(\partial \Lambda \cap \Lambda) = \Phi(\partial \Lambda) \cap tK = \Sigma \cap tK = \Sigma \cap tK = [-t,t] \times [-t,t] \times \{0\}$.

We define $P : \mathbb{R}^3 \to \mathbb{R}^2$ as the canonical projection onto the first two components and $\hat{\Phi} := P\Phi$ with $\Lambda \cap \partial \Lambda$ as its domain of definition. Note that by the above considerations $\hat{\Phi} : \Lambda \cap \partial \Lambda \to [-t,t] \times [-t,t] \times \{0\}$ is a bi-Lipschitz mapping with $\hat{\Phi}(x) = 0 \in \mathbb{R}^2$. Let $[s_-,s_+] \subset [-1,1]$ be the maximal interval containing $0$, such that $\hat{\Phi}\kappa^{-1}([s_-,s_+]) \subset [-t,t] \times [-t,t]$ and denote the set $\hat{\Phi}\kappa^{-1}([s_-,s_+])$ by $C$. It is not hard to see, that $C$ is the connected component of $\hat{\Phi}(\Lambda \cap \Upsilon \cap (\partial \Lambda \setminus \Upsilon))$, which contains $0 \in \mathbb{R}^2$ within $\hat{\Phi}(\Lambda \cap \partial \Lambda) = [-t,t] \times [-t,t]$. We claim: $\hat{\Phi}(\Lambda \cap \Upsilon \cap (\partial \Lambda \setminus \Upsilon)) \setminus C$ has a positive distance to $0 \in \mathbb{R}^2$. (1.5.3)

In fact, the elements of $\hat{\Phi}(\Lambda \cap \Upsilon \cap (\partial \Lambda \setminus \Upsilon)) \setminus C$ correspond to numbers from the set $[-t,t] \times [-t,t] \times \{0\}$ in particular, $\hat{\Phi}(\Lambda \cap \partial \Lambda)$ which contains $0 \in \mathbb{R}^2$ is the image of this set under $\hat{\Phi}$ and $\kappa$. Let $\Theta$ be the (bi-Lipschitz) mapping $\hat{\Phi}\kappa^{-1} : [s_-,s_+] \to C$. We will identify $[s_-,s_+]$ by means of the (bi-Lipschitz) embedding $\mathbb{R} \ni x \mapsto (x,0) \in \mathbb{R}^2$ with the set $[s_-,s_+] \times \{0\}$. Then by a deep lying theorem of Tukia, cf. [Tuk80, Thm. B], there exists a bi-Lipschitz extension of $\Theta$ which maps $\mathbb{R}^2$ onto itself that we will denote by $\Theta$. Note that $\Theta$ maps $[s_-,s_+] \times \{0\}$ onto $C$ and, in particular, $\Theta(0) = 0 \in \mathbb{R}^2$. As $\hat{\Phi}(\Lambda \cap \partial \Lambda) = [-t,t] \times [-t,t]$ is open in $\mathbb{R}^2$ and due to (1.5.3) one finds an $\epsilon \in ]0,\min\{s_-,s_+\}]$, such that $\hat{\Phi}(\Lambda \cap \Upsilon \cap (\partial \Lambda \setminus \Upsilon)) \setminus C \cap \Theta([-\epsilon,\epsilon] \times [-\epsilon,\epsilon]) = \emptyset$ (1.5.4) and simultaneously $\Theta([-\epsilon,\epsilon] \times [-\epsilon,\epsilon]) \subset [-t,t] \times [-t,t]$ holds, see Figure 1.3.
1. Hölder Continuity for Second Order Elliptic Problems With Nonsmooth Data

This $\epsilon$ will provide us a suitable neighborhood $\mathcal{U}$ of $x$ to complete our task. In fact, we set $\mathcal{Z} := \Theta([-\epsilon, \epsilon] \times [-\epsilon, \epsilon])$ and $\mathcal{U} := \Theta^{-1}(\mathcal{Z} \times [-\epsilon, \epsilon])$. For the bi-Lipschitz mapping $\Psi$ we define first the mapping $\xi$ on $\mathcal{Z} \times [-\epsilon, \epsilon]$ by

$$\xi(y_1, y_2, y_3) := (\Theta^{-1}(y_1, y_2), y_3)$$

and then set $\Psi := \frac{1}{\epsilon}(\xi \circ \Phi)$ on $\mathcal{U}$. Note that $\Phi(\mathcal{U})$ is exactly $\mathcal{Z} \times [-\epsilon, \epsilon]$ by construction, which implies that everything is well defined. Furthermore, $\mathcal{U}$ is obviously a neighborhood of $x$ and since $\xi$ and $\Phi$ are bi-Lipschitz mappings, $\Psi$ is of the same quality.

Regarding the desired mapping properties of $\Psi$, we already see

$$\Psi(x) = \xi(\Phi(x))/\epsilon = \xi(0)/\epsilon = (\Theta^{-1}(0, 0), 0)/\epsilon = 0$$

and

$$\epsilon \Psi(\mathcal{U}) = \xi(\mathcal{Z} \times [-\epsilon, \epsilon]) = \Theta^{-1}(\mathcal{Z}) \times [-\epsilon, \epsilon] = \epsilon K.$$

Since $\mathcal{U} \subset \mathcal{O}$, we get

$$\epsilon \Psi(\mathcal{U} \cap \Lambda) = \xi((\mathcal{Z} \times [-\epsilon, \epsilon] \cap \Phi(\Lambda \cap \mathcal{O})) = \epsilon K \cap \xi(K)$$

and thus $\epsilon \Psi(\mathcal{U} \cap \Lambda) = \epsilon K \cap (\Theta^{-1}([-1, 1]^2) \times [-1, 0])$.

Observing $\Theta([-\epsilon, \epsilon] \times [-\epsilon, \epsilon]) \subset ]-t, t]\times ]-t, t[ \subset ]-1, 1]^2$ and thus $]-\epsilon, \epsilon[ \times ]-\epsilon, \epsilon[ \subset \Theta^{-1}([-1, 1]^2)$, this yields

$$\Psi(\mathcal{U} \cap \Lambda) = K.$$  \hspace{1cm} (1.5.5)

For the boundary of $\Lambda$ we get by analogous considerations

$$\Psi(\mathcal{U} \cap \partial \Lambda) = \Sigma,$$  \hspace{1cm} (1.5.6)

so the only thing left to prove is $\Psi(\mathcal{U} \cap \Sigma) = \Sigma_0$.

First, we focus on the interface $\mathcal{U} \cap \Sigma \cap (\partial \Lambda \setminus \Sigma)$ and show that this is mapped to the line $]-1, 1[ \times \{0\} \times \{0\}$, i.e. the boundary of $\Sigma_0$ in $\Sigma$. Then, in a second step, we will show that $\Psi(\mathcal{U} \cap \Sigma)$ must be exactly one of the half squares $\Sigma_0$ or $-\Sigma_0$. We first observe

$$\Theta^{-1}(\mathcal{Z} \cap \Phi(\mathcal{X} \cap \Sigma \cap (\partial \Lambda \setminus \Sigma)))$$

$$= \Theta^{-1}(\mathcal{Z} \cap \Phi(\mathcal{X} \cap \Sigma \cap (\partial \Lambda \setminus \Sigma)) \setminus C) \cup \Theta^{-1}(\mathcal{Z} \cap \Phi(\mathcal{X} \cap \Sigma \cap (\partial \Lambda \setminus \Sigma)) \cap C).$$

Figure 1.3.: $\hat{\Phi}(\mathcal{X} \cap \partial \Lambda)$ and $\Theta^{-1}(\hat{\Phi}(\mathcal{X} \cap \partial \Lambda))$
Now, the left part of this union is empty thanks to (1.5.4). Using $\mathcal{Z} = \Theta([-\epsilon, \epsilon] \times [-\epsilon, \epsilon])$ and $\mathcal{C} \subset \hat{\Phi}(\mathcal{X} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y}))$, we thus obtain

$$
\Theta^{-1}((\mathcal{Z} \cap \hat{\Phi}(\mathcal{X} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})))) = \{[-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \} \cap \Theta^{-1}(\mathcal{C})
= \{[-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \} \cap \{[-\epsilon, \epsilon] \times \{0\}\}
= [-\epsilon, \epsilon] \times \{0\}.
$$

(1.5.7)

Having in mind that $\Phi(\mathcal{X} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})) = \hat{\Phi}(\mathcal{X} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})) \times \{0\}$ and $\mathcal{U} = \Phi^{-1}([\epsilon] \times [-\epsilon, \epsilon])$, this implies

$$
e\Psi(\mathcal{U} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})) = \xi(\Phi(\mathcal{X} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})) \cap \Phi(\mathcal{U}))
= \Theta^{-1}(\hat{\Phi}(\mathcal{X} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})) \cap \mathcal{Z}) \times \{0\}
= [-\epsilon, \epsilon] \times \{0\} \times \{0\}
$$

and thus

$$
\Psi(\mathcal{U} \cap \mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})) = [-1, 1] \times \{0\} \times \{0\}.
$$

(1.5.8)

Now, we claim:

(*) (1.5.6) and (1.5.8) imply, that $\Psi(\mathcal{U} \cap \mathcal{Y})$ is either $[-1, 1] \times -1, 0] \times \{0\}$ or $[-1, 1] \times \{0\} \times [0, 1] \times \{0\}$.

Firstly, (1.5.6) and (1.5.8) imply that at least one of the two sets in this claim must contain a point from $\Psi(\mathcal{U} \cap \mathcal{Y})$. Let in this spirit $\mathcal{A}$ be any of the two sets $[-1, 1] \times -1, 0] \times \{0\}$ or $[-1, 1] \times \{0\} \times [0, 1] \times \{0\}$, which contains at least one point from $\Psi(\mathcal{U} \cap \mathcal{Y})$. Both the sets $\Psi(\mathcal{U} \cap \mathcal{Y})$ and $\Psi(\mathcal{U} \cap \mathcal{Y}^o)$ are open in $\Psi(\mathcal{U} \cap \partial \Lambda)$ and, consequently, the – mutually disjoint – sets $\Psi(\mathcal{U} \cap \mathcal{Y}) \cap \mathcal{A}$ and $\Psi(\mathcal{U} \cap \partial \Lambda \setminus \mathcal{Y})^o \cap \mathcal{A}$ are open in $\mathcal{A}$. Since by (1.5.8) no points from $\Psi(\mathcal{U} \cap \mathcal{Y})$ can lie in $\mathcal{A}$, we have, according to (1.5.2) and (1.5.6), the identity

$$
(\Psi(\mathcal{U} \cap \mathcal{Y}) \cap \mathcal{A}) \cup (\Psi(\mathcal{U} \cap \partial \Lambda \setminus \mathcal{Y})^o \cap \mathcal{A}) = \Psi(\mathcal{U} \cap \partial \Lambda) \cap \mathcal{A} = \mathcal{A}.
$$

(1.5.9)

$\mathcal{A}$ is connected, therefore (1.5.9) can only be true if $\Psi(\mathcal{U} \cap \partial \Lambda \setminus \mathcal{Y})^o \cap \mathcal{A} = \emptyset$. This means: if any of the two sets $[-1, 1] \times -1, 0] \times \{0\}$ or $[-1, 1] \times \{0\} \times [0, 1] \times \{0\}$ contains a point from $\Psi(\mathcal{U} \cap \mathcal{Y})$, then it is a subset of $\Psi(\mathcal{U} \cap \mathcal{Y})$. But then the other cannot contain a point from $\Psi(\mathcal{U} \cap \mathcal{Y})$, because in this case it also would be a subset of $\Psi(\mathcal{U} \cap \mathcal{Y})$, which cannot be true in view of (1.5.8) and our supposition that $\partial \Lambda \setminus \mathcal{Y}$ is the closure of its interior. This proves the claim (*).

Together with (1.5.5) this gives $\Psi(\mathcal{U} \cap (\Lambda \cup \mathcal{Y})) = K_+ \cup \Sigma_0$ or $\Psi(\mathcal{U} \cap (\Lambda \cup \mathcal{Y})) = K_- \cup (-\Sigma_0)$. In the first case we have finished the proof, in the second we compose $\Psi$ with a reflection at the $x$-$z$-plane to conclude.

**Corollary 1.5.5.** If $\Lambda \subset \mathbb{R}^3$ is a Lipschitzian polyhedron and $\mathcal{Y} \cap (\partial \Lambda \setminus \mathcal{Y})$ is a finite union of line segments, then $\Lambda \cup \mathcal{Y}$ is regular.

**Remark 1.5.6.** Theorem 1.5.4 makes precise an old suggestion of Gröger, see [Grö89, Remark 1]. Unfortunately, the given intrinsic characterization is restricted to the dimensions 2 and 3, because there is no analogue of the Tukia theorem in dimensions above 2. Nevertheless, the by far most important cases concerning applications are covered.
1.6. Application to semilinear elliptic optimal control problems

In the subsequent we will employ the results of the previous sections, in particular Theorem 1.3.3, to derive necessary and sufficient optimality conditions for the following semilinear elliptic optimal control problem with pointwise state and control constraints and jumping boundary conditions:

\[
\begin{align*}
\text{(P)} & \quad \begin{cases}
\text{minimize} & J(y, u) := \int_{\Omega} L(x, y(x)) \, dx + \int_{\Gamma} l(x, y(x), u(x)) \, d\sigma \\
\text{subject to} & -\nabla \cdot a \nabla y + b(x, y) = f \quad \text{in } \Omega \\
& \partial_n y = u \quad \text{on } \Gamma \\
& y = 0 \quad \text{on } \partial \Omega \setminus \Gamma \\
& \text{and } u_{\min}(x) \leq u(x) \leq u_{\max}(x) \quad \text{a.e. on } \Gamma \\
& g(x, y(x)) \leq 0 \quad \text{for all } x \in \overline{\Omega}.
\end{cases}
\end{align*}
\]

As already mentioned in the introduction, necessary and sufficient optimality conditions for semilinear elliptic control problems have been addressed by numerous authors before (cf. for instance [Cas93, CTU00, CdlRT] and the references therein). In particular, we refer to the recent contribution of Casas et al. [CdlRT], where an optimal control problem is analyzed that is very similar to (P), but does not contain mixed boundary conditions. However, as we will see in the following, with the results of Section 1.3 at hand, the analysis of [CdlRT] can easily be adapted to (P).

Note that mixed boundary conditions play an important role in various applications. A typical example is the optimal control of an electric potential in a conducting material by adjusting the direct current inducing the potential. In the stationary case, this problem is modelled by the electrostatic equation, an elliptic PDE with homogeneous Neumann boundary conditions at isolated surfaces, homogeneous Dirichlet conditions at the anode and inhomogeneous Neumann boundary conditions at the cathode, where the control enters the system (see for instance [DHK03]). Hence, the arising problem is covered by the general problem (P). We point out that state-constrained optimal control problems with mixed boundary conditions and distributed control can be discussed analogously to the following investigation of (P). However, to keep the discussion concise, we do not consider distributed controls here.

In addition to Assumption 1.3.2 for the coefficient function $a$, we require the following conditions to be satisfied by the quantities in (P):

**Assumption 1.6.1.** The domain $\Omega \subset \mathbb{R}^d$, $d \leq 4$, is a bounded Lipschitz domain, $\Gamma \subset \partial \Omega$ is an open part of its boundary and $\partial \Omega \setminus \Gamma$ has positive measure. Moreover, $\Omega \cup \Gamma$ is regular in the sense of Gröger (cf. Theorems 1.5.2 and 1.5.4 for the two and three dimensional case). The function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable w.r.t. the second variable and monotone increasing, i.e., $\frac{\partial b}{\partial y}(x, y) \geq 0$ a.e. in $\Omega$. Furthermore, there is an $s > d/2$ such that

\[
b(\cdot, 0) \in L^s(\Omega) \quad \text{and} \quad f \in L^s(\Omega).
\]

Moreover, for all $M > 0$ there is a constant $C_{b,M} > 0$, such that

\[
\begin{align*}
\left| \frac{\partial b}{\partial y}(x, y_1) \right| + \left| \frac{\partial^2 b}{\partial y^2}(x, y_1) \right| & \leq C_{b,M} \quad \text{and} \\
\left| \frac{\partial^2 b}{\partial y^2}(x, y_2) - \frac{\partial^2 b}{\partial y^2}(x, y_1) \right| & \leq C_{b,M} \left| y_2 - y_1 \right|
\end{align*}
\]
for almost all \( x \in \Omega \) and all \( y_1, y_2 \in \mathbb{R} \) with \( |y_1|, |y_2| \leq M \).

**Assumption 1.6.2.** The function \( l : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function of class \( C^2 \) w.r.t. the second and third variables. In addition, \( l \) is convex w.r.t. the third variable. Moreover, \( l(\cdot, 0, 0) \in L^1(\Gamma) \) and for all \( M > 0 \) there exist a constant \( C_{l,M} > 0 \) and a function \( \psi_{l,M} \in L^2(\Gamma) \) with

\[
\frac{\partial l}{\partial y}(x, y_1, u_1) + \frac{\partial l}{\partial u}(x, y_1, u_1) \leq \psi_{l,M}(x), \quad |D^2_{(y,u)}l(x, y_1, u_1)| \leq C_{l,M} \nonumber \tag{1.6.1}
\]

for almost all \( x \in \Gamma \) and all \( |y_1|, |y_2|, |u_1|, |u_2| \leq M \). Here, \( D^2_{(y,u)}l \) denotes the Hessian of \( l \) w.r.t. \( (y, u) \). Furthermore, \( L : \Omega \times \mathbb{R} \to \mathbb{R} \) fulfills analogous conditions, i.e. it is of class \( C^2 \) w.r.t. the second variable, \( L(\cdot, 0, 0) \in L^1(\Omega) \) and for all \( M > 0 \) there exist \( C_{L,M} > 0 \) and \( \psi_{L,M} \in L^2(\Omega) \) with

\[
\frac{\partial L}{\partial y}(x, y_1) \leq \psi_{L,M}(x), \quad \frac{\partial^2 L}{\partial y^2}(x, y_1) \leq C_{L,M} \nonumber \tag{1.6.2}
\]

for almost all \( x \in \Omega \) and all \( |y_1|, |y_2| \leq M \).

**Assumption 1.6.3.** The bounds in the control constraints satisfy \( u_{\min}, u_{\max} \in L^\infty(\Gamma) \) with \( u_{\min}(x) < u_{\max}(x) \) a.e. in \( \Omega \). Moreover, \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is continuous and twice continuously differentiable w.r.t. the second variable. In addition, \( \frac{\partial g}{\partial y} \) and \( \frac{\partial^2 g}{\partial y^2} \) are continuous on \( \Omega \times \mathbb{R} \), and \( g(x, 0) < 0 \) is satisfied on \( \partial \Omega \setminus \Gamma \).

Note that the last conditions in Assumption 1.6.3 allow for the existence of a Slater point, which is essential for the derivation of first-order necessary conditions (see Assumption 1.6.16 below).

### 1.6.1. Discussion of the state equation

We start the discussion of (P) with the analysis of the state equation, i.e.

\[
-\nabla \cdot a \nabla y + b(x, y) = f \quad \text{in } \Omega \nonumber \tag{1.6.3}
\]

\[
\partial_n y = u \quad \text{on } \Gamma \nonumber
\]

\[
y = 0 \quad \text{on } \partial \Omega \setminus \Gamma. \nonumber
\]

**Definition 1.6.4.** Let \( q \in [2, \infty] \) and \( s, r \in \mathbb{R} \) satisfy \( s > 1 \) and let \( r > 1 \), if \( d = q = 2 \), and \( s \geq dq/(d+q) \) and \( r \geq (d-1)q/d \), otherwise. Moreover, let \( \varphi \in L^s(\Omega) \) and \( \psi \in L^r(\Gamma) \) be given. Then we denote the elements of \( W^{-1,s}_\Gamma(\Omega) \), associated to \( \varphi \) and \( \psi \), by \( \hat{\varphi} \) and \( \hat{\psi} \), i.e.

\[
\langle \hat{\varphi}, w \rangle_{W^{-1,s}_\Gamma(\Omega)} := \int_\Omega \varphi \, w \, dx, \quad \langle \hat{\psi}, w \rangle_{W^{-1,s}_\Gamma(\Omega)} := \int_\Gamma \psi \, w \, d\sigma, \quad w \in W^{1,q}_\Gamma(\Omega). \nonumber \tag{1.6.4}
\]

Since embedding and trace theorems guarantee \( w \in L^s(\Omega) \) and \( \tau \, w \in L^r(\Gamma) \), if \( w \in W^{1,q}_\Gamma(\Omega) \), the integrals in (1.6.4) are finite and, hence, \( \hat{\varphi} \) and \( \hat{\psi} \) are well defined.
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**Definition 1.6.5.** Suppose that \( f \in L^s(\Omega) \), \( s > 2d/(d + 2) \), and \( u \in L^r(\Gamma) \), \( r > (2d - 2)/d \). Then a function \( y \in W^{-1,2}_r(\Omega) \cap L^\infty(\Omega) \) is said to be a solution of (1.6.3), if it fulfills the operator equation

\[
-\nabla \cdot a \nabla y + \tilde{b}(y) = \tilde{f} + \tilde{u} \quad \text{in} \quad W^{-1,2}_r(\Omega),
\]

where \( \tilde{f}, \tilde{u} \in W^{-1,2}_r(\Omega) \) are defined according to Definition 1.6.4 and \( \tilde{b} : L^\infty(\Omega) \to W^{-1,2}_r(\Omega) \) is analogously given by

\[
(\tilde{b}(y), w)_{W^{-1,2}_r(\Omega)} := \int_\Omega b(x, y(x)) w(x) \, dx, \quad w \in W^{-1,2}_r(\Omega).
\]

Note that, due to Assumption 1.6.1, the Nemyzki operator \( \Phi_b(y) := b(\cdot, y(\cdot)) \) is continuous from \( L^\infty(\Omega) \) to \( L^s(\Omega) \), \( s > 2d/(d + 2) \), so \( \tilde{b} \) is well defined.

**Theorem 1.6.6.** Let \( f \in L^s(\Omega) \) with \( s > d/2 \) and \( u \in L^r(\Gamma) \) with \( r > d - 1 \). Then, under Assumption 1.6.1, there exists a unique solution of (1.6.3) in the sense of Definition 1.6.5. Moreover, there is an \( \alpha > 0 \) such that this solution belongs to \( C^{\alpha}(\Omega) \).

**Proof.** The existence of a unique solution in \( W^{-1,2}_r(\Omega) \cap L^\infty(\Omega) \) is standard (cf. for instance [Cas93] or [AR97] and the references therein). For convenience of the reader, we recall the main arguments. First, one considers a modified nonlinearity given by

\[
b_k(y) := \begin{cases} 
    b(k), & \text{if } y > k, \\
    b(y), & \text{if } -k \leq y \leq k, \\
    b(-k), & \text{if } y < -k,
\end{cases}
\]

with some \( k > 0 \). For a nonlinearity of this form, Browder and Minty’s theorem for monotone operators immediately implies the existence of a unique solution in \( W^{-1,2}_r(\Omega) \). Then a classical argument in the spirit of Stampacchia [Sta65] yields

\[
\|y\|_{L^\infty(\Omega)} \leq c_\infty \left( \|f\|_{L^s(\Omega)} + \|u\|_{L^r(\Gamma)} + 1 \right)
\]

with a constant \( c_\infty \) independent of \( f \) and \( k \). It is easily verified that the mixed boundary conditions do not influence the analysis in [Sta65].

Hence, if we choose \( k \geq \|y\|_{L^\infty(\Omega)} \), then the solution of the truncated problem coincides with the one of (1.6.3). It remains to verify the Hölder continuity of \( y \), which follows from Theorem 1.3.3 together with a classical bootstrapping argument. To see this, rewrite (1.6.5) as

\[
(-\nabla \cdot a \nabla + 1)y = g
\]

with \( g := \tilde{f} + \tilde{u} - \tilde{b}(y) + \tilde{\gamma} \) and \( \tilde{\gamma} \) according to Definition 1.6.4. Due to \( f, b(y) \in L^s(\Omega), s > d/2 \), \( u \in L^r(\Gamma), r > d - 1 \), and \( y \in L^\infty(\Omega) \), Sobolev embedding theorems give that \( g \in W^{-1,q}_r(\Omega) \) for a \( q > d \). Therefore, Theorem 1.3.3 implies \( y \in C^{\alpha}(\Omega) \).

**Definition 1.6.7.** For the rest of this section, let \( s > d/2 \) and \( r > d - 1 \) be fixed, but arbitrary. Moreover, \( f \) is a fixed inhomogeneity in \( L^s(\Omega) \) (cf. Assumption 1.6.1). Based on Theorem 1.6.6, we introduce the control-to-state operator \( S : L^r(\Gamma) \to W^{-1,2}_r(\Omega) \cap C^{\alpha}(\Omega) \), mapping \( u \) to the solution of (1.6.3).

**Lemma 1.6.8.** Suppose that there is a sequence \( \{u_k\} \) converging weakly to \( u \) in \( L^r(\Gamma) \). Then \( S(u_k) \to S(u) \) in \( W^{-1,2}_r(\Omega) \cap C^{\alpha}(\Omega) \).
Proof. With Theorem 1.3.3 at hand, the arguments are standard (cf. for instance [CdLRT]). Nevertheless, let us recall the basic ideas. In all what follows we use the notation \( y_k := S(u_k) \) and \( y := S(u) \). The weak convergence of \( \{u_k\} \) implies the uniform boundedness of this sequence in \( L^r(\Gamma) \) giving in turn that \( \{y_k\} \) is uniformly bounded in \( C(\Omega) \). Hence, \( \{y_k\} \) and \( \{\Phi_b(y_k)\} \), with \( \Phi_b \) as defined above, converge weakly in \( L^s(\Omega) \) with \( s > d/2 \), to some \( z_\Phi \) and \( z_y \), respectively. Now define the sequence \( \{g_k\} \) in \( W^{-1,2}_r(\Omega) \) by \( g_k := \bar{f} + \bar{u}_k - \bar{b}(y_k) + \bar{y} \). Due to the compact embedding \( L^s(\Omega) \hookrightarrow W^{-1,2}_r(\Omega) \) and the compactness of the trace operator \( \tau_r : W^{-1,q}_r(\Omega) \to L^r(\Gamma) \), weak convergences of \( \{u_k\} \), \( \{y_k\} \), and \( \{\Phi_b(y_k)\} \) imply strong convergence of \( \{g_k\} \) in \( W^{-1,q}_r(\Omega) \) to \( g := \bar{f} + \bar{u} - \bar{z}_\Phi + \bar{z}_y \), where \( \bar{z}_\Phi \) and \( \bar{z}_y \) again denote the elements in \( W^{-1,q}_r(\Omega) \) associated to \( z_\Phi \) and \( z_y \), respectively. Now consider again the auxiliary equation (1.6.6) with \( g_k \) as inhomogeneity. Theorem 1.3.3 then implies

\[
y_k \to \eta := (-\nabla \cdot a \nabla + 1)^{-1} g \quad \text{in} \quad W^{1,2}_r(\Omega) \cap C^\alpha(\Omega).
\]

This in particular guarantees \( y_k \to \eta \) in \( L^\infty(\Omega) \) and, hence, \( \Phi_b(y_k) \to \Phi_b(\eta) \) in \( L^s(\Omega) \), \( s > d/2 \), as well as \( \bar{y}_k \to \bar{\eta} \) and \( \bar{b}(y_k) \to \bar{b}(\eta) \) in \( W^{-1,2}_r(\Omega) \). Consequently, \( \eta \) is the solution of (1.6.3) associated to \( u \), which implies \( y_k \to \bar{y} \) in \( W^{1,2}_r(\Omega) \cap C^\alpha(\Omega) \). \( \square \)

Now, we turn to the linearized version of (1.6.3). Given a \( \bar{y} \in L^\infty(\Omega) \), the linearized state equation reads as

\[
-\nabla \cdot a \nabla \bar{y} + \bar{b}'(\bar{y}) \bar{y} = \bar{h} \quad \text{in} \quad W^{-1,2}_r(\Omega),
\]

where \( \bar{h} \in W^{-1,2}_r(\Omega) \) and \( \bar{b}'(\bar{y}) : W^{-1,2}_r(\Omega) \to W^{-1,2}_r(\Omega) \) is defined by

\[
\langle \bar{b}'(\bar{y}) \bar{y}, w \rangle_{W^{-1,2}_r(\Omega)} := \int_\Omega \frac{\partial b}{\partial y}(x, \bar{y}(x)) \bar{y}(x) w(x) \ dx, \quad w \in W^{1,2}_r(\Omega).
\]

Note that Assumption 1.6.1 implies \( \frac{\partial b}{\partial y}(x, \bar{y}(x)) \in L^\infty(\Omega) \) and that, due to the monotonicity of \( b, \frac{\partial b}{\partial y}(x, \bar{y}(x)) \geq 0 \) holds true a.e. in \( \Omega \). Hence, an immediate consequence of Corollary 1.3.6 is the following

**Lemma 1.6.9.** Let \( \bar{y} \in L^\infty(\Omega) \) be given. For every \( \bar{h} \in W^{-1,2}_r(\Omega) \) there is a unique solution \( y \in W^{1,2}_r(\Omega) \) of (1.6.7). Furthermore, if \( \bar{h} \in W^{-1,q}_r(\Omega) \) for some \( q > d \), then \( y \in W^{1,2}_r(\Omega) \cap C^\alpha(\Omega) \) for some \( \alpha > 0 \).

In view of Assumption 1.6.1, the Nemyzki operator \( \Phi_b(y) = b(\cdot, y(\cdot)) \) clearly is twice continuously Fréchet differentiable in \( L^\infty(\Omega) \). Thus, together with Lemma 1.6.9, the implicit function theorem implies the following result (for a detailed proof see for instance [CM02a]).

**Theorem 1.6.10.** Under Assumption 1.6.1 the control-to-state operator \( S \) is twice continuously Fréchet differentiable from \( L^r(\Gamma) \) to \( W^{1,2}_r(\Omega) \cap C^\alpha(\Omega) \). Its first derivative at \( \bar{u} \in L^r(\Gamma) \) in direction \( h \in L^r(\Gamma) \) solves

\[
-\nabla \cdot a \nabla \eta + \bar{b}'(\bar{y}) \eta = \bar{h} \quad \text{in} \quad W^{-1,2}_r(\Omega),
\]

where \( \bar{y} = S(\bar{u}) \) and \( \bar{h} \) denotes the element of \( W^{-1,2}_r(\Omega) \) associated to \( h \). Furthermore, \( \eta = S''(\bar{u})[h_1, h_2], h_i \in L^r(\Gamma), i = 1, 2, \) is the solution of

\[
-\nabla \cdot a \nabla \eta + \bar{b}'(\bar{y}) \eta = -\bar{b}''(\bar{y}) y_1 y_2 \quad \text{in} \quad W^{-1,2}_r(\Omega)
\]

with \( y_i = S''(\bar{u}) h_i, i = 1, 2, \) i.e. the solution of (1.6.8), and

\[
\langle \bar{b}''(\bar{y}) y_1 y_2, w \rangle_{W^{-1,2}_r(\Omega)} := \int_\Omega \frac{\partial^2 b}{\partial y^2}(x, \bar{y}(x)) y_1(x) y_2(x) w(x) \ dx, \quad w \in W^{1,2}_r(\Omega).
\]
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Note that, due to \( h_i \in L^r(\Gamma), \ i = 1, 2 \), Lemma 1.6.9 yields \( y_i \in L^\infty(\Omega) \). In addition, Assumption 1.6.1 implies \( \frac{\partial^2 \tilde{b}}{\partial y_i \partial y_j}(\cdot, \tilde{y}(\cdot)) \in L^r(\Omega) \), so \( \tilde{b}''(\tilde{y})y_1y_2 \) is well defined.

### 1.6.2. An adjoint equation involving measures

It is well known that the Lagrange multipliers associated to pointwise state constraints are in general only regular Borel measures, which appear as inhomogeneity in the adjoint equation (cf. for instance [Cas93]). Before we are in the position to discuss such an equation, the set \( D_q \), introduced in Corollary 1.3.7, has to be investigated in more detail. Recall that \( D_q \) denotes the domain of the maximal restriction of \(-\nabla \cdot a \nabla + 1\) to \( W_\Gamma^{-1,q}(\Omega) \). Throughout this section we use the abbreviation \( A_V := -\nabla \cdot a \nabla + V \), where \( V \in L^\infty(\Omega) \) denotes a given, non-negative function. Since \(|\partial \Omega \setminus \Gamma| > 0\) by Assumption 1.6.1, \( A_V : D_q \to W_\Gamma^{-1,q}(\Omega) \) is continuously invertible.

**Lemma 1.6.11.** The space \( (D_q, \| \cdot \|_{D_q} := \| -\nabla \cdot a \nabla \cdot \|_{W_\Gamma^{-1,q}(\Omega)} \) is a Banach space. Moreover, \( W_\Gamma^{1,q}(\Omega) \) is dense in \( D_q \).

**Proof.** Clearly, since \( A_1 := -\nabla \cdot a \nabla + 1 \) is linear and injective, \( \| \cdot \|_{D_q} \) indeed represents a norm on \( D_q \) and is equivalent to the graph norm on \( D_q \). Moreover, the completeness of \( D_q \) follows from the closedness of \( A_1 : D_q \to W_\Gamma^{-1,q}(\Omega) \), which holds since \( A_1 \) is continuously invertible by Corollary 1.3.6. In order to show the density of \( W_\Gamma^{1,q}(\Omega) \), consider the restriction of \( A_1 \) to functions in \( W_\Gamma^{1,q}(\Omega) \), denoted by \( \tilde{A}_1 : W_\Gamma^{1,q}(\Omega) \to W_\Gamma^{-1,q}(\Omega) \). Due to the continuous invertibility of \( A_1 \), the space \( W_\Gamma^{1,q}(\Omega) \) is dense in \( D_q \), if \( R(\tilde{A}_1) \) is dense in \( W_\Gamma^{-1,q}(\Omega) \), where \( R(\tilde{A}_1) \) denotes the range of \( \tilde{A}_1 \). Since \( R(\tilde{A}_1) = \ker(\tilde{A}_1^*)^\perp \), \( R(\tilde{A}_1) \) is dense in \( W_\Gamma^{-1,q}(\Omega) \), iff \( \ker(\tilde{A}_1^*) = 0 \), where \( \tilde{A}_1^* \) is defined by

\[
\langle \tilde{A}_1^* v, w \rangle_{W_\Gamma^{-1,q}} := \int_\Omega (a^T \nabla v \cdot \nabla w + vw) \, dx, \quad v \in W_\Gamma^{1,q'}(\Omega), w \in W_\Gamma^{1,q}(\Omega).
\]

Clearly, \( \tilde{A}_1^* \) is injective due to the coercivity of the associated bilinear form, giving in turn the assertion. \( \square \)

Next, let us consider the following PDE

\[
\begin{align*}
-\nabla \cdot a^T \nabla p + V p &= \mu_\Omega & \text{in } \Omega \\
\partial_n p &= \mu_\Gamma & \text{on } \Gamma \\
p &= 0 & \text{on } \partial \Omega \setminus \Gamma,
\end{align*}
\]  

(1.6.10)

where \( \Omega \) and \( a \) are supposed to fulfill the assumptions of Theorem 1.3.3. Moreover, \( V \) is a fixed, but arbitrary non-negative function in \( L^\infty(\Omega) \). Furthermore, the inhomogeneity \( \mu \) is given in \( M(\overline{\Omega}) \) which is the space of regular Borel measures that can be identified with the dual of \( C(\overline{\Omega}) \) by means of the Riesz representation theorem. Moreover, \( \mu_\Omega \) and \( \mu_\Gamma \) denote the restrictions of \( \mu \) to \( \Omega \) and \( \Gamma \), respectively. In view of formal integration by parts, we define solutions to (1.6.10) as follows.

**Definition 1.6.12.** A function \( p \in W_\Gamma^{1,q'(\Omega)} \), \( q' = q/(q - 1) < d/(d - 1) \), is said to be a solution of (1.6.10), if the equation

\[
\int_\Omega (a^T \nabla p \cdot \nabla w + V pw) \, dx = \langle \mu, w \rangle_{M(\overline{\Omega})} \quad \forall w \in W_\Gamma^{1,q}(\Omega)
\]

(1.6.11)

is satisfied.
Lemma 1.6.13. Let \( V \in L^\infty(\Omega) \), \( V(x) \geq 0 \) a.e. in \( \Omega \) and \( \mu \in \mathcal{M}(\overline{\Omega}) \) be given. Then there is a unique solution to (1.6.10) in the sense of Definition 1.6.12. This solution satisfies
\[
\|p\|_{W^{-1,q'}(\Omega)} \leq \gamma \|\mu\|_{\mathcal{M}(\overline{\Omega})}
\]
with a constant \( \gamma > 0 \) independent of \( \mu \).

Proof. The variational formulation (1.6.11) is equivalent to
\[
\langle \hat{A}_V^* p, w \rangle_{W^{-1,q'}(\Omega)} = \langle p, \hat{A}_V w \rangle_{W^{-1,q'}(\Omega)} = \langle \mu, w \rangle_{\mathcal{M}(\overline{\Omega})} \quad \forall w \in W^{-1,q}(\Omega),
\]
where, as above, \( \hat{A}_V : W^{-1,q}(\Omega) \rightarrow W^{-1,q}(\Omega) \) denotes the restriction of \( A_V \) to \( W^{-1,q}(\Omega) \). Since \( W^{-1,q}(\Omega) \) is dense in \( D_q \) by Lemma 1.6.11, we see that (1.6.12) is equivalent to
\[
\langle p, A_V w \rangle_{W^{-1,q'}(\Omega)} = \langle \mu, w \rangle_{\mathcal{M}(\overline{\Omega})} \quad \forall w \in D_q \quad \Leftrightarrow \quad A_V^* p = \hat{\mu} \quad \text{in} \ D_q^*.
\]
Here, \( D_q^* \) denotes the dual to \( D_q \). Moreover, \( \hat{\mu} \) is the element in \( D_q^* \) associated to \( \mu \in \mathcal{M}(\overline{\Omega}) \) by
\[
\langle \hat{\mu}, w \rangle_{D_q} = \langle \mu, w \rangle_{\mathcal{M}(\overline{\Omega})} = \int_{\overline{\Omega}} w(x) \, d\mu \quad \forall w \in D_q,
\]
which is well defined due to \( D_q \hookrightarrow C(\overline{\Omega}) \) by Theorem 1.3.3. Now, Corollary 1.3.6 guarantees \( (A_V^*)^{-1} \in \mathcal{B}(D_q^*, W^{-1,q}(\Omega)) \). Hence, due to the equivalence of \( A_V^* p = \hat{\mu} \) to (1.6.11), there is a unique solution in the sense of the above definition and it holds
\[
\|p\|_{W^{-1,q'}(\Omega)} \leq \|(A_V)^{-1}\|_{\mathcal{B}(W^{-1,q}(\Omega), D_q)} \|\hat{\mu}\|_{D_q^*}
\]
\[
\leq \gamma \sup_{w \in D_q, w \neq 0} \frac{\langle \hat{\mu}, w \rangle_{D_q}}{\|w\|_{D_q}} \leq \gamma \sup_{w \in C(\overline{\Omega}), w \neq 0} \frac{\langle \mu, w \rangle_{\mathcal{M}(\overline{\Omega})}}{\|w\|_{C(\overline{\Omega})}} = \gamma \|\mu\|_{\mathcal{M}(\overline{\Omega})},
\]
which gives the assertion. \( \square \)

1.6.3. Necessary and sufficient optimality conditions

With the above results, the analysis of this section is along the lines of [CdIRT]. Hence, we shorten the description, if the arguments are analogous to the ones in [CdIRT]. First, let us introduce the reduced objective functional \( j : L^\infty(\Gamma) \rightarrow \mathbb{R} \) and the Lagrange function \( \mathcal{L} : L^\infty(\Gamma) \times \mathcal{M}(\overline{\Omega}) \rightarrow \mathbb{R} \) by
\[
j(u) := J(S(u), u) = \int_\Omega L(x, S(u)(x)) \, dx + \int_{\Gamma} l(x, S(u)(x), u(x)) \, d\sigma,
\]
(1.6.13)
\[
\mathcal{L}(u, \mu) := j(u) + \int_{\overline{\Omega}} g(x, S(u)(x)) \, d\mu.
\]
(1.6.14)

**Lemma 1.6.14.** Let \( \mu \in \mathcal{M}(\overline{\Omega}) \) be arbitrary. Then the Lagrange function is twice continuously Fréchet differentiable w.r.t. \( u \) from \( L^\infty(\Gamma) \) to \( \mathbb{R} \). If \( \mu_{\partial \Omega \setminus \Gamma} = 0 \), then its first partial derivative at \( \bar{u} \) in direction \( h \in L^\infty(\Gamma) \) is given by
\[
\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \mu)h = \int_{\Gamma} \left[ \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{u}(x)) + p(x) \right] h(x) \, d\sigma,
\]
(1.6.15)
where \( \bar{y} = S(\bar{u}) \) and \( p \in W^{1,q}_r(\Omega) \) solves
\[
-\nabla \cdot a^T \nabla p + \frac{\partial b}{\partial y}(\bar{y})p = \frac{\partial L}{\partial y}(\bar{y}) + \frac{\partial q}{\partial y}(\bar{y})\mu \quad \text{in } \Omega
\]
\[
\partial_n p = \frac{\partial l}{\partial y}(\bar{y},\bar{u}) + \frac{\partial q}{\partial y}(\bar{y})\mu \quad \text{on } \Gamma \quad (1.6.16)
\]
\[
p = 0 \quad \text{on } \partial \Omega \setminus \Gamma
\]
in the sense of Definition 1.6.12. Moreover, the second derivative of \( L \) at \( \bar{u} \) in directions \( h_1, h_2 \in L^\infty(\Gamma) \) is given by
\[
\frac{\partial^2 L}{\partial u^2}(\bar{u},\mu)h_1h_2 = \int_\Omega \left[ \frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u})y_1y_2 - p \frac{\partial^2 b}{\partial y^2}(x,\bar{y})y_1y_2 \right] dx
\]
\[
+ \int_\Gamma \left[ \frac{\partial^2 l}{\partial y^2}(x,\bar{y},\bar{u})y_1y_2 + \frac{\partial^2 l}{\partial y\partial u}(x,\bar{y},\bar{u})(y_1h_2 + y_2h_1) \right. \\
\left. + \frac{\partial^2 l}{\partial u^2}(x,\bar{y},\bar{u})h_1h_2 \right] d\sigma + \int_\Pi \frac{\partial^2 g}{\partial y^2}(x,\bar{y},\bar{u})y_1y_2 d\mu
\]
with \( y_i = S'(\bar{u})h_i, \ i = 1,2 \).

**Proof.** The arguments are standard (cf. \cite{CM02a}). Nevertheless, we shortly describe the derivation of (1.6.15) to see how the associated theory is influenced by the analysis of the adjoint equation as carried out in Section 1.6.2. The differentiability of \( L \) is an immediate consequence of Assumptions 1.6.1–1.6.3 and Theorem 1.6.10. Concerning the explicit form of \( \frac{\partial L}{\partial u} \), the chain rule yields
\[
\frac{\partial L}{\partial u}(\bar{u},\mu)h = \int_\Omega \frac{\partial L}{\partial y}(\bar{y},\bar{u})y dx
\]
\[
+ \int_\Gamma \left( \frac{\partial l}{\partial y}(\bar{y},\bar{u})y + \frac{\partial l}{\partial u}(\bar{y},\bar{u})h \right) d\sigma + \langle \mu, \frac{\partial q}{\partial y}(\bar{y})y \rangle_{M(\Pi)}
\]
(1.6.18)

with \( \bar{y} = S(\bar{u}) \) and \( y = \frac{\partial S}{\partial u}(\bar{u})h \), i.e. \( y \in D_q \) solves \( A'(\bar{y})y = -\nabla \cdot a \nabla y + \hat{b}'(\bar{y})y = \hat{h} \) in \( W^{-1,q}_r(\Omega) \). According to the analysis of Section 1.6.2, the variational formulation of (1.16) is equivalent to
\[
\langle A'(\bar{y})p, v \rangle_{D_q^*} = \int_\Omega \frac{\partial L}{\partial y}(\bar{y},\bar{u})v dx + \int_\Gamma \frac{\partial l}{\partial y}(\bar{y},\bar{u})v d\sigma + \langle \mu, \frac{\partial q}{\partial y}(\bar{y})v \rangle_{M(\Pi)} \quad \forall \ v \in D_q
\]
(cf. the proof of Lemma 1.6.13). Choosing \( v = y \) as test function yields
\[
\int_\Omega \frac{\partial L}{\partial y}(\bar{y},\bar{u})y dx + \int_\Gamma \frac{\partial l}{\partial y}(\bar{y},\bar{u})y d\sigma + \langle \mu, \frac{\partial q}{\partial y}(\bar{y})y \rangle_{M(\Pi)}
\]
\[
= \langle A'(\bar{y})p, y \rangle_{D_q^*} = \langle A'(\bar{y})y, p \rangle_{W^{-1,q}_r(\Omega)} = \langle \hat{h}, p \rangle_{W^{-1,q}_r(\Omega)} = \int_\Gamma h p d\sigma.
\]
Inserting this into (1.6.18) gives (1.6.15). Finally, (1.6.17) follows from an analogous argument. \( \square \)
Definition 1.6.15. A function \( u \in L^\infty(\Gamma) \) is called feasible for (P), if it fulfills
\[
\begin{align*}
    u_{\min}(x) \leq u(x) \leq u_{\max}(x) & \quad \text{a.e. on } \Gamma \\
    g(x, S(u)(x)) \leq 0 & \quad \text{for all } x \in \overline{\Omega}.
\end{align*}
\]

Under Assumptions 1.6.1–1.6.3 there is at least one (global) solution of (P), provided that a feasible function exists (see [CM02, Thm. 8] for the proof).

Let us now turn to necessary optimality conditions for (P). It is well known that certain constraint qualifications are required to discuss pointwise inequality constraints on the state as they occur in (P). Here, we rely on the following linearized Slater condition.

Definition 1.6.16. Let \( \tilde{u} \in L^\infty(\Gamma) \) be feasible for (P). We say that the linearized Slater condition is fulfilled at \( \tilde{u} \), if there exists a function \( \tilde{u} \in L^\infty(\Gamma) \), such that
\[
\begin{align*}
    u_{\min}(x) \leq \tilde{u}(x) & \leq u_{\max}(x) \quad \text{a.e. on } \Gamma, \quad (1.6.19) \\
    g(x, \tilde{g}(x)) + \frac{\partial g}{\partial y}(x, \tilde{y}(x))y(x) < 0 & \quad \text{for all } x \in \overline{\Omega}, \quad (1.6.20)
\end{align*}
\]
where \( \tilde{y} = S(\tilde{u}) \) and \( \tilde{y} = S'(\tilde{u})(\tilde{u} - \tilde{u}) \).

Note that, due to \( \tilde{y}|_{\partial \Omega \setminus \Gamma} = 0 \), (1.6.20) yields \( g(x, \tilde{g}(x)) = g(x, 0) < 0 \) for all \( x \in \partial \Omega \setminus \Gamma \) which is guaranteed by Assumption 1.6.3. With the existence and regularity results for the state and the adjoint equation obtained before, the theory of first-order necessary conditions for (P) is standard. For the corresponding theorem, we define the Hamiltonian associated to (P), denoted by \( H : \Gamma \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \):
\[
H(x, y, u, p) := l(x, y, u) + p(u - b(x, y)).
\]

The definition of \( H \) allows to formulate the first-order necessary conditions in form of Pontryagin’s principle. For the corresponding proof we refer to [BC95].

Theorem 1.6.17. Suppose that \( \tilde{u} \in L^\infty(\Gamma) \) is a local solution of (P) in the topology of \( L^\infty(\Gamma) \), i.e., \( j(\tilde{u}) \leq j(u) \) for all feasible \( u \) with \( \| u - \tilde{u} \|_{L^\infty(\Gamma)} \leq \varepsilon \). Furthermore, denote the state associated to \( \tilde{u} \) by \( \tilde{y} \in W^{1,2}_r(\Omega) \cap C^0(\Omega) \). Moreover, let Assumptions 1.6.1–1.6.3 hold and let the linearized Slater condition be satisfied at \( \tilde{u} \). Then there exist a function \( p \in W^{1,q}_r(\Omega) \) and a Borel measure \( \mu \in \mathcal{M}(\overline{\Omega}) \), such that the adjoint equation (1.6.16) is fulfilled in the sense of Definition 1.6.12 and it holds
\[
\int_{\overline{\Omega}} (v(x) - g(x, \tilde{y}(x)))d\mu(x) \leq 0 \quad \text{for all } v \in C(\overline{\Omega}) \text{ with } v(x) \leq 0 \forall x \in \overline{\Omega} \quad (1.6.21)
\]
\[
H(x, \tilde{y}(x), \tilde{u}(x), p(x)) = \min_{t \in U_{ad}(x, \tilde{u}(x))} H(x, \tilde{y}(x), t, p(x)) \quad \text{a.e. on } \Gamma, \quad (1.6.22)
\]
where \( U_{ad}(x, \tilde{u}(x)) := [\max\{u_{\min}(x), \tilde{u}(x) - \varepsilon\}, \min\{u_{\max}(x), \tilde{u}(x) + \varepsilon\}] \).

In [Cas93], it is shown that the Lagrange multiplier associated to the state constraints is concentrated in the Borel set \{ \( x \in \overline{\Omega} : g(x, \tilde{y}(x)) = 0 \) \}, such that \( \mu_{\partial \Omega \setminus \Gamma} = 0 \) since \( g(x, 0) < 0 \) on \( \partial \Omega \setminus \Gamma \) according to Assumption 1.6.3. Hence, we obtain homogeneous Dirichlet boundary conditions on \( \partial \Omega \setminus \Gamma \) also in the adjoint equation (1.6.16). The first-order necessary conditions in Theorem 1.6.17 can also be formulated in terms of the Lagrangian (see [CdlRT] for details). In all what follows, let \( \tilde{u} \) again be a fixed local optimum with associated state \( \tilde{y} \), adjoint state \( p \), and Lagrange multiplier \( \mu \) such that (1.6.21) and (1.6.22) are fulfilled. For the statement of second-order sufficient conditions accounting for strongly active sets, we have to restrict to
the two dimensional case, since the underlying analysis heavily relies on the assumption that $S : L^2(\Omega) \to C(\overline{\Omega})$ (see [CdlRT, Sections 4 and 6.3]). In view of Theorem 1.6.6, this is not fulfilled in the three and four dimensional case. We start with the definition of the critical cone associated to $\bar{u}$:

$$C(\bar{u}) := \{ h \in L^2(\Gamma) : h \text{ satisfies conditions (a), (b), and (c)}\},$$

where

$$h(x) \begin{cases} 
\geq 0, & \text{if } \bar{u}(x) = u_{\text{min}}(x), \\
\leq 0, & \text{if } \bar{u}(x) = u_{\text{max}}(x), \\
= 0, & \text{if } \frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), p(x)) = 0,
\end{cases}$$

and $y = S'(\bar{u})h$. Note that $y \in C(\overline{\Omega})$ in the two dimensional case. Moreover, the derivative of the Hamiltonian is given by $\frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), p(x)) = \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{u}(x)) + p$.

Now, we are in the position to state the second-order sufficient conditions for (P). With the above results, in particular Lemma 1.6.8, the corresponding proof is completely analogous to the one presented in [CdlRT].

**Theorem 1.6.18.** Let $d = 2$, let Assumptions 1.6.1–1.6.3 be satisfied and suppose that $\bar{u} \in L^\infty(\Gamma)$ with associated state $\bar{y} \in W^{1,2}_0(\Omega) \cap C^\alpha(\Omega)$ is feasible for (P). Moreover, let $p \in W^{1,d}_\Gamma(\Omega)$ and $\mu \in M(\Omega)$ exist such that (1.6.16), (1.6.21), and (1.6.22) are satisfied. In addition, it is assumed that there are two constants $\omega, \tau > 0$ with

$$\frac{\partial^2 l}{\partial u^2}(x, \bar{y}(x), \bar{u}(x)) \geq \omega \quad \text{if} \quad \left| \frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), p(x)) \right| \leq \tau, \quad \text{a.e. on } \Gamma \quad (1.6.23)$$

$$\frac{\partial^2 L}{\partial u^2}(\bar{u}, \mu) h^2 > 0 \quad \text{for all } h \in C(\bar{u}) \setminus \{0\}. \quad (1.6.24)$$

Then there exist $\varepsilon, \delta > 0$, such that

$$j(u) \geq j(\bar{u}) + \frac{\delta}{2} \| u - \bar{u} \|^2_{L^2(\Gamma)}$$

for all feasible $u \in L^\infty(\Gamma)$ with $\| u - \bar{u} \|_{L^\infty(\Gamma)} < \varepsilon$.

Note that, according to Lemma 1.6.14, $L$ is only continuously differentiable from $L^\infty(\Gamma)$ to $\mathbb{R}$. However, it is straightforward to see that Assumptions 1.6.1–1.6.3 ensure that $\frac{\partial^2 L}{\partial u^2}$ and $\frac{\partial^2 L}{\partial u \partial \mu}$ can be extended from $L^\infty(\Gamma)$ to $L^2(\Gamma)$ using (1.6.15) and (1.6.17). This extension is also used in (1.6.24). Note further that the sufficient conditions (1.6.23) and (1.6.24) are natural in the sense that they are comparatively close to the necessary optimality conditions (see [CdlRT, Remark 4.2] for details).

**Remark 1.6.19.** We point out that the second-order analysis can be extended to the three dimensional case if distributed controls are applied instead of boundary control, since $L^2(\Omega) \hookrightarrow W^{-1,q}_\Gamma(\Omega)$ and thus continuous states are obtained with controls in $L^2(\Omega)$ (see [CdlRT, Thm. 4.1]). Nevertheless, up to the authors’ best knowledge, there is no proof of second-order conditions accounting for strongly active sets in case of pointwise state constraints and boundary controls in three dimensions.
References (Article 1)


References (Article 1)


References (Article 1)


[HR08] R. Haller-Dintelmann and J. Rehberg: Maximal parabolic regularity for divergence operators including mixed boundary conditions, WIAS-Preprint no. 1288, 2008.


References (Article 1)


2. Maximal Parabolic Regularity for Divergence Operators Including Mixed Boundary Conditions

by Robert Haller-Dintelmann and Joachim Rehberg

Abstract

We show that elliptic second order operators $A$ of divergence type fulfill maximal parabolic regularity on distribution spaces, even if the underlying domain is highly non-smooth and $A$ is complemented with mixed boundary conditions. Applications to quasilinear parabolic equations with non-smooth data are presented.

2.1. Introduction

It is known that divergence operators fulfill maximal parabolic regularity on $L^p$ spaces – even if the underlying domain is non-smooth, the coefficients are discontinuous and the boundary conditions are mixed, see [Are97] and also [HR06]. This provides a powerful tool for the treatment of linear and nonlinear parabolic equations in $L^p$ spaces, see [Pru02, CL94, MERS04, HR06]. The only disadvantage of this concept is that the appearing Neumann conditions have to be homogeneous and that distributional right hand sides (e.g. surface densities) are not admissible. Confronted with these phenomena, it seems an adequate alternative to consider the equations in distribution spaces, what we will do in this paper. Pursuing this idea, one has, of course, to prove that the occurring elliptic operators satisfy parabolic regularity on those spaces in an appropriate sense.

In fact, we show that, under very mild conditions on the domain $\Omega$, the Dirichlet boundary part $\partial \Omega \setminus \Gamma$ and the coefficient function, elliptic divergence operators satisfy maximal parabolic regularity on a huge variety of spaces, among which are Sobolev, Besov and Lizorkin-Triebel spaces, provided that the differentiability index is between 0 and $-1$ (cf. Theorem 2.5.16).

We consider this as the first main result of this work, also interesting in itself. Up to now, the only existing results for mixed boundary conditions in distribution spaces (apart from the Hilbert space situation) are, to our knowledge, that of Groger [Grö92] and the recent one of Griepentrog [Gri05]. Concerning the Dirichlet case, compare [Byu07] and references therein.

Having this first result at hand, the second aim of this work is the treatment of quasilinear parabolic equations of the formal type

$$\left\{ \begin{array}{l}
(F(u))' - \nabla \cdot G(u)\mu \nabla u = R(t, u), \\
u(T_0) = u_0,
\end{array} \right. \quad (2.1.1)$$

combined with mixed, nonlinear boundary conditions:

$$\nu \cdot G(u)\mu \nabla u + b(u) = g \text{ on } \Gamma \quad \text{and} \quad u = 0 \text{ on } \partial \Omega \setminus \Gamma. \quad (2.1.2)$$

Let us point out some ideas, which will give a certain guideline for the paper: Our analysis is based on a regularity result for the square root $(-\nabla \cdot \mu \nabla)^{1/2}$ on $L^p$ spaces. It has already been
2. Maximal Regularity for Operators with Mixed Boundary Conditions

remarked in the introduction of [AT01] that estimates between \( \|(-\nabla \cdot \mu \nabla)^{1/2} f\|_p \) and \( \|\nabla f\|_p \) should provide powerful tools for the treatment of elliptic and parabolic problems involving divergence form operators. It seems, however, that this idea has not yet been developed to its full strength, cf. [ERS07, Ch. 5].

Originally, our strategy for proving maximal parabolic regularity for divergence operators on \( H^{-1,q}_\Gamma \) was to show an analog of the central result of [AT01], this time in case of mixed boundary conditions, namely that

\[
(-\nabla \cdot \mu \nabla + 1)^{-1/2} : L^q \rightarrow H^{1,q}_\Gamma
\]

provides a topological isomorphism for suitable \( q \). This would give the possibility of carrying over the maximal parabolic regularity, known for \( L^q \), to the dual of \( H^{1,q}_\Gamma \), because, roughly spoken, \((-\nabla \cdot \mu \nabla + 1)^{-1/2}\) commutes with the corresponding parabolic solution operator. Unfortunately, we were only able to prove the continuity of (2.1.3) within the range \( q \in [2, \infty[ \), due to a result of Duong and McIntosh [DM99], but did not succeed in proving the continuity of the inverse in general. Let us explicitly mention that the proof of the isomorphism property of (2.1.3) would be a great achievement. In particular, this would allow here to avoid the localization procedure we had to introduce in Section 2.5 in order to prove maximal parabolic regularity, and to generalize our results to higher dimensions. The isomorphism property is known for the Hilbert space case \( L^2 \) (see [AKM06]) in case of mixed boundary conditions and even complex coefficients, but the proof fundamentally rests on the Hilbert space structure, so that we do not see a possibility of directly generalizing this to the \( L^p \) case.

It turns out, however, that (2.1.3) provides a topological isomorphism, if \( \Omega \cup \Gamma \) is the image under a volume-preserving and bi-Lipschitz mapping of one of Gröger’s model sets [Grö89], describing the geometric configuration in neighborhoods of boundary points of \( \Omega \). Thus, in these cases one may carry over the maximal parabolic regularity from \( L^q \) to \( H^{-1,q}_\Gamma \). Knowing this, we localize the linear parabolic problem, use the 'local' maximal parabolic information and interpret this again in the global context at the end. Interpolation with the \( L^p \) result then yields maximal parabolic regularity on the corresponding interpolation spaces.

Let us explicitly mention that the concept of Gröger’s regular sets, where the domain itself is a Lipschitz domain, seems adequate to us, because it covers many realistic geometries that fail to be domains with Lipschitz boundary. The price one has to pay is that the problem of optimal elliptic regularity becomes much more delicate and, additionally, trace theorems for this situation are scarcely to be found in the literature.

The strategy for proving that (2.1.1), (2.1.2) admit a unique local solution is as follows. We reformulate (2.1.1) into a usual quasilinear equation, where the time derivative directly affects the unknown function. Assuming additionally that the elliptic operator \(-\nabla \cdot \mu \nabla + 1 : H^{1,q}_\Gamma \rightarrow H^{-1,q}_\Gamma\) provides a topological isomorphism for a \( q \) larger than the space dimension \( d \), the existence and uniqueness results for abstract quasilinear equations of Prüss (see [Prü02], see also [CL94]) apply to the resulting quasilinear parabolic equation. The detailed discussion how to assure all requirements of [Prü02], including the adequate choice of the Banach space, is presented in Section 6. The crucial point is that the linear elliptic operator which corresponds to the initial value satisfies maximal parabolic regularity, which has been proved before. Let us further emphasize that the presented setting allows for coefficient functions that really jump at hetero interfaces of the material and permits mixed boundary conditions, as well as domains which do not possess a Lipschitz boundary, see Section 2.7. It is well known that this is required when modelling real world problems, see e.g. [Som56, CJ88] for problems from thermodynamics or [FGR90, BHR05] concerning biological models. Last but not least, heterostructures are the determining features of many fundamental effects in semiconductors, see for instance [Sel84, BHK03, Kap99].
One further advantage is that nonlinear, nonlocal boundary conditions are admissible in our concept, despite the fact that the data is highly non-smooth, compare [Ama88]. The calculus of maximal parabolic $L^p([T_0, T]; X)$ regularity is preferable to the concept of Hölder continuity in time, because it allows for reaction terms $R$ which discontinuously depend on time. This is important in many examples (see [UT01, HT99, KRS05]), in particular in the control theory of parabolic equations. Alternatively, the reader should think e.g. of a manufacturing process for semiconductors, where light is switched on/off at a sharp time point and, of course, parameters in the chemical process then change abruptly. It is remarkable that, nevertheless, the solution is Hölder continuous simultaneously in space and time, see Corollary 2.6.16 below.

We finish these considerations by looking at the special case of semilinear problems. It turns out that here satisfactory results may be achieved even without the additional continuity condition on $-\nabla \cdot \mu \nabla + 1$ mentioned above, see Corollary 2.6.17.

In Section 2.7 we give examples for geometries, Dirichlet boundary parts and coefficients in three dimensions for which our additional supposition, the isomorphy $-\nabla \cdot \mu \nabla + 1 : H^1_{\Gamma \cdot q} \rightarrow H^{-1}_{\Gamma \cdot q}$ really holds for a $q > d$. In Subsection 2.7.3 we take a closer look at the special geometry of two crossing beams, which provides a geometrically easy example of a domain $\Omega$ that does not have a Lipschitz boundary and thus cannot be treated by former theories, but which is covered by our results.

Finally, some concluding remarks are given in Section 2.8.

### 2.2. Notation and general assumptions

Throughout this article the following assumptions are valid.

- $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain and $\Gamma$ is an open subset of $\partial \Omega$.
- The coefficient function $\mu$ is a Lebesgue measurable, bounded function on $\Omega$ taking its values in the set of real, symmetric, positive definite $d \times d$ matrices, satisfying the usual ellipticity condition.

**Remark 2.2.1.** Concerning the notions 'Lipschitz domain' and 'domain with Lipschitz boundary' (synonymous: strongly Lipschitz domain) we follow the terminology of Grisvard [Gri85], see also [Maz85].

For $\varsigma \in [0, 1]$ and $1 < q < \infty$ we define $H^{\varsigma, q}_{\Gamma \cdot q}(\Omega)$ as the closure of

$$C^{\infty}_{\Gamma \cdot q}(\Omega) := \{ \psi|_{\Omega} : \psi \in C^{\infty}(\mathbb{R}^d), \text{ supp}(\psi) \cap (\partial \Omega \setminus \Gamma) = \emptyset \} \quad (2.2.1)$$

in the Sobolev space $H^{\varsigma, q}(\Omega)$. Of course, if $\Gamma = \emptyset$, then $H^{\varsigma, q}_{\Gamma \cdot q}(\Omega) = H^{\varsigma, q}(\Omega)$ and if $\Gamma = \partial \Omega$, then $H^{\varsigma, q}_{\Gamma \cdot q}(\Omega) = H^{\varsigma, q}(\Omega)$. This last point follows from the fact that $\Omega$, as a Lipschitz domain, admits a continuous extension operator from $H^{1, q}_{\Gamma \cdot q}(\Omega)$ into $H^{1, q}_{\Gamma \cdot q}(\mathbb{R}^d)$, see [Giu94, Thm. 3.10].

Thus, the set $C^{\infty}(\Omega) := \{ \psi|_{\Omega} : \psi \in C^{\infty}(\mathbb{R}^d) \}$ is dense in $H^{1, q}(\Omega)$. Concerning the dual of $H^{\varsigma, q}_{\Gamma \cdot q}(\Omega)$, we have to distinguish between the space of linear and the space of anti-linear forms on this space. We define $H^{-\varsigma, q}_{\Gamma \cdot q}(\Omega)$ as the space of continuous, linear forms on $H^{\varsigma, q}_{\Gamma \cdot q}(\Omega)$ and $\tilde{H}^{-\varsigma, q}_{\Gamma \cdot q}(\Omega)$ as the space of anti-linear forms on $H^{\varsigma, q}_{\Gamma \cdot q}(\Omega)$ if $1/q + 1/q' = 1$. Note that $L^p$ spaces may be viewed as part of $\tilde{H}^{-\varsigma, q}_{\Gamma \cdot q}$ for suitable $\varsigma, q$ via the identification of an element $f \in L^p$ with the anti-linear form $H^{-\varsigma, q}_{\Gamma \cdot q} \ni \psi \mapsto \int_{\Omega} \Gamma \tilde{\psi} \, dx$.

If misunderstandings are not to be expected, we drop the $\Omega$ in the notation of spaces, i.e. function spaces without an explicitly given domain are to be understood as function spaces on $\Omega$. 


By $K$ we denote the open unit cube in $\mathbb{R}^d$, by $K-$ the lower half cube $K \cap \{x : x_d < 0\}$, by $\Sigma = K \cap \{x : x_d = 0\}$ the upper plate of $K-$ and by $\Sigma_0$ the left half of $\Sigma$, i.e. $\Sigma_0 = \Sigma \cap \{x : x_{d-1} < 0\}$.

As in the preceding paragraph, we will throughout the paper use $x, y, \ldots$ for vectors in $\mathbb{R}^d$, whereas the components of $x$ will be denoted by italics $x_1, x_2, \ldots, x_d$ or in three dimensions also by $x, y, z$.

If $B$ is a closed operator on a Banach space $X$, then we denote by $\text{dom}_X(B)$ the domain of this operator. $\mathcal{L}(X,Y)$ denotes the space of linear, continuous operators from $X$ into $Y$; if $X = Y$, then we abbreviate $\mathcal{L}(X)$. Furthermore, we will write $\langle \cdot, \cdot \rangle_X$ for the dual pairing of elements of $X$ and the space $X'$ of anti-linear forms on $X$.

Finally, the letter $c$ denotes a generic constant, not always of the same value.

### 2.3. Preliminaries

In this section we will properly define the elliptic divergence operator and afterwards collect properties of the $L^p$ realizations of this operator which will be needed in the subsequent chapters. First of all we establish the following extension property for function spaces on Lipschitz domains, which will be used in the sequel.

**Proposition 2.3.1.** There is a continuous extension operator $\text{Ext} : L^1(\Omega) \to L^1(\mathbb{R}^d)$, whose restriction to any space $H^{1,q}(\Omega)$ ($q \in [1,\infty]$) maps this space continuously into $H^{1,q}(\mathbb{R}^d)$. Moreover, $\text{Ext}$ maps $L^p(\Omega)$ continuously into $L^p(\mathbb{R}^d)$ for $p \in [1,\infty]$.

**Proof.** The assertion is proved for the spaces $H^{1,q}$ in [Giu94, Thm. 3.10] see also [Maz85, Ch. 1.1.16]. Inspecting the corresponding proofs (which are given via localization, Lipschitz diffeomorphism and symmetric reflection) one easily recognizes that the extension mapping at the same time continuously extends the $L^p$ spaces. \qed

Let us introduce an assumption on $\Omega$ and $\Gamma$ which will define the geometrical framework relevant for us in the sequel.

**Assumption 2.3.2.**

a) For any point $x \in \partial \Omega$ there is an open neighborhood $\Upsilon_x$ of $x$ and a bi-Lipschitz mapping $\phi_x$ from $\Upsilon_x$ into $\mathbb{R}^d$, such that $\phi_x((\Omega \cup \Gamma) \cap \Upsilon_x) = \alpha K_-$ or $\alpha(K_- \cup \Sigma)$ or $\alpha(K_- \cup \Sigma_0)$ for some positive $\alpha = \alpha(x)$.

b) Each mapping $\phi_x$ is, in addition, volume-preserving.

**Remark 2.3.3.** Assumption 2.3.2 a) exactly characterizes Gröger’s regular sets, introduced in his pioneering paper [Grö89]. Note that the additional property ‘volume-preserving’ also has been required in several contexts (see [GGKR02] and [Grö92]).

It is not hard to see that every Lipschitz domain and also its closure is regular in the sense of Gröger, the corresponding model sets are then $K_-$ or $K_- \cup \Sigma$, respectively, see [Gri85, Ch 1.2]. A simplifying topological characterization of Gröger’s regular sets for $d = 2$ and $d = 3$ will be given in Section 2.8.

In particular, all domains with Lipschitz boundary (strongly Lipschitz domains) satisfy Assumption 2.3.2: if, after a shift and an orthogonal transformation, the domain lies locally beyond a graph of a Lipschitz function $\psi$, then one can define $\phi(x_1, \ldots, x_d) = (x_1 - \psi(x_2, \ldots, x_d), x_2, \ldots, x_d)$. Obviously, the mapping $\phi$ is then bi-Lipschitz and the determinant of its Jacobian is identically 1. For further examples see Section 2.7.

Next we have to introduce a boundary measure on $\partial \Omega$. Since in our context $\Omega$ is not necessarily a domain with Lipschitz boundary, this is not canonic. Let, according to the definition of a Lipschitz domain, for every point $x \in \partial \Omega$ an open neighborhood $\Upsilon_x$ of $x$ and a bi-Lipschitz
function $\phi : \Upsilon \to \mathbb{R}^d$ be given, which satisfy $\phi_x(\Upsilon_x \cap \Omega) = K_x$, $\phi_x(\Upsilon_x \cap \partial \Omega) = \Sigma$ and $\phi_x(x) = 0$. Let $\Upsilon_{x_1}, \ldots, \Upsilon_{x_l}$ be a finite subcovering of $\partial \Omega$. Define on $\partial \Omega \cap \Upsilon_{x}$ the measure $\sigma_j$ as the $\phi_x^{-1}$-image of the $(d - 1)$-dimensional Lebesgue measure on $\Sigma$. Clearly, this measure is a positive, bounded Radon measure. Finally, define the measure $\sigma$ on $\partial \Omega$ by

$$\int_{\partial \Omega} f \, d\sigma := \sum_{j=1}^l \int_{\partial \Omega \cap \Upsilon_{x_j}} f \, d\sigma_j, \quad f \in C(\partial \Omega).$$

Clearly, $\sigma$ also is a bounded, positive Radon measure. Furthermore, it is not hard to see that the measure $\sigma$ – simultaneously viewed as a measure on $\mathbb{R}^d$ – satisfies

$$\sup_{x \in \mathbb{R}^d} \sup_{r \in [0,1]} \sigma(B(x, r)) r^{1-d} < \infty,$$

where, here and in the sequel, $B(x, r)$ denotes the ball centered at $x$ with radius $r$, compare [JW84, Ch. II.1], in particular Example 1 there.

Later we will repeatedly need the following interpolation results from [GGKR02].

**Proposition 2.3.4.** Let $\Omega$ and $\Gamma$ satisfy Assumption 2.3.2 a) and let $\theta \in [0, 1]$.

i) Then for $q_0, q_1 \in [1, \infty]$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ one has

$$H_\Gamma^{\theta, q} = [L^{q_0}, H_\Gamma^{1, q_1}]_{\theta}, \quad \text{if } \theta \neq \frac{1}{q},$$

$$H_\Gamma^{-\theta, q} = [L^{q_0}, H_\Gamma^{-1, q_1}]_{\theta}, \quad \text{if } \theta \neq 1 - \frac{1}{q},$$

and

$$H_\Gamma^{\pm 1, q} = [H_\Gamma^{\pm 1, q_0}, H_\Gamma^{\pm 1, q_1}]_{\theta}. \quad (2.3.3)$$

ii) If additionally Assumption 2.3.2 b) is fulfilled and $\frac{1}{q} \neq \theta \neq 1 - \frac{1}{q}$, then

$$H_\Gamma^{\pm \theta, q} = [H_\Gamma^{-1, q_0}, H_\Gamma^{1, q_1}]_{\frac{1}{\pm q}}. \quad (2.3.4)$$

**Corollary 2.3.5.** Under the same assumptions as for (2.3.3) one has

$$\hat{H}_\Gamma^{-1, q} = [\hat{H}_\Gamma^{-1, q_0}, \hat{H}_\Gamma^{-1, q_1}]_{\theta}. \quad (2.3.5)$$

**Proof.** (2.3.5) may be deduced from (2.3.3) by means of the retraction/coretraction theorem (see [Tri78, Ch. 1.2.4]), where the coretraction is the mapping which assigns to $f \in \hat{H}_\Gamma^{1, r}$ the linear form $H_\Gamma^{1, r'} \ni \psi \mapsto \langle f, \psi \rangle_{\hat{H}_\Gamma^{-1, r'}}$. \hfill $\square$

Having this at hand, we can prove the following trace theorem.

**Theorem 2.3.6.** Assume $q \in [1, \infty]$ and $\theta \in \left[\frac{1}{q}, 1\right]$. Let $\Pi$ be a Lipschitz hypersurface in $\Omega$ and let $\varpi$ be any measure on $\Pi$ which satisfies

$$\sup_{x \in \mathbb{R}^d} \sup_{r \in [0,1]} \varpi(B(x, r)) r^{1-d} < \infty.$$

Then the trace operator $\text{Tr}$ from $H^{\theta, q}(\Omega)$ to $L^q(\Pi, \varpi)$ is continuous.
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Proof. Since $\Omega$ is an extension domain for $H^{1,q}$ and $L^q$ simultaneously, one has the inequality

$$\|u\|_{L^{q}(\Omega, \varpi)} = \|u\|_{L^{q}(\Pi, \varpi)} \leq c \|u\|_{H^{1,q}(\Omega)}^{\frac{1}{q}} \|u\|_{L^{q}(\Omega)}^{\frac{1}{q}} \leq c \|u\|_{H^{1,q}(\Omega)}, \quad u \in H^{1,q}(\Omega),$$  \hspace{1cm} (2.3.6)

for $q \in [1, \infty[$, see [Maz85, Ch. 1.4.7]. But due to a general interpolation principle (see [BS88, Ch. 5, Prop. 2.10]) this yields a continuous mapping

$$\left( L^q(\Omega), H^{1,q}(\Omega) \right) := u \mapsto \|u\|_{L^q(\Pi, \varpi)}. \hspace{1cm} (2.3.7)$$

Since $\Omega$ is a Lipschitz domain, (2.3.1) in view of $\theta > 1/q$ in particular yields the equality $H^{\theta,q}(\Omega) = \left[ L^q(\Omega), H^{1,q}(\Omega) \right]_\theta$. Thus, we have the continuous embedding

$$H^{\theta,q}(\Omega) = \left[ L^q(\Omega), H^{1,q}(\Omega) \right]_\theta \hookrightarrow \left( L^q(\Omega), H^{1,q}(\Omega) \right),$$

see [Tri78, Ch. 1.10.3, Thm. 1 and Ch. 1.3.3]. This, together with (2.3.7), proves the theorem. \hfill \Box

We define the operator $A : H^{1,2}_{\Gamma} \to \check{H}^{-1,2}_{\Gamma}$ by

$$\langle A\psi, \varphi \rangle_{H^{1,2}_{\Gamma}} := \int_{\Gamma} \mu \nabla \psi \cdot \nabla \varphi \, dx + \int_{\Gamma} \kappa \psi \varphi \, d\sigma, \quad \psi, \varphi \in H^{1,2}_{\Gamma}, \hspace{1cm} (2.3.8)$$

where $\kappa \in L^\infty(\Gamma, d\sigma)$. Note that in view of (2.3.6) the form in (2.3.8) is well defined.

In the special case $\kappa = 0$, we write more suggestively $-\nabla \cdot \mu \nabla$ instead of $A$.

The $L^2$ realization of $A$, i.e. the maximal restriction of $A$ to the space $L^2$, we denote by the same symbol $A$; clearly this is identical with the operator which is induced by the form on the right hand side of (2.3.8). If $B$ is a selfadjoint operator on $L^2$, then by the $L^p$ realization of $B$ we mean its restriction to $L^p$ if $p > 2$ and the $L^p$ closure of $B$ if $p \in [1, 2[$.

We decided not to use different symbols for all these (and lateron also other) realizations of our operators in this paper, since we think that the gain in exactness would be largely outweighed by the resulting complexity of notation. Naturally, this means that we have to pay attention to domains even more thoroughly.

Remark 2.3.7. Following [Ouh05, Ch. 1.4.2] (see also [Ber86, Ch. 1]), we did not define $A$ as an operator with values in the space of linear forms on $H^{1,2}_{\Gamma}$, but in the space of anti-linear forms. This guarantees that the restriction of this operator to $L^2$ equals the usual selfadjoint operator that is induced by the sesquilinear form in (2.3.8), which is crucial for our analysis.

In this spirit, the duality between $\check{H}^{-1,2}_{\Gamma}$ and $H^{1,q}_{\Gamma}$ is to be considered as the extended $L^2$ duality $L^2 \times L^2 \ni \langle \psi, \varphi \rangle \to \int_{\Omega} \psi \varphi \, dx$, where $L^2$ acts as the set of anti-linear forms on itself. Especially, all occurring adjoint operators are to be understood with respect to this dual pairing.

First, we collect some basic facts on $A$.

Proposition 2.3.8. i) $\nabla \cdot \mu \nabla$ generates an analytic semigroup on $\check{H}^{-1,2}_{\Gamma}$.

ii) $-\nabla \cdot \mu \nabla$ is selfadjoint on $L^2$ and bounded by $0$ from below. The restriction of $-A$ to $L^2$ is densely defined and generates an analytic semigroup there.

iii) If $\lambda > 0$ then the operator $(-\nabla \cdot \mu \nabla + \lambda)^{1/2} : H^{1,2}_{\Gamma} \to L^2$ provides a topological isomorphism; in other words: the domain of $(-\nabla \cdot \mu \nabla + \lambda)^{1/2}$ on $L^2$ is the form domain $H^{1,2}_{\Gamma}$.

iv) The form domain $H^{1,2}_{\Gamma}$ is invariant under multiplication with functions from $H^{1,q}$, if $q > d$. 

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v) Assume \( \varkappa \geq 0 \). Then, under Assumption 2.3.2 a), for all \( p \in [1, \infty[ \) the operator \( -A \) generates a semigroup of contractions on \( L^p \). Additionally, it satisfies

\[
\| (A + \lambda)^{-1} \|_{\mathcal{L}(L^p)} \leq \frac{c}{|\lambda|}, \quad \Re \lambda \geq 0.
\]

vi) Under Assumption 2.3.2 a) \( \text{dom} \tilde{H}^{-1,q}_{\Gamma}(-\nabla \cdot \mu \nabla) \) embeds compactly into \( \tilde{H}^{-1,q}_{\Gamma} \) for every \( q \in [2, \infty[, \ i.e. \) the resolvent of \( (-\nabla \cdot \mu \nabla) \) is compact on \( \tilde{H}^{-1,q}_{\Gamma} \).

Proof. i) is proved in [Ouh05, Thm. 1.55], see also [GR89].

ii) The first assertion follows from a classical representation theorem for forms, see [Kat80, Ch. VI.2.1]. Secondly, one verifies that the form \( \tilde{H}^{1,2}_{\Gamma} \ni \psi \mapsto \int_{\Omega} \psi \cdot \mu \nabla \psi \cdot \nabla \psi + \psi \psi \ dx \) with arbitrarily small relative bound. In fact, thanks to (2.3.6),

\[
\left| \int_{\Gamma} \varkappa |\psi|^2 d\sigma \right| \leq \| \varkappa \|_{L^\infty(\Gamma)} \| \psi \|_{L^2(\partial \Omega)}^2 \leq \| \varkappa \|_{L^\infty(\Gamma)} \| \psi \|_{H^{1,2}_{\Gamma}(\partial \Omega)} \| \psi \|_{L^2(\Omega)}
\]

\[
\leq c \| \psi \|_{H^{1,2}_{\Gamma}(\partial \Omega)}^2 + \frac{1}{c} \| \varkappa \|_{L^\infty(\Gamma)} \| \psi \|_{L^2(\Omega)}^2.
\]

Thus, the form (2.3.8) is also closed on \( H^{1,2}_{\Gamma} \) and sectorial. Moreover, the operator \( -A \) generates an analytic semigroup by the representation theorem for sectorial forms, see also [Kat80, Ch. VI.2.1].

iii) This follows from the second representation theorem of forms (see [Kat80, Ch. VI.2.6]), applied to the operator \( -\nabla \cdot \mu \nabla + \lambda \).

iv) First, for \( u \in C^\infty_{\Gamma} \) and \( v \in C^\infty \) the product \( uv \) is obviously in \( C^\infty_{\Gamma} \subseteq H^{1,2}_{\Gamma} \). But, by definition of \( H^{1,2}_{\Gamma} \), the set \( C^\infty_{\Gamma} \) (see (2.2.1)) is dense in \( H^{1,2}_{\Gamma} \) and \( C^\infty \) is dense in \( H^{1,q}_{\Gamma} \).

Thus, the assertion is implied by the continuity of the mapping

\[
H^{1,2}_{\Gamma} \times H^{1,q}_{\Gamma} \ni (u, v) \mapsto uv \in H^{1,2}_{\Gamma},
\]

because \( H^{1,2}_{\Gamma} \) is closed in \( H^{1,2}_{\Gamma} \).

v) This is proved in [GKR01, Thm. 4.11, Thm. 5.2].

vi) The operator \( (-\nabla \cdot \mu \nabla + 1)^{-1} \) has the following - continuous - mapping properties

\[
(-\nabla \cdot \mu \nabla + 1)^{-1} : \tilde{H}^{-1,2}_{\Gamma} \rightarrow H^{1,2}_{\Gamma} \hookrightarrow L^2
\]

and

\[
(-\nabla \cdot \mu \nabla + 1)^{-1} : \tilde{H}^{-1,q}_{\Gamma} \rightarrow L^\infty \hookrightarrow L^{d+1} \quad \text{for } q \geq d + 1
\]

(see [Gri02]). This shows that the resolvent is compact for \( q = 2 \) and for \( q \geq d + 1 \). If one takes in (2.3.10) \( q = d + 1 \) and interpolates between (2.3.9) and (2.3.10), one obtains a continuous mapping \( (-\nabla \cdot \mu \nabla + 1)^{-1} : \tilde{H}^{-1,q}_{\Gamma} \rightarrow L^q \) for every \( q \in [2, d + 1[, \) see Corollary 2.3.5.
2. Maximal Regularity for Operators with Mixed Boundary Conditions

One essential instrument for our subsequent considerations are (upper) Gaussian estimates.

**Theorem 2.3.9.** The semigroup generated by $\nabla \cdot \mu \nabla$ in $L^2$ satisfies upper Gaussian estimates, precisely:

$$(e^{t \nabla \cdot \mu \nabla} f)(x) = \int_{\Omega} K_t(x,y) f(y) \, dy, \quad x \in \Omega, \ f \in L^2,$$

for some measurable function $K_t : \Omega \times \Omega \to \mathbb{R}_+$ and for all $\epsilon > 0$ there exist constants $c, b > 0$, such that

$$0 \leq K_t(x,y) \leq \frac{c}{t^{d/2}} e^{-b|x-y|^2 t}, \ t > 0, \ a.a. \ x, y \in \Omega. \quad (2.3.11)$$

This follows from the following simplified version of Theorem 6.10 in [Ouh05] (see also [AtE97]).

**Proposition 2.3.10** (Ouhabaz). Assume that $-\nabla \cdot \omega \nabla$, with $\omega \in L^\infty(\Omega; L^1(\mathbb{R}^d))$ uniformly elliptic, is defined on the form domain $V \subseteq H^{1,2}$ that satisfies

a) $V$ is closed in $H^{1,2}$,

b) $H^{1,2}_0 \subseteq V$,

c) $V$ has the $L^1$-$H^{1,2}$ extension property,

d) $u \in V$ implies $\text{sign}(u) \inf(1, |u|) \in V$, where $\text{sign}(u) = u/|u|$ if $u \neq 0$ and $\text{sign}(u) = 0$ else.

e) $u \in V$ implies $e^\psi u \in V$ for every $\psi \in C^\infty(\mathbb{R}^d)$, such that $\psi$ and $|\nabla \psi|$ are bounded in $\mathbb{R}^d$.

Then $e^{t \nabla \cdot \omega \nabla}$ satisfies an upper Gaussian estimate as in (2.3.11).

**Proof of Theorem 2.3.9.** We have to verify conditions a) – e) from Proposition 2.3.10 for $V = H^{1,2}$. a) and b) are obvious. For c) see Proposition 2.3.1 and d) is covered by [Ouh05, Proposition 4.11]. Finally, e) follows from Proposition 2.3.8 iv).

Another notion in our considerations will be the bounded holomorphic functional calculus that we want to introduce briefly. Let $X$ be a Banach space and $-B$ the generator of a bounded analytic semigroup on $X$. Denoting, for $\kappa \in [0, \pi]$, $\Sigma_\kappa := \{ z \in \mathbb{C} \setminus \{ 0 \} : |\text{arg}(z)| < \kappa \}$, we then have for some $\theta \in [0, \pi/2]$

$\sigma(B) \subseteq \Sigma_\theta \cup \{ 0 \}$ and $\| R(\lambda, B) \|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \ \lambda \in \mathbb{C} \setminus \Sigma_\theta$.

Following [McI86] (see also [DHP03]), for any angle $\kappa \in [0, \pi]$ we define the function spaces

$\mathcal{H}^\infty_0(\Sigma_\kappa) := \{ \psi : \Sigma_\kappa \to \mathbb{C}, \ \text{holomorphic and bounded} \}$ and

$\mathcal{H}^\infty(\Sigma_\kappa) := \{ \psi \in \mathcal{H}^\infty(\Sigma_\kappa) : \ \text{there exist } C, \epsilon > 0 \text{ s.t. } |\psi(z)| \leq C \frac{|z|^r}{(1 + |z|^{2\epsilon})} \},$

both equipped with the norm $\| \psi \|_{\mathcal{H}^\infty(\Sigma_\kappa)} := \sup_{z \in \Sigma_\kappa} |\psi(z)|$. Then for $\psi \in \mathcal{H}^\infty_0(\Sigma_\kappa)$ with $\kappa > \theta$, we may compute $\psi(B)$, using the Cauchy integral formula

$$\psi(B) = \frac{1}{2\pi i} \int_{\Sigma} \psi(z) R(z, B) \, dz,$$
where the path is given by the two rays \( t e^{\pm i \varphi}, t > 0 \), for some \( \theta < \varphi < \kappa \). Note that this integral is absolutely convergent in \( \mathcal{L}(X) \). We now say that \( B \) has a bounded \( \mathcal{H}^\infty \)-calculus, if there is a constant \( C \geq 0 \), such that

\[
\|\psi(B)\|_{\mathcal{L}(X)} \leq C\|\psi\|_{\mathcal{H}_0^\infty}, \quad \psi \in \mathcal{H}_0^\infty(\Sigma_\kappa),
\]

for some \( \kappa > \theta \). The infimum of all angles \( \kappa \), for which this holds, is called the \( \mathcal{H}^\infty \)-angle \( \varphi_B^\infty \) of \( B \).

If \( B \) admits a bounded \( \mathcal{H}^\infty \)-calculus for some \( \kappa > \theta \), then the mapping \( \mathcal{H}_0^\infty(\Sigma_\kappa) \ni \psi \mapsto \psi(B) \in \mathcal{L}(X) \) can be extended uniquely to an algebra homomorphism between \( \mathcal{H}_0^\infty(\Sigma_\kappa) \) and \( \mathcal{L}(X) \).

**Proposition 2.3.11.** Let \( \partial \Omega \setminus \Gamma \) have nonzero boundary measure. Then the following assertions hold for every \( p \in [1, \infty] \).

i) For sufficiently small \( \gamma > 0 \), the operator \( -\nabla \cdot \mu \nabla - \gamma \) has a bounded \( \mathcal{H}^\infty \)-calculus on \( L^p \) with \( \mathcal{H}^\infty \)-angle \( \varphi_{-\nabla \cdot \mu \nabla - \gamma}^\infty = 0 \).

ii) The set \( \{(-\nabla \cdot \mu \nabla)^{is} : s \in \mathbb{R}\} \) forms a strongly continuous group on \( L^p \) admitting the estimate

\[
\|(\nabla \cdot \mu \nabla)^{is}\|_{\mathcal{L}(L^p)} \leq c_p e^{\epsilon|s|} \theta, \quad s \in \mathbb{R},
\]

with \( 0 \leq \theta < \pi/2 \).

**Proof.** Since the boundary measure of \( \partial \Omega \setminus \Gamma \) is nonzero, the operator \( -\nabla \cdot \mu \nabla \) is continuously invertible in \( L^2 \), i.e. \( 0 \) does not belong to the spectrum. Hence, for sufficiently small \( \gamma > 0 \), \( -\nabla \cdot \mu \nabla - \gamma \) is still self-adjoint and bounded by \( 0 \) from below, cf. Proposition 2.3.8 ii). Thus, for every \( \delta \geq 0 \) the operator \( -\nabla \cdot \mu \nabla - \gamma + \delta \) has a bounded \( \mathcal{H}^\infty \)-calculus on \( L^2 \) with \( \mathcal{H}^\infty \)-angle \( 0 \). Furthermore, taking \( \delta > \gamma \), the semigroup generated by \( \nabla \cdot \mu \nabla + \gamma - \delta \) obeys the Gaussian estimate (2.3.11) with \( \epsilon = 0 \). Thus, \( -\nabla \cdot \mu \nabla - \gamma + \delta \) also has a bounded \( \mathcal{H}^\infty \)-calculus on \( L^p \) with \( \mathcal{H}^\infty \)-angle \( 0 \) for all \( 1 < p < \infty \) by [DR96].

In order to eliminate the ‘+ \( \delta \)’, we observe that the spectrum of \( -\nabla \cdot \mu \nabla \) is \( p \)-independent, thanks to the Gaussian estimates, see [Kun99]. Thus, also in \( L^p \) the spectrum of \( -\nabla \cdot \mu \nabla - \gamma \) is contained in the positive real axis. It was shown in [KKW06, Prop. 6.10], that in such a case, we may shift back the operator without losing the bounded \( \mathcal{H}^\infty \)-calculus, as long as the spectrum does not reach zero. This shows i).

As the functions \( z \mapsto z^s \) belong to \( \mathcal{H}_0^\infty(\Sigma_\phi) \) for every \( s \in \mathbb{R} \) and every \( \phi \in [0, \pi] \), part i) of this proof yields \( (-\nabla \cdot \mu \nabla)^{is} \in \mathcal{L}(L^p) \) with \( \|(\nabla \cdot \mu \nabla)^{is}\|_{\mathcal{L}(L^p)} \leq c \) for all \( -1 \leq s \leq 1 \). Thus, ii) follows by [Ama95, Thm. III.4.7.1 and Cor. III.4.7.2]. \( \square \)

### 2.4. Mapping properties for \((-\nabla \cdot \mu \nabla)^{1/2}\)

In this chapter we prove that, under certain topological conditions on \( \Omega \) and \( \Gamma \), the mapping

\[
(-\nabla \cdot \mu \nabla)^{1/2} : H^{1,q}_\Gamma \to L^q
\]

is a topological isomorphism for \( q \in \{1, 2\} \). We abbreviate \( -\nabla \cdot \mu \nabla \) by \( A_0 \) throughout this chapter. Let us introduce the following

**Assumption 2.4.1.** There is a bi-Lipschitz, volume-preserving mapping \( \phi \) from a neighborhood of \( \overline{\Omega} \) into \( \mathbb{R}^d \) such that \( \phi(\Omega \cup \Gamma) = \alpha K_- \) or \( \alpha(K_- \cup \Sigma) \) or \( \alpha(K_- \cup \Sigma_0) \) for some \( \alpha > 0 \).
Remark 2.4.2. It is known that for a bi-Lipschitz mapping the property of being volume-preserving is equivalent to the property that the absolute value of the determinant of the Jacobian is one almost everywhere (see [EG92, Ch. 3]).

The main results of this section are the following two theorems.

Theorem 2.4.3. Under the general assumptions made in Section 2 the following holds true: If \( \partial \Omega \setminus \Gamma \) has nonzero boundary measure, then, for every \( q \in [1, 2] \), the operator \( A_0^{-1/2} \) is a continuous operator from \( L^q \) into \( H^{1,q}_\Gamma \). Hence, it continuously maps \( \hat{H}^{-1,q}_\Gamma \) into \( L^q \) for any \( q \in [2, \infty] \).

Theorem 2.4.4. If in addition Assumption 2.4.1 is fulfilled and \( q \in [1, 2] \), then \( A_0^{-1/2} \) maps \( \hat{H}^{1,q}_\Gamma \) continuously into \( L^q \). Hence, it continuously maps \( L^q \) into \( \hat{H}^{-1,q}_\Gamma \) for any \( q \in [2, \infty] \).

Remark 2.4.5. In both theorems the second assertion follows from the first by the selfadjointness of \( A_0 \) on \( L^2 \) and duality (see Remark 2.3.7); thus we focus on the proof of the first assertions in the sequel.

Let us first prove the continuity of the operator \( A_0^{-1/2} : L^q \to H^{1,q}_\Gamma \). In order to do so, we observe that this follows, whenever

1. The Riesz transform \( \nabla A_0^{-1/2} \) is a bounded operator on \( L^q \), and, additionally,

2. \( A_0^{-1/2} \) maps \( L^q \) into \( H^{1,q}_\Gamma \).

The first item can be deduced from the following result of Duong and McIntosh (see [DM99, Thm. 2]) that is even true in a much more general setting.

Proposition 2.4.6. Let \( B \) be a positive, selfadjoint operator on \( L^2 \), having the space \( W \) as its form domain and admitting the estimate \( \| \nabla \psi \|_{L^2} \leq c \| B^{1/2} \psi \|_{L^2} \) for all \( \psi \in W \). Assume that \( W \) is invariant under multiplication by bounded functions with bounded, continuous first derivatives and that the kernel \( K_t \) of the semigroup \( e^{-tB} \) satisfies bounds

\[
|K_t(x,y)| \leq \frac{C}{t^{d/2}} \left( 1 + \frac{|x-y|^2}{t} \right)^{-\beta} \tag{2.4.1}
\]

for some \( \beta > d/2 \). Then the operator \( \nabla B^{-1/2} \) is of weak type \((1,1)\), and, thus can be extended from \( L^2 \) to a bounded operator on \( L^q \) for all \( q \in [1, 2] \).

Proof of Theorem 2.4.3. According to Theorem 2.3.9 the semigroup kernels corresponding to the operator \( A_0 \) satisfy the estimate (2.3.11). Thus, considering the operator \( A_0 + \epsilon \) for some \( \epsilon > 0 \), the corresponding kernels satisfy again (2.3.11), but without the factor \( e^{\epsilon t} \) now. Next, we verify that \( B := A_0 + \epsilon \) and \( W := H^{1,2}_\Gamma \) satisfy the assumptions of Proposition 2.4.6. By Proposition 2.3.8, \( W = H^{1,2}_\Gamma \) is the domain for \((A_0 + \epsilon)^{1/2}\), thus \( \| \nabla \psi \|_{L^2} \leq c \| (A_0 + \epsilon)^{1/2} \psi \|_{L^2} \) holds for all \( \psi \in W \). The invariance property of \( W \) under multiplication is ensured by Proposition 2.3.8. Concerning the bound (2.4.1), it is easy to see that the resulting Gaussian bounds from Theorem 2.3.9 are even much stronger, since the function \( r \mapsto (1 + r)^{\beta} e^{-br} \), \( r \geq 0 \), is bounded for every \( \beta > 0 \). All this shows that \((A_0 + \epsilon)^{-1/2} : L^q \to H^{1,q}_\Gamma \) is continuous for \( q \in [1, 2] \).

Writing

\[
A_0^{-1/2} = (A_0 + \epsilon)^{-1/2}(A_0 + \epsilon)^{1/2}A_0^{-1/2},
\]

the assertion 1. follows, if we know that \((A_0 + \epsilon)^{1/2}A_0^{-1/2} : L^q \to L^q \) is continuous. In order to see this, choose \( \epsilon \) so small that Proposition 2.3.11 i) ensures a bounded \( \mathcal{H}^\infty \)-calculus on
L^q for $A_0 - \epsilon$, and observe that the function $z \mapsto (z + 2\epsilon)^{1/2}(z + \epsilon)^{-1/2}$ is in $\mathcal{H}(\Sigma_\phi)$ for any $\phi \in [0, \pi[$.

It remains to show 2. The first point makes clear that $A_0^{-1/2}$ maps $L^q$ continuously into $H^{1,q}$, thus one has only to verify the correct boundary behavior of the images. If $f \in L^2 \hookrightarrow L^q$, then one has $A_0^{-1/2}f \in H^{1,2} \hookrightarrow H^{1,q}$. Thus, the assertion follows from 1. and the density of $L^2$ in $L^q$.

**Remark 2.4.7.** Theorem 2.4.3 is not true for other values of $q$ in general: If it were, then, due to the case $q \leq 2$ and duality, $A_0^{-1/2} : H^{-1,q} \rightarrow L^q$ and $A_0^{-1/2} : L^q \rightarrow H^{1,q}$ would be continuous for a $q > 2$. But for any $q > 2$ one can find a coefficient function $\mu$ such that the corresponding operator $A_0^{-1}$ does not map $H^{-1,q}$ into $H^{1,q}$, see [Mey63, EKRS07, ERS07], see also [Aus07] and the references therein.

It follows the proof of Theorem 2.4.4. It will be deduced from the subsequent deep result on divergence operators with Dirichlet boundary conditions and some permanence principles.

**Proposition 2.4.8** (Auscher/Tchamitchian, [AT01]). Let $q \in [1, \infty[$ and $\Omega$ be a strongly Lipschitz domain. Then the root of the operator $A_0$, combined with a homogeneous Dirichlet boundary condition, maps $H^{1,q}_0(\Omega)$ continuously into $L^q(\Omega)$.

For further reference we mention the following immediate consequence of Theorem 2.4.3 and Proposition 2.4.8.

**Corollary 2.4.9.** Under the hypotheses of Proposition 2.4.8 the operator $A_0^{-1/2}$ provides a topological isomorphism between $L^q$ and $H^{1,q}_0$, if $q \in [1, 2]$.

In view of Assumption 2.4.1 it is a natural idea to reduce our considerations to the three model constellations mentioned there. In order to do so, we have to show that the assertion of Theorem 2.4.4 is invariant under volume-preserving bi-Lipschitz transformations of the domain.

**Proposition 2.4.10.** Assume that $\phi$ is a mapping from a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^d$ that is additionally bi-Lipschitz. Let us denote $\phi(\Omega) = \Omega_\Delta$ and $\phi(\Gamma) = \Gamma_\Delta$. Define for any function $f \in L^1(\Omega_\Delta)$

$$(\Phi f)(x) = f(\phi(x)) = (f \circ \phi)(x), \quad x \in \Omega.$$

Then

i) The restriction of $\Phi$ to any $L^p(\Omega_\Delta)$, $1 \leq p < \infty$, provides a linear, topological isomorphism between this space and $L^p(\Omega)$.

ii) For any $p \in [1, \infty[$, the mapping $\Phi$ induces a linear, topological isomorphism

$$\Phi_p : H^{1,p}_0(\Omega_\Delta) \rightarrow H^{1,p}_0(\Omega).$$

iii) $\Phi_*^p$ is a linear, topological isomorphism between $\tilde{H}^{-1,p}_0(\Omega)$ and $\tilde{H}^{-1,p}_0(\Omega_\Delta)$ for any $p \in [1, \infty[$.

iv) One has

$$\Phi_*^p A_0 \Phi_p = -\nabla \cdot \mu_\Delta \nabla$$

with

$$\mu_\Delta(y) = \frac{1}{|\det(D\phi)(\phi^{-1}(y))|} (D\phi)(\phi^{-1}(y)) \mu(\phi^{-1}(y)) (D\phi)^T(\phi^{-1}(y))$$

for almost all $y \in \Omega_\Delta$. Here, $D\phi$ denotes the Jacobian of $\phi$ and $\det(D\phi)$ the corresponding determinant.
v) $\mu_\Delta$ also is bounded, Lebesgue measurable, elliptic and takes real, symmetric matrices as values.

vi) The restriction of $\Phi_2^*\Phi$ to $L^2(\Omega_\Delta)$ equals the multiplication operator which is induced by the function $|\det(D\phi)(\phi^{-1}(\cdot))|^{-1}$. Consequently, if $|\det(D\phi)| = 1$ a.e., then the restriction of $\Phi_2^*\Phi$ to $L^2(\Omega_\Delta)$ is the identity operator on $L^2(\Omega_\Delta)$, or, equivalently, $(\Phi_2^*)^{-1}|_{L^2(\Omega_\Delta)} = \Phi|_{L^2(\Omega_\Delta)}$.

**Proof.** For i) see [Maz85, Ch. 1.1.7]. The proof of ii) is contained in [GGKR02, Thm. 2.10] and iii) follows from ii) by duality (see Remark 2.3.7). Assertion iv) is well known, see Proposition 2.4.10, we get the following operator equation on $f$

\[\Phi_2^*\Phi f, g)_{H_\Delta^{1,2}(\Omega_\Delta)} = (f \circ \Phi, g \circ \Phi)_{H_\Delta^{1,2}(\Omega)} = \int_\Omega f(\phi(x))\overline{\gamma}(\phi(x)) \, dx\]

\[= \int_{\Omega_\Delta} f(y)\overline{\gamma}(y) \frac{1}{|\det(D\phi)(\phi^{-1}(y))|} \, dy.\]

Thus, the anti-linear form $\Phi_2^*\Phi f$ on $H_\Delta^{1,2}(\Omega_\Delta)$ is represented by $|\det(D\phi)(\phi^{-1}(\cdot))|^{-1} \in L^\infty(\Omega_\Delta)$.

**Lemma 2.4.11.** Let $p \in [1, \infty]$. Suppose further that $\partial\Omega \setminus \Gamma$ does not have boundary measure zero and that $|\det(D\phi)| = 1$ almost everywhere in $\Omega$. Then, in the notation of the preceding proposition, the operator $(-\nabla \cdot \mu_\Delta \nabla)^{1/2}$ maps $H_\Delta^{1,p}(\Omega_\Delta)$ continuously into $L^p(\Omega_\Delta)$ if and only if $A_0^{1/2}$ maps $H_\Delta^{1,p}(\Omega)$ continuously into $L^p(\Omega)$.

**Proof.** We will employ the formula

\[B^{-1/2} = \frac{1}{\pi} \int_0^\infty t^{-1/2}(B + t)^{-1} \, dt, \tag{2.4.4}\]

$B$ being a positive operator on a Banach space $X$, see [Tri78, Ch. 1.14/1.15] or [Paz83, Ch. 2.6]. Obviously, the integral converges in the $L(X)$-norm.

It is clear that our hypotheses of $\partial\Omega \setminus \Gamma$ not having boundary measure zero implies that $\partial\Omega_\Delta \setminus \Gamma_\Delta$ also has positive boundary measure. Thus, both, $A_0$ and $-\nabla \cdot \mu_\Delta \nabla$ do not have spectrum in zero and are positive operators in the sense of [Tri78, Ch. 1.14] on any $L^p$ (see Proposition 2.3.8). From (2.4.2) and vi) of the preceding proposition one deduces

\[\Phi_2^*(A_0 + t)\Phi_2 = -\nabla \cdot \mu_\Delta \nabla + t \tag{2.4.5}\]

for every $t > 0$. This leads to

\[\Phi_2^{-1}(A_0 + t)^{-1}\Phi_2^* = \left(-\nabla \cdot \mu_\Delta \nabla + t\right)^{-1}.\]

Restricting this last equation to elements from $L^2(\Omega_\Delta)$ and making once more use of vi) in Proposition 2.4.10, we get the following operator equation on $L^2(\Omega_\Delta)$,

\[\Phi^{-1}(A_0 + t)^{-1}\Phi|_{L^2(\Omega_\Delta)} = \left(-\nabla \cdot \mu_\Delta \nabla + t\right)^{-1}.\]

Integrating this equation with weight $\frac{t^{-1/2}}{\pi}$, one obtains, according to (2.4.4),

\[\Phi^{-1}A_0^{-1/2}\Phi|_{L^2(\Omega_\Delta)} = \left(-\nabla \cdot \mu_\Delta \nabla\right)^{-1/2}, \tag{2.4.6}\]
again as an operator equation on \( L^2(\Omega) \). We recall that the operators \( A_0^{-1/2} : L^2(\Omega) \to H^1_{\Gamma_0} (\Omega), (\nabla \cdot \mu_\delta \nabla)^{-1/2} : L^2(\Omega) \to H^1_{\Gamma_0} (\Omega) \), \( \Phi_2 : H^1_{\Gamma_0} (\Omega) \to H^1_{\Gamma_0} (\Omega) \) and \( \Phi : L^2(\Omega) \to L^2(\Omega) \) all are topological isomorphisms. In particular, for any \( f \in L^2(\Omega) \) the element \( A_0^{-1/2} \Phi f \) is from \( H^1_{\Gamma_0} (\Omega) \). Thus, we may write (2.4.6) as

\[
\Phi_2^{-1} A_0^{-1/2} \Phi \|_{L^2(\Omega)} = (\nabla \cdot \mu_\delta \nabla)^{-1/2}
\]

and afterwards invert (2.4.7). We get the following operator equation on \( H^1_{\Gamma_0} (\Omega) \):

\[
\Phi^{-1} A_0^{1/2} \Phi_2 = (\nabla \cdot \mu_\delta \nabla)^{1/2}.
\]

In the sequel we make use of the fact that \( \Phi_\rho : H^1_{\Gamma_0} (\Omega) \to H^1_{\Gamma_0} (\Omega) \) and \( \Phi : L^p(\Omega) \to L^p(\Omega) \) are topological isomorphisms for all \( p \in [1, \infty] \). Thus, first considering the case \( p \in [1, 2] \) and assuming that \( A_0^{1/2} \) maps \( H^1_{\Gamma_0} (\Omega) \) continuously into \( L^p(\Omega) \), we may estimate for all \( \psi \in H^1_{\Gamma_0} (\Omega) \)

\[
\| (\nabla \cdot \mu_\delta \nabla)^{1/2} \psi \|_{L^p(\Omega)} = \| \Phi^{-1} A_0^{1/2} \Phi_2 \psi \|_{L^p(\Omega)}
\]

\[
\leq \| \Phi^{-1} \|_{L(L^p(\Omega); L^p(\Omega))} \| A_0^{1/2} \|_{L(L^p(\Omega); L^p(\Omega))} \| \Phi_2 \psi \|_{H^1_{\Gamma_0} (\Omega)}.
\]

Observing that \( \Phi_2 \) is only the restriction of \( \Phi_\rho \), one may estimate the last factor in (2.4.8):

\[
\| \Phi_2 \psi \|_{H^1_{\Gamma_0} (\Omega)} \leq \| \Phi_\rho \|_{L(L^p(\Omega); H^1_{\Gamma_0} (\Omega))} \| \psi \|_{H^1_{\Gamma_0} (\Omega)}.
\]

This means that \((\nabla \cdot \mu_\delta \nabla)^{1/2}\) maps \( H^1_{\Gamma_0} (\Omega) \), equipped with the induced \( H^1_{\Gamma_0} (\Omega) \)-norm, continuously into \( L^p(\Omega) \) and, consequently, extends to a continuous mapping from the whole \( H^1_{\Gamma_0} (\Omega) \) into \( L^p(\Omega) \) by density.

If \( p \in [2, \infty] \), one has the same estimates (2.4.8) and (2.4.9), in this case only for elements \( \psi \in H^1_{\Gamma_0} (\Omega) \subseteq H^1_{\Gamma_0} (\Omega) \).

Finally, the equivalence stated in the assertion follows by simply interchanging the roles of \( \mu \) and \( \mu_\delta \).

**Remark 2.4.12.** It is the property of 'volume-preserving' which leads, due to vi) of Proposition 2.4.10, to (2.4.5) and then to (2.4.6) and thus allows to hide the complicated geometry of the boundary in \( \Phi \) and \( \mu_\delta \).

It turns out that 'bi-Lipschitz' together with 'volume-preserving' is not a too restrictive condition. In particular, there are such mappings – although not easy to construct – which map the ball onto the cylinder, the ball onto the cube and the ball onto the half ball, see [GHKR05], see also [FP92]. The general message is that this class has enough flexibility to map 'non-smooth objects' onto smooth ones.

Lemma 2.4.11 allows to reduce the proof of Theorem 2.4.4 to \( \Omega = \alpha K \) and the three cases \( \Gamma = \emptyset \), \( \Gamma = \alpha \Sigma \) or \( \Gamma = \alpha \Sigma_0 \). The first case, \( \Gamma = \emptyset \), is already contained in Proposition 2.4.8.

In order to treat the second one, we will use a reflection argument. To this end we define for any \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) the symbol \( x_- := (x_1, \ldots, x_{d-1}, -x_d) \) and for a \( d \times d \) matrix \( \omega \), the matrix \( \omega^- \) by

\[
\omega^- := \begin{cases} 
\omega_{j,k}, & \text{if } j, k < d, \\
-w_{j,k}, & \text{if } j = d \text{ and } k \neq d \text{ or } k = d \text{ and } j \neq d, \\
\omega_{j,k}, & \text{if } j = k = d.
\end{cases}
\]
2. Maximal Regularity for Operators with Mixed Boundary Conditions

Corresponding to the coefficient function $\mu$ on $K_-$, we then define the coefficient function $\hat{\mu}$ on $K$ by

$$\hat{\mu}(x) := \begin{cases} 
\mu(x), & \text{if } x \in K_-, \\
(\mu(x_-))^{-1}, & \text{if } x_- \in K_-, \\
0, & \text{if } x \in \Sigma.
\end{cases}$$

Finally, we define for $\varphi \in L^1(K)$ the reflected function $\varphi_-$ by $\varphi_-(x) = \varphi(x_-)$ and, using this, the extension and restriction operators

$$\mathcal{E} : L^1(K_-) \to L^1(K), \quad (\mathcal{E} f)(x) = \begin{cases} 
f(x), & \text{if } x \in K_-, \\
f(x_-), & \text{if } x_- \in K_-, 
\end{cases}$$

$$\mathcal{G} : \tilde{H}^{-1,2}(K_-) \to \tilde{H}^{-1,2}(K), \quad (\mathcal{G} f, \varphi)_{\tilde{H}^{-1,2}(K_-)} = (f, \varphi|_{K_-} + \varphi_-|_{K_-})_{\tilde{H}^{-1,2}(K_-)},$$

$$\mathcal{R} : L^1(K) \to L^1(K_-), \quad \mathcal{R} f = f|_{K_-}.$$

**Proposition 2.4.13.**

i) If $\psi \in H_0^{1,2}(K_-)$ satisfies $A_0 \psi = f \in \tilde{H}^{-1,2}(K_-)$, then

$$-\nabla \cdot \hat{\mu} \nabla \psi = \mathcal{G} f \in \tilde{H}^{-1,2}(K).$$

ii) The operator $\mathcal{G} : \tilde{H}^{1,2}_\Sigma(K_-) \to \tilde{H}^{-1,2}(K)$ is continuous.

**Proof.**

i) It is known that $\mathcal{G} \hat{\varphi}$ belongs to $H_0^{1,2}(K)$, see [Giu94, Lemma 3.4]. Thus, the assertion is obtained by the definitions of $\mathcal{G} \hat{\varphi}$, $\mathcal{G} f$, $A_0$, $-\nabla \cdot \hat{\mu} \nabla$ and straightforward calculations, based on Proposition 2.4.10 when applied to the transformation $x \mapsto x_-$.

ii) The operator under consideration is the adjoint of $H_0^{1,2}(K) \ni \varphi \mapsto (\varphi|_{K_-} + \varphi_-|_{K_-}) \in H_0^{1,2}(K)$.

We are now in the position to prove Theorem 2.4.4 for the case $\Gamma = \alpha \Sigma$. Up to a homothety we may focus on the case $\alpha = 1$. First, we note that for any function $\varphi \in L^2(K_-)$ one finds $\mathcal{E} \varphi = \mathcal{G} \varphi$, where we identified the functions $\varphi$ and $\mathcal{E} \varphi$ with the corresponding regular distributions. Thus, one obtains from Proposition 2.4.13 i) that $A_0 + t) \psi = f \in \tilde{H}^{-1,2}(K_-)$ implies

$$(-\nabla \cdot \hat{\mu} \nabla + t) \mathcal{E} \psi = \mathcal{G} f,$$

or, equivalently,

$$\mathcal{E} \psi = (\nabla \cdot \hat{\mu} \nabla + t)^{-1} \mathcal{G} f$$

for every $t \in [0, \infty]$. Expressing $\psi = (A_0 + t)^{-1} f$, this yields

$$\mathcal{E} (A_0 + t)^{-1} f = (\nabla \cdot \hat{\mu} \nabla + t)^{-1} \mathcal{G} f.$$

Multiplying this by $\frac{t^{-1/2}}{\pi}$ and integrating over $t$, one obtains in accordance with (2.4.4)

$$\mathcal{E} A_0^{-1/2} f = (\nabla \cdot \hat{\mu} \nabla)^{-1/2} \mathcal{G} f, \quad f \in \tilde{H}^{-1,2}(K_-).$$

(2.4.10)

Applying the restriction operator $\mathcal{R}$ to both sides of (2.4.10), we get

$$A_0^{-1/2} \mathcal{R} f = \mathcal{R} (\nabla \cdot \hat{\mu} \nabla)^{-1/2} \mathcal{G} f, \quad f \in \tilde{H}^{-1,2}(K_-).$$

(2.4.11)

Considering in particular elements $f \in L^2(K_-)$ and taking for these into account $\mathcal{E} f = \mathcal{G} f$, (2.4.11) implies

$$A_0^{-1/2} f = \mathcal{R} (\nabla \cdot \hat{\mu} \nabla)^{-1/2} \mathcal{E} f, \quad f \in L^2(K_-).$$

(2.4.12)
Since both operators $-A_0$ and $\nabla \cdot \hat{\mu} \nabla$ generate contraction semigroups on any $L^p$, and $0$ does not belong to the spectrum for both of them, the operators $A_0^{-1/2}$ and $(\nabla \cdot \hat{\mu} \nabla)^{-1/2}$ are bounded also on $L^p(K_-)$ and $L^p(K)$, respectively. Hence, (2.4.12) remains true for any $f \in L^p(K_-)$ with $p \in [1, 2]$. Now, on one hand it is clear that $\mathcal{E}(L^p(K_-))$ equals the symmetric part of $L^p(K)$, i.e. the set of functions which satisfy $\varphi = \varphi_-$. Using the definition of the coefficient function $\hat{\mu}$ and formula (2.4.2), one recognizes that the resolvent of $-\nabla \cdot \hat{\mu} \nabla$ commutes with the mapping $\varphi \mapsto \varphi_-$. Again exploiting formula (2.4.4), this shows that $(\nabla \cdot \hat{\mu} \nabla)^{-1/2}$ also commutes with the mapping $\varphi \mapsto \varphi_-$. Thus, $(\nabla \cdot \hat{\mu} \nabla)^{-1/2}$ maps the set of symmetric functions, satisfying $\varphi = \varphi_-$, into itself and also the set of antisymmetric functions, satisfying $\varphi = -\varphi_-$. Consequently, $(-\nabla \cdot \hat{\mu} \nabla)^{-1/2} = \mathcal{E}(L^p(K_-))$ must equal the symmetric part of $H_0^{1,p}(K)$ because $(\nabla \cdot \hat{\mu} \nabla)^{-1/2}$ is a surjection onto the whole $H_0^{1,p}(K)$ by Corollary 2.4.9. But, it is known (see [Giu94, Thm. 3.10]) that for any given function $h \in H_0^{1,p}(K_-)$ the symmetric extension belongs to $H_0^{1,p}(K)$. Thus $\mathcal{R}(-\nabla \cdot \hat{\mu} \nabla)^{-1/2} \mathcal{E} = A_0^{-1/2}$ is a surjection onto $H_0^{1,p}(K_-)$. Since, by Theorem 2.4.3 $A_0^{-1/2} : L^p(K_-) \to H_0^{1,p}(K_-)$ is continuous, the continuity of the inverse is implied by the open mapping theorem.

In order to prove the same for the third model constellation, i.e. $\Gamma = \Sigma_0$, we show

**Lemma 2.4.14.** For every $\alpha > 0$ there is a volume-preserving, bi-Lipschitz mapping $\phi : \mathbb{R}^d \to \mathbb{R}^d$ that maps $\alpha(K_- \cup \Sigma_0)$ onto $\alpha(K_- \cup \Sigma)$.

**Proof.** Up to a homothety we may focus on the case $\alpha = 1$. Let us first consider the case $d = 2$. We define on the lower halfspace $\{(x, y) \in \mathbb{R}^2 : y \leq 0\}$

$$
\rho_1(x, y) := \begin{cases} 
(x - y/2, y/2), & \text{if } x \leq 0, \ y \geq x, \\
(x/2, -x/2 + y), & \text{if } x \leq 0, \ y < x, \\
(x/2, x/2 + y), & \text{if } x > 0, \ y < -x, \\
(x + y/2, y/2), & \text{if } x > 0, \ y \geq -x.
\end{cases}
$$

Observing that $\rho_1$ acts as the identity on the $x$-axis, we may define $\rho_1$ on the upper halfspace $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ by $\rho_1(x, y) = (x_0, -y_0)$ with $(x_0, y_0) = \rho_1(x, -y)$. In this way we obtain a globally bi-Lipschitz transformation $\rho_1$ from $\mathbb{R}^2$ onto itself that transforms $K_- \cup \Sigma_0$ onto the triangle shown in Figure 2.1.

![Figure 2.1: $K_- \cup \Sigma_0$ and $\rho_1(K_- \cup \Sigma_0)$](image-url)
Next we define the bi-Lipschitz mapping \( \rho_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
\rho_2(x, y) := \begin{cases} 
(x, x + 2y + 1), & \text{if } x \leq 0, \\
(x, -x + 2y + 1), & \text{if } x > 0,
\end{cases}
\]
in order to get the geometric constellation in Figure 2.2.

![Figure 2.2: \( \rho_2(\rho_1(K_- \cup \Sigma_0)) \)]

If \( \rho_3 \) is the (clockwise) rotation of \( \pi/4 \), we thus achieved that \( \rho := \rho_3 \rho_2 \rho_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) is bi-Lipschitzian and satisfies
\[
\rho(K_- \cup \Sigma_0) = \left\{ (x, y) \in \mathbb{R}^2 : -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} < y \leq \frac{1}{\sqrt{2}} \right\}.
\]
Let \( \rho_4 : \mathbb{R}^2 \to \mathbb{R}^2 \) be the affine mapping \( (x, y) \mapsto (\sqrt{2}x, \sqrt{2}y - \frac{1}{2}) \). Then \( \phi = \phi_2 := \rho_4 \rho \) maps \( K_- \cup \Sigma_0 \) bi-Lipschitzian onto \( K_- \cup \Sigma \) in the 2-d case. As is easy to check, the modulus of the determinant of the Jacobian is identically one a.e. Hence, \( \phi_2 \) is volume-preserving.

If \( d \geq 3 \), one simply puts \( \phi(x_1, \ldots, x_d) := (x_1, \ldots, x_{d-2}, \phi_2(x_{d-1}, x_d)) \).

Thus, the proof of Theorem 2.4.4 in the case \( \Gamma = \alpha \Sigma_0 \) results from the case \( \Gamma = \alpha \Sigma \), Lemma 2.4.11 and Lemma 2.4.14.

**Remark 2.4.15.** Let us mention that Lemma 2.4.11, only applied to \( \Omega = K \) and \( \Gamma = \emptyset \) (the pure Dirichlet case) already provides a zoo of geometries which is not covered by [AT01]. Notice in this context that the image of a strongly Lipschitz domain under a bi-Lipschitz transformation needs not to be a strongly Lipschitz domain at all, cf. Subsection 2.7.3, see also [Gri85, Ch. 1.2].

### 2.5. Maximal parabolic regularity for \( A \)

In this section we intend to prove the first main result of this work announced in the introduction. Let us first recall the notion of maximal parabolic \( L^s \) regularity.

**Definition 2.5.1.** Let \( 1 < s < \infty \), let \( X \) be a Banach space and let \( J := [T_0, T] \subseteq \mathbb{R} \) be a bounded interval. Assume that \( B \) is a closed operator in \( X \) with dense domain \( D \) (in the sequel always equipped with the graph norm). We say that \( B \) satisfies maximal parabolic \( L^s(J; X) \)
2.5. Maximal parabolic regularity for $A$

regularity, if for any $f \in L^s(J;X)$ there exists a unique function $u \in W^{1,s}(J;X) \cap L^s(J;D)$ satisfying

$$u' + Bu = f, \quad u(T_0) = 0,$$

where the time derivative is taken in the sense of $X$-valued distributions on $J$ (see [Ama95, Ch. III.1]).

**Remark 2.5.2.**

i) It is well known that the property of maximal parabolic regularity of an operator $B$ is independent of $s \in ]1, \infty[$ and the specific choice of the interval $J$ (cf. [Dor93]). Thus, in the following we will say for short that $B$ admits maximal parabolic regularity on $X$.

ii) If an operator satisfies maximal parabolic regularity on a Banach space $X$, then its negative generates an analytic semigroup on $X$ (cf. [Dor93]). In particular, a suitable left half plane belongs to its resolvent set.

iii) If $X$ is a Hilbert space, the converse is also true: The negative of every generator of an analytic semigroup on $X$ satisfies maximal parabolic regularity, cf. [deS64] or [Dor93].

iv) If $-B$ is a generator of an analytic semigroup on a Banach space $X$, we define

$$B\left(\frac{\partial}{\partial t} + B\right)^{-1}: C(J; \text{dom}_X(B)) \rightarrow L^s(J;X)$$

by

$$\left(\left(\frac{\partial}{\partial t} + B\right)^{-1} f\right)(t) := B \int_t^{T_0} e^{(s-t)B} f(s) \, ds.$$ 

Then, by definition of the distributional time derivative, it is easy to see that $B$ has maximal parabolic regularity on $X$ if and only if the operator $B\left(\frac{\partial}{\partial t} + B\right)^{-1}$ continuously extends to an operator from $L^s(J;X)$ into itself.

v) Observe that

$$W^{1,s}(J;X) \cap L^s(J;D) \hookrightarrow C(J; (X,D)_{1-\frac{1}{s},s}). \quad (2.5.1)$$

Let us first formulate the following lemma, needed in the sequel.

**Lemma 2.5.3.** Suppose that $X,Y$ are Banach spaces, which are contained in a third Banach space $Z$ with continuous injections. Let $B$ be a linear operator on $Z$ whose restriction to each of the spaces $X,Y$ induce closed, densely defined operators there. Assume that the induced operators fulfill maximal parabolic regularity on $X$ and $Y$, respectively. Then $B$ satisfies maximal parabolic regularity on each of the interpolation spaces $[X,Y]_\theta$ and $(X,Y)_{\theta,s}$ with $\theta \in ]0,1[, s \in ]1, \infty[$.

**Proof.** By supposition, $(X,Y)$ forms an interpolation couple. In this case it is known (see [Tri78, Ch. 1.18.4]) that one has for any $\theta \in ]0,1[$ and any $s \in ]1, \infty[$ the interpolation identities

$$[L^s(J;X), L^s(J;Y)]_\theta = L^s(J;[X,Y]_\theta) \quad (2.5.2)$$

and

$$(L^s(J;X), L^s(J;Y))_{\theta,s} = L^s(J;(X,Y)_{\theta,s}). \quad (2.5.3)$$

Due to Remark 2.5.2 ii), $-B$ generates an analytic semigroup on $X$ and $Y$, respectively. Obviously, the corresponding resolvent estimates are maintained under real and complex
interpolation, so \(-B\) also generates an analytic semigroup on the corresponding interpolation spaces. Taking into account (2.5.2) or (2.5.3) and invoking Remark 2.5.2 iv), the operators

\[
B(\frac{\partial}{\partial t} + B)^{-1} : L^s(J; X) \to L^s(J; X)
\]

and

\[
B(\frac{\partial}{\partial t} + B)^{-1} : L^s(J; Y) \to L^s(J; Y)
\]

are continuous, if \(s \in [1, \infty[\). Thus, interpolation together with (2.5.2) ((2.5.3), respectively) tells us that \(B(\frac{\partial}{\partial t} + B)^{-1}\) also maps \(L^s(J; [X,Y]_\theta)\) and \(L^s(J; (X,Y)_{\theta,s})\) continuously into itself. So the assertion follows again by Remark 2.5.2 iv).

This lemma will lead to the main result of this section, maximal regularity of \(A\) in various distribution spaces, as soon as we can show this in the space \(\tilde{H}^{-1,q}_\Gamma\), what we will do now. Precisely, we will show the following result.

**Theorem 2.5.4.** Let \(\Omega, \Gamma\) fulfill Assumption 2.3.2 and set \(q_{\text{iso}} := \sup M_{\text{iso}}\), where

\[
M_{\text{iso}} := \{ q \in [2, \infty[ : -\nabla \cdot \mu \nabla + 1 : H^{1,q}_\Gamma \to \tilde{H}^{-1,q}_\Gamma \text{ is a topological isomorphism} \}.
\]

Then \(-\nabla \cdot \mu \nabla\) satisfies maximal parabolic regularity on \(\tilde{H}^{-1,q}_\Gamma\) for all \(q \in [2, q_{\text{iso}}[,\) where by \(r^*\) we denote the Sobolev conjugated index of \(r\), i.e.

\[
r^* = \begin{cases} 
\infty, & \text{if } r \geq d, \\
\left(\frac{1}{r} - \frac{1}{d}\right)^{-1}, & \text{if } r \in [1, d[.
\end{cases}
\]

**Remark 2.5.5.**

i) If \(\Omega, \Gamma\) fulfill Assumption 2.3.2 a), then \(q_{\text{iso}} > 2\), see [GR89] and also [Grö89].

ii) It is clear by Lax-Milgram and interpolation (see Proposition 2.3.4 and Corollary 2.3.5) that \(M_{\text{iso}}\) is the interval \([2, q_{\text{iso}}]\) or \([2, q_{\text{iso}}[\). Moreover, it can be concluded from a deep theorem of Sneiberg [Sne74] (see also [Aus07, Lemma 4.16]) that the second case cannot occur.

In a first step we show

**Theorem 2.5.6.** Let \(\Omega, \Gamma\) fulfill Assumption 2.4.1. Then \(-\nabla \cdot \mu \nabla\) satisfies maximal parabolic regularity on \(\tilde{H}^{-1,q}_\Gamma\) for all \(q \in [2, \infty[\).

This will be a consequence of the following lemma.

**Lemma 2.5.7.** Let \(\Omega, \Gamma\) satisfy Assumption 2.4.1. Then for all \(q \in [2, \infty[\) the set \(\{(-\nabla \cdot \mu \nabla)^s : s \in \mathbb{R}\}\) forms a strongly continuous group on \(\tilde{H}^{-1,q}_\Gamma\), satisfying the estimate

\[
\|(-\nabla \cdot \mu \nabla)^s\|_{L(\tilde{H}^{-1,q}_\Gamma)} \leq c e^{s|\vartheta|}, \quad s \in \mathbb{R}, \quad (2.5.4)
\]

for some \(\vartheta \in [0, \frac{\pi}{2}[\).

Moreover, we have the following resolvent estimate

\[
\|(-\nabla \cdot \mu \nabla + \lambda)^{-1}\|_{L(\tilde{H}^{-1,q}_\Gamma)} \leq \frac{c}{1 + |\lambda|}, \quad \text{Re } \lambda \geq 0. \quad (2.5.5)
\]

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Proof. We first note that Assumption 2.4.1 in particular implies that the Dirichlet boundary part $\partial \Omega \setminus \Gamma$ has non-zero boundary measure. Thus, by Proposition 2.3.11 i), we may fix some $\epsilon > 0$, such that $-\nabla \cdot \mu \nabla - \epsilon$ has a bounded $\mathcal{H}_\infty$-calculus on $L^q$. Since the functions $z \mapsto (z + \epsilon)^{1/2} (z + \epsilon)^{i s} (z + \epsilon)^{-1/2}$, $s \in \mathbb{R}$, and $z \mapsto (z + \epsilon + \lambda)^{-1} = (z + \epsilon)^{1/2} (\lambda + z + \epsilon)^{-1/2}$, $\Re \lambda \geq 0$, are in $\mathcal{H}_\infty(\Sigma_\phi)$ for all $\phi \in [0, \pi]$, one has the operator identities

$$( -\nabla \cdot \mu \nabla )^{i s} = ( -\nabla \cdot \mu \nabla )^{1/2} ( -\nabla \cdot \mu \nabla )^{i s} ( -\nabla \cdot \mu \nabla )^{-1/2}, \quad s \in \mathbb{R},$$

and

$$( -\nabla \cdot \mu \nabla + \lambda )^{-1} = ( -\nabla \cdot \mu \nabla )^{1/2} ( -\nabla \cdot \mu \nabla + \lambda )^{-1} ( -\nabla \cdot \mu \nabla )^{-1/2}, \quad \Re \lambda \geq 0,$$

on $L^q$. Under Assumption 2.4.1 $( -\nabla \cdot \mu \nabla )^{1/2}$ is a topological isomorphism between $L^q$ and $\dot{H}^{-1,q}_\Gamma$ for every $q \in [2, \infty[$, thanks to Theorem 2.4.3 and Theorem 2.4.4. Thus, one can estimate for every $f \in L^q$

$$\| ( -\nabla \cdot \mu \nabla )^{i s} f \|_{\dot{H}^{-1,q}_\Gamma} \leq \| ( -\nabla \cdot \mu \nabla )^{1/2} \|_{L^q(\dot{H}^1, L^q)} \| ( -\nabla \cdot \mu \nabla )^{i s} \|_{L^q(\dot{H}^{-1,q}_\Gamma)} \| ( -\nabla \cdot \mu \nabla )^{-1/2} \|_{L^q(\dot{H}^{-1,q}_\Gamma, L^q)} \| f \|_{L^q}.$$

Since $L^q$ is dense in $\dot{H}^{-1,q}_\Gamma$, this inequality extends to all of $\dot{H}^{-1,q}_\Gamma$. Together with Proposition 2.3.11 ii) this yields the estimate (2.5.4), which also implies the group property, see [Ama95, Thm. III.4.7.1 and Cor. III.4.7.2].

(2.5.5) is proved analogously to (2.5.4), only using (2.5.7) instead of (2.5.6) and the corresponding resolvent estimate in $L^q$, cf. Proposition 2.3.8 v) (note that here $-\nabla \cdot \mu \nabla$ is continuously invertible).

It follows the proof of Theorem 2.5.6: By Theorems 2.4.3 and 2.4.4, $\dot{H}^{-1,q}_\Gamma$ is an isomorphic image of the UMD space $L^q$ and, hence, a UMD space itself. Since by Lemma 2.5.7 the operator $-\nabla \cdot \mu \nabla$ generates an analytic semigroup and has bounded imaginary powers with the right bound, maximal parabolic regularity follows by the Dore-Venni result [DV87].

Now we intend to 'globalize' Theorem 2.5.6, in other words: We prove that $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $\dot{H}^{-1,q}_\Gamma$ for suitable $q$ if $\Omega$, $\Gamma$ satisfy only Assumption 2.3.2, i.e. if $\alpha K_\omega$, $\alpha (K_\omega \cup \Sigma)$ and $\alpha (K_\omega \cup \Sigma_0)$ need only to be model sets for the constellation around boundary points. Obviously, then the variety of admissible $\Omega$’s and $\Gamma$’s increases considerably, in particular, $\Gamma$ may have more than one connected component.

2.5.1. Auxiliaries

We continue with some results which in essence allow to restrict distributions to subdomains and, on the other hand, to extend them to a larger domain – including the adequate boundary behavior.

Lemma 2.5.8. Let $\Omega, \Gamma$ satisfy Assumption 2.3.2 and let $\Upsilon \subseteq \mathbb{R}^d$ be open, such that $\Omega_\bullet := \Omega \cap \Upsilon$ is also a Lipschitz domain. Furthermore, we put $\Gamma_\bullet := \Gamma \cap \Upsilon$ and fix an arbitrary function $\eta \in C^\infty_0(\mathbb{R}^d)$ with supp($\eta$) $\subseteq \Upsilon$. Then for any $q \in [1, \infty[$ we have the following assertions.

i) If $v \in H^{1,q}_\Gamma(\Omega)$, then $\eta v|_{\Omega_\bullet} \in H^{1,q}_{\Gamma_\bullet}(\Omega_\bullet)$ and the mapping

$$H^{1,q}_\Gamma(\Omega) \ni v \mapsto \eta v|_{\Omega_\bullet} \in H^{1,q}_{\Gamma_\bullet}(\Omega_\bullet)$$

is continuous.
ii) Let for any \( v \in L^1(\Omega) \) the symbol \( \tilde{v} \) indicate the extension of \( v \) to \( \Omega \) by zero. Then the mapping

\[
H^{1,q}_\Gamma(\Omega) \ni v \mapsto \tilde{v}
\]

has its image in \( H^{1,q}_\Gamma(\Omega) \) and is continuous.

Proof. For the proof of both items we will employ the following well known set inclusion (cf. [Die71, Ch. 3.8])

\[
(\partial \Omega \cap \Upsilon) \cup (\Omega \cap \partial \Upsilon) \subseteq \partial \Omega_* \subseteq (\partial \Omega \cap \Upsilon) \cup (\overline{\Omega} \cap \partial \Upsilon).
\]  

(2.5.8)

i) First one observes that the multiplication with \( \eta \) combined with the restriction is a continuous mapping from \( H^{1,q}_\Gamma(\Omega) \) into \( H^{1,q}_\Gamma(\Omega) \). Thus, we only have to show that the image is contained in \( H^{1,q}_\Gamma(\Omega) \), which, in turn, is sufficient to show for elements of the dense subset

\[
\{ v|_\Omega : v \in C^\infty(\mathbb{R}^d), \text{supp}(v) \cap (\partial \Omega \setminus \Gamma) = \emptyset \}
\]

only. By (2.5.8) we get for such functions

\[
\text{supp}(\eta v) \cap (\partial \Omega_* \setminus \Gamma_*) \subseteq \text{supp}(\eta) \cap \text{supp}(v) \cap \left( (\partial \Omega \cap \Upsilon) \cup (\overline{\Omega} \cap \partial \Upsilon) \right) \setminus (\Gamma \cap \Upsilon).
\]

Since \( (\partial \Omega \cap \Upsilon) \cap (\Gamma \cap \Upsilon) = \emptyset \), we see

\[
\left( (\partial \Omega \cap \Upsilon) \cup (\overline{\Omega} \cap \partial \Upsilon) \right) \setminus (\Gamma \cap \Upsilon) = \left( (\partial \Omega \cap \Upsilon) \setminus (\Gamma \cap \Upsilon) \right) \cup \left( (\overline{\Omega} \cap \partial \Upsilon) \setminus (\Gamma \cap \Upsilon) \right)
\]

This, together with \( \text{supp}(\eta) \subseteq \Upsilon \), yields

\[
\text{supp}(\eta v) \cap (\partial \Omega_* \setminus \Gamma_*) \subseteq \text{supp}(\eta) \cap \text{supp}(v) \cap (\partial \Omega \setminus \Gamma) = \emptyset.
\]

ii) Let \( v \in C^\infty(\mathbb{R}^d) \) with \( \text{supp}(v) \cap (\partial \Omega_* \setminus \Gamma_*) = \emptyset \). Since by the left hand side of (2.5.8) we have

\[
\partial \Omega_* \setminus \Gamma_* \supseteq (\partial \Omega \cap \Upsilon) \setminus \Gamma_* = \Upsilon \cap (\partial \Omega \setminus \Gamma),
\]

it follows \( \text{supp}(v) \cap (\Upsilon \cap (\partial \Omega \setminus \Gamma)) = \emptyset \). Combining this with \( \text{supp}(\eta) \subseteq \Upsilon \), we obtain

\[
\text{supp}(\eta v) \cap (\partial \Omega \setminus \Gamma) = \text{supp}(\eta v) \cap (\Upsilon \cap (\partial \Omega \setminus \Gamma)) = \emptyset,
\]

so \( \eta v|_\Omega \in H^{1,q}_\Gamma(\Omega) \). Furthermore, it is not hard to see that \( \| \eta v \|_{H^{1,q}_\Gamma(\Omega)} \leq c_\eta \| v \|_{H^{1,q}_\Gamma(\Omega)} \), where the constant \( c_\eta \) is independent from \( v \). Thus, the assertion follows, since \( \{ v|_\Omega : v \in C^\infty(\mathbb{R}^d), \text{supp}(v) \cap (\partial \Omega_* \setminus \Gamma_*) = \emptyset \} \) is dense in \( H^{1,q}_\Gamma(\Omega) \) and \( H^{1,q}_\Gamma(\Omega) \) is closed in \( H^{1,q}_\Gamma(\Omega) \).

\[ \square \]

Lemma 2.5.9. Let \( \Omega, \Gamma, \Upsilon, \eta, \Omega_* \) and \( \Gamma_* \) be as in the preceding lemma, but assume \( \eta \) to be real valued. Denote by \( \mu_* \) the restriction of the coefficient function \( \mu \) to \( \Omega_* \) and assume \( v \in H^{1,q}_\Gamma(\Omega) \) to be the solution of

\[
-\nabla \cdot \mu \nabla v = f \in \tilde{H}^{-1,2}_\Gamma(\Omega).
\]

Then the following holds true:

i) For all \( q \in [1, \infty] \) the anti-linear form

\[
f_* : w \mapsto \langle f, \eta w \rangle_{\tilde{H}^{-1,2}_\Gamma(\Omega)}
\]

(where \( \eta w \) again means the extension of \( \eta w \) by zero to the whole \( \Omega \) is well defined and continuous on \( \tilde{H}^{-1,2}_\Gamma(\Omega) \)), whenever \( f \) is an anti-linear form from \( \tilde{H}^{-1,q}_\Gamma(\Omega) \). The mapping \( \tilde{H}^{-1,q}_\Gamma(\Omega) \ni f \mapsto f_* \in \tilde{H}^{-1,q}_\Gamma(\Omega) \) is continuous.
ii) If we denote the anti-linear form
\[ H^{1,2}_{\Gamma^*}(\Omega) \ni w \mapsto \int_{\Omega} v \mu \cdot \nabla \eta \cdot \nabla w \, dx \]
by \( I_v \), then \( u := \eta v|_{\Omega^*} \) satisfies
\[-\nabla \cdot \mu v \nabla u = -\mu v|_{\Omega^*} \cdot \nabla \eta|_{\Omega^*} + I_v + f^* .\]

iii) For every \( q \geq 2 \) and all \( r \in [2, q^*] \) (\( q^* \) denoting again the Sobolev conjugated index of \( q \)) the mapping
\[ H^{1,q}_{\Gamma}(\Omega) \ni v \mapsto -\mu v|_{\Omega^*} \cdot \nabla \eta|_{\Omega^*} + I_v + f_v^* \in \tilde{H}^{-1,r}_{\Gamma^*}(\Omega^*) \quad (2.5.9) \]
is well defined and continuous.

Proof. i) The mapping \( f \mapsto f^* \) is the adjoint to \( v \mapsto \tilde{\eta} v \) which maps by the preceding lemma \( H^{1,q}_{\Gamma^*}(\Omega^*) \) continuously into \( H^{1,q}_{\Gamma^*}(\Omega) \).

ii) For every \( w \in H^{1,2}_{\Gamma^*}(\Omega^*) \) we have
\[ \langle -\nabla \cdot \mu v \nabla u, w \rangle_{\tilde{H}^{-1,2}_{\Gamma^*}(\Omega^*)} = \int_{\Omega} \mu v|_{\Omega^*} \cdot \nabla w \, dx \]
\[ = -\int_{\Omega} \mu v|_{\Omega^*} \cdot \nabla w \, dx + \int_{\Omega} \mu v|_{\Omega^*} \cdot \nabla \eta \, dx + \int_{\Omega} \mu v \cdot \nabla \tilde{\eta} \, dx \quad (2.5.10) \]

An application of the definitions of \( I_v \) and \( f_v^* \) yields the assertion.

iii) We regard the terms on the right hand side of (2.5.9) from left to right: \( |\nabla \eta| \in L^\infty(\Omega^*) \) and \( |\mu v|_{\Omega^*} \in L^q(\Omega^*) \), consequently \( \mu v|_{\Omega^*} \cdot \nabla \eta|_{\Omega^*} \in L^q(\Omega^*) \). This gives by Sobolev embedding and duality \( \mu v|_{\Omega^*} \cdot \nabla \eta|_{\Omega^*} \in (H^{1,q'}(\Omega^*))' \hookrightarrow \tilde{H}^{-1,r}_{\Gamma^*}(\Omega^*) \). On the other hand, we have \( v \in H^{1,q}_{\Gamma^*}(\Omega^*) \hookrightarrow L^q(\Omega) \). Thus, concerning \( I_v \), we can estimate
\[ |\langle I_v, w \rangle_{\tilde{H}^{-1,1,r}_{\Gamma^*}(\Omega^*)}| \leq \| v \|_{L^q(\Omega^*)} \| \mu \|_{L^\infty(\Omega^*)} \| \nabla \eta \|_{L^q(\Omega^*)} \| w \|_{H^{1,q'}(\Omega^*)}, \]
what implies the assertion. \( \square \)

Remark 2.5.10. It is the lack of integrability for the gradient of \( v \) (see the counterexample in [ERS07, Ch. 4]) together with the quality of the needed Sobolev embeddings which limits the quality of the correction terms. In the end it is this effect which prevents the applicability of the localization procedure in Subsection 2.5.2 in higher dimensions – at least when one aims at a \( q > d \).

Remark 2.5.11. If \( v \in L^2(\Omega) \) is a regular distribution, then \( v^* \) is the regular distribution \((\eta v)|_{\Omega^*}\).

Lemma 2.5.12. Let in the terminology of Lemma 2.5.9 \( \chi \in C^\infty(\mathbb{R}^d) \) be a function with \( \text{supp}(\chi) \subseteq \Sigma \) and \( \chi \equiv 1 \) in a neighborhood of \( \text{supp}(\eta) \). Furthermore, for \( q \in [1, \infty] \), we define for every \( f \in \tilde{H}^{-1,q}_{\Gamma^*}(\Omega) \) the element \( f^* \in \tilde{H}^{-1,q}_{\Gamma^*}(\Omega) \) by
\[ \langle f^*, \psi \rangle_{\tilde{H}^{-1,q}_{\Gamma^*}(\Omega)} := \langle f, (\chi \psi)|_{\Omega^*} \rangle_{\tilde{H}^{-1,q}_{\Gamma^*}(\Omega^*)}, \]
\( \psi \in H^{1,q}_{\Gamma^*}(\Omega) \). (The definition is justified by Lemma 2.5.8.) Then

i) For every \( f \in \tilde{H}^{-1,q}_{\Gamma^*}(\Omega^*) \) one has \( f^* \in \tilde{H}^{-1,q}_{\Gamma^*}(\Omega) \), and the mapping
\[ \tilde{H}^{-1,q}_{\Gamma^*}(\Omega^*) \ni f \mapsto f^* \in \tilde{H}^{-1,q}_{\Gamma^*}(\Omega) \]
is continuous.
ii) For any \( f \in \dot{H}^{-1,q}_r(\Omega) \) one has the identity \( (f^*)^* = \eta f \in \dot{H}^{-1,q}_r(\Omega) \).

iii) If \( v \in H^{1,2}_r(\Omega) \) and \( -\nabla \cdot \mu \nabla (\eta v|_{\Gamma_*}) \in \dot{H}^{-1,q}_r(\Omega_*), \) then
\[
(\nabla \cdot \mu \nabla (\eta v|_{\Gamma_*}))^* = -\nabla \cdot \mu \nabla (\eta v) \in \dot{H}^{-1,q}_r(\Omega).
\]

Proof. i) The mapping \( f \mapsto f^* \) is the adjoint to \( H^{1,q}_r(\Omega) \ni v \mapsto (\chi v)|_{\Gamma_*} \) which acts continuously into \( \dot{H}^{-1,q}_r(\Omega_*), \) see Lemma 2.5.8.

ii) We only need to prove the assertion for \( f \in L^q(\Omega) \), because \( L^q(\Omega) \) is dense in \( \dot{H}^{-1,q}_r(\Omega) \) and the mappings \( \dot{H}^{-1,q}_r(\Omega) \ni f \mapsto (f^*)^* \in \dot{H}^{-1,q}_r(\Omega) \) and \( \dot{H}^{-1,q}_r(\Omega) \ni f \mapsto \eta f \in \dot{H}^{-1,q}_r(\Omega) \) are both continuous. For \( f \in L^q(\Omega) \) the assertion follows directly from the definitions of \( f^* \) and \( f^*. \)

iii) For any \( \psi \in H^{1,q}_r(\Omega) \) we have
\[
\langle (\nabla \cdot \mu \nabla (\eta v|_{\Gamma_*}))^*, \psi \rangle_{\dot{H}^{-1,q}_r(\Omega)} = \langle -\nabla \cdot \mu \nabla (\eta v|_{\Gamma_*}), (\chi \psi)|_{\Gamma_*} \rangle_{\dot{H}^{-1,q}_r(\Omega)}
\]
\[
= \int_{\Omega_*} \mu \nabla (\eta v) \cdot \nabla (\chi \psi) \, dx
\]
\[
= \int_{\Omega} \mu \nabla (\eta v) \cdot \nabla \psi \, dx = \langle -\nabla \cdot \mu \nabla (\eta v), \psi \rangle_{\dot{H}^{-1,q}_r(\Omega)},
\]

because \( \eta \equiv 0 \) on \( \Omega \setminus \Upsilon \) and \( \chi \equiv 1 \) on \( \text{supp}(\eta) \).

\(\square\)

### 2.5.2. Core of the proof of Theorem 2.5.4

We are now in the position to start the proof of Theorem 2.5.4. We first note that in any case the operator \( -\nabla \cdot \mu \nabla \) admits maximal parabolic regularity on the Hilbert space \( \dot{H}^{-1,2}_r \), since its negative generates an analytic semigroup on this space by Proposition 2.3.8, cf. Remark 2.5.2 iii). Thus, defining
\[
M_{\text{MR}} := \{ q \geq 2 : -\nabla \cdot \mu \nabla \text{ admits maximal regularity on } \dot{H}^{-1,q}_r \}
\]
and \( q_{\text{MR}} := \sup M_{\text{MR}} \), yields \( q_{\text{MR}} \geq 2 \). In the same way as for \( q_{\text{iso}} \) and using Lemma 2.5.3, we see by interpolation that \( M_{\text{MR}} \) is \( \{2\} \) or an interval with left endpoint 2. Our aim is to show that in fact \( q_{\text{MR}} \geq q_{\text{iso}}^* \), so we assume that \( q_{\text{MR}} < q_{\text{iso}}^* \). The main step towards a contradiction is contained in the following lemma.

**Lemma 2.5.13.** Let \( \Omega, \Gamma, \Upsilon, \eta, \Omega_* \), \( \Gamma_* \), \( \mu_* \) be as before. Assume that \( -\nabla \cdot \mu_* \nabla \) satisfies maximal parabolic regularity on \( \dot{H}^{-1,q}_r(\Omega) \) for all \( q \in [2, \infty[ \) and that \( -\nabla \cdot \mu \nabla \) satisfies maximal parabolic regularity on \( \dot{H}^{-1,q}_r(\Omega) \) for some \( q \in [2, q_{\text{iso}}] \). If \( r \in [q, q^*] \) and \( G \in L^s(J; \dot{H}^{-1,q}_r(\Omega)) \), then the unique solution \( V \in W^{1,s}(J; \dot{H}^{-1,q}(\Omega)) \cap L^s(J; \text{dom}_\dot{H}^{-1,q}(\Omega)(-\nabla \cdot \mu \nabla)) \) of
\[
V' - \nabla \cdot \mu \nabla V = G, \quad V(T_0) = 0,
\]

even satisfies
\[
\eta V \in W^{1,s}(J; \dot{H}^{-1,q}(\Omega)) \cap L^s(J; \text{dom}_\dot{H}^{-1,q}(\Omega)(-\nabla \cdot \mu \nabla)).
\]
Proof. \( V \in L^s(J; \text{dom}_{\tilde{H}^{-1, r}_\Gamma}(-\nabla \cdot \mu \nabla)) \) implies, due to our supposition \( q \in [2, q_{iso}] \) and Remark 2.5.5 ii, \( V \in L^s(J; \tilde{H}^{1, q}_\Gamma(\Omega)) \). Of course, equation (2.5.11) is to be read as follows: For almost all \( t \in J \) it holds \(-\nabla \cdot \mu \nabla V(t) = G(t) - V'(t) \), where \( V' \) is the derivative in the sense of \( \tilde{H}^{-1, q}_\Gamma \)-valued distributions. Hence, Lemma 2.5.9 ii) implies for almost all \( t \in J \)

\[
(V'(t))_\Omega - \nabla \cdot \mu_\Omega \nabla ((\eta V(t))|_{\Omega_\tau}) = -\mu_\Omega \nabla V(t)|_{\Omega_\tau} \cdot \nabla \eta|_{\Omega_\tau} + I_{V(t)} + (G(t))_\Omega. \tag{2.5.12}
\]

Since by Lemma 2.5.9 i) the mapping \( \tilde{H}^{-1, r}_\Gamma(\Omega) \ni f \mapsto f_\Omega \in \tilde{H}^{-1, r}_\Gamma(\Omega_\tau) \) is continuous, we have \( (G(\cdot))_\Omega \in L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \). Moreover, the property \( V \in L^s(J; \tilde{H}^{1, q}_\Gamma(\Omega)) \) and iii) of Lemma 2.5.9 assure \(-\mu_\Omega \nabla V(\cdot)|_{\Omega_\tau} \cdot \nabla \eta|_{\Omega_\tau} + I_{V(\cdot)} \in L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \). Thus, the right hand side of (2.5.12) is contained in \( L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \). Let us next inspect the term \((V'(t))_\Omega\); Since \( \tilde{H}^{-1, q}_\Gamma(\Omega) \ni w \mapsto w_\Omega \in \tilde{H}^{-1, q}_\Gamma(\Omega_\tau) \) is linear and continuous, it equals \((V_\Omega)'(t)\). But by Remark 2.5.11 the function \( t \mapsto V_\Omega(t) \) is identical to the function \( t \mapsto (\eta V(\cdot))|_{\Omega_\tau} \). Hence, \( (\eta V(\cdot))|_{\Omega_\tau} \) satisfies the following equation in \( \tilde{H}^{-1, q}_\Gamma(\Omega_\tau) \):

\[
(\eta V(\cdot))_\Omega'(t) - \nabla \cdot \mu_\Omega \nabla ((\eta V(\cdot))|_{\Omega_\tau}) = -\mu_\Omega \nabla V(t)|_{\Omega_\tau} \cdot \nabla \eta|_{\Omega_\tau} + I_{V(t)} + (G(t))_\Omega. \tag{2.5.13}
\]

By supposition, \(-\nabla \cdot \mu_\Omega \nabla \) fulfills maximal parabolic regularity in \( \tilde{H}^{-1, r}_\Gamma(\Omega_\tau) \). As the right hand side of (2.5.13) is in fact from \( L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \), this implies that there is a unique function \( U \in W^{1, s}(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \cap L^s(J; \text{dom}_{\tilde{H}^{-1, r}_\Gamma(\Omega_\tau)}(-\nabla \cdot \mu_\Omega \nabla)) \) which satisfies \( U(T_0) = 0 \) and

\[
U'(t) - \nabla \cdot \mu_\Omega \nabla (U(t)) = -\mu_\Omega \nabla V(t)|_{\Omega_\tau} \cdot \nabla \eta|_{\Omega_\tau} + I_{V(t)} + (G(t))_\Omega. \tag{2.5.14}
\]

as an equation in \( L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \). However, this last equation can (by the embedding \( \tilde{H}^{-1, r}_\Gamma(\Omega_\tau) \to \tilde{H}^{-1, q}_\Gamma(\Omega_\tau) \)) also be read as an equation in \( L^s(J; \tilde{H}^{-1, q}_\Gamma(\Omega_\tau)) \). Since the solution is unique in \( L^s(J; \tilde{H}^{-1, q}_\Gamma(\Omega_\tau)) \), (2.5.13) and (2.5.14) together imply \( U = (\eta V(\cdot))|_{\Omega_\tau} \) and, consequently,

\[
(V(\cdot))_\Omega = (\eta V(\cdot))|_{\Omega_\tau} \in W^{1, s}(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)) \cap L^s(J; \text{dom}_{\tilde{H}^{-1, r}_\Gamma(\Omega_\tau)}(-\nabla \cdot \mu_\Omega \nabla)), \tag{2.5.15}
\]

see Remark 2.5.11.

We now aim at a re-interpretation of this regularity in terms of the space \( W^{1, s}(J; \tilde{H}^{-1, r}_\Gamma(\Omega)) \cap L^s(J; \text{dom}_{\tilde{H}^{-1, r}_\Gamma(\Omega)}(-\nabla \cdot \mu_\Omega \nabla)) \). Observe that (2.5.15) implies that \(-\nabla \cdot \mu_\Omega \nabla ((\eta V(\cdot))|_{\Omega_\tau}) \in L^s(J; \tilde{H}^{-1, q}_\Gamma(\Omega_\tau)) \). Applying Lemma 2.5.12 iii), this gives

\[
-\nabla \cdot \mu \nabla (\eta V(\cdot)) \in L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega)). \tag{2.5.16}
\]

Obviously, \( V \in L^s(J; \tilde{H}^{1, q}_\Gamma) \) yields \( \eta V \in L^s(J; \tilde{H}^{1, q}_\Gamma) \), while \( r \in ]q, q^*[ \) implies the embedding \( \tilde{H}^{1, q}_\Gamma \hookrightarrow L^r \hookrightarrow \tilde{H}^{-1, r}_\Gamma \). Hence, one obtains

\[
\eta V \in L^s(J; \tilde{H}^{1, q}_\Gamma) \hookrightarrow L^s(J; \tilde{H}^{-1, r}_\Gamma). \tag{2.5.17}
\]

Combining this with (2.5.16), we find

\[
\eta V(\cdot) \in L^s(J; \text{dom}_{\tilde{H}^{-1, r}_\Gamma(\Omega)}(-\nabla \cdot \mu \nabla)).
\]

On the other hand, (2.5.15) implies

\[
((V(\cdot))_\tau)' \in L^s(J; \tilde{H}^{-1, r}_\Gamma(\Omega_\tau)).
\]
2. Maximal Regularity for Operators with Mixed Boundary Conditions

By Lemma 2.5.12 i), we have $((V(\cdot))\ast)^{\ast} \in L^s(J; \mathring{H}^{-1,r}_T(\Omega))$. But as before $(((V(\cdot))\ast)^{\ast})^{\ast}$ equals $((V(\cdot))\ast)^{\ast}$, which, by Lemma 2.5.12 ii), is $(\eta V(\cdot))^{\ast}$. Summing up, we get

$$(\eta V(\cdot))^{\ast} \in L^s(J; \mathring{H}^{-1,r}_T(\Omega)).$$

Taking into account (2.5.17) again, this gives

$$\eta V(\cdot) \in W^{1,s}(J; \mathring{H}^{-1,r}_T(\Omega)),$$

what proves the lemma. \hfill \Box

**Proof of Theorem 2.5.4.** For every $x \in \Omega$ let $\Xi_x \subseteq \Omega$ be an open cube, containing $x$. Furthermore, let for any point $x \in \partial\Omega$ an open neighborhood be given according to the assumption of the theorem (see Assumption 2.3.2). Possibly shrinking this neighborhood to a smaller one, one obtains a new neighborhood $\Upsilon_x$, and a bi-Lipschitz, volume-preserving mapping $\phi_x$ from a neighborhood of $\overline{\Xi}_x$ into $\mathbb{R}^d$ such that $\phi_x(\Upsilon_x \cap (\Omega \cup \Gamma)) = \beta K_\varepsilon, \beta(K_\varepsilon \cup \Sigma)$ for some $\beta = \beta(x) > 0$.

Obviously, the $\Xi_x$ and $\Upsilon_x$ together form an open covering of $\overline{\Omega}$. Let $\Xi_{x_1}, \ldots, \Xi_{x_k}$, together with $\Upsilon_{x_{k+1}}, \ldots, \Upsilon_{x_k}$ be a finite subcovering and $\eta_1, \ldots, \eta_l$ a $C^\infty$ partition of unity, subordinate to this subcovering. Set $\Omega_j := \Xi_{x_j} = \Upsilon_{x_j} \cap \Omega$ for $j \in \{1, \ldots, k\}$ and $\Omega_j := \Upsilon_{x_j} \cap \Omega$ for $j \in \{k+1, \ldots, l\}$. Moreover, set $\Gamma_j := \emptyset$ for $j \in \{1, \ldots, k\}$ and $\Gamma_j := \Upsilon_{x_j} \cap \Gamma$ for $j \in \{k+1, \ldots, l\}$.

Denoting the restriction of $\mu$ to $\Omega_j$ by $\mu_j$, each operator $-\nabla \cdot \mu_j \nabla$ satisfies maximal parabolic regularity in $\mathring{H}^{-1,q}_{1,j}(\Omega_j)$ for all $q \in [2, \infty]$ and all $j$, according to Theorem 2.5.6.

Assuming now $q_{MR} < q_{iso}$, we may choose some $q \in [2, q_{iso}]$ with $q_{MR} < q^*$. In order to see this, we first observe that

$$p \leq q \iff p^* \leq q^* \quad (2.5.18)$$

holds, whenever $p^* < \infty$. Setting $q = \max\left\{2, \frac{d\tilde{q}}{(d+\tilde{q})}\right\}$ for some $\tilde{q} \in [q_{MR}, q_{iso}]$, this, together with $(d\tilde{q}/(d+\tilde{q}))^* = \tilde{q}$, yields immediately that $q^* = \max\left\{2^*, \tilde{q}\right\} \geq \tilde{q} > q_{MR}$. Furthermore, again by (2.5.18), we have $q < q_{iso}$, since $q^* < q_{iso}$ and finally $q \geq 2$ is guaranteed by the choice of $q$. Having the so chosen $q$ at hand, we take some $r \in \max\{q, q_{MR}\}$, $q^*[t]$, which is possible due to $q < q^*$. Now, let $G \in L^s(J; \mathring{H}^{-1,r}_T(\Omega))$ be given. Then by Lemma 2.5.13 the unique solution $V \in W^{1,s}(J; \mathring{H}^{-1,q}_T(\Omega)) \cap L^s(J; dom_{\mu_j}(-\nabla \cdot \mu_j \nabla))$ of (2.5.11) satisfies $\eta_j V \in W^{1,s}(J; \mathring{H}^{-1,r}_T(\Omega)) \cap L^s(J; dom_{\mu_j}(-\nabla \cdot \mu_j \nabla))$ for every $j$. This implies maximal parabolic regularity for $-\nabla \cdot \mu \nabla$ on $\mathring{H}^{-1,r}_T$, in contradiction to $r > q_{MR}$. Thus we have $q_{MR} \geq q_{iso}$ and the proof is finished. \hfill \Box

**Remark 2.5.14.** Note that Theorem 2.5.4 already yields maximal regularity of $-\nabla \cdot \mu \nabla$ on $\mathring{H}^{-1,q}_T$ for all $q \in [2, 2^*[ without any additional information on $dom_{\mu_j}(-\nabla \cdot \mu_j \nabla)$ nor on $dom_{\mu_j}(-\nabla \cdot \mu_j \nabla)$.

In the 2-d case this already implies maximal regularity for every $q \in [2, \infty[$. Taking into account Remark 2.5.5 i), without further knowledge on the domains we get in the 3-d case every $q \in [2, 6 + \varepsilon[$ and in the 4-d case every $q \in [2, 4 + \varepsilon[$, where $\varepsilon$ depends on $\Omega, \Gamma, \mu$.

### 2.5.3. The operator $A$

Next we carry over the maximal parabolic regularity result, up to now proved for $-\nabla \cdot \mu \nabla$ on the spaces $\mathring{H}^{-1,q}_T$, to the operator $A$ and to a much broader class of distribution spaces. For this we need the following perturbation result.
Lemma 2.5.15. Suppose $q \geq 2$, $\varsigma \in [1 - \frac{1}{q}, 1]$ and $\mathcal{X} \in L^\infty(\Gamma, d\sigma)$ and let $\Omega, \Gamma$ satisfy Assumption 2.3.2. If we define the mapping $Q : \text{dom}_{\bar{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla) \rightarrow \bar{H}^{-\varsigma,q}_\Gamma$ by

$$
(Q\psi, \varphi)_{\bar{H}^{-\varsigma,q}_\Gamma} := \int_\Gamma \mathcal{X} \psi \overline{\varphi} d\sigma, \quad \varphi \in H^{\varsigma,q}_\Gamma
$$

then $Q$ is well defined and continuous. Moreover, it is relatively bounded with respect to $-\nabla \cdot \mu \nabla$, when considered on the space $\bar{H}^{-\varsigma,q}_\Gamma$, and the relative bound may be taken arbitrarily small.

Proof. One has for every $\psi \in \text{dom}_{\bar{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow \text{dom}_{\bar{H}^{-1,2}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow H^{1,2}_\Gamma$

$$
\|Q\psi\|_{\bar{H}^{-\varsigma,q}_\Gamma} = \sup_{\|\varphi\|_{\bar{H}^{\varsigma,q}_\Gamma} = 1} |(Q\psi, \varphi)_{\bar{H}^{-\varsigma,q}_\Gamma}| = \sup_{\|\varphi\|_{\bar{H}^{\varsigma,q}_\Gamma} = 1} \left| \int_\Gamma \mathcal{X} \psi \overline{\varphi} d\sigma \right|
$$

\quad \leq \|\mathcal{X}\|_{L^\infty(\Gamma, d\sigma)} \|\psi\|_{L^2(\partial\Omega, d\sigma)} \sup_{\|\varphi\|_{\bar{H}^{\varsigma,q}_\Gamma} = 1} \|\varphi\|_{L^2(\partial\Omega, d\sigma)},
$$

where the last factor is finite according to Theorem 2.3.6. Let us first consider the case $q = 2$. Then (2.5.19) can be further estimated (see (2.3.6))

$$
\leq c\|\psi\|_{L^2(\partial\Omega, d\sigma)} \leq c\|\psi\|_{\bar{H}^{1,2}_\Gamma} \leq c\|\psi\|_{\bar{H}^{1,2}_\Gamma} \leq c\|\psi\|_{\bar{H}^{1,2}_\Gamma} + \frac{c}{\epsilon}\|\psi\|_{\bar{H}^{1,2}_\Gamma}
$$

by Young’s inequality. Taking into account $\text{dom}_{\bar{H}^{-1,2}_\Gamma}(-\nabla \cdot \mu \nabla) = H^{1,2}_\Gamma$, this proves the case $q = 2$. Concerning the case $q > d$, we make use of the embedding

$$
\text{dom}_{\bar{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow \text{dom}_{\bar{H}^{-1,2}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow C^\alpha(\Omega) \quad \text{for some } \alpha = \alpha(q) > 0,
$$

if $q > d$ (see [Gri02]). Thus, for $q > d + \frac{1}{2}$ the term $\|\psi\|_{L^2(\partial\Omega, d\sigma)}$ in (2.5.19) can be estimated by $(\sigma(\partial\Omega))^\frac{1}{2}\|\psi\|_{C(\overline{\Omega})}$, which shows, due to (2.5.20), the asserted continuity of $Q$, if $q > d + \frac{1}{2}$. Since $\text{dom}_{\bar{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow C^\alpha(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact and $C(\overline{\Omega}) \hookrightarrow \bar{H}^{-\varsigma,q}_\Gamma$ is continuous and injective, we may apply Ehrlich’s lemma (see [Wlo87, Ch. I, Prop. 7.3]) and estimate

$$
\|\psi\|_{C(\overline{\Omega})} \leq \epsilon\|\psi\|_{\text{dom}_{\bar{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla)} + \beta(\epsilon)\|\psi\|_{\bar{H}^{-\varsigma,q}_\Gamma}, \quad \psi \in \text{dom}_{\bar{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla),
$$

for arbitrary $\epsilon > 0$. Together with (2.5.19) this yields the second assertion for $q > d + \frac{1}{2}$. Concerning the remaining case $q \in [2, d + \frac{1}{2}]$, we employ the representation

$$
\bar{H}^{-1,2}_\Gamma = [\bar{H}^{-1,2d}_\Gamma, \bar{H}^{-1,2}_\Gamma)_0 \quad \text{with } \theta = \frac{1}{q}\cdot \frac{2d - q}{d - 1}
$$

(see Corollary 2.3.5) and we will invest the knowledge $\text{dom}_{\bar{H}^{-1,2d}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow L^\infty$ and $\text{dom}_{\bar{H}^{-1,2}_\Gamma}(-\nabla \cdot \mu \nabla) = H^{1,2}_\Gamma$. Clearly, (2.5.21) implies

$$
\text{dom}_{\bar{H}^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla) = [\text{dom}_{\bar{H}^{-1,2d}_\Gamma}(-\nabla \cdot \mu \nabla), \text{dom}_{\bar{H}^{-1,2}_\Gamma}(-\nabla \cdot \mu \nabla)]_0.
$$

Taking $q = 2d$ in (2.5.20) and combining this with the embedding $C^\alpha \hookrightarrow L^r$ for any finite $r$, (2.5.22) yields

$$
\text{dom}_{\bar{H}^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow [L^r, H^{1,2}_\Gamma)_0 = H^{0, \frac{2}{q} - \delta(r, \theta)}_\Gamma,
$$

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where $\delta(r, \theta) \not\subset 0$ for $r \to \infty$, see Proposition 2.3.4. If $q \in \left]2, d + \frac{1}{2}\right]$, then it is clear from the definition of $\theta$ that $\theta \geq \frac{1}{\eta}$, $d - \frac{3}{2} > \frac{1}{\eta}$. On the other hand, one easily verifies $p \in \left[q, q^{2(\frac{d-1}{d-\frac{3}{2}})}\right]$. Thus, choosing $r$ large enough, one gets for every $q \in \left]2, d + \frac{1}{2}\right]$ a continuous embedding

$$\text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(-\nabla \cdot \mu \nabla) \hookrightarrow H^{\frac{1}{2} - \frac{1}{q}, q}_{\Gamma},$$

what gives a compact embedding

$$\text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(-\nabla \cdot \mu \nabla) \hookrightarrow \text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(-\nabla \cdot \mu \nabla) \hookrightarrow H^{\frac{1}{2} - \frac{1}{q}, q}_{\Gamma}. \quad (2.5.23)$$

Due to Theorem 2.3.6, the term $\|\psi\|_{L^q(\partial \Omega, d\sigma)}$ in (2.5.19) may be estimated by $c\|\psi\|_{H^{\frac{1}{2} - \frac{1}{q}, q}_{\Gamma}}$. But, in view of the compactness of the mapping (2.5.23) and the continuity of the injection $H^{\frac{1}{2} - \frac{1}{q}, q}_{\Gamma} \hookrightarrow \tilde{H}^{-1,q}_{\Gamma}$ one may also here apply Ehrling’s lemma and estimate

$$\|\psi\|_{H^{\frac{1}{2} - \frac{1}{q}, q}_{\Gamma}} \leq c\|\psi\|_{\text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(-\nabla \cdot \mu \nabla) + \beta(\epsilon)\|\psi\|_{\tilde{H}^{-1,q}_{\Gamma}}}$$

for $\epsilon$ arbitrarily small. Together with (2.5.19) this shows the assertion in the last case. □

**Theorem 2.5.16.** Suppose $q \geq 2$, $\kappa \in L^\infty(\Gamma, d\sigma)$ and let $\Omega, \Gamma$ satisfy Assumption 2.3.2.

i) If $\kappa \in \left]1 - \frac{1}{q}, 1\right]$, then $\text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(-\nabla \cdot \mu \nabla) = \text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(A)$.

ii) If $\kappa \in \left]1 - \frac{1}{q}, 1\right]$ and $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $\tilde{H}^{-1,q}_{\Gamma}$, then $A$ also does.

iii) The operator $A$ satisfies maximal parabolic regularity on $L^2$. If $\kappa \geq 0$, then $A$ satisfies maximal parabolic regularity on $L^p$ for all $p \in \left]1, \infty\right]$.

iv) Suppose that $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $\tilde{H}^{-1,q}_{\Gamma}$. Then $A$ satisfies maximal parabolic regularity on any of the interpolation spaces

$$[L^2, \tilde{H}^{-1,q}_{\Gamma}|_{\theta}, \quad \theta \in [0, 1],$$

or

$$(L^2, \tilde{H}^{-1,q}_{\Gamma})_{\theta,s}, \quad \theta \in [0, 1], \quad s \in [1, \infty[.$$

Let $\kappa \geq 0$ and $p \in \left]1, \infty\right[$ in case of $d = 2$ or $p \in \left[(\frac{1}{2} + \frac{1}{q})^{-1}, \infty\right[$ if $d \geq 3$. Then $A$ also satisfies maximal parabolic regularity on any of the interpolation spaces

$$[L^p, \tilde{H}^{-1,q}_{\Gamma}|_{\theta}, \quad \theta \in [0, 1],$$

or

$$(L^p, \tilde{H}^{-1,q}_{\Gamma})_{\theta,s}, \quad \theta \in [0, 1], \quad s \in [1, \infty[. \quad (2.5.24)$$

**Proof.**

i) By Lemma 2.5.15, if $\psi \in \text{dom}_{\tilde{H}^{-1,q}_{\Gamma}}(-\nabla \cdot \mu \nabla)$, then $Q\psi$ is well defined and one has the equality $A\psi = -\nabla \cdot \mu \nabla \psi + Q\psi$ by definition of $A$. Thus, the assertion follows from the relative boundedness with relative bound smaller than $1$, shown in Lemma 2.5.15, and a classical perturbation theorem, see [Kat80, Ch. IV.1].
ii) The assertion is also proved by means of a -- highly nontrivial -- perturbation theorem (see [KW01]), which states that, if \( X \) is a UMD space and a densely defined, closed operator \( B \) satisfies maximal parabolic regularity on \( X \), then \( B + B_0 \) also satisfies maximal parabolic regularity on \( X \), provided \( \text{dom}_X(B_0) \supseteq \text{dom}_X(B) \) and \( B_0 \) is relatively bounded with respect to \( B \) with arbitrarily small relative bound. In our case, \( H^{-1,q}_\Gamma \) is -- as the dual of the closed subspace \( H^{-1,q}_\Gamma \) of the UMD space \( H^{-1,q} \) -- itself a UMD space, see [Ama95, Ch. III.4.5] and [Are04, Ch. 6.1]. \( H^{-1,q}_\Gamma \) is the isometric image of \( \hat{H}^{-1,q}_\Gamma \) under the mapping which assigns to \( f \in \hat{H}^{-1,q}_\Gamma \) the linear form \( \hat{H}^{-1,q}_\Gamma \ni \psi \mapsto (f, \psi)_{H^{-1,q}_\Gamma} \).

Hence, \( \hat{H}^{-1,q}_\Gamma \) is also a UMD space. Finally, \( H^{-1,q}_\Gamma \) is a complex interpolation space between the UMD space \( \hat{H}^{-1,q}_\Gamma \) and the UMD space \( L^q \) (see Remark 2.5.17 below), and consequently also a UMD space. Hence, an application of Lemma 2.5.15 yields the result.

iii) The first assertion follows from Proposition 2.3.8 ii) and Remark 2.5.2 iii). The second is shown in [GKR01, Thm. 7.4].

iv) Under the given conditions on \( p \), we have the embedding \( L^p \hookrightarrow \hat{H}^{-1,2}_\Gamma \). Thus, the assertion follows from the preceding points and Lemma 2.5.3.

**Remark 2.5.17.** The interpolation spaces \( [L^p, H^{-1,q}_\Gamma]_\theta (\theta \in [0,1]) \) and \( (L^p, H^{-1,q}_\Gamma)_{\theta,s} (\theta \in [0,1], s \in [1,\infty]) \) are characterized in [GGKR02], see in particular Remark 3.6. Identifying each \( f \in L^q \) with the anti-linear form \( L^q \ni \psi \mapsto \int_{\Omega} f \psi \, dx \) and using again the retraction/coretraction theorem with the coretractions from Corollary 2.3.5, one easily identifies the interpolation spaces in (2.5.24) and (2.5.25). In particular, this yields \( [L^{\infty}, \hat{H}^{-1,q}_\Gamma]_\theta = \hat{H}^{-\theta,q}_\Gamma \) if \( \theta \neq 1 - \frac{1}{q} \).

**Corollary 2.5.18.** Let \( \Omega \) and \( \Gamma \) satisfy Assumption 2.3.2. The operator \(-A\) generates analytic semigroups on all spaces \( \hat{H}^{-1,q}_\Gamma \) if \( q \in [2, q_{\text{im}}] \) and on all the interpolation spaces occurring in Theorem 2.5.16, there \( q \) also taken from \( [2, q_{\text{im}}] \). Moreover, if \( \kappa \geq 0 \), the following resolvent estimates are valid:

\[
\|(A + 1 + \lambda)^{-1}\|_{L(\hat{H}^{-1,q}_\Gamma)} \leq \frac{C_q}{1 + |\lambda|}, \quad \text{Re} \lambda \geq 0. \tag{2.5.26}
\]

**Proof.** The first assertion is implied by Theorem 2.5.4 and Remark 2.5.2 ii), which gives (2.5.26) for \( \lambda \in \gamma + \Sigma_q \) with a fixed \( \gamma \in \mathbb{R} \) and fixed \( \kappa > \pi/2 \). On the other hand, the resolvent of \( A_0 \) is compact (see Proposition 2.3.8), what, due to Lemma 2.5.15, remains true also for \( A \), see [Kat80, Ch. IV.1]. Since no \( \lambda \) with \( \text{Re} \lambda \leq 0 \) is an eigenvalue,

\[
\sup_{\lambda \in \{\lambda : \text{Re} \lambda \geq 0\} \setminus (\gamma + \Sigma_q)} (|\lambda| + 1)\| (A + 1 + \lambda)^{-1} \|_{L(\hat{H}^{-1,q}_\Gamma)} < \infty,
\]

because \( \{\lambda : \text{Re} \lambda \geq 0\} \setminus (\gamma + \Sigma_q) \) is compact. 

\[
\square
\]

### 2.6. Nonlinear parabolic equations

In this chapter we will apply maximal parabolic regularity for the treatment of quasilinear parabolic equations which are of the (formal) type (2.1.1). Concerning all the occurring operators we will formulate precise requirements in Assumption 2.6.11 below.

The outline of the chapter is as follows: First we give a motivation for the choice of the Banach space we will regard (2.1.1)/(2.1.2) in. Afterwards we show that maximal parabolic regularity,
combined with regularity results for the elliptic operator, allows to solve this problem. Below we will transform (2.1.1)/(2.1.2) to a problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
u'(t) + B(u(t))u(t) = S(t, u(t)), \quad t \in J, \\
u(T_0) = u_0.
\end{array} \right.
\end{aligned}
\] (2.6.1)

To give the reader already here an idea what properties of the operators \(-\nabla \cdot G(u)\mu \nabla\) and of the corresponding Banach space are required, we first quote the result on existence and uniqueness for abstract quasilinear parabolic equations (due to Clément/Li [CL94] and Prüss [Prü02]) on which our subsequent considerations will base.

**Proposition 2.6.1.** Suppose that \(B\) is a closed operator on some Banach space \(X\) with dense domain \(D\), which satisfies maximal parabolic regularity on \(X\). Suppose further \(u_0 \in (X, D)_{1-\frac{1}{s}}\) and \(B : J \times (X, D)_{1-\frac{1}{s}} \to \mathcal{L}(D, X)\) to be continuous with \(B = B(T_0, u_0)\). Let, in addition, \(S : J \times (X, D)_{1-\frac{1}{s}} \to X\) be a Carathéodory map and assume the following Lipschitz conditions on \(B\) and \(S\):

1. **(B) For every \(M > 0\) there exists a constant \(C_M > 0\), such that for all \(t \in J\)
   \[
   \|B(t, u) - B(t, \tilde{u})\|_{\mathcal{L}(D, X)} \leq C_M \|u - \tilde{u}\|_{(X, D)_{1-\frac{1}{s}}},
   \]
   if \(\|u\|_{(X, D)_{1-\frac{1}{s}}}, \|\tilde{u}\|_{(X, D)_{1-\frac{1}{s}}} \leq M\).

2. **(R) \(S(\cdot, 0) \in L^s(J; X)\) and for each \(M > 0\) there is a function \(h_M \in L^s(J)\), such that
   \[
   \|S(t, u) - S(t, \tilde{u})\|_X \leq h_M(t) \|u - \tilde{u}\|_{(X, D)_{1-\frac{1}{s}}},
   \]
   holds for a.a. \(t \in J\), if \(\|u\|_{(X, D)_{1-\frac{1}{s}}}, \|\tilde{u}\|_{(X, D)_{1-\frac{1}{s}}} \leq M\).

Then there exists \(T^* \in J\), such that (2.6.1) admits a unique solution \(u\) on \([T_0, T^*[; X\) satisfying
\[
u \in W^{1,s}([T_0, T^*[; X) \cap L^s([T_0, T^*[; D).\]

**Remark 2.6.2.** Up to now we were free to consider complex Banach spaces. But the context of equations like (2.1.1) requires real spaces, in particular in view of the quality of the superposition operator \(\mathcal{F}\). Therefore, from this moment on we use the real versions of the spaces. In particular, \(H^{\psi, q}_F\) is now understood as the dual of the real space \(H^{\psi, q}_F\) and clearly can be identified with the set of anti-linear forms on the complex space \(H^{\psi, q}_F\) that take real values when applied to real functions.

Fortunately, the property of maximal parabolic regularity is maintained for the restriction of the operator \(A\) to the real spaces in case of a real function \(\kappa\), as \(A\) then commutes with complex conjugation.

We will now give a motivation for the choice of the Banach space \(X\) we will use later. It is not hard to see that \(X\) has – in view of the applicability of Proposition 2.6.1 – to fulfill the subsequent demands:

a) The operators \(A\), or at least the operators \(-\nabla \cdot \mu \nabla\), defined in (2.3.8), must satisfy maximal parabolic regularity on \(X\).

b) As in the classical theory (see [LSU67], [GS81], [Str81] and references therein) quadratic gradient terms of the solution should be admissible for the right hand side.
c) The operators \(-\nabla \cdot \mathcal{G}(u)\mu \nabla\) should behave well concerning their dependence on \(u\), see condition (B) above.

d) \(X\) has to contain certain measures, supported on Lipschitz hypersurfaces in \(\Omega\) or on \(\partial \Omega\) in order to allow for surface densities on the right hand side or/and for inhomogeneous Neumann conditions.

The condition in a) is assured by Theorem 2.5.4 and Theorem 2.5.16 for a great variety of Banach spaces, among them candidates for \(X\). Requirement b) suggests that one should have
\[
dom_X(-\nabla \cdot \mu \nabla) \hookrightarrow H^{1,q}_\Gamma \quad \text{and} \quad L^\frac{d}{q} \hookrightarrow X.
\]
Since \(-\nabla \cdot \mu \nabla\) maps \(H^{1,q}_\Gamma\) into \(H^{-1,q}_\Gamma\), this altogether leads to the necessary condition
\[
L^\frac{d}{q} \hookrightarrow X \hookrightarrow H^{-1,q}_\Gamma. \tag{2.6.2}
\]

Sobolev embedding shows that \(q\) cannot be smaller than the space dimension \(d\). Taking into account d), it is clear that \(X\) must be a space of distributions which (at least) contains surface densities. In order to recover the desired property \(\dom_X(-\nabla \cdot \mu \nabla) \hookrightarrow H^{1,q}_\Gamma\) from the necessary condition in (2.6.2), we make for all what follows this general

**Assumption 2.6.3.** There is a \(q > d\), such that \(-\nabla \cdot \mu \nabla + 1: H^{1,q}_\Gamma \to H^{-1,q}_\Gamma\) is a topological isomorphism.

**Remark 2.6.4.** For \(q \geq 4\) Assumption 2.6.3 is generically false in case of mixed boundary conditions, see [Sha68] for the famous counterexample. Moreover, even in the Dirichlet case, when the domain \(\Omega\) has only a Lipschitz boundary or the coefficient function \(\mu\) is constant within layers, one cannot expect \(q \geq 4\), see [JK95] and [EKRS07]. This is the reason, why all our following considerations are restricted to two or three dimensions. Of course, these are the most relevant ones when treating real world applications.

In Section 2.7 we will present examples for domains \(\Omega\), coefficient functions \(\mu\) and Dirichlet boundary parts \(\Omega \setminus \Gamma\), for which Assumption 2.6.3 is fulfilled.

In all what follows the space dimension \(d\) is either 2 or 3 and from now on we fix one \(q \in ]d, 4[\) for which Assumption 2.6.3 holds.

As a first step we will show that Assumption 2.6.3 carries over to a broad class of modified operators.

**Lemma 2.6.5.** Assume that \(\xi\) is a real valued, uniformly continuous function on \(\Omega\) that admits a lower bound \(\xi > 0\). Then the operator \(-\nabla \cdot \xi \mu \nabla + 1\) also is a topological isomorphism between \(H^{1,q}_\Gamma\) and \(H^{-1,q}_\Gamma\).

**Proof.** We identify \(\xi\) with its (unique) continuous continuation to the closure \(\overline{\Omega}\) of \(\Omega\). Furthermore, we observe that for any coefficient function \(\omega\) the inequality
\[
\|\nabla \cdot \omega \nabla\|_{\mathcal{L}(H^{1,q}_\Gamma, H^{-1,q}_\Gamma)} \leq \|\omega\|_{L^\infty(\Omega; L^2(\mathbb{R}^d))} \tag{2.6.3}
\]
holds true. Next, by Assumption 2.6.3 and Corollary 2.5.18 it is clear that
\[
\sup_{y \in \overline{\Omega}} \|(-\nabla \cdot \xi(y)\mu \nabla + 1)^{-1}\|_{\mathcal{L}(H^{1,q}_\Gamma, H^{-1,q}_\Gamma)} \leq \frac{1}{\xi} \sup_{y \in \overline{\Omega}} \|(-\nabla \cdot \mu \nabla + (\xi(y))^{-1})^{-1}\|_{\mathcal{L}(H^{1,q}_\Gamma, H^{-1,q}_\Gamma)} =: \gamma
\]
is finite. Let for any \(x \in \overline{\Omega}\) a ball \(B_x\) around \(x\) be given, such that
\[
\gamma \sup_{y \in B_x \cap \overline{\Omega}} |\xi(x) - \xi(y)|\|\mu\|_{L^\infty(\Omega; L^2(\mathbb{R}^d))} < 1. \tag{2.6.4}
\]
Then, we choose a finite subcovering \(B_{x_1}, \ldots, B_{x_k}\) of \(\overline{\Omega}\) and a partition of unity \(\eta_1, \ldots, \eta_k\) subordinate to this subcovering, and we set \(\Lambda_x := B_x \cap \Omega\).
Assume that $f \in H^{1,q}_\Gamma \subseteq H^{1,2}_\Gamma$ and $v \in H^{1,2}_\Gamma$ is a solution of $-\nabla \cdot \xi \mu \nabla v + v = f$. Then a calculation, completely analogous to (2.5.10) (choose there $\Upsilon$ so big that $\overline{\Omega} \subseteq \Upsilon$) shows that the function $u := \eta_j v$ satisfies the equation

$$-\nabla \cdot \xi \mu \nabla u + u = \eta_j f - \xi \mu \nabla v \cdot \nabla \eta_j + I_j$$

(2.6.5)

in $H^{-1,2}_\Gamma$, where $I_j$ is the distribution $w \mapsto \int_{\Omega} v \xi \mu \nabla \eta_j \cdot \nabla w \, dx$. Then applying Lemma 2.5.9 iii) with the same 'big' $\Upsilon$, we get that the right hand side of (2.6.5) is from $H^{-1,q}_\Gamma$, since $f \in H^{-1,q}_\Gamma$. If we define the function $\xi_j$ on $\Omega$ by

$$\xi_j(y) = \begin{cases} \xi(y), & \text{if } y \in \Lambda_{x_j} \\ \xi(x_j), & \text{elsewhere in } \Omega, \end{cases}$$

then $u = \eta_j v$ satisfies besides (2.6.5) also the equation

$$-\nabla \cdot \xi_j \mu \nabla u + u = \eta_j f - \xi \mu \nabla v \cdot \nabla \eta_j + I_j,$$

because $\xi_j = \xi$ on the support of $u$. But we have, according to (2.6.3) and (2.6.4)

$$\|(-\nabla \cdot \xi_j \mu \nabla + 1 - (-\nabla \cdot \xi(x_j) \mu \nabla + 1))(-\nabla \cdot \xi(x_j) \mu \nabla + 1)^{-1}\|_{L(H^{-1,q}_\Gamma)}$$

$$\leq \| -\nabla \cdot \xi_j \mu \nabla + 1 - (-\nabla \cdot \xi(x_j) \mu \nabla + 1)\|_{L(H^{-1,q},H^{-1,q}_\Gamma)}$$

$$\cdot \|(-\nabla \cdot \xi(x_j) \mu \nabla + 1)^{-1}\|_{L(H^{-1,q},H^{-1,q}_\Gamma)}$$

$$\leq \gamma \sup_{y \in \Lambda_{x_j}} |\xi(x_j) - \xi(y)|\|\mu\|_{L^\infty(\Omega;L(\mathbb{R}))} < 1.$$  

Thus, by a classical perturbation result (see [Kat80, Ch. IV.1]), the operator $-\nabla \cdot \xi_j \mu \nabla + 1$ also provides a topological isomorphism between $H^1_{\Gamma,q}$ and $H^{-1,1}_{\Gamma,q}$. Hence, for every $j$ we have $\eta_j v \in H^1_{\Gamma,q}$, and, hence, $v \in H^1_{\Gamma,q}$. So the assertion is implied by the open mapping theorem. ☐

In this spirit, one could now suggest $X := H^{-1,q}_{\Gamma}$ to be a good choice for the Banach space, but in view of condition (R) the right hand side of (2.6.1) has to be a continuous mapping from an interpolation space $(\text{dom}_X(A),X)_{1-\frac{1}{q},s}$ into $X$. Chosen $X := H^{-1,q}_{\Gamma}$, for elements $\psi \in (\text{dom}_X(A),X)_{1-\frac{1}{q},s} = (H^1_{\Gamma,q},H^{-1,q}_{\Gamma})_{1-\frac{1}{q},s}$ the expression $|\nabla \psi|^2$ cannot be properly defined and, if so, will not lie in $H^{-1,1}_{\Gamma,q}$ in general. This shows that $X := H^{-1,q}_\Gamma$ is not an appropriate choice, but we will see that $X := H^{-\varsigma,q}_\Gamma$, with $\varsigma$ properly chosen, is.

**Lemma 2.6.6.** Put $X := H^{-\varsigma,q}_\Gamma$ with $\varsigma \in [0,1] \setminus \left\{ \frac{1}{q}, 1 - \frac{1}{q} \right\}$. Then

i) For every $\tau \in \left(\frac{1}{2}, 1\right]$, there is a continuous embedding $(X,\text{dom}_X(\nabla \cdot \mu \nabla))_{\tau,1} \hookrightarrow H^1_{\Gamma,q}$.

ii) If $\varsigma \in \left(\frac{1}{q}, 1\right]$, then $X$ has a predual $X^* = H^{\varsigma,q'}_\Gamma$ which admits the continuous, dense injections $H^1_{\Gamma,q'} \hookrightarrow X^* \hookrightarrow L^\infty(\mathbb{R}^d)$ that by duality clearly imply (2.6.2). Furthermore, $H^1_{\Gamma,q}$ is a multiplier space for $X^*$.

**Proof.** i) $-\nabla \cdot \mu \nabla$ satisfies resolvent estimates

$$\|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}\|_{L(\mathbb{R})} \leq \frac{c}{1 + \lambda}, \quad \lambda \in [0, \infty[,$$

(2.6.6)
if \( Y = H^{1,q}_r \) or \( Y = L^q \), see Corollary 2.5.18. In view of (2.3.2) then (2.6.6) also holds for \( X \). This enables us to define fractional powers for \(-\nabla \cdot \mu \nabla + 1\) on each of the occurring spaces. According to (2.3.4) and Assumption 2.6.3 one has
\[
H^{1,q}_r = [H^{1,q}_r, H^{1,q}_r]_{\frac{1}{q}} = [H^{1,q}_r, \text{dom}_{H^{1,q}_r}(-\nabla \cdot \mu \nabla + 1)]_{\frac{1}{q}} \\
\hookrightarrow \text{dom}_{H^{1,q}_r}((-\nabla \cdot \mu \nabla + 1)^\alpha),
\]
if \( \alpha \in [0, \frac{1}{p-1}] \), see [Tri78, Ch. 1.15.2]. Thus, \((-\nabla \cdot \mu \nabla + 1)^\alpha \in \mathcal{L}(H^{-1,q}_r, H^{-1,q}_r)\), if \( \alpha \in [0, \frac{1}{p-1}] \).

Consequently, we can estimate
\[
\|(-\nabla \cdot \mu \nabla + 1)^{\alpha-1}\|_{\mathcal{L}(H^{-1,q}_r, H^{-1,q}_r)} \\
\leq \|(-\nabla \cdot \mu \nabla + 1)^{\alpha}\|_{\mathcal{L}(H^{-1,q}_r, H^{-1,q}_r)} \|(-\nabla \cdot \mu \nabla + 1)^{-1}\|_{\mathcal{L}(H^{-1,q}_r, H^{-1,q}_r)} < \infty.
\]
Clearly, this means \( \text{dom}_{H^{-1,q}_r}((-\nabla \cdot \mu \nabla + 1)^{1-\alpha}) \hookrightarrow H^{1,q}_r \). Putting \( \tau := 1 - \alpha \), this implies
\[
(H^{-1,q}_r, \text{dom}_{H^{-1,q}_r}((-\nabla \cdot \mu \nabla + 1)^{1-\alpha}))_{\tau,1} \hookrightarrow \text{dom}_{H^{-1,q}_r}((-\nabla \cdot \mu \nabla + 1)^\tau) \hookrightarrow H^{1,q}_r
\]
for \( \tau \in [\frac{1}{p-1}, 1] \), see [Tri78, Ch. 1.15.2].

ii) The first assertion is clear by Sobolev embedding. The second follows from known multiplier results, see [Gri85, Ch. 1.4] or [MS85]. \(\square\)

Next we will consider requirement c), see condition (B) in Proposition 2.6.1.

**Lemma 2.6.7.** Let \( q \) be a number from Assumption 2.6.3 and let \( X \) be a Banach space with predual \( X_* \) that admits the continuous and dense injections
\[
H^{1,q}_r \hookrightarrow X_* \hookrightarrow L^\frac{q}{r}'. \tag{2.6.7}
\]

i) If \( \xi \in H^{1,q}_r \) is a multiplier on \( X_* \), then \( \text{dom}_X(-\nabla \cdot \mu \nabla) \hookrightarrow \text{dom}_X(-\nabla \cdot \xi \mu \nabla) \).

ii) If \( H^{1,q}_r \) is a multiplier space for \( X_* \), then the (linear) mapping \( H^{1,q}_r \ni \xi \mapsto -\nabla \cdot \xi \mu \nabla \in \mathcal{L}(\text{dom}_X(-\nabla \cdot \mu \nabla), X) \) is well defined and continuous.

**Proof.** The supposition \( q > d \geq 2 \) and (2.6.7) imply the existence of a continuous and dense injection \( H^{1,2}_r \hookrightarrow X_* \). Thus, it is not hard to see that \( \psi \) belongs to \( \text{dom}_X(-\nabla \cdot \mu \nabla) \) iff the linear form
\[
\varphi \mapsto \int_\Omega \nabla \psi \cdot \mu \nabla \varphi \ d\text{x}
\]
is continuous on \( H^{1,2}_r \), when \( H^{1,2}_r \) is equipped with the \( X_* \) topology. We denote the set \( H^{1,2}_r \cap \{ \varphi \in X_* : \| \varphi \|_{X_*} = 1 \} \) by \( \mathcal{M} \). Assuming \( \psi \in \text{dom}_X(-\nabla \cdot \mu \nabla) \), we can estimate
\[
\| -\nabla \cdot \xi \mu \nabla \psi \|_X = \sup_{\varphi \in \mathcal{M}} \left| \int_\Omega \xi \mu \nabla \psi \cdot \nabla \varphi \ d\text{x} \right| \\
\leq \sup_{\varphi \in \mathcal{M}} \left| \int_\Omega \nabla \psi \cdot \mu \nabla (\xi \varphi) \ d\text{x} \right| + \sup_{\varphi \in \mathcal{M}} \left| \int_\Omega \nabla \psi \cdot \mu \varphi \nabla \xi \ d\text{x} \right| \tag{2.6.8} \\
\leq \| \psi \|_{\text{dom}_X(-\nabla \cdot \mu \nabla)} \sup_{\varphi \in \mathcal{M}} \| \xi \varphi \|_{X_*} + \| \psi \|_{H^{1,q}_r} \| \mu \|_{L^\infty} \| \xi \|_{H^{1,q}_r} \sup_{\varphi \in \mathcal{M}} \| \varphi \|_{L^\frac{q}{r}'}.
\]
We observe that the supposition $H_{\Gamma}^{1,q'} \hookrightarrow X_*$ together with Assumption 2.6.3 leads to the continuous embedding $dom_X(-\nabla \cdot \mu \nabla) \hookrightarrow H^{1,q}$. Thus, (2.6.8) is not larger than

$$m_\xi \|\psi\|_{dom_X(-\nabla \cdot \mu \nabla)} + \|\xi\|_{H^{1,q}} \|\mu\|_{L^\infty} \text{Emb}(dom_X(-\nabla \cdot \mu \nabla), H^{1,q}) \text{Emb}(X_*, L_{-\nabla \cdot \psi}^\prime) \|\psi\|_{dom_X(-\nabla \cdot \mu \nabla)},$$

where $m_\xi$ denotes the norm of the multiplier on $X_*$ induced by $\xi$ and $\text{Emb}(\cdot, \cdot)$ stands again for the corresponding embedding constants.

Assertion ii) also results from the estimates in the proof of i).

**Corollary 2.6.8.** If $\xi$ additionally to the hypotheses of Lemma 2.6.7 i) has a positive lower bound, then

$$dom_X(-\nabla \cdot \xi \mu \nabla) = dom_X(-\nabla \cdot \mu \nabla).$$

**Proof.** According to Lemma 2.6.7 i) one has only to show $dom_X(-\nabla \cdot \xi \mu \nabla) \hookrightarrow dom_X(-\nabla \cdot \mu \nabla)$. By Lemma 2.6.5 we have $dom_{H^{-1,q}}(-\nabla \cdot \xi \mu \nabla) = H_{\Gamma}^{1,q}$. Thus, one can apply Lemma 2.6.7 to the situation $\mu = \xi \mu$ and $\xi = \frac{1}{\xi}$.

Next we will show that functions on $\partial \Omega$ or on a Lipschitz hypersurface, which belong to a suitable summability class, can be understood as elements of the distribution space $H_{\Gamma}^{-\varsigma,q}$.

**Theorem 2.6.9.** Assume $q \in [1, \infty]$, $\varsigma \in \left[1 - \frac{1}{q}, 1\right] \setminus \{1\}$ and let $\Pi, \varpi$ be as in Theorem 2.3.6. Then the adjoint trace operator $(\text{Tr})^*$ maps $L^q(\Pi)$ continuously into the space $(H^{\varsigma,q'}(\Omega))^\prime \hookrightarrow H_{\Gamma}^{-\varsigma,q}.

**Proof.** The result is obtained from Theorem 2.3.6 by duality.

**Remark 2.6.10.** Here we restricted the considerations to the case of Lipschitz hypersurfaces, since this is the most essential insofar as it gives the possibility of prescribing jumps in the normal component of the current $j := \mathcal{G}(u)\mu \nabla u$ along hypersurfaces where the coefficient function jumps. This case is of high relevance in view of applied problems and has attracted much attention also from the numerical point of view, see e.g. [AL02], [CC98] and references therein.

In fact, it is possible to include much more general sets where distributional right hand sides live. For the identification of (singular) measures as distributions on lower dimensional sets, see also [Zie89, Ch. 4] and [JW84, Ch. VI.]. We did not make explicit use of this here, because at present we do not see direct applications.

From now on we fix once and for all a number $\varsigma \in \left[\max\{1 - \frac{1}{q}, \frac{d}{2}\}, 1\right]$ and set for all what follows $X := H_{\Gamma}^{-\varsigma,q}$.

Next we introduce the requirements on the data of problem (2.1.1)/(2.1.2).

**Assumption 2.6.11. Op)** For all what follows we fix a number $s > \frac{2}{1-\varsigma}$.

**Su)** There exists $f \in C^2(\mathbb{R})$, positive, with strictly positive derivative, such that $\mathcal{F}$ is the superposition operator induced by $f$.

**Ga)** The mapping $\mathcal{G} : H^{1,q} \rightarrow H^{1,q}$ is locally Lipschitz continuous.

**Gb)** For any ball in $H^{1,q}$ there exists $\delta > 0$, such that $\mathcal{G}(u) \geq \delta$ for all $u$ from this ball.

**Ra)** The function $\mathcal{R} : J \times H^{1,q} \rightarrow X$ is of Carathéodory type, i.e. $\mathcal{R}(\cdot, u)$ is measurable for all $u \in H^{1,q}$ and $\mathcal{R}(t, \cdot)$ is continuous for a.a. $t \in J$.
2.6. Nonlinear parabolic equations

\textbf{Rb}) $R(\cdot,0) \in L^s(J;X)$ and for $M > 0$ there exists $h_M \in L^s(J)$, such that
\[ \|R(t,u) - R(t,\tilde{u})\|_X \leq h_M(t)\|u - \tilde{u}\|_{H^{1,s}}, \quad t \in J, \]
provided $\max(\|u\|_{H^{1,s}}, \|\tilde{u}\|_{H^{1,s}}) \leq M$.

\textbf{BC}) $b$ is an operator of the form $b(u) = Q(b_\circ(u))$, where $b_\circ$ is a (possibly nonlinear), locally Lipschitzian operator from $C(\overline{\Omega})$ into itself (see Lemma 2.5.15).

\textbf{Gg}) $g \in L^q(\Gamma)$.

\textbf{IC}) $u_0 \in (X,\text{dom}_X(-\nabla \cdot \mu \nabla))_{1-\frac{1}{q},s}$.

\textbf{Remark 2.6.12.} At the first glance the choice of $s$ seems indiscriminate. The point is, however, that generically in applications the explicit time dependence of the reaction term $R$ is essentially bounded. Thus, in view of condition \textbf{Rb}) it is justified to take $s$ as any arbitrarily large number, whose magnitude needs not to be controlled explicitly, see Example 2.7.5.

Note that the requirement on $G$ allows for nonlocal operators. This is essential if the current depends on an additional potential governed by an auxiliary equation, what is usually the case in drift-diffusion models, see [Ama93], [GG96] or [Sel84].

The conditions \textbf{Ra}) and \textbf{Rb}) are always satisfied if $R$ is a mapping into $L^{q/2}$ with the analog boundedness and continuity properties, see Lemma 2.6.6 ii).

The estimate in (2.5.19) shows that $Q$ in fact is well defined on $C(\overline{\Omega})$, therefore condition \textbf{BC}) makes sense, see also (2.5.20). In particular, $b_\circ$ may be a superposition operator, induced by a $C^q(\mathbb{R})$ function. Let us emphasize that in this case the inducing function needs not to be positive. Thus, non-dissipative boundary conditions are included.

Finally, the condition \textbf{IC}) is an 'abstract' one and hardly to verify, because one has no explicit characterization of $(X,\text{dom}_X(-\nabla \cdot \mu \nabla))_{1-\frac{1}{q},s}$ at hand. Nevertheless, the condition is reproduced along the trajectory of the solution by means of the embedding (2.5.1).

In order to solve (2.1.1)/(2.1.2), we will consider instead (2.6.1) with
\[ B(u) := -\nabla \cdot \frac{G(u)}{F'(u)} \mu \nabla \]
and the right hand side $S$
\[ S(t,u) := \frac{R(t,u)}{F'(u)} + \left(\nabla \frac{1}{F'(u)}\right) \cdot \left(G(u)\mu \nabla u\right) - \frac{Q(b_\circ(u))}{F'(u)} + \frac{(\text{Tr})^\ast g}{F'(u)}, \quad (2.6.10) \]
seeking the solution in the space $W^{1,s}(J;X) \cap L^s(J;\text{dom}_X(-\nabla \cdot \mu \nabla))$.

\textbf{Remark 2.6.13.} Let us explain this reformulation: as is well known in the theory of boundary value problems, the boundary condition (2.1.2) is incorporated by introducing the boundary terms $-\varepsilon b_\circ(u)$ and $g$ on the right hand side. In order to understand both as elements from $X$, we write $Q(b_\circ(u))$ and $(\text{Tr})^\ast g$, see Lemma 2.5.15 and Theorem 2.6.9. On the other hand, our aim was to eliminate the nonlinearity under the time derivation: we formally differentiate $(F(u))' = F'(u)u'$ and afterwards divide the whole equation by $F'(u)$. Finally, we employ the equation
\[ -\frac{1}{F'(u)} \nabla \cdot G(u) \mu \nabla u = -\nabla \cdot \frac{G(u)}{F'(u)} \mu \nabla u - \left(\nabla \frac{1}{F'(u)}\right) \cdot \left(G(u)\mu \nabla u\right), \quad (2.6.11) \]
which holds for any $u \in \text{dom}_X(-\nabla \cdot G(u)\mu \nabla) = \text{dom}_X(-\nabla \cdot G(u))$ as an equation in $X$, compare Lemma 2.6.6 ii) and Corollary 2.6.8.
Theorem 2.6.14. Let \( d \in \{2, 3\} \), let Assumption 2.6.3 be satisfied and assume that the data of the problem satisfy Assumption 2.6.11. Then (2.6.1) has a local in time, unique solution in \( W^{1,s}(J; X) \cap L^s(J; \text{dom}_X (-\nabla \cdot \mu \nabla)) \), provided that \( \mathcal{B} \) and \( \mathcal{S} \) are given by (2.6.9) and (2.6.10), respectively.

Proof. First of all we note that, due to Op), \( 1 - \frac{1}{s} > \frac{1}{p} \). Thus, if \( \tau \in ]\frac{1}{p}, 1 - \frac{1}{s}[ \) by a well known interpolation result (see [Tri78, Ch. 1.3.3]) and Lemma 2.6.6 i) we have

\[
(X, \text{dom}_X (-\nabla \cdot \mu \nabla))_{1 - \frac{1}{s}, s} \hookrightarrow (X, \text{dom}_X (-\nabla \cdot \mu \nabla))_{\tau, 1} \hookrightarrow H^{1,q}.
\]

(2.6.12)

Hence, by IC), \( u_0 \in H^{1,q} \). Consequently, due to the suppositions on \( \mathcal{F} \) and \( \mathcal{G} \), both the functions \( \frac{\mathcal{G}(u_0)}{\mathcal{F}(u_0)} \) and \( \frac{\mathcal{F}(u_0)}{\mathcal{G}(u_0)} \) belong to \( H^{1,q} \) and are bounded from below by a positive constant. Denoting \( -\nabla \cdot \frac{\mathcal{G}(u_0)}{\mathcal{F}(u_0)} \mu \nabla \) by \( B \), Corollary 2.6.8 gives \( \text{dom}_X (-\nabla \cdot \mu \nabla) = \text{dom}_X (B) \). This implies \( u_0 \in (X, \text{dom}_X (B))_{1 - \frac{1}{s}, s} \). Furthermore, the so defined \( B \) has maximal parabolic regularity on \( X \), thanks to (2.5.24) in Theorem 2.5.16 with \( p = q \).

Condition (B) from Proposition 2.6.1 is implied by Lemma 2.6.7 ii) in cooperation with Lemma 2.6.6 ii), the fact that the mapping \( H^{1,q} \ni \phi \mapsto \frac{\mathcal{G}(\phi)}{\mathcal{F}(\phi)} \in H^{1,q} \) is boundedly Lipschitz and (2.6.12).

It remains to show that the ‘new’ right hand side \( \mathcal{S} \) satisfies condition (R) from Proposition 2.6.1. We do this for every term in (2.6.10) separately, beginning from the left: concerning the first, one again uses (2.6.12), the asserted conditions Ra) and Rb) on \( \mathcal{R} \), the local Lipschitz continuity of the mapping \( H^{1,q} \ni u \mapsto \frac{\mathcal{F}(u)}{\mathcal{G}(u)} \in H^{1,q} \) and the fact that \( H^{1,q} \) is a multiplier space over \( X \). The second term can be treated in the same spirit, if one takes into account the embedding \( L^{q/2} \hookrightarrow X \) and applies Hölder’s inequality. The assertion for the last two terms results from (2.6.12), the assumptions BC)/Gg), Lemma 2.5.15 and Theorem 2.6.9.

Remark 2.6.15. According to (2.6.11) it is clear that the solution \( u \) satisfies the equation

\[
\mathcal{F}'(u)u' - \nabla \cdot \mathcal{G}(u) \mu \nabla u + Q(b_u(u)) = \mathcal{R}(t, u) + (\text{Tr})^* g
\]

(2.6.13)
as an equation in \( X \). Note that, if \( \mathcal{R} \) takes its values only in the space \( L^{q/2} \hookrightarrow X \), then – in the light of Lemma 2.5.15 – the elliptic operators incorporate the boundary conditions (2.1.2) in a generalized sense, see [GGZ74, II.2] or [Cia78, Ch. 1.2].

The remaining problem is to identify \( \mathcal{F}'(u)u' \) with \( (\mathcal{F}(u))' \) where the prime has to be understood as the distributional derivative with respect to time. This identification (technically rather involved) is proved in [HR06] for the case where the Banach space \( X \) equals \( L^{q/2} \), but can be carried over to the case \( X = H^{1,s-q} \) word by word.

We will now show that the solution \( u \) is Hölder continuous simultaneously in space and time, even more:

Corollary 2.6.16. There exist \( \alpha, \beta > 0 \) such that the solution \( u \) of (2.6.13) belongs to the space \( C^\beta(J; H^{1/3,q}(\Omega)) \hookrightarrow C^\beta(J; C^{\alpha}(\Omega)) \).

Proof. During this proof we write for short \( D := \text{dom}_X (B) \). A straightforward application of Hölder’s inequality yields the embedding

\[
W^{1,s}(J; X) \hookrightarrow C^\delta(J; X) \quad \text{with} \quad \delta = 1 - \frac{1}{s}.
\]

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Take $\lambda$ from the interval $\left(\frac{1}{2}, 1\right)$, which is nonempty in view of Op. Using Lemma 2.6.6 i) and the reiteration theorem for real interpolation, one can estimate

$$\frac{\|u(t_1) - u(t_2)\|_{H^{1,q}}}{|t_1 - t_2|^{\frac{1}{2}(1-\lambda)}} \leq c \frac{\|u(t_1) - u(t_2)\|_{(X,D)_{\lambda(1-\frac{1}{2}),1}}}{|t_1 - t_2|^{\frac{1}{2}(1-\lambda)}} \leq c \frac{\|u(t_1) - u(t_2)\|_{1}}{|t_1 - t_2|^{\frac{1}{2}(1-\lambda)}} \leq c \left(\frac{\|u(t_1) - u(t_2)\|_{X}}{|t_1 - t_2|} \right)^{1-\lambda} \left(2 \sup_{t \in J} \|u(t)\|_{(X,D)_{1-\frac{1}{2},s}}\right)^{\lambda}.$$  

Finally, we will have a closer look at the semilinear case. It turns out that one can achieve satisfactory results here without Assumption 2.6.3, and thus also without the corresponding restriction on the space dimension, at least when the nonlinear term depends only on the function itself and not on its gradient.

**Theorem 2.6.17.** Assume that $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $H^{-1,q}_\Gamma$ for some $q > d$. Suppose further that the function $\mathcal{R} : J \times C(\overline{\Omega}) \to H^{-1,q}_\Gamma$ is of Carathéodory type, i.e. $\mathcal{R}(\cdot, u)$ is measurable for all $u \in C(\overline{\Omega})$ and $\mathcal{R}(t, \cdot)$ is continuous for a.a. $t \in J$ and, additionally, obeys the following condition: $\mathcal{R}(\cdot, 0) \in L^s(J; H^{-1,q}_\Gamma)$ and for all $M > 0$ there exists $h_M \in L^s(J)$, such that

$$\|\mathcal{R}(t, u) - \mathcal{R}(t, \tilde{u})\|_{H^{-1,q}_\Gamma} \leq h_M(t)\|u - \tilde{u}\|_{C(\overline{\Omega})}, \quad t \in J.$$  

Then the equation

$$u' - \nabla \cdot \mu \nabla u = \mathcal{R}(t, u), \quad u(T_0) = 0$$

admits exactly one local in time solution.

**Proof.** It is clear that $\mathcal{R}$ satisfies the abstract conditions on the reaction term, posed in Proposition 2.6.1, if we can show $[H^{-1,q}_\Gamma, dom_{H^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla)]_{\theta} \hookrightarrow C(\overline{\Omega})$ for some large $\theta \in [0,1]$. This we will do: using the embedding $dom_{H^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla) \hookrightarrow C^\alpha$ for some positive $\alpha$ (see [Gri02]) and the reiteration theorem for complex interpolation, one can write

$$[H^{-1,q}_\Gamma, dom_{H^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla)]_{\theta} \hookrightarrow [H^{-1,2,q}_\Gamma, \frac{1}{2}, C^\alpha]_{2\theta-1} = [L^2, C^\alpha]_{2\theta-1}.$$  

But based on the results of Triebel [Tri78a], in [GKR01, Ch. 7] it is shown that this last space continuously embeds into another Hölder space, if $\theta$ is chosen large enough. 

**2.7. Examples**

In this section we describe geometric configurations for which our Assumption 2.6.3 holds true and we present concrete examples of mappings $\mathcal{G}$ and reaction terms $\mathcal{R}$ fitting into our framework. Another part of this section is then devoted to the special geometry of two crossing beams that is interesting, since this is not a domain with Lipschitz boundary, but it falls into the scope of our theory, as we will show.
2. Maximal Regularity for Operators with Mixed Boundary Conditions

2.7.1. Geometric constellations

While our results in Sections 2.4 and 2.5 on the square root of $-\nabla \cdot \mu \nabla$ and maximal parabolic regularity are valid in the general geometric framework of Assumption 2.3.2, we additionally had to impose Assumption 2.6.3 for the treatment of quasilinear equations in Section 2.6. Here we shortly describe geometric constellations, in which this additional condition is satisfied. Let us start with the observation that the 2-d case is covered by Remark 2.5.5 i).

Admissible three-dimensional settings may be described as follows.

**Proposition 2.7.1.** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a $q > 3$ such that $-\nabla \cdot \mu \nabla + 1$ is a topological isomorphism from $H^{1,q}_\Gamma$ onto $H^{-1,q}_\Gamma$, if one of the following conditions is satisfied:

i) $\Omega$ has a Lipschitz boundary. $\Gamma = \emptyset$ or $\Gamma = \partial \Omega$. $\Omega_o \subseteq \Omega$ is another domain which is $C^1$ and which does not touch the boundary of $\Omega$. $\mu|_{\Omega_o} \in \text{BUC}(\Omega_o)$ and $\mu|_{\Omega_o \setminus \overline{\Omega_o}} \in \text{BUC}(\Omega \setminus \overline{\Omega_o})$.

ii) $\Omega$ has a Lipschitz boundary, $\Gamma = \emptyset$. $\Omega_o \subseteq \Omega$ is a Lipschitz domain, such that $\partial \Omega_o \cap \Omega$ is a $C^1$ surface and $\partial \Omega$ and $\partial \Omega_o$ meet suitably (see [ERS07] for details). $\mu|_{\Omega_o} \in \text{BUC}(\Omega_o)$ and $\mu|_{\Omega_o \setminus \overline{\Omega_o}} \in \text{BUC}(\Omega \setminus \overline{\Omega_o})$.

iii) $\Omega$ is a three dimensional Lipschitzian polyhedron. $\Gamma = \emptyset$. There are hyperplanes $H_1, \ldots, H_n$ in $\mathbb{R}^3$ which meet at most in a vertex of the polyhedron such that the coefficient function $\mu$ is constantly a real, symmetric, positive definite $3 \times 3$ matrix on each of the connected components of $\Omega \cup \bigcup_{l=1}^n H_l$. Moreover, for every edge on the boundary, induced by a hetero interface $H_l$, the angles between the outer boundary plane and the hetero interface do not exceed $\pi$ and at most one of them may equal $\pi$.

iv) $\Omega$ is a convex polyhedron, $\Gamma \cap (\partial \Omega \setminus \Gamma)$ is a finite union of line segments. $\mu \equiv 1$.

v) $\Omega \subseteq \mathbb{R}^3$ is a prismatic domain with a triangle as basis. $\Gamma$ equals either one half of one of the rectangular sides or one rectangular side or two of the three rectangular sides. There is a plane which intersects $\Omega$ such that the coefficient function $\mu$ is constant above and below the plane.

vi) $\Omega$ is a bounded domain with Lipschitz boundary. Additionally, for each $x \in \Gamma \cap (\partial \Omega \setminus \Gamma)$ the mapping $\phi_x$ defined in Assumption 2.3.2 is a $C^1$-diffeomorphism from $\Upsilon_x$ onto its image. $\mu \in \text{BUC}(\Omega)$.

The assertions i) and ii) are shown in [ERS07], while iii) is proved in [EKRS07] and iv) is a result of Dauge [Dau92]. Recently, v) was obtained in [HKR08] and vi) will be published in a forthcoming paper.

**Corollary 2.7.2.** The assertion remains true, if there is a finite open covering $\Upsilon_1, \ldots, \Upsilon_l$ of $\overline{\Omega}$, such that each of the pairs $\Omega_j := \Upsilon_j \cap \Omega$, $\Gamma_j := \Gamma \cap \Upsilon_j$ fulfills one of the points i) – vi).

**Proof.** The corollary can be proved by means of Lemma 2.5.9 and Lemma 2.5.8.

**Remark 2.7.3.** Proposition 2.7.1 together with Corollary 2.7.2 provides a huge zoo of geometries and boundary constellations, for which $-\nabla \cdot \mu \nabla$ provides the required isomorphism. We intend to complete this in the future.
2.7.2. Nonlinearities and reaction terms

The most common case is that where $\mathcal{F}$ is the exponential or the Fermi-Dirac distribution function $\mathcal{F}_{1/2}$ given by

$$
\mathcal{F}_{1/2}(t) := \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{s}}{1 + e^{-t}} ds
$$

and $\mathcal{G}$ also is a Nemyctkii operator of the same type. In phase separation problems, a rigorous formulation as a minimal problem for the free energy reveals that $\mathcal{G} = \mathcal{F}'$ is appropriate. This topic has been thoroughly investigated in [Qua92], [QRV99], [GL97], and [GL98], see also [GS03] and [Gri04]. It is noteworthy that in this case $\frac{\mathcal{F}}{\mathcal{G}} \equiv 1$ (we conjecture that this is not accidental) and the evolution equation (2.1.1) leads not to a quasilinear equation (2.6.1) but to one which is only semilinear. We consider this as a hint for the adequateness of our treatment of the parabolic equations.

As a second example we present a nonlocal operator arising in the diffusion of bacteria; see [CC04], [CL99] and references therein.

**Example 2.7.4.** Let $\eta$ be a continuously differentiable function on $\mathbb{R}$ which is bounded from above and below by positive constants. Assume $\varphi \in L^2(\Omega)$ and define

$$
\mathcal{G}(u) := \eta \left( \int_\Omega u \varphi \, dx \right), \quad u \in H^{1,q}.
$$

Now we give two examples for mappings $\mathcal{R}$.

**Example 2.7.5.** Assume that $[T_0, T[ = \bigcup_{l=1}^j [t_l, t_{l+1}]$ is a (disjoint) decomposition of $[T_0, T]$ and let for $l \in \{1, \ldots, j\}$

$$
Z_l : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}
$$

be a function which satisfies the following condition: For any compact set $K \subseteq \mathbb{R}$ there is a constant $L_K$ such that for any $a, \tilde{a} \in K$, $b, \tilde{b} \in \mathbb{R}^d$ the inequality

$$
|Z_l(a, b) - Z_l(\tilde{a}, \tilde{b})| \leq L_K |a - \tilde{a}|_\mathbb{R} \left( |b|_\mathbb{R}^2 + |\tilde{b}|_\mathbb{R}^2 \right) + L_K |b - \tilde{b}|_\mathbb{R}^d \left( |b|_\mathbb{R}^2 + |\tilde{b}|_\mathbb{R}^d \right)
$$

holds. We define a mapping $Z : [T_0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ by setting

$$
Z(t, a, b) := Z_l(t, a, b), \quad \text{if} \quad t \in [t_l, t_{l+1}].
$$

The function $Z$ defines a mapping $\mathcal{R} : [T_0, T] \times H^{1,q} \rightarrow L^{q/2}$ in the following way: If $\psi$ is the restriction of an $\mathbb{R}$-valued, continuously differentiable function on $\mathbb{R}^d$ to $\Omega$, then we put

$$
\mathcal{R}(t, \psi)(x) = Z(t, \psi(x), (\nabla \psi)(x)) \quad \text{for} \quad x \in \Omega
$$

and afterwards extend $\mathcal{R}$ by continuity to the whole set $[T_0, T] \times H^{1,q}$.

**Example 2.7.6.** Assume $\iota : \mathbb{R} \rightarrow [0, \infty)$ to be a continuously differentiable function. Furthermore, let $T : H^{1,q} \rightarrow H^{1,q}$ be the mapping which assigns to $v \in H^{1,q}$ the solution $\varphi$ of the elliptic problem (including boundary conditions)

$$
-\nabla \cdot \iota(v) \nabla \varphi = 0. \quad (2.7.1)
$$

If one defines

$$
\mathcal{R}(v) = \iota(v) |\nabla(T(v))|^2,
$$

then, under reasonable suppositions on the data of (2.7.1), the mapping $\mathcal{R}$ satisfies Assumption $\text{Ra}$.

This second example comes from a model which describes electrical heat conduction; see [AC94] and the references therein.
2.7.3. An unorthodox example: two crossing beams

Finally, we want to present in some detail the example of two beams, mentioned in the introduction, which is not a domain with Lipschitz boundary, and, hence, not covered by former theories. Consider in $\mathbb{R}^3$ the set

$$B_\infty := \{ -10, 10 \times -1, 1 \times -2, 0 \} \cup \{ -1, 1 \times -10, 10 \times 0, 2 \} \cup \{ -1, 1 \times -1, 1 \times \{ 0 \} \},$$

together with a $3 \times 3$ matrix $\mu_1$, considered as the coefficient matrix on the first beam, and another $3 \times 3$ matrix $\mu_2$, considered as the coefficient function on the other beam. Both matrices are assumed to be real, symmetric and positive definite. If one defines the coefficient function $\mu$ as $\mu_1$ on the first beam, and as $\mu_2$ on the other, then, due to Proposition 2.7.1 iii),

$$-\nabla \cdot \mu \nabla : H_0^{1,q} \to H^{-1,q}$$

provides a topological isomorphism for some $q > 3$, if one can show that $B_\infty$ is a Lipschitz domain. In fact, we will show more, namely:

**Lemma 2.7.7.** $B_\infty$ fulfills Assumption 2.3.2.

**Proof.** For all points $x \in \partial \Omega$ the existence of a corresponding neighborhood $\Upsilon_x$ and a mapping $\Phi_x$ can be deduced easily, except for the points $x$ from the set

$$\text{Sing} := \{ (-1, -1, 0), (-1, 1, 0), (1, -1, 0), (1, 1, 0) \}.$$

In fact, for all points $x \in B_\infty \setminus \text{Sing}$ there is a neighborhood $\Upsilon_x$, such that either $B_\infty \cap \Upsilon_x$ or $\Upsilon_x \setminus B_\infty$ is convex and, hence, a domain with Lipschitz boundary. Thus, these points can be treated as in Remark 2.3.3.

Exemplarily, we aim at a suitable transformation in a neighborhood of the point $(1, -1, 0)$; the construction for the other three points is – mutatis mutandis – the same. For doing so, we first shift $B_\infty$ by the vector $(-1, 1, 0)$, so that the transformed point of interest becomes the origin. Now we apply the transformation $\phi^\Delta$ on $\mathbb{R}^3$ that is given in Figure 2.3. The following

![Figure 2.3](image)

Figure 2.3.: Cut through $B_\infty + (-1, 1, 0)$ at a plane $y = \delta$ (for $\delta > 0$ small) and the transformation $\phi^\Delta$

is straightforward to verify:

- Both transformations coincide on the plane $\{ x : z = x \}$ and thus together define a globally bi-Lipschitz mapping $\phi^\Delta : \mathbb{R}^3 \to \mathbb{R}^3$, which, additionally, is volume-preserving.
2.8. Concluding Remarks

- The intersection of $\phi \left( B_{\infty} + (-1, 1, 0) \right)$ with a sufficiently small, paraxial cube $\epsilon K$ around 0 equals the set

$$\{ x : -\epsilon < x < 0, -\epsilon < y < \epsilon, -\epsilon < z < 0 \} \cup \{ x : 0 \leq x < \epsilon, 0 < y < \epsilon, -\epsilon < z < 0 \}.$$  

(To prove the latter, note that the $y$-component is left invariant under $\phi$ and that $\phi$ acts in the plane $y = 0$ as follows: the vector $(0, 1)$ is mapped onto $(-1, 0)$ and the vector $(-1, 0)$ onto $(0, -1)$. Finally, the vector $(1, 0)$ is left invariant.) Next we introduce the mapping $\phi_{\Delta}$ which is defined as the linear mapping $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the set $\{ x : -x < y \}$ and as the identity on the set $\{ x : -x \geq y \}$, see Figure 2.4.

![Figure 2.4: Cut through $\phi \left( B_{\infty} + (-1, 1, 0) \right)$ at a plane $z = -\delta$ in a neighborhood of 0 ($\delta > 0$ sufficiently small)](image)

One directly verifies that $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acts as the identity on the set $\{ x : -x = y \}$; thus $\phi_{\Delta}$ in fact is a bi-Lipschitz, volume-preserving mapping from $\mathbb{R}^3$ onto itself. After this transformation the resulting object, intersected with a sufficiently small paraxial cube $\epsilon K$, equals the convex set

$$\{ x : -\epsilon < x < \epsilon, 0 < y < \epsilon, -\epsilon < z < 0 \}.$$  

Here again Remark 2.3.3 applies, what finishes the proof.

2.8. Concluding Remarks

**Remark 2.8.1.** The reader may have asked himself why we restricted the considerations to real, symmetric coefficient functions $\mu$. The answer is twofold: first, we need at all costs Gaussian estimates for our techniques and it is known that these are not available for complex coefficients in general, see [ADT96] and also [Dav97]. Additionally, Proposition 2.4.8 also rests on this supposition. On the other hand, in the applications we have primarily in mind this condition is satisfied.

**Remark 2.8.2.** Under the additional Assumption 2.6.3, Theorem 2.5.4 implies maximal parabolic regularity for $-\nabla \cdot \mu \nabla$ on $H^{-1,q}_I$ for every $q \in [2, \infty[$, as in the 2-d case.
Besides, the question arises whether the limitation for the exponents, caused by the localization procedure, is principal in nature or may be overcome when applying alternative ideas and techniques (cf. Theorem 2.4.4). We do not know the answer at present.

**Remark 2.8.3.** We considered here only the case of one single parabolic equation, but everything can be carried over in a straightforward way to the case of diagonal systems; 'diagonal' in this case means that the function $\mathcal{G}$ is allowed to depend on the vector $u = (u_1, \ldots, u_n)$ of solutions and the right hand side also. In the same spirit one can treat triagonal systems.

**Remark 2.8.4.** Inspecting Proposition 2.6.1, one easily observes that in fact an additional $t$-dependence of the function $\mathcal{G}$ would be admissible. We did not carry this out here for the sake of technical simplicity.

**Remark 2.8.5.** In (2.1.2) we restricted our setting to the case where the Dirichlet boundary condition is homogeneous. It is straightforward to generalize this to the case of inhomogeneous Dirichlet conditions by splitting off the inhomogeneity, see [GGZ74, Ch. II.2] or [Cia78, Ch. 1.2], see also [HR06] where this has been carried out in detail in the case of parabolic systems.

**Remark 2.8.6.** If one knows a priori that the right hand side of (2.1.1) depends Hölder continuously on the time variable $t$, then one can use other local existence and uniqueness results for abstract parabolic equations, see e.g. [Lun84] for details. In this case the solution $u$ is even strongly differentiable in the space $X$ (with continuous derivative), what may lead to a better justification of time discretization then, compare [AS04] and references therein.

**Remark 2.8.7.** Let us explicitly mention that Assumption 2.6.3 is not always fulfilled in the 3-d case. First, there is the classical counterexample of Meyers, see [Mey63], a simpler (and somewhat more striking) one is constructed in [EKRS07], see also [ERS07]. The point, however, is that not the mixed boundary conditions are the obstruction but a somewhat 'irregular' behavior of the coefficient function $\mu$ in the inner of the domain. If one is confronted with this, spaces with weight may be the way out.

**Remark 2.8.8.** In two and three space dimensions one can give the following simplifying characterization for a set $\Omega \cup \Gamma$ to be regular in the sense of Gröger, i.e. to satisfy Assumption 2.3.2 a), see [HMR]:

If $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $\Gamma \subset \partial \Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger iff $\partial \Omega \setminus \Gamma$ is the finite union of (non-degenerate) closed arc pieces.

In $\mathbb{R}^3$ the following characterization can be proved, heavily resting on a deep result of Tukia [Tuk80]:

If $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain and $\Gamma \subset \partial \Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger iff the following two conditions are satisfied:

i) $\partial \Omega \setminus \Gamma$ is the closure of its interior (within $\partial \Omega$).

ii) for any $x \in \Gamma \cap (\partial \Omega \setminus \Gamma)$ there is an open neighborhood $\mathcal{U} \ni x$ and a bi-Lipschitz mapping $\kappa : \mathcal{U} \cap \Gamma \cap (\partial \Omega \setminus \Gamma) \to ]-1, 1[^3$. 

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3. Elliptic Model Problems Including Mixed Boundary Conditions and Material Heterogeneities

by Robert Haller-Dintelmann, Hans-Christoph Kaiser and Joachim Rehberg

Abstract

We present model problems in three dimensions, where the operator $-\nabla \cdot \mu \nabla$ maps the Sobolev space $W^{1,p}_\Gamma(\Omega)$ isomorphically onto $W^{-1,p}_\Gamma(\Omega)$ for a $p > 3$. The emphasis is here on the case where different boundary conditions meet material heterogeneities.

3.1. Introduction

Many elliptic problems originating from science, engineering, and technology exhibit mixed boundary conditions and non-smooth material parameters, see [Ama93] and the references cited there. For instance, in the simulation of operation and fabrication of semiconductor devices one is regularly confronted with heterogeneous materials in the volume and on the boundary (contacts), see [Sc] while dealing with elliptic and parabolic equations as mathematical models, see [Ga93]. However, not much is known concerning maximal regularity for elliptic operators which include mixed boundary conditions. Moreover, most of this is restricted to Hilbert space scales, see e.g. [Sav97], [RS89], [HS04], [EF99], [Bro94], [BLM99]. Unfortunately, the Hilbert space $H^{3/2}$ is a principle threshold for mixed elliptic, second order problems at least in the case when the Dirichlet and Neumann boundary part meet on smooth parts of the boundary, see [Sha68] and also [Sav97]. Thus, within this scale one cannot expect an embedding of the domains of these operators in $L^\infty$ (or even in $C^\alpha$) in case of three or more space dimensions. But exactly this is desirable in view of nonlinear, in particular quasilinear problems, see [Reh05], [HR06], [MERS04]. Concerning optimal regularity in non-Hilbert spaces there are the results of [Sha63], [Sha68], [BdV74], [Grö89], [Dau92], [GR01], [Gri02]; for the pure Dirichlet or pure Neumann case see [JK95] and [Zan00], respectively. Gröger proved in [Grö89] that under only $L^\infty$ (and ellipticity) assumptions on the coefficient function $\mu$, the Lipschitz property of the domain $\Omega$ and very weak assumptions on the Dirichlet boundary part $\partial \Omega \setminus \Gamma$ the operator

$$-\nabla \cdot \mu \nabla : W^{1,p}_\Gamma(\Omega) \to W^{-1,p}_\Gamma(\Omega)$$

is a topological isomorphism for a certain $p > 2$ ($W^{1,p}_\Gamma(\Omega)$ denoting the subspace of $W^{1,p}(\Omega)$ including a trace zero condition on the Dirichlet boundary part $\partial \Omega \setminus \Gamma$, and $W^{-1,p}_\Gamma(\Omega)$ the dual of $W^{1,p}_\Gamma(\Omega)$). This result has found numerous applications within the treatment of applied problems. Nevertheless, it is well known that under these general assumptions one can only expect that $p$ exceeds 2 arbitrarily little. This is the reason why the applications of [Grö89]...
remained restricted to two dimensional problems. Because the demand for three dimensional modelling and simulation steadily increases, the question arises under which assumptions the isomorphism property of (3.1.1) can be obtained for a \( p > 3 \) and, in particular, whether this is true with mixed boundary conditions. Dauge proved in [Dau92] that if the domain is a convex polyhedron and the border between Dirichlet and Neumann boundary part consists of (finitely many) line segments, then the Laplacian provides a topological isomorphism between \( W^{1,p}_\Gamma \) and \( W^{-1,p}_\Gamma \) for some \( p > 3 \). In this paper we generalise this to prototypical situations where mixed boundary conditions and heterogeneous, anisotropic coefficient functions occur simultaneously. Thus, this calculus allows for jumps in the conormal derivative of solutions across internal interfaces. This means, e.g. in electrostatics, that the jump in the normal component of the displacement \( \nu_+ \varepsilon \nabla \varphi - \nu_- \varepsilon \nabla \varphi \) across a prescribed interface equals the surface charge density on the interface, and this surface charge density is represented by a distribution on the underlying domain \( \Omega \).

In view of an adequate localisation principle, see [Grö89], the geometric constellations we investigate may be viewed as local constituents of rather complex global settings. Since the knowledge of the singularity of solutions is crucial for the efficiency of numerical methods, there exist of course several numerical approaches to determine singular exponents of concrete anisotropic problems, see [LS87], [CDL01], [SY96] and the references therein. For a more general numerical approach to heterogeneous elliptic problems see for instance [AL02], [HLLW05], [CZ98], [Wan04] and the references cited there.

In detail, our results are as follows:

**Theorem 3.1.1.** Let \( \Lambda \subset \mathbb{R}^2 \) be an open triangle, let further \( P \) be the center of one of its sides and \( \Upsilon \) the open leg between \( P \) and one of its neighbouring vertices. Define \( \Pi \equiv \Lambda \times [-1,1] \) and the boundary part \( \Sigma \) as \( \Upsilon \times [-1,1] \). Suppose \( \Xi \) to be a plane within \( \mathbb{R}^3 \) that intersects \( \{ P \} \times [-1,1] \) in exactly one point. Assume that the elliptic coefficient function \( \mu \) takes its values in the set of real, symmetric, positive definite \( 3 \times 3 \) matrices and is constant on both components of \( \Pi \setminus \Xi \). Then there is a \( p > 3 \) such that

\[
-\nabla : \mu \nabla : W^{1,p}_\Sigma(\Pi) \rightarrow W^{-1,p}_\Sigma(\Pi)
\]

(3.1.2)

is a topological isomorphism.

![Figure 3.1.](image-url) The model domains \( \Pi \) under consideration in Theorem 3.1.1 (left) and Theorem 3.1.2 (right) with generating triangle \( \Lambda \), Neumann boundary \( \Sigma \) (hatched area) and material interface \( \Xi \) (shaded area) cutting the domain.
Theorem 3.1.2. Let $\Lambda \subset \mathbb{R}^2$ be an open triangle, $\Upsilon$ be one of its open sides or $\partial \Lambda \setminus \Upsilon$ one of its closed sides. Define $\Pi \overset{\text{def}}{=} \Lambda \times [-1,1]$ and the boundary part $\Sigma$ as $\Upsilon \times [-1,1]$. Let further $\Xi$ be a plane the intersection of which with the boundary of $\Sigma$ consists of exactly two points. Assume that the elliptic coefficient function $\mu$ takes its values in the set of real, symmetric, positive definite $3 \times 3$ matrices and is constant on both components of $\Pi \setminus \Xi$. Then there is a $p > 3$ such that (3.1.2) is a topological isomorphism.

Corollary 3.1.3. Let $\Lambda$, $\Upsilon$ and $\Pi$ be as in Theorem 3.1.2. Let $\Sigma$ be $\Upsilon \times [-1,1]$ combined with the ground plate or/and the upper plate. Let further $\Xi$ be a plane as in Theorem 3.1.2 which does neither touch the upper/lower plate and let $\mu$ be as in Theorem 3.1.2. Then the conclusion of Theorem 3.1.2 also holds.

Remark 3.1.4. The supposition that the plane $\Xi$ has only a finite intersection with edges where the Dirichlet boundary part meets with the complementing boundary part is crucial. If this is not the case, a bimaterial outer edge (see Definition 3.4.5 below) with mixed boundary conditions occurs, for which the appearing singularities may be arbitrarily large, see Remark 3.9.5 in the appendix.

Remark 3.1.5. Let us further mention that $\Pi \cup \Sigma$ in Theorem 3.1.1 can be taken as Gröger’s third model set, see [Grö89], and thus Theorem 3.1.1 can be viewed as a regularity assertion for Gröger’s third model constellation if the coefficient function has a discontinuity along a plane.

Operators of type (3.1.1) — which may be seen as the principal part of the (Dirichlet)-homogenization of an elliptic operator — are of fundamental significance in many application areas. This is the case not only in mechanics (see [LS87, Ch. IV/V]), thermodynamics (see [Som56]), and electrodynamic (see [Som52]) of heterogeneous media, but also in mining, multiphase flow, mathematical biology (see [FGR90], [BHR05]), and semiconductor device simulation (see [Sel84], [Ga93], [GH97]), in particular quantum electronics (see [WV91], [BKPR03], [KKF06], [WHLL03], [Wan04], [LLVS01]).

The nonhomogeneous coefficient function $\mu$ represents varying material properties as the context requires. It may be thermal conductivity in a heat equation (see [Som56, §21]), or dielectric permittivity in a Poisson equation, or diffusivity in a transport equation (see for instance [Sel84, §2.2] for carrier continuity equations), or effective electron mass in a Schrödinger equation (see [KKF06]).

Let us emphasise that the matrices which constitute the coefficient function $\mu$ may be not diagonal and, in particular, not multiples of the identity, see [Ama93] and [LS87, Ch. IV/V]. This is motivated by the applications, for instance in heat conduction, see [Som56, §21.B]. On the other hand anisotropic coefficients are absolutely necessary in view of (local) deformation and transformation of the domain in the localisation procedure, see Proposition 3.5.3. It should be noted that in case of an essentially anisotropic coefficient matrix $\mu$ the generic properties of the elliptic operator differ dramatically from the case of a scalar coefficient, see [EKRS07, Remark 5.1], [ERS07, §4], and [Sav98, Ch. 5].

The outline of the paper is as follows: in the next section we will introduce some notation. In Section 3.3 the strategy of proof is explained. Section 3.4 contains some preliminaries which establish the connection between the regularity of the solution and the edge singularities. In Section 3.5 we collect some auxiliary results which justify at the end the transformation of the problem to a Dirichlet one. Section 3.6 is devoted to the core of the proof of Theorem 3.1.1, essentially based on the discussion of the edge singularities. In Section 3.7 we give the proofs of Theorem 3.1.2 and of Corollary 3.1.3. Some concluding remarks are given in Section 3.8. The appendix in Section 3.9 finishes the paper by establishing the required estimates for the occurring singularities for geometric edges and bimaterial outer edges.
3. Elliptic Model Problems

3.2. Notation

Throughout this paper $\Omega \subset \mathbb{R}^d$ always denotes a bounded Lipschitz domain (see [Gri85] for the definition) and $\Gamma \subset \partial \Omega$ is an open part of its boundary. $W^{1,p}(\Omega)$ denotes the (complex) Sobolev space on $\Omega$ consisting of those $L^p(\Omega)$ functions whose first order distributional derivatives also belong to $L^p(\Omega)$ (see [Gri85] or [Maz85]). Note that $\Omega$ enjoys the extension property for $W^{1,p}(\Omega)$ in view of being a bounded Lipschitz domain, see [Giu94, Thm. 3.10] or [Maz85, Ch. 1.1.16]. Thus, $W^{1,p}(\Omega)$ is identical with the completion of the set $\{v|_{\Omega} : v \in C^\infty(\mathbb{R}^d)\}$ with respect to the norm $\|v\|_{W^{1,p}} \defeq \left(\int_{\Omega} (|\nabla v|^p + |v|^p) \, dx\right)^{1/p}$. We use the symbol $W^{-1,p}(\Omega)$ for the closure of

$$\{v|_{\Omega} : v \in C^\infty(\mathbb{R}^d), \text{supp } v \cap (\partial \Omega \setminus \Gamma) = \emptyset\}$$

in $W^{1,p}(\Omega)$. If $\Gamma = \emptyset$ we write as usual $W^{1,p}_0(\Omega)$ instead of $W^{1,p}(\Omega)$. $W^{-1,p}_0(\Omega)$ denotes the dual to $W^{1,p}_0(\Omega)$ and $W^{-1,p}(\Omega)$ denotes the dual to $W^{1,p}(\Omega)$, when $\frac{1}{p} + \frac{1}{p'} = 1$ holds. If $\Omega$ is understood, then we sometimes abbreviate $W^{\pm 1,p}_\Gamma$, $W^{1,p}_0$ and $W^{-1,p}$, respectively. $(\cdot, \cdot)_X$ always indicates the duality between a Banach space $X$ and its dual; in case of $X = \mathbb{C}^d$ we mostly write $(\cdot, \cdot)$. If $\omega$ is a Lebesgue measurable, essentially bounded function on $\Omega$ taking its values in the set of real, symmetric $d \times d$ matrices, then we define $-\nabla \cdot \omega \nabla$ by

$$(\omega v, \nabla w)_{W^{-1,2}(\Omega)} \defeq \int_{\Omega} \omega \nabla v \cdot \nabla w \, dx ; \quad v, w \in W^{-1,2}(\Omega). \tag{3.2.1}$$

The maximal restriction of $-\nabla \cdot \omega \nabla$ to any of the spaces $W^{-1,p}_\Gamma(\Omega)$ ($p > 2$) we will denote by the same symbol. Finally, we define for any two complex numbers $\sigma, \lambda$

$$\sigma^\lambda \defeq \exp(\lambda \log |\sigma| + i \lambda \arg \sigma), \quad \arg \sigma \in [-\pi, \pi]; \tag{3.2.2}$$

and for $\iota, \vartheta \in [-\pi, \pi]$ with $\iota < \vartheta$ we define the sector

$$K_{\iota, \vartheta} \defeq \{r \cos \vartheta, r \sin \vartheta) : r > 0, \theta \in [\iota, \vartheta]\}.$$

3.3. Strategy of proof

Because the core of the proof of Theorem 3.1.1 is very technical, we will give here an exposition of the ideas behind it for the convenience of the reader. Clearly, the problem is a mixed boundary value problem with discontinuous coefficients on a convex polyhedron. One should expect that this could – in principle – be treated as in [Dau92] (see also [MR03]), where the following is shown for the mixed problem with Hölder continuous coefficients:

If for any edge point or vertex $x$ a certain spectrum $\sigma(x)$ satisfies

$$p < \inf_{z \in \sigma(x), \text{Re } z \in [0,1]} \frac{2}{1 - \text{Re } z} \quad \text{for any edge point } x \tag{3.3.1}$$

and, additionally,

$$p < \inf_{z \in \sigma(x), \text{Re } z \in [0,1]} \frac{3}{1 - \text{Re } z} \quad \text{for any vertex } x, \tag{3.3.2}$$

then the associated differential operator with Hölder continuous coefficients provides a topological isomorphism between $W^{1,p}_\Gamma$ and $W^{-1,p}_\Gamma$. Here, the spectrum $\sigma(x)$ is the spectrum
of an associated (generalized) Sturm-Liouville operator, if \( x \) is a point from an edge, see the next section for details, and in the case of a vertex it is the spectrum of an associated Laplace-Beltrami operator, see [Dau92] for details. The problem is that it is already difficult to determine the spectrum of this Laplace-Beltrami operator if the coefficient function is constant, and we have no idea how to do this in the case of heterogeneous materials.

Fortunately, there is a way out of this dilemma: for Dirichlet problems a deep idea of Maz’ya [MERS04] permits to restrict the investigation to the edge singularities as far as the integrability of the gradient of the solution up to an index \( p > 3 \) is concerned, see Proposition 3.4.6 below. (This heavily rests on the a priori known Hölder continuity of the solution, see [LU68, Ch. III.14].) So we may circumvent the analysis of the vertex singularities, if we can transform the problem to an equivalent one with Dirichlet boundary conditions. The strategy of proof is thus the following. We first deform the problem via a bi-Lipschitz (in fact: piecewise linear) mapping, such that in the resulting polyhedron the Neumann boundary part is a complete side of it, see Figure 3.2. When doing so, we have to show that under this deformation the occurring spaces \( W^{1,p}_\Gamma \) and \( W^{-1,p}_\Gamma \) are suitably mapped on spaces of the same quality and that, additionally, the differential operator goes over into a similar one (see Proposition 3.5.3). In a second step we reflect the problem across the Neumann boundary part and identify the detailed structure of the resulting (Dirichlet) problem (see Proposition 3.5.4). One especially obtains additional edges in the interior of the polyhedron coming from the transform and the reflection process (see also Figure 3.2). The main part of the proof is then to show, that all edges fulfill the supposition of Proposition 3.4.6, which stands in an obvious relation to (3.3.1). Regrettably, this latter is a touchy business, heavily resting on an adequate reformulation of the transmission conditions for the Sturm-Liouville problem (see Subsection 3.6.2) and, finally, on the sophisticated estimates for the geometrical and bimaterial outer edges, which we present in Section 3.9.

### 3.4. Edge singularities

In this section we first recall the optimal regularity result from [MERS04] for heterogeneous Dirichlet problems on polyhedral domains and explain how to identify the occurring edge singularities.

![Figure 3.2.: Model domain, original (left) and transformed (right), with Neumann boundary (hatched area) and material interface (shaded area). N.B. the material interface is flexed after transformation, and there is an additional material interface framed by the dashed lines after transformation. (\( \Lambda \), \( \Upsilon \), and \( \Sigma \) keep their names after transformation.)](image)
3. Elliptic Model Problems

Definition 3.4.1. Let numbers $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$ be given and, additionally, real, positive definite $2 \times 2$ matrices $\rho^1, \ldots, \rho^n$. We introduce on $]0, \theta_n[ \setminus \{\theta_1, \ldots, \theta_{n-1}\}$ coefficient functions $b_0, b_1, b_2$ the restrictions of which to the interval $]\theta_j, \theta_{j+1}[, j = 0, \ldots, n-1$, are given by

\[
\begin{align*}
  b_0(\theta) &= \rho^1_{11} \cos^2 \theta + 2\rho^1_{12} \sin \theta \cos \theta + \rho^1_{22} \sin^2 \theta, \\
  b_1(\theta) &= (\rho^2_{22} - \rho^1_{11}) \sin \theta \cos \theta + \rho^1_{12}(\cos^2 \theta - \sin^2 \theta), \\
  b_2(\theta) &= \rho^1_{11} \sin^2 \theta - 2\rho^1_{12} \sin \theta \cos \theta + \rho^1_{22} \cos^2 \theta.
\end{align*}
\]

(3.4.1)

If $\theta_n \neq \theta_0 + 2\pi$, then we define the space $H$ as $W^{1,2}(\theta_0, \theta_n]$, else as the periodic Sobolev space $W^{1,2}(\theta_0, \theta_n] \cap \{ \psi : \psi(\theta_0) = \psi(\theta_n) \}$ (which clearly may be identified with the Sobolev space $W^{1,2}(S^1)$ on the unit circle $S^1$). For every $\lambda \in \mathbb{C}$ we define the quadratic form $t_\lambda$ on $H$ by

\[
t_\lambda[\psi] \overset{\text{def}}{=} \int_{\theta_0}^{\theta_n} b_2 \psi' \overline{\psi'} + \lambda b_1 \psi \overline{\psi} - \lambda b_0 \psi \overline{\psi} - \lambda^2 b_0 \psi \overline{\psi} \, d\theta
\]

(3.4.2)

and $\mathcal{A}_\lambda$ as the operator which is induced by $t_\lambda$ on $L^2(\theta_0, \theta_n]$.

Remark 3.4.2. It is easy to check that $b_2 \geq \frac{(\rho^1_{12})^2}{\rho^1_{22}}$. From this it is straightforward to see that each form $t_\lambda$ is sectorial, what is also true for $\mathcal{A}_\lambda$, see [Kat80, Ch. VI].

Definition 3.4.3. Let $\Omega \subset \mathbb{R}^3$ be a polyhedron which, additionally, is a Lipschitz domain and $\{\Omega_k\}_k$ a (finite, disjoint) polyhedral partition of $\Omega$. Let $\mu$ be a matrix function on $\Omega$ which is constant on each $\Omega_k$ and takes real, symmetric, positive definite $3 \times 3$ matrices as values. Take any edge $E$ of any of the $\Omega_k$’s and consider an arbitrary inner point $P$ of this edge. Choose a new orthogonal coordinate system $(x, y, z)$ with origin at the point $P$ such that the direction of $E$ coincides with the $z$-axis. We denote by $\mathcal{O}_E$ the corresponding orthogonal transformation matrix and by $\mu_{E,p}$ the piecewise constant matrix function which coincides in a neighbourhood of $P$ with $\mathcal{O}_E \mu(\mathcal{O}_E^{-1}(x + P)) \mathcal{O}_E^{-1}$ and which satisfies

\[
\mu_{E,p}(tx, ty, z) = \mu_{E,p}(x, y, 0), \quad \text{for all } (x, y, z) \in \mathbb{R}^3, \ t > 0.
\]

(3.4.3)

By $\mu_{E}(\cdot, \cdot)$ we denote the upper left $2 \times 2$ block of $\mu_{E,p}(\cdot, \cdot, 0)$.

Remark 3.4.4. There exist angles $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$, such that $\mu_E$ is constant on each of the sectors $K_{\theta_j}^{\theta_{j+1}}$ and takes real, symmetric, positive definite matrices as values. Note that $\theta_n = \theta_0 + 2\pi$ if $\mu_E$ corresponds to an interior edge $E$, otherwise $\mu_E$ is given on an infinite sector $K^\theta_{\theta_0}$ which coincides near $P$ with the intersection of (the transformed) $\Omega$ with the $x$-$y$-plane.

Definition 3.4.5. We call an edge $E$ of $\Omega$ a geometric edge if $E \subset \partial \Omega$ and all inner points of $E$ belong to the closure of exactly one sub-polyhedron $\Omega_k$. Further, we say that $E$ is a bimaterial outer edge if $E \subset \partial \Omega$ and the function $\mu_E$ takes exactly two different values.

We proceed by quoting the central linear regularity result [MERS04, Thm. 2.3], by means of which our regularity results will be deduced:

Proposition 3.4.6. Let $\Omega$, $\{\Omega_k\}_k$ and $\mu$ as in Definition 3.4.3. For any edge $E$ let $\mu_E$ be the $2 \times 2$ matrix valued function on $K^\theta_{\theta_0}$ in the sense of Definition 3.4.3. If for every edge $E$ the thus induced operators $\mathcal{A}_\lambda$ on $L^2(\theta_0, \theta_n]$ have a trivial kernel for all $\lambda$ with $\text{Re} \lambda \in [0, 1/3 + \epsilon]$ ($\epsilon > 0$ arbitrarily small), then there is a $p > 3$ such that

\[
- \nabla \cdot \mu \nabla : W^{1,p}_0(\Omega) \rightarrow W^{-1,p}(\Omega)
\]

(3.4.4)

is a topological isomorphism.
Remark 3.4.7. Unfortunately, there are some errors in the paper [MERS04], cf. also [EKRS07, Remark 2.2]. First, the assertion of [MERS04, Thm. 2.3] that the exponent \( p \) can be taken from the interval \([2, 2/(1 - \lambda_T)]\) is erroneous, since the assumptions of [MERS04, Thm. 2.4] have to be taken into account. The correct formulation of the linear regularity result proved in [MERS04] is given in Proposition 3.4.6 above. Furthermore, the signs in formulas for the coefficients of certain generalized Sturm-Liouville equations are not correct, in detail: in [MERS04, p. 240] there is a wrong sign in the formula for the Mellin transform formulas for the coefficients of certain generalized Sturm-Liouville equations are not correct, the result proved in [MERS04] is given in Proposition 3.4.6 above. Furthermore, the signs in Thm. 2.4 have to be taken into account. The correct formulation of the linear regularity principle (to determine the critical values \( \lambda \)) to determine the critical values \( \lambda \) of them. In the next sections we will do this for all edges resulting from our problems.

Thus the question arises how to find the parameters \( \lambda \) for which the operator \( A_\lambda \) only has a trivial kernel. One proceeds as follows: standard arguments show that any function \( u \) from the kernel of the operator \( A_\lambda \) obeys the differential equation

\[(b_2u')' + \lambda (b_1u') + \lambda b_1u' + \lambda^2 b_0u = 0 \quad (3.4.5)\]

on each of the intervals \( ]\theta_j, \theta_{j+1}[ \) and, additionally, in every point \( \theta \in \{\theta_1, \ldots, \theta_{n-1}\} \) the transmission conditions

\[ [u]_{\theta} = 0, \quad [b_2u' + \lambda b_1u]_{\theta} = 0 \quad (3.4.6)\]

have to be satisfied. (As usual, \([u]_{\theta}\) stands for \( \lim_{\theta^-} u - \lim_{\theta^+} u \)). In order to find the critical parameters \( \lambda \), one employs the elementary solutions of the differential equation (3.4.5) on each of the subintervals \( ]\theta_j, \theta_{j+1}[ \)

\[ \theta \mapsto e^{-i\lambda \theta} (\alpha e^{2i\theta} + 1)^\lambda, \quad \theta \mapsto e^{i\lambda \theta} (\bar{\alpha} e^{-2i\theta} + 1)^\lambda \]

which were announced in the pioneering paper [CDL01] (see also [MERS04, Ch. 3.6] for further details). The complex number \( \alpha = \alpha_j \) is determined by the matrix

\[ m = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \rho_{11}^j & \rho_{12}^j \\ \rho_{12}^j & \rho_{22}^j \end{pmatrix} \]

as

\[ \alpha = \frac{i(m_{22} - D_m^{1/2}) - m_{12}}{i(m_{22} + D_m^{1/2}) + m_{12}}, \quad (3.4.7)\]

where \( D_m \) denotes the determinant of the matrix \( m \).

Remark 3.4.8. Because \( m_{22} \) is positive, \( \alpha \) necessarily satisfies \( 0 \leq |\alpha| < 1 \). Moreover, if

\[ \begin{pmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{12} & \bar{m}_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & -m_{12} \\ -m_{12} & m_{22} \end{pmatrix}, \]

then \( \bar{\alpha} = \bar{\alpha} \).

Making on any interval \( ]\theta_j, \theta_{j+1}[ \) an ansatz

\[ u_j(\theta) \overset{\text{def}}{=} c_{j,+} e^{-i\lambda \theta} (\alpha_j e^{2i\theta} + 1)^\lambda + c_{j,-} e^{i\lambda \theta} (\bar{\alpha}_j e^{-2i\theta} + 1)^\lambda, \quad (3.4.8)\]

these functions automatically satisfy (3.4.5), while the boundary conditions together with the transmission conditions (3.4.6) for \( \theta = \theta_j \ (j \in \{1, \ldots, n-1\}) \) lead to a \( 2n \times 2n \) homogeneous linear system for the coefficients \( c_{j,+}, c_{j,-} \). The usual criterion for the (nontrivial) solvability of this system gives the characteristic equation of the problem (3.4.5, 3.4.6) and allows (in principle) to determine the critical values \( \lambda \) — or at least to give estimates for the real part of them. In the next sections we will do this for all edges resulting from our problems.
3. Elliptic Model Problems

3.5. Auxiliary Results

Lemma 3.5.1. In the terminology from above let \( \lambda \) with \( \Re \lambda \in ]0,1[ \) be a number such that there exists a (nontrivial) function \( v_\lambda \in \mathcal{H} \) from the kernel of \( A_\lambda \), see Definition 3.4.1. Let \( \omega_13, \omega_23 \) and \( \omega_33 \) be real valued, bounded, measurable functions on \( K^{\theta}_{0n} \) and define the coefficient function \( \omega \) on \( K^{\theta}_{0n} \equiv K^{\theta}_{0n} \times \mathbb{R} \) by

\[
\omega(x, y, z) \equiv \begin{pmatrix}
\rho_{11}^2 \\
\rho_{12}^2 \\
\omega_{13}(x, y) \\
\omega_{23}(x, y) \\
\omega_{33}(x, y)
\end{pmatrix}, \quad \text{if} \quad (x, y) \in K^{\theta}_{1j+1}.
\tag{3.5.1}
\]

Then there is a compactly supported element \( f \in W^{-1,6}(K^{\theta}_{0n}) \) such that the — also compactly supported — variational solution \( v \in W^{1,2}(K^{\theta}_{0n}) \) of \( \nabla \cdot \omega \nabla v = f \) on \( K^{\theta}_{0n} \) does not belong to \( W^{1,\frac{3}{2}+\varepsilon}(K^{\theta}_{0n}) \).

Proof. It is not hard to calculate that the function \( \psi_0 \) given by

\[
\psi_0(x, y) = (x^2 + y^2)^{\lambda/2} v_\lambda(\arg(x + iy))
\tag{3.5.2}
\]

belongs to \( W^{1,p}(K^{\theta}_{0n}) \) if \( p \in \left[ 2, \frac{2}{1 - \Re \lambda} \right] \) but not to \( W^{1,\frac{3}{2}+\varepsilon}(K^{\theta}_{0n}) \). (Recall that \( v_\lambda \) does not vanish identically on \( ]\theta_0, \theta_n[ \).) By construction of \( A_\lambda \), the function \( \psi_0 \) satisfies

\[
-\nabla \cdot \rho \nabla \psi_0 = 0.
\tag{3.5.3}
\]

in the distributional sense, see [MERS04]. We define now the function \( \psi \) by \( \psi(x, y, z) \equiv \psi_0(x, y) \) and notice that \( \psi \) belongs to \( W^{1,p}(K^{\theta}_{0n}) \) for \( p \in \left[ 2, \frac{2}{1 - \Re \lambda} \right] \) but not to the space \( W^{1,\frac{3}{2}+\varepsilon}(K^{\theta}_{0n}) \). Suppose \( \varphi = \varphi_1 \otimes \varphi_2 \) with \( \varphi_1 \in C_0^\infty(K^{\theta}_{0n}) \) and \( \varphi_2 \in C_0^\infty(\mathbb{R}) \), then

\[
\int_{K^{\theta}_{0n}} \langle \omega \nabla \psi, \nabla \varphi \rangle_{C^1} \, dx \, dy = \int_{\mathbb{R}} \int_{K^{\theta}_{0n}} \langle \rho \nabla \psi_0, \nabla \varphi_1 \rangle_{C^2} \, dx \, dy \, \varphi_2(z) \, dz \quad + \quad \int_{K^{\theta}_{0n}} \langle \omega_{13} \frac{\partial \psi_0}{\partial x} + \omega_{23} \frac{\partial \psi_0}{\partial y} \rangle \varphi_1 \, dx \, dy \int_{\mathbb{R}} \frac{\partial \varphi_2}{\partial z} \, dz.
\tag{3.5.4}
\]

The first addend vanishes by (3.5.3) and the second by \( \varphi_2 \in C_0^\infty(\mathbb{R}) \). The set of \( \varphi \)'s with the above tensor product structure is total in \( C_0^\infty(K^{\theta}_{0n}) \), therefore (3.5.4) is also zero for any \( \varphi \) from this latter space. Let \( \eta \) be a function from \( C_0^\infty(\mathbb{R}^3) \) which equals 1 in a neighbourhood of \( 0 \in \mathbb{R}^3 \) and which vanishes outside a ball \( B \). Then one calculates for any \( \varphi \in C_0^\infty(K^{\theta}_{0n}) \)

\[
\int_{K^{\theta}_{0n}} \langle \omega \nabla (\eta \psi), \nabla \varphi \rangle \, dx = - \int_{K^{\theta}_{0n}} \varphi \langle \omega \nabla \psi, \nabla \eta \rangle \, dx \quad + \quad \int_{K^{\theta}_{0n}} \psi \langle \omega \nabla \eta, \nabla \varphi \rangle \, dx \quad + \quad \int_{K^{\theta}_{0n}} \langle \omega \nabla \psi, (\nabla (\eta \varphi)) \rangle \, dx.
\tag{3.5.5}
\]

\( \int_{K^{\theta}_{0n}} \langle \omega \nabla \psi, (\nabla (\eta \varphi)) \rangle \, dx \) vanishes because (3.5.4) always is zero if \( \varphi \in C_0^\infty(K^{\theta}_{0n}) \). On the other hand, it is not hard to see that the other two addends on the right hand side define — in their dependence on \( \varphi \) — continuous linear forms on \( W^{1,6/5}(K^{\theta}_{0n}) \), namely: the property \( \psi \in W^{1,2}(K^{\theta}_{0n}) \) and the compact support property of \( \eta \) imply \( \langle \omega \nabla \psi, \nabla \eta \rangle \in L^2(K^{\theta}_{0n}) \). Combining
3.5. Auxiliary Results

Concerning the second addend, one easily estimates

\[
\int_{\mathcal{K}^d_{\theta_0}} \psi(\omega \nabla \eta, \nabla \varphi) \, dx \leq \|\omega \nabla \eta\|_{L^\infty(\mathcal{K}^d_{\theta_0})} \|\psi\|_{L^6(\mathcal{B} \cap \mathcal{K}^d_{\theta_0})} \|\varphi\|_{W^{1,6/5}(\mathcal{K}^d_{\theta_0})}
\]

\[
\leq \|\omega \nabla \eta\|_{L^\infty(\mathcal{K}^d_{\theta_0})} \|\varphi\|_{W^{1,2}(\mathcal{B} \cap \mathcal{K}^d_{\theta_0})} \|\varphi\|_{W^{1,6/5}(\mathcal{K}^d_{\theta_0})}.
\]

Thus, setting \( v \overset{\text{def}}{=} \eta \psi \), one obtains the assertion. \( \square \)

**Remark 3.5.2.** If \( \theta_n = \theta_0 + 2\pi \), then \( \mathcal{K}^d_{\theta_0} = \mathbb{R}^2, \mathcal{K}^d_{\theta_0} = \mathbb{R}^3 \) and, hence, \( W^{1,p}(\mathcal{K}^d_{\theta_0}) = W^{1,p}(\mathbb{R}^3) \).

**Proposition 3.5.3.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain and \( \Gamma \) be an open subset of its boundary. Assume that \( \phi \) is a mapping from a neighbourhood of \( \Omega \) into \( \mathbb{R}^d \) which is bi-Lipschitz. Let us denote \( \phi(\Omega) = \Omega_* \) and \( \phi(\Gamma) = \Gamma_* \). Then

i) For any \( p \overset{\in}{\in} ]1, \infty[ \), \( \phi \) induces a linear, topological isomorphism

\[
\Psi_p : W^{1,p}(\Omega_*) \to W^{1,p}(\Omega)
\]

which is given by \( (\Psi_p f)(x) = f(\phi(x)) = (f \circ \phi)(x) \).

ii) \( \Psi_p^* \) is a linear, topological isomorphism between \( W^{-1,p}(\Omega) \) and \( W^{-1,p}(\Omega_*) \).

iii) If \( \omega \) is a bounded measurable function on \( \Omega_* \), taking its values in the set of \( d \times d \) matrices, then

\[
\Psi_p^* \nabla \cdot \omega \nabla \psi = \nabla \cdot \omega_* \nabla \psi
\]

with

\[
\omega_* (y) = (D \phi)(\phi^{-1}(y)) \omega(\phi^{-1}(y)) (D \phi)^T (\phi^{-1}(y)) \frac{1}{|\det(D \phi)(\phi^{-1}(y))|}.
\]

\( (D \phi \) denotes the Jacobian of \( \phi \) and \( \det(D \phi) \) the corresponding determinant). If, in particular, \( -\nabla \cdot \omega \nabla : W^{1,p}(\Omega) \to W^{-1,p}(\Omega) \) is a topological isomorphism, then \( -\nabla \cdot \omega_* \nabla : W^{1,p}(\Omega_*) \to W^{-1,p}(\Omega_*) \) also is (and vice versa).

**Proof.** The proof of i) is contained in [GGKR02, Thm. 2.10]. ii) follows from i) by duality. We prove iii): For \( f \in W^{1,p}_{r_*}(\Omega_*) \), \( g \in W^{1,p}_{r_*}(\Omega_*) \) we get by the change of variables formula:

\[
\langle -\Psi_p^* \nabla \cdot \omega \nabla (\Psi_p f), g \rangle_{W^{-1,p}_{r_*}(\Omega_*)} = \langle -\nabla \cdot \omega \nabla (\Psi_p f), \Psi_p^* g \rangle_{W^{-1,p}_{r_*}(\Omega)}
\]

\[
= \langle -\nabla \cdot \omega \nabla (f \circ \phi), g \circ \phi \rangle_{W^{-1,p}_{r_*}(\Omega)}
\]

\[
= \int_{\Omega} \langle \omega(x) \nabla (f \circ \phi)(x), \nabla (g \circ \phi)(x) \rangle \, dx
\]

\[
= \int_{\Omega} \langle \omega(x) (D \phi)^T (x) (\nabla f)(\phi(x)), (D \phi)^T (x) (\nabla g)(\phi(x)) \rangle \, dx
\]

\[
= \int_{\Omega} \langle (D \phi)(x) \omega(x) (D \phi)^T (x)(\nabla f)(\phi(x)), (\nabla g)(\phi(x)) \rangle \frac{|\det(D \phi)(x)|}{|\det(D \phi)(\phi^{-1}(y))|} \, dx
\]

\[
= \int_{\Omega_*} \langle (D \phi)(\phi^{-1}(y)) \omega(\phi^{-1}(y)) (D \phi)^T (\phi^{-1}(y)) \nabla f(y), \nabla g(y) \rangle \, dy
\]

\[
= \langle -\nabla \cdot ((D \phi)(\phi^{-1}(\cdot)) \omega(\phi^{-1}(\cdot)) (D \phi)^T (\phi^{-1}(\cdot)) \nabla f), g \rangle_{W^{-1,p}_{r_*}(\Omega_*)}.
\]
3. Elliptic Model Problems

The essential point is that $\phi$ — as a Lipschitz continuous function — is differentiable almost everywhere and its (weak) derivative is essentially bounded (see [EG92, Ch. 4.2.3]). The last assertion follows from i), ii) and (3.5.6).

Proposition 3.5.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex, polygonal domain and $\Gamma$ be an open subset of $\partial \Omega$ such that $\Omega \cap \{(x,0,z) : x, z \in \mathbb{R}\} = \Gamma$. Let for any $x = (x,y,z)$ the symbol $x_-$ denote the element $(x,-y,z)$ and define $\bar{\Omega}$ as the interior of $\Omega \cup \{x : x_- \in \Omega\} \cup \bar{\Gamma}$.

If $\omega$ is a bounded, measurable function on $\Omega$ taking its values in the set of real, symmetric $3 \times 3$ matrices, then we define

$$\hat{\omega}(x) \overset{\text{def}}{=} \begin{cases} \omega(x), & \text{if } x \in \Omega, \\ \begin{pmatrix} \omega_{11}(x_-) & -\omega_{12}(x_-) & \omega_{13}(x_-) \\ -\omega_{12}(x_-) & \omega_{22}(x_-) & -\omega_{23}(x_-) \\ \omega_{13}(x_-) & -\omega_{23}(x_-) & \omega_{33}(x_-) \end{pmatrix}, & \text{if } x_- \in \Omega. \end{cases} \quad (3.5.8)$$

i) If $\psi \in W^{1,2}_\Gamma(\Omega)$ satisfies the equation $-\nabla \cdot \omega \nabla \psi = f \in W^{-1,2}_\Gamma(\Omega)$, then the equation $-\nabla \cdot \hat{\omega} \nabla \hat{\psi} = \hat{f} \in W^{-1,2}_\Gamma(\Omega)$ holds for $\hat{\psi}$ with

$$\hat{\psi}(x) = \begin{cases} \psi(x), & \text{if } x \in \Omega, \\ \psi(x_-), & \text{if } x_- \in \Omega \end{cases}$$

and $\hat{f}$ defined by $\langle \hat{f}, \varphi \rangle_{W^{-1,2}_\Gamma(\Omega)} \overset{\text{def}}{=} \langle f, \varphi|_{\Omega} + \varphi_-|_{\Omega} \rangle_{W^{1,2}_\Gamma(\Omega)}$. For $\varphi \in W^{1,1}(\bar{\Omega})$ the function $\varphi_-$ is defined by $\varphi_-(x) \overset{\text{def}}{=} \varphi(x_-)$.

ii) Moreover, if $f \in W^{-1,p}_\Gamma(\Omega)$, then $\hat{f} \in W^{-1,p}(\bar{\Omega})$; and if $-\nabla \cdot \hat{\omega} \nabla : W^{1,p}_0(\bar{\Omega}) \to W^{-1,p}(\bar{\Omega})$ is a topological isomorphism, then $-\nabla \cdot \omega \nabla : W^{1,p}_0(\Omega) \to W^{-1,p}(\bar{\Omega})$ also is.

Proof. i) It is known that $\hat{\psi}$ belongs to $W^{1,p}_\Gamma(\bar{\Omega})$, see [Gin94, Lemma 3.4]. Thus, i) is obtained by the definitions of $\hat{\psi}, \hat{f}, -\nabla \cdot \omega \nabla, -\nabla \cdot \hat{\omega} \nabla$ and straightforward calculations, based on Proposition 3.5.3 when applied to the transformation $x \mapsto x_-$. ii) The operator $f \mapsto \hat{f}$ is the adjoint to $\varphi \mapsto \langle \varphi|_{\Omega} + \varphi_-|_{\Omega} \rangle$. The latter maps each $W^{1,p}_0(\bar{\Omega})$ continuously into $W^{1,p}_\Gamma(\Omega)$ for any $p \in ]1,\infty[$. The last statement is then implied by the preceding ones and the definition of $\hat{\psi}$.

Remark 3.5.5. The proposition is mutatis mutandis true for the reflection at other planes.

Remark 3.5.6. In fact it can be shown that $-\nabla \cdot \omega \nabla : W^{1,p}_\Gamma(\Omega) \to W^{-1,p}_\Gamma(\Omega)$ is a topological isomorphism, if and only if $-\nabla \cdot \hat{\omega} \nabla$ is a topological isomorphism between the symmetric part of $W^{1,p}_0(\bar{\Omega})$ and the symmetric part of $W^{-1,p}(\bar{\Omega})$. The point is that this is not of use to us here because the reduction procedure of [MERS04] applies to the whole space and not only to the symmetric parts of the spaces (see also Remark 3.6.4).

In the sequel we will transform our model problems which include mixed boundary conditions to the case of Dirichlet conditions — which are imposed in Proposition 3.4.6. In essence, this happens via a linear transformation leading to a peculiar triangle, a bi-Lipschitz transformation and a reflection argument. All of this is carried out in the next section.
3.6. Proof of Theorem 3.1.1

3.6.1. Transformation of the problem

Proposition 3.5.3 allows us in a first step to reduce the case of an arbitrary triangle $\Lambda$ to that one where $\Lambda$ is the triangle with the vertices $(1, -1), (-1, 1), (1/2, 1/2)$ and, additionally, $\Upsilon$ is the line segment between $(0, 0)$ and $(1, -1)$, see Figure 3.1 and Figure 3.3. Namely, first one shifts the triangle such that $P$ becomes the origin. Let $P_1$ denote the vertex where (the shifted) $\Upsilon$ ends and $P_2$ the vertex which does not touch $-\Upsilon$. We now transform $\mathbb{R}^2$ under the linear mapping which assigns $P_1$ to $(1, -1)$ and $P_2$ to $(1/2, 1/2)$. Extending this mapping to $\mathbb{R}^3$ by letting the $z$-component invariant, one obtains the special geometric constellation of Figure 3.3 stated above. Clearly, the transformed plane $\Xi$ maintains the properties demanded in the suppositions of Theorem 3.1.1. In particular, we denote the point, where the (transformed) plane intersects the $z$-axis, by $P_3$. In a natural sense we may speak of an upper half space $\mathcal{G}_u$ and a lower half space $\mathcal{G}_l$ (each on one side of the intersecting plane $\Xi$), where the coefficient function $\mu$ takes the values

$$\mu^+ = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \quad \text{on } \mathcal{G}_u, \quad \mu^- = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \quad \text{on } \mathcal{G}_l. \quad (3.6.1)$$

We transform the problem via the bi-Lipschitz transformation (see Figure 3.4)

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0, 0 \\ (0, \sqrt{2}) \end{pmatrix} \rightarrow \begin{pmatrix} (1/\sqrt{2}, 0) \\ (0, 0) \end{pmatrix}$$

Figure 3.4.: The piecewise linear transformation $\phi$ in the $x$-$y$-plane. Triangle $\Lambda$ before (left) and after (right) transformation by $\phi$. 

Figure 3.3.: Transformation of a model domain (see the left side of Figure 3.1) to the generic domain, with Neumann boundary (hatched area) and material interface (shaded area). ($\Lambda$, $\Upsilon$, $\Sigma$, and $\Xi$ keep their names after transformation.)
Thus, we end up with a Dirichlet problem on \( \hat{\Pi} \). By Proposition 3.5.4 it suffices to show that

\[
\phi \equiv \begin{cases} 
\left( \begin{array}{ccc} 
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & \sqrt{2} & 0 \\
0 & 0 & 1 \\
\sqrt{2} & 0 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & 0 & 1 
\end{array} \right) & \text{on } \{(x,y,z) : y > x\} \\
\left( \begin{array}{ccc} 
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33} \\
a_{22} & a_{23} & a_{33} \\
a_{11} + 2a_{12} + a_{13} & a_{12} + a_{22} & a_{11} + a_{22} + a_{23} \\
a_{12} + a_{11} & a_{12} + a_{13} & a_{11} + a_{13} + a_{23} \\
a_{13} & a_{23} & a_{33} \\
a_{13} & a_{23} & a_{33} \\
a_{11} + a_{13} + a_{23} & a_{11} + a_{23} & a_{11} + a_{23} + a_{33} \\
\end{array} \right) & \text{on } \{(x,y,z) : y \leq x\}. 
\end{cases}
\]

(Please notice that the determinants of both matrices in (3.6.2) equal 1). \( \phi(\Lambda) \) is again a triangle — denoted by \( \Lambda_\phi \) — and has now the vertices \((0,0), (0,\sqrt{2}), (\sqrt{2},0)\), while the new domain is \( \Pi_\phi = \Lambda_\phi \times [-1,1] \). \( \Upsilon \) equals the subinterval \([0, \sqrt{2}]\) of the \( x \)-axis. The image \( \Xi_\phi \) of \( \Pi \cap \Xi \) consists of two triangles having one common edge \( E_\phi \subset \{(x,x,z) : x > 0, z \in \mathbb{R}\} \). (Of course, if \( \Xi \) was orthogonal to the \( z \)-axis, then both triangles are also orthogonal to the \( z \)-axis.) Clearly, the Neumann boundary part is now the rectangle with the vertices \((0,0,-1), (0,0,1), (\sqrt{2},0,-1), (\sqrt{2},0,1)\). The transformed matrix (see Proposition 3.5.3)

\[
\left( \begin{array}{ccc} 
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & \sqrt{2} & 0 \\
0 & 0 & 1 \\
\sqrt{2} & 0 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & 0 & 1 
\end{array} \right) 
\]

is calculated as

\[
\left( \begin{array}{ccc} 
a_{11} + 2a_{12} + a_{13} & a_{12} + a_{22} & a_{11} + a_{22} + a_{23} \\
a_{12} + a_{22} & 2a_{22} / \sqrt{2}a_{23} & a_{33} \\
a_{11} + a_{13} + a_{23} & \sqrt{2}a_{23} & a_{33} \\
\end{array} \right),
\]

while the transformed matrix

\[
\left( \begin{array}{ccc} 
\sqrt{2} & 0 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & 0 & 1 \\
\end{array} \right) 
\]

is calculated as

\[
\left( \begin{array}{ccc} 
2a_{11} & a_{12} + a_{13} & \sqrt{2}a_{13} \\
a_{12} + a_{11} & a_{12} + a_{13} & a_{11} + a_{13} + a_{23} \\
a_{13} & a_{23} & a_{33} \sqrt{2}a_{23} & a_{33} \end{array} \right). \]

(and analogously for the matrix \( b \)). We reflect the problem at the \( x \)-\( z \)-plane in the spirit of Proposition 3.5.4 and obtain a new triangle \( \Lambda \) with the vertices \((0,0,\sqrt{2}), (\sqrt{2},0), (0,-\sqrt{2})\), a new domain \( \Pi \equiv \Lambda \times [-1,1] \) and the coefficient function \( \mu \) on \( \Pi \) is defined as in (3.5.8). Thus, we end up with a Dirichlet problem on \( \Pi \). By Proposition 3.5.4 it suffices to show that

\[ -\nabla \cdot \hat{\mu} \nabla : W^{1,p}_0(\hat{\Pi}) \rightarrow W^{-1,p}(\hat{\Pi}) \]

is a topological isomorphism for a \( p > 3 \). For this, however, we may apply Proposition 3.4.6: we are done if we are able to show that for all edges \( E \) the induced operators \( A_\lambda \) have a trivial kernel for all \( \lambda \) with \( \text{Re} \lambda \in [0,1/3 + \epsilon] \) (\( \epsilon > 0 \) arbitrarily small). The occurring edges \( E \) are the following, see also Figure 3.5:

- geometric edges,
- bimaterial outer edges,
- the edges \( E^+_z \) and \( E^-_z \) lying between \( P_0 \) and \((0,0,1)\), or between \( P_0 \) and \((0,0,-1)\), respectively,
- the edge \( E_{xz} \), which is the intersection of the (transformed) \( \Xi \) with the \( x \)-\( z \)-plane,
- \( E_\phi \) and the reflected \( E_\phi \).
3.6. Proof of Theorem 3.1.1

3.6.2. Reformulation of the transmission conditions

The aim of this subsection is to express the transmission conditions for the ansatz functions (see (3.4.6)) in a condensed manner in terms of \( \alpha_j, \alpha_{j+1}, \theta_j \).

**Lemma 3.6.1.** Let \( \alpha \) be defined by (3.4.7), and

\[
m = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11}^j & \rho_{12}^j \\ \rho_{12}^j & \rho_{22}^j \end{pmatrix},
\]

where \( c_+, c_- \) are arbitrary complex constants. Further, let \( b_1, b_2 \) be defined as in (3.4.1). Then

\[
b_2(\theta)u'(\theta) + \lambda b_1(\theta)u(\theta) = -iD_m^{1/2} \lambda \left[ c_+ e^{-i\lambda \theta} (\alpha e^{2\theta} + 1)^\lambda + c_- e^{i\lambda \theta} (\bar{\alpha} e^{-2\theta} + 1)^\lambda \right],
\]

where \( D_m \) again denotes the determinant of the matrix \( m \).

**Proof.** First, one easily verifies

\[
u'(\theta) = c_+ e^{-i\lambda \theta} (\alpha e^{2\theta} + 1)^\lambda \frac{1 - \alpha e^{2\theta}}{1 + \alpha e^{2\theta}} + c_- e^{i\lambda \theta} (\bar{\alpha} e^{-2\theta} + 1)^\lambda i \lambda \frac{1 - \bar{\alpha} e^{-2\theta}}{1 + \bar{\alpha} e^{-2\theta}}.
\]

Next we want to prove

\[-b_2(\theta)i \frac{1 - \alpha e^{2\theta}}{1 + \alpha e^{2\theta}} + b_1(\theta) = -iD_m^{1/2}.
\]

For this we calculate

\[i \frac{1 - \alpha e^{2\theta}}{1 + \alpha e^{2\theta}} = i \frac{e^{-2\theta} - \alpha}{e^{-2\theta} + \alpha} \]

Figure 3.5.: Generic model domain after transformation by \( \phi \) with Neumann boundary (hatched area), material interface (shaded area), and the edges \( E^+_z, E_{xz}, E_{\phi} \), as well as bimaterial outer edges. (\( \Sigma \) keeps its name after transformation with \( \phi \).)
3. Elliptic Model Problems

and abbreviate the denominator \( i(m_{22} + D_m^{1/2}) + m_{12} \) of \( \alpha \) by \( N_\alpha \). One has
\[
e^{-2i\theta} - \alpha = (e^{-2i\theta} [i(m_{22} + D_m^{1/2}) + m_{12}] - i(m_{22} - D_m^{1/2}) + m_{12})/N_\alpha
\]
and
\[
e^{-2i\theta} + \alpha = (e^{-2i\theta} [i(m_{22} + D_m^{1/2}) + m_{12}] + i(m_{22} - D_m^{1/2}) - m_{12})/N_\alpha,
\]
what leads to
\[
\frac{1 - \alpha e^{-2i\theta}}{1 + \alpha e^{-2i\theta}} = \frac{i m_{12}(e^{-2i\theta} + 1) + m_{22}i(e^{-2i\theta} - 1) + D_m^{1/2}i(e^{-2i\theta} + 1)}{m_{12}(e^{-2i\theta} - 1) + m_{22}i(e^{-2i\theta} + 1) + D_m^{1/2}i(e^{-2i\theta} - 1)}.
\]
(3.6.12)

We augment the last fraction by \( \frac{\sin \theta}{e^{i\theta}} \), exploiting the equation
\[
e^{-2i\theta} + 1 \sin \theta = \frac{e^{-i\theta}(e^{-i\theta} + e^{i\theta})}{e^{-i\theta}(e^{-i\theta} - e^{i\theta})} \sin \theta = \frac{\cos \theta}{-i} = i \cos \theta,
\]
the right hand side of (3.6.12) becomes
\[
\frac{im_{12} \cos \theta + im_{22} \sin \theta - D_m^{1/2} \cos \theta}{m_{12} \sin \theta - m_{22} \cos \theta + iD_m^{1/2} \sin \theta} = \frac{-m_{12} \cos \theta - m_{22} \sin \theta - iD_m^{1/2} \cos \theta}{m_{12} \sin \theta - m_{22} \cos \theta + iD_m^{1/2} \sin \theta} = \frac{(-m_{12} \cos \theta - m_{22} \sin \theta - iD_m^{1/2} \cos \theta)(m_{12} \sin \theta - m_{22} \cos \theta - iD_m^{1/2} \sin \theta)}{(m_{12} \sin \theta - m_{22} \cos \theta)^2 + D_m \sin^2 \theta} = \frac{(m_{22} - m_{11}) \cos \theta \sin \theta + m_{12}(\cos^2 \theta - \sin^2 \theta) + iD_m^{1/2}}{m_{11} \sin^2 \theta - 2m_{12} \cos \theta \sin \theta + m_{22} \cos^2 \theta} = \frac{b_1(\theta) + iD_m^{1/2}}{b_2(\theta)}.
\]
(3.6.13)

Thus, (3.6.10) holds true. By complex conjugation one obtains from (3.6.10)
\[
b_2(\theta) i \frac{1 - \bar{\alpha} e^{-2i\theta}}{1 + \bar{\alpha} e^{-2i\theta}} + b_1(\theta) = iD_m^{1/2}.
\]
(3.6.14)

(3.6.10) and (3.6.14) together with (3.6.9) give the assertion (3.6.8).

\[\square\]

**Corollary 3.6.2.** Let \( u \) be the function on \([\theta_0, \theta_n]\) which coincides on \([\theta_j, \theta_{j+1}]\) with \( u_j \) defined in (3.4.8).

i) Assume first \( j \in \{1, \ldots, n-1\} \) and let \( D_j \) and \( D_{j+1} \) denote the determinants of the matrices
\[
\begin{pmatrix}
\rho_{11}^j & \rho_{12}^j \\
\rho_{21}^j & \rho_{22}^j
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\rho_{11}^{j+1} & \rho_{12}^{j+1} \\
\rho_{21}^{j+1} & \rho_{22}^{j+1}
\end{pmatrix},
\]
respectively. If we abbreviate \( \alpha \overset{\text{def}}{=} \alpha_j \) and \( \beta \overset{\text{def}}{=} \alpha_{j+1} \), then the transmission conditions in the point \( \theta = \theta_j \),
\[
[u]_\theta = [b_2 u' + \lambda b_1 u]_\theta = 0
\]
(3.6.15)

express as
\[
c_{j+} e^{-i\lambda \theta} (\alpha e^{-2i\theta} + 1) + c_{j-} e^{-i\lambda \theta} (\alpha e^{-2i\theta} + 1)
\]
\[
= c_{j+} e^{-i\lambda \theta} (\beta e^{-2i\theta} + 1) + c_{j-} e^{-i\lambda \theta} (\beta e^{-2i\theta} + 1)
\]
(3.6.16)
Next we consider the edges $E$ or geometric edges and bimaterial outer edges we show in the appendix that the operators

$$3.6.3. \text{Discussion of the edge singularities}$$

Thus, one has to consider the ansatz functions (see Remark 3.4.8)

$$D_j^{1/2}\left[c_{j+1}e^{-i\lambda\theta}(\alpha e^{2i\theta} + 1)^\lambda - c_{j-1}e^{i\lambda\theta}(\alpha e^{-2i\theta} + 1)^\lambda\right]$$

$$= D_j^{1/2}\left[c_{j-1}e^{-i\lambda\theta}(\beta e^{2i\theta} + 1)^\lambda - c_{j+1}e^{i\lambda\theta}(\beta e^{-2i\theta} + 1)^\lambda\right], \quad (3.6.17)$$

respectively. Thus, in case of $D_j = D_{j+1}$ for (3.6.15) it is necessary and sufficient that

$$c_{j+1}(\alpha e^{2i\theta} + 1)^\lambda = c_{j+1}(\beta e^{-2i\theta} + 1)^\lambda \quad (3.6.18)$$

and

$$c_{j-1}(\alpha e^{-2i\theta} + 1)^\lambda = c_{j-1}(\beta e^{-2i\theta} + 1)^\lambda \quad (3.6.19)$$

hold.

ii) Assume now $\theta_n = \theta_0 + 2\pi$. Then the corresponding transmission conditions in $\theta_0$ express as

$$c_{1+}e^{-i\lambda\theta_0}(\alpha_1 e^{2i\theta_0} + 1)^\lambda + c_{1-}e^{i\lambda\theta_0}(\alpha_1 e^{-2i\theta_0} + 1)^\lambda$$

$$= c_{n+}e^{-i\lambda\theta_0}(\alpha_n e^{2i\theta_0} + 1)^\lambda + c_{n-}e^{i\lambda\theta_0}(\alpha_n e^{-2i\theta_0} + 1)^\lambda \quad (3.6.20)$$

and

$$D_n^{1/2}\left[c_{n-}e^{-i\lambda\theta_0}(\alpha_n e^{2i\theta_0} + 1)^\lambda - c_{n+}e^{i\lambda\theta_0}(\alpha_n e^{-2i\theta_0} + 1)^\lambda\right]$$

$$= D_n^{1/2}\left[c_{n+}e^{-i\lambda\theta_0}(\alpha_n e^{2i\theta_0} + 1)^\lambda - c_{n-}e^{i\lambda\theta_0}(\alpha_n e^{-2i\theta_0} + 1)^\lambda\right], \quad (3.6.21)$$

respectively.

### 3.6.3. Discussion of the edge singularities

For geometric edges and bimaterial outer edges we show in the appendix that the operators $A_\lambda$ have a trivial kernel if Re $\lambda \in [0, 1/2]$. Next we consider the edges $E_+^*$ and $E_-^*$: starting with $E_+^*$, one has to deal with the coefficient matrices

$$m \equiv \begin{pmatrix} a_{11} + 2a_{12} + a_{22} & a_{12} - a_{22} \\ -a_{12} - a_{22} & 2a_{22} \end{pmatrix} \quad \text{if } \theta \in ]-\pi/2, -\pi/4[$$

$$\hat{m} \equiv \begin{pmatrix} 2a_{11} & -a_{12} - a_{11} \\ -a_{12} - a_{11} & a_{11} + 2a_{12} + a_{22} \end{pmatrix} \quad \text{if } \theta \in ]-\pi/4, 0[$$

$$\hat{o} \equiv \begin{pmatrix} 2a_{11} & a_{12} + a_{11} \\ a_{12} + a_{11} & a_{11} + 2a_{12} + a_{22} \end{pmatrix} \quad \text{if } \theta \in ]0, \pi/4[$$

$$o \equiv \begin{pmatrix} a_{11} + 2a_{12} + a_{22} & 2a_{12} + a_{22} \\ a_{12} + a_{22} & 2a_{22} \end{pmatrix} \quad \text{if } \theta \in ]\pi/4, \pi/2[.$$

Thus, one has to consider the ansatz functions (see Remark 3.4.8)

$$w \equiv c_+e^{-i\lambda}(\alpha e^{2i\theta} + 1)^\lambda + c_-e^{i\lambda}(\alpha e^{-2i\theta} + 1)^\lambda \quad \text{on } ]-\pi/2, -\pi/4[$$

$$u \equiv d_+e^{-i\lambda}(\beta e^{2i\theta} + 1)^\lambda + d_-e^{i\lambda}(\beta e^{-2i\theta} + 1)^\lambda \quad \text{on } ]-\pi/4, 0[$$

$$v \equiv d_+e^{-i\lambda}(\beta e^{2i\theta} + 1)^\lambda + d_-e^{i\lambda}(\beta e^{-2i\theta} + 1)^\lambda \quad \text{on } ]0, \pi/4[$$

$$w \equiv c_+e^{-i\lambda}(\alpha e^{2i\theta} + 1)^\lambda + c_-e^{i\lambda}(\alpha e^{-2i\theta} + 1)^\lambda \quad \text{on } ]\pi/4, \pi/2[,$$
with \( \alpha \) defined by (3.4.7) (and \( \beta \) analogously from the entries of the matrix \( o \)). Please notice that the determinants of the matrices \( m, \tilde{m}, \alpha, \tilde{\alpha} \) all equal the determinant of the matrix

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{pmatrix},
\]

the value of which we denote by \( D \) in this proof. Taking this into account, the transmission conditions in \( \theta = -\pi/4 \) read in view of (3.6.18)/(3.6.19)

\[
\hat{c}_+(1 - i\tilde{\alpha})^\lambda = \hat{d}_+(1 - i\tilde{\beta})^\lambda
\]

(3.6.22) and

\[
\hat{c}_-(1 + i\alpha)^\lambda = \hat{d}_-(1 + i\beta)^\lambda.
\]

(3.6.23) Analogously, the transmission conditions in 0 equivalently express as

\[
\hat{d}_+(1 + \beta)^\lambda = d_+(1 + \beta)^\lambda
\]

(3.6.24) and

\[
\hat{d}_-(1 + \beta)^\lambda = d_-(1 + \beta)^\lambda,
\]

(3.6.25) while those in \( \pi/4 \) can be written as

\[
d_+(1 + i\beta)^\lambda = c_+(1 + i\alpha)^\lambda
\]

(3.6.26) and

\[
d_-(1 - i\tilde{\beta})^\lambda = c_-(1 - i\tilde{\alpha})^\lambda.
\]

(3.6.27) The boundary condition \( u(\pi/2) = w(\pi/2) = 0 \) leads to

\[
c_+e^{-i\lambda\pi/2}(1 - \alpha)^\lambda + c_-e^{i\lambda\pi/2}(1 - \tilde{\alpha})^\lambda = 0,
\]

(3.6.28) or, in other words,

\[
c_+ = -c_-e^{i\lambda\pi}(1 - \tilde{\alpha})^\lambda
\]

(3.6.29) while the boundary condition \( u(-\pi/2) = \tilde{w}(-\pi/2) = 0 \) gives

\[
\hat{c}_+e^{i\lambda\pi/2}(1 - \tilde{\alpha})^\lambda + \hat{c}_-e^{-i\lambda\pi/2}(1 - \alpha)^\lambda = 0,
\]

or, alternatively,

\[
\hat{c}_- = -\hat{c}_+e^{i\lambda\pi}(1 - \tilde{\alpha})^\lambda
\]

(3.6.30) Combining (3.6.29), (3.6.27), (3.6.25), (3.6.23), (3.6.30), (3.6.22), (3.6.24), (3.6.26), one ends up with the characteristic equation for \( \lambda \):

\[
e^{2i\lambda\pi/2}(1 - \alpha)^\lambda (1 - i\tilde{\beta})^\lambda (1 + \beta)^\lambda (1 + i\alpha)^\lambda (1 - \alpha)^\lambda (1 - i\tilde{\alpha})^\lambda (1 + \beta)^\lambda (1 + i\alpha)^\lambda (1 - i\tilde{\alpha})^\lambda (1 + \beta)^\lambda (1 + i\tilde{\beta})^\lambda (1 + \beta)^\lambda (1 + i\tilde{\beta})^\lambda = e^{2i\lambda\pi}(1 - \alpha)^\lambda (1 - i\tilde{\alpha})^\lambda (1 + \beta)^\lambda (1 + i\alpha)^\lambda (1 - \alpha)^\lambda (1 - i\tilde{\alpha})^\lambda (1 + \beta)^\lambda (1 + i\alpha)^\lambda (1 - i\tilde{\alpha})^\lambda (1 + \beta)^\lambda (1 + i\tilde{\beta})^\lambda (1 + \beta)^\lambda (1 + i\tilde{\beta})^\lambda = 1.
\]

(3.6.31) Let us remark that \( c_+ \) cannot vanish unless also the other coefficients vanish. Moreover, we notice that all the terms \( 1 + i\alpha, \frac{1}{1 - i\alpha}, 1 - \tilde{\alpha}, \frac{1}{1 - i\tilde{\alpha}}, 1 + \beta, \frac{1}{1 + \beta}, 1 - i\beta, \frac{1}{1 + i\beta} \) have positive real part because \( |\alpha|, |\beta| < 1 \). Hence, we have

\[
\frac{(1 + i\alpha)^\lambda}{(1 - \alpha)^\lambda} = \left(\frac{1 + i\alpha}{1 - \alpha}\right)^\lambda \quad \text{and} \quad \frac{(1 - \tilde{\alpha})^\lambda}{(1 - i\tilde{\alpha})^\lambda} = \left(\frac{1 - \tilde{\alpha}}{1 - i\tilde{\alpha}}\right)^\lambda,
\]

(3.6.32)
as well as
\[
\frac{(1 + \beta)^\lambda}{(1 + i\beta)^\lambda} = \left(\frac{1 + \beta}{1 + i\beta}\right)^\lambda \quad \text{and} \quad \frac{(1 - i\beta)^\lambda}{(1 + \beta)^\lambda} = \left(\frac{1 - i\beta}{1 + \beta}\right)^\lambda, \tag{3.6.33}
\]
if \(\Re \lambda \leq 1\). Further, observing the relations
\[
\left(\frac{1 + i\alpha}{1 - \alpha}\right) = \frac{1}{\bar{1 - \alpha}} \quad \text{and} \quad \left(\frac{1 + \beta}{1 + i\beta}\right) = \frac{1}{1 - i\beta}, \tag{3.6.34}
\]
and putting \(\sigma = \frac{1 + i\alpha}{1 - \alpha}\) and \(\kappa = \frac{1 + \beta}{1 + i\beta}\), this altogether enables us to rewrite (3.6.31) as
\[
e^{2i\lambda(\sigma + 2(\sigma + \kappa))} = 1. \tag{3.6.35}
\]
It is obvious that all \(\lambda\) satisfying (3.6.35) must be real. Our claim is now: \(\sigma + \kappa\) equals \(\pi/2\) or \(-3\pi/2\). For this, we mention that, by definition, \(\sigma, \kappa \in [-\pi, \pi]\); thus the claim is true, if we can show
\[
1 + i\alpha \quad 1 + \beta = i. \tag{3.6.36}
\]
This we will do now: exploiting the definitions of \(\alpha, \beta\) we get
\[
\frac{1 + i\alpha}{1 - \alpha} = \frac{1}{2} \frac{D^{1/2} + m_{12} - m_{22} + i(D^{1/2} - m_{22} - m_{12})}{m_{12} + iD^{1/2}} = \frac{1}{2m_{11}} \left(D^{1/2} + m_{11} - m_{12} + i(D^{1/2} - m_{11} + m_{12})\right) = \frac{1}{2m_{11}} \left(D^{1/2} + \frac{a_{11} - a_{22}}{2} + i(D^{1/2} + \frac{a_{22} - a_{11}}{2})\right). \nonumber
\]
Analogously, we calculate:
\[
\frac{1 + \beta}{1 + i\beta} = \frac{2i\alpha_{22}}{D^{1/2} + a_{12} - a_{22} + i(D^{1/2} + a_{22} - a_{12})} = \frac{2i\alpha_{22}}{D^{1/2} + \frac{a_{11} - a_{22}}{2} + i(D^{1/2} + \frac{a_{22} - a_{11}}{2})}. \nonumber
\]
Taking into account \(\alpha_{22} = m_{11}\), this gives (3.6.36). Hence, the transcendental equation (3.6.35) for \(\lambda\) reads in any case as \(e^{4\pi\lambda} = 1\). Trivially, the smallest positive \(\lambda\) possible is \(\lambda_0 = 1/2\). Thus, the edge \(E^+_x\) meets the preconditions of Proposition 3.4.6. The considerations for the edge \(E^-_x\) are the same, word by word.

Next we consider the edge \(E_{xz}\), lying in the \(x-z\)-plane. The coefficient matrices belonging to its neighbouring sectors are
\[
Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} a_{11} + 2a_{12} + a_{22} \\ a_{12} + 2a_{22} \\ a_{31} + a_{23} \end{pmatrix} \begin{pmatrix} a_{12} + a_{22} \\ 2a_{22} \\ \sqrt{2}a_{23} \end{pmatrix}, \tag{3.6.37}
\]
\[
R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} b_{11} + 2b_{12} + b_{22} \\ b_{12} + 2b_{22} \\ b_{31} + b_{23} \end{pmatrix} \begin{pmatrix} b_{12} + b_{22} \\ 2b_{22} \\ \sqrt{2}b_{23} \end{pmatrix}, \tag{3.6.38}
\]
if \(y > 0\) and their reflected counterparts
\[
\hat{Q} = \begin{pmatrix} q_{11} & -q_{12} & q_{13} \\ -q_{21} & q_{22} & -q_{23} \\ q_{31} & -q_{32} & q_{33} \end{pmatrix} \quad \text{and} \quad \hat{R} = \begin{pmatrix} r_{11} & -r_{12} & r_{13} \\ -r_{21} & r_{22} & -r_{23} \\ r_{31} & -r_{32} & r_{33} \end{pmatrix},
\]
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if \( y < 0 \), (see (3.6.3), (3.6.5) and Proposition 3.5.4). According to Proposition 3.4.6 one has to perform a rotation in the \( x-z \)-plane which moves the edge \( E_{zz} \) to the \( z \)-axis. This means, one has to consider the matrices

\[
\begin{pmatrix}
\cos \zeta & 0 & -\sin \zeta \\
0 & 1 & 0 \\
\sin \zeta & 0 & \cos \zeta
\end{pmatrix}
\begin{pmatrix}
\cos \zeta & 0 & \sin \zeta \\
0 & 1 & 0 \\
-\sin \zeta & 0 & \cos \zeta
\end{pmatrix},
\]

\( M \) taken as \( Q, R, \tilde{Q}, \tilde{R} \), respectively and \( \zeta \) being the angle between the edge \( E_{zz} \) and the \( z \)-axis. A straightforward calculation shows that the resulting upper \( 2 \times 2 \) blocks look alike

\[
\begin{pmatrix}
s_{11} & -s_{12} \\
-s_{12} & s_{22}
\end{pmatrix}
\begin{pmatrix}
t_{11} & -t_{12} \\
t_{12} & t_{22}
\end{pmatrix}
\]

(3.6.39)

Hence (see Remark 3.4.8), the corresponding numbers \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are related by \( \alpha_1 = \overline{\alpha}_4 \) and \( \alpha_2 = \pi_3 \). In the sequel we employ the numbers \( \alpha \) for \( \alpha_3 \) and \( \beta \) for \( \alpha_4 \). In this notation we show:

**Lemma 3.6.3.** Assume the existence of complex numbers \( c_+, c_-, d_+, d_- \) such that

\[
\begin{align*}
\tilde{v} &\overset{\text{def}}{=} \hat{d}_+ e^{-i\lambda} (\beta e^{2i} + 1)^\lambda + \hat{d}_- e^{i\lambda} (\beta e^{-2i} + 1)^\lambda & \text{on } & \lbrack -\pi, -\zeta \rbrack \\
\tilde{w} &\overset{\text{def}}{=} \hat{c}_+ e^{-i\lambda} (\alpha e^{2i} + 1)^\lambda + \hat{c}_- e^{i\lambda} (\alpha e^{-2i} + 1)^\lambda & \text{on } & \lbrack -\zeta, 0 \rbrack \\
w &\overset{\text{def}}{=} c_+ e^{-i\lambda} (\alpha e^{2i} + 1)^\lambda + c_- e^{i\lambda} (\alpha e^{-2i} + 1)^\lambda & \text{on } & \lbrack 0, \zeta \rbrack \\
v &\overset{\text{def}}{=} d_+ e^{-i\lambda} (\beta e^{2i} + 1)^\lambda + d_- e^{i\lambda} (\beta e^{-2i} + 1)^\lambda & \text{on } & \lbrack \zeta, \pi \rbrack.
\end{align*}
\]

obeys the transmission conditions in \( -\pi, -\zeta, 0, \zeta \). Then \( \text{Re} \lambda \not\in \lbrack 0, 1/2 \rbrack \).

**Proof.** The transmission condition \( \lbrack b_2 u' + \lambda b_1 u \rbrack_0 = 0 \) together with Corollary 3.6.2 (see in particular (3.6.17)) implies

\[
\hat{c}_+ (\overline{\alpha} + 1)^\lambda - \hat{c}_- (\alpha + 1)^\lambda = c_+ (\alpha + 1)^\lambda - c_- (\overline{\alpha} + 1)^\lambda. \tag{3.6.40}
\]

On the other hand, the transmission condition for \( b_2 u' + \lambda b_1 u \) in \( -\pi/\pi \) (see (3.6.21)) gives

\[
\hat{d}_+ e^{i\pi} (\overline{\beta} + 1)^\lambda - \hat{d}_- e^{-i\pi} (\beta + 1)^\lambda = d_+ e^{-i\pi} (\beta + 1)^\lambda - d_- e^{i\pi} (\overline{\beta} + 1)^\lambda. \tag{3.6.41}
\]

Let us first consider the case where

\[
\hat{c}_+ = c_-, \quad \hat{c}_- = c_+, \quad \hat{d}_+ = d_-, \quad \hat{d}_- = d_. \tag{3.6.42}
\]

Inserting these relations in (3.6.40) and (3.6.41) one obtains that both sides of (3.6.40) and (3.6.41) in fact have to vanish. But this means in view of Lemma 3.6.1 nothing else but

\[
b_2 (\theta) u' (\theta) + \lambda b_1 (\theta) u (\theta) = 0 \quad \text{for} \quad \theta = 0, \pi.
\]

Thus, the restriction of \( u \) to the interval \( \lbrack 0, \pi \rbrack \) leads to a bimaterial problem including a Neumann condition on both interval ends. Then \( \text{Re} \lambda \not\in \lbrack 0, 1/2 \rbrack \), see Theorem 3.9.2 below.

Assume now that (3.6.42) is not satisfied. Then we introduce the function

\[
\tilde{u} \overset{\text{def}}{=} \begin{cases}
\hat{v} \overset{\text{def}}{=} \hat{c}_- e^{-i\lambda} (\alpha e^{2i} + 1)^\lambda + \hat{c}_+ e^{i\lambda} (\overline{\alpha} e^{-2i} + 1)^\lambda & \text{on } \lbrack 0, \zeta \rbrack \\
\hat{v} \overset{\text{def}}{=} \hat{d}_- e^{-i\lambda} (\beta e^{2i} + 1)^\lambda + \hat{d}_+ e^{i\lambda} (\overline{\beta} e^{-2i} + 1)^\lambda & \text{on } \lbrack \zeta, \pi \rbrack.
\end{cases} \tag{3.6.43}
\]
on $[0, \pi]$ and consider the function
\[
u_* \overset{\text{def}}{=} \nu|_{[0, \pi]} - \bar{u}
\]
\[
= \begin{cases}
(c_+ - \hat{c}_+\alpha e^{2i\nu} + 1)\lambda + (c_+ - \hat{c}_+\alpha e^{-2i\nu} + 1)\lambda & \text{on } [0, \zeta[ \\
(d_+ - \hat{d}_+\beta e^{2i\nu} + 1)\lambda + (d_+ - \hat{d}_+\beta e^{-2i\nu} + 1)\lambda & \text{on } \zeta, \pi].
\end{cases}
\]

It is straightforward to verify that the condition $[u]_0 = 0$ implies $\nu_*(0) = 0$ and the periodicity condition in $-\pi/\pi$ yields $\nu_*(\pi) = 0$. Next we intend to show the transmission conditions $[\nu_*]_{\zeta} = [b_2(\theta)\bar{u} + \lambda b_1(\theta)\bar{u}]_{\zeta} = 0$. Because we already know by supposition $[\bar{u}]_{\zeta} = [b_2(\theta)\bar{u} + \lambda b_1(\theta)\bar{u}]_{\zeta} = 0$, it remains to show $[\bar{u}]_{\zeta} = [b_2(\theta)\bar{u} + \lambda b_1(\theta)\bar{u}]_{\zeta} = 0$. One easily verifies $[\bar{u}]_{\zeta} = [u]_{-\zeta}$, and the latter is zero by supposition. Finally, by Lemma 3.6.1 we have
\[
[b_2(\theta)\bar{u} + \lambda b_1(\theta)\bar{u}]_{\zeta} = \begin{cases}
(b_2\bar{u} + \lambda b_1\bar{u})_{\zeta} - (b_2\bar{v} + \lambda b_1\bar{v})_{\zeta} \\
= i\lambda D_\alpha(\hat{c}_-e^{-i\lambda\zeta}(\alpha e^{2i\nu} + 1)\lambda - \hat{c}_+e^{i\lambda\zeta}(\alpha e^{-2i\nu} + 1)\lambda)
+ i\lambda D_\beta(\hat{d}_-e^{-i\lambda\zeta}(\beta e^{2i\nu} + 1)\lambda - \hat{d}_+e^{i\lambda\zeta}(\beta e^{-2i\nu} + 1)\lambda)
= - (b_2\bar{u}' + \lambda b_1\bar{u})_{\zeta} + (b_2\bar{v}' + \lambda b_1\bar{v})_{\zeta}
= [b_2\bar{u}' + \lambda b_1\bar{u}]_{-\zeta}.
\end{cases}
\]

But the right hand side of this equation is zero in view of the transmission condition $[b_2u' + \lambda b_1u]_{-\zeta} = 0$. Thus, this second case leads to a bimaterial Dirichlet problem, for which also Theorem 3.9.2 gives $\text{Re} \lambda \notin [0, 1/2]$.

**Remark 3.6.4.** The key point is here that we can invest estimates of $\text{Re} \lambda$ for bimaterial edges in both, the Neumann and Dirichlet case (see Theorem 3.9.2 below), despite the fact that in the original problem the edge is situated on a Neumann boundary plane.

It remains to consider the edge $E_\phi$ (and its reflected counterpart). Let first $t \in \mathbb{R}$ be a number such that $(0, 0, t) + E_\phi$ has its endpoint in $0 \in \mathbb{R}^3$ and $\mathcal{O}$ be a rotation of the plane $\{(x, x, z) : x, z \in \mathbb{R}\}$ which transforms $((0, 0, t) + E_\phi)$ to the z-axis. Suppose that for one $\lambda$ with $\text{Re} \lambda \in [0, 1/2]$ there is a (nontrivial) function $v_\lambda$ from the kernel of the resulting operator $\mathcal{A}_\lambda$. If one takes the coefficient function defined in (3.5.1) as
\[
\omega(x, y, z) \overset{\text{def}}{=} \begin{pmatrix}
\rho_1^1 & \rho_1^2 & \rho_1^3 \\
\rho_2^1 & \rho_2^2 & \rho_2^3 \\
\rho_3^1 & \rho_3^2 & \rho_3^3
\end{pmatrix}
\]
if $(x, y) \in K^{\delta_{j+1}}_{\theta_j 1}$, \hspace{1cm} (3.6.44)

then, by Lemma 3.5.1, there is a compactly supported element $f \in W^{-1,6}(\mathbb{R}^3)$ such that the — also compactly supported — variational solution $v \in W^{1,2}(\mathbb{R}^3)$ of $-\nabla \cdot \omega \nabla v = f$ does not belong to $W^{1,4}(\mathbb{R}^3)$. Because the support of $v$ is compact, it can then (the more) not belong to $W^{1,6}(\mathbb{R}^3)$. Now we evoke the transformations $\mathcal{O}$, the shift $(0, 0, t)$ and $\phi$. Applying Proposition 3.5.3, one obtains a $f_\phi \in W^{-1,6}(\mathbb{R}^3)$ and a $v_\phi \in W^{1,2}(\mathbb{R}^3) \setminus W^{1,4}(\mathbb{R}^3)$ satisfying $-\nabla \cdot \omega_r \nabla v_\phi = f_\phi$, or, equivalently, $-\nabla \cdot \omega_r \nabla v_\phi + v_\phi = f_\phi + v_\phi \in W^{-1,6}(\mathbb{R}^3)$. It is not hard to see that the matrix valued function $\omega_r$ equals above $\Xi$ the matrix $\mu^+$ and below $\Xi$ the matrix $\mu^-$ (see (3.6.1)). But a result of [ERS07, Thm. 3.11], see also [AT98, Ch. 4.5], says that
\[
-\nabla \cdot \omega_r \nabla + 1 : W^{1,p}(\mathbb{R}^3) \to W^{-1,p}(\mathbb{R}^3)
\]
is a topological isomorphism for any $p \in ]1, \infty[$. This contradicts the above supposition. The proof for the reflected $E_\phi$ runs along the same lines; thus the proof of Theorem 3.1.1 is complete.
3. Elliptic Model Problems

3.7. Proof of Theorem 3.1.2 and of Corollary 3.1.3

First we consider the case where Υ is one side of the triangle Λ. Modulo an affine transformation in \( \mathbb{R}^2 \) we may focus on the case where Υ is identical with the interval \([0,1]\) on the \(x\)-axis, see Proposition 3.5.3. We reflect \( \Pi \) symmetrically at the \(x-z\)-plane and obtain a domain \( \hat{\Pi} \) and a reflected coefficient function \( \hat{\mu} \). The resulting boundary conditions are then homogeneous Dirichlet on all \( \partial \hat{\Pi} \). By Proposition 3.5.4 it is sufficient to show that

\[
-\nabla \cdot \hat{\mu} \nabla : W^{1,p}_0(\hat{\Pi}) \to W^{-1,p}(\hat{\Pi})
\]

is a topological isomorphism for \( p > 3 \). Of course, we will again apply Proposition 3.4.6 and have, hence, to discuss the edge singularities. The occurring edges are:

i) geometric edges,

ii) bimaterial outer edges,

iii) the intersection of the \(x-z\)-plane with \( \Xi \), in particular, the parts of the \(z\)-axis below and above the intersection point with \( \Xi \) is a bimaterial outer edge.

For all these edges we already know that the corresponding operators \( A_\lambda \) have a trivial kernel provided \( \text{Re}\ \lambda \in [0,1/2] \); namely: the claim for geometric edges and bimaterial outer edges is shown in the next section (see Theorem 3.9.1 and Theorem 3.9.2) while the situation of iii) is exactly the same as treated in Lemma 3.6.3.

Let us now regard the second case: modulo an affine transformation in \( \mathbb{R}^2 \) we may restrict ourselves to the case where \( \Upsilon \) is the union of the interval \([0,1]\) on the \(x\)-axis and the interval \([0,1]\) on the \(y\)-axis. Again we reflect the problem at the \(x-z\)-plane, but afterwards a second time at the \(y-z\)-plane. Thus, we end up with a Dirichlet problem on \( ˘\Pi \equiv V \times [-1,1], \) where \( V \subset \mathbb{R}^2 \) is the square with the vertices \((0,1),(1,0),(0,-1),(-1,0)\). Denoting the new coefficient function by \( ˘\mu \), it suffices by Proposition 3.5.4 to show that

\[
-\nabla \cdot ˘\mu \nabla : W^{1,p}_0(˘\Pi) \to W^{-1,p}(˘\Pi)
\]

is a topological isomorphism for \( p > 3 \). According to Proposition 3.4.6, it remains to show that for every edge \( E \) the kernels of the corresponding operators \( A_\lambda \) are trivial if \( \text{Re}\ \lambda \in [0,1/3 + \epsilon] \) (\( \epsilon \) arbitrarily small). If \((0,0,t)\) is the intersection point of \( \Xi \) with the \(z\)-axis, then the occurring edges are:

i) geometric edges,

ii) bimaterial outer edges,

iii) \( \{(0,0,s) : s \in [-1,t]\} \),

iv) \( \{(0,0,s) : s \in [t,1]\} \),

v) the intersection of the \(x-z\)-plane with \( \Xi \),

vi) the intersection of the \(x-z\)-plane with the reflected \( \Xi \),

vii) the intersection of the \(y-z\)-plane with \( \Xi \),

viii) the intersection of the \(y-z\)-plane with the reflected \( \Xi \).
Thus, from this point on we are in the same situation as in the discussion for the edge $\cap$, but has only a Dirichlet condition on its upper (ground) plate. The latter is already treated in Theorem 3.1.2. The first can be reflected at the upper (ground) plate and one ends up again with the setting which is treated in Theorem 3.1.2.

3.8. Concluding remarks

The geometric and bimaterial outer edges are treated in the appendix. (iii), (iv), (v), (vi) lead again to a constellation (3.6.39), which was treated in Lemma 3.6.3. This is also true for (vii) and (viii), but requires here an additional moment’s thought: let us denote the value of the coefficient function $\mu$ above $\Xi$ by $\mu^+$ and below $\Xi$ by $\mu^-$. Concerning (vii), the 'reflected matrices' then equal

$$\begin{pmatrix} \mu_{11}' & -\mu_{12}' & -\mu_{13}' \\ -\mu_{12}' & \mu_{22}' & \mu_{23}' \\ -\mu_{13}' & \mu_{23}' & \mu_{33}' \end{pmatrix}$$ and $$\begin{pmatrix} \mu_{11}'' & -\mu_{12}'' & -\mu_{13}'' \\ -\mu_{12}'' & \mu_{22}'' & \mu_{23}'' \\ -\mu_{13}'' & \mu_{23}'' & \mu_{33}'' \end{pmatrix}.$$ 

We perform now a rotation within the $x$-$y$-plane which transforms the (positive) $y$-axis into the (positive) $x$-axis and the (positive) $x$-axis into the negative $y$-axis; clearly the transformed edge lies then in the $x$-$z$-plane. One obtains the transformed coefficient matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11}' & \mu_{12}' & \mu_{13}' \\ \mu_{12}' & \mu_{22}' & \mu_{23}' \\ \mu_{13}' & \mu_{23}' & \mu_{33}' \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mu_{22}' & -\mu_{12}' & \mu_{23}' \\ -\mu_{12}' & \mu_{11}' & -\mu_{13}' \\ \mu_{23}' & -\mu_{13}' & \mu_{33}' \end{pmatrix},$$

while the reflected matrices transform as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11}' & -\mu_{12}' & -\mu_{13}' \\ -\mu_{12}' & \mu_{22}' & \mu_{23}' \\ -\mu_{13}' & \mu_{23}' & \mu_{33}' \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mu_{22}' & \mu_{12}' & \mu_{23}' \\ \mu_{12}' & \mu_{11}' & \mu_{13}' \\ \mu_{23}' & \mu_{13}' & \mu_{33}' \end{pmatrix}.$$ 

Thus, from this point on we are in the same situation as in the discussion for the edge $E_{xz}$ (see page 125) and everything runs completely the same way. (viii) is analogous to (vii).

We come to the proof of Corollary 3.1.3: because we demanded that the plane $\Xi$ should not touch the upper plate nor the ground plate it is possible to divide the problem by a suitable partition of unity into one which affects the upper (lower) part and is separated from $\Xi$ and one which contains $\Pi \cap \Xi$ but has only a Dirichlet condition on its upper (ground) plate. The latter is already treated in Theorem 3.1.2. The first can be reflected at the upper (ground) plate and one ends up again with the setting which is treated in Theorem 3.1.2.

3.8. Concluding remarks

The results of this paper easily carry over to problems with Robin boundary conditions. Indeed, one can prove that if $\varphi$ is the surface measure on $\partial \Omega$ and $\kappa \in L^\infty(\Gamma, d\varphi)$, then the linear map $T : W^{1,p}_\Gamma(\Pi) \to W^{-1,p}_\Gamma(\Pi)$ given by

$$\langle T\psi, \varphi \rangle_{W^{-1,p}_\Gamma} = \int_\Gamma \kappa \psi \varphi \, d\varphi$$

(and representing the Robin boundary condition) is infinitesimally small with respect to the operator $\nabla \cdot \mu \nabla$. Thus, the domains of both operators are the same by classical perturbation theory, see [Kat80, Ch. IV.1].
The reader has possibly asked himself why the results are deduced from [MERS04] and why the concept of that paper does not work for boundary conditions which are not Dirichlet. One problem consists in finding an adequate energy space in case of edges on Neumann boundary parts which, additionally, has to be in correspondence with the properties of the Mellin transform. Our attempts to find such an energy space have failed up to now. Alternatively, the question arises whether it is possible to discard vertices from the analysis – by reflection arguments – also for the Neumann case. It turns out that this can be done in relevant cases, but seems to be delicate in general.

In principle it is possible to generalize our results to the case where not only one plane intersects the domain, but several do. In order to classify the singularities stemming from the additional inner edges (where the planes meet) one can apply the result [EKRS07, Thm. 2.5]. We have not carried out this here only for technical simplicity, see also [Kne04].

3.9. Appendix: The transcendental equation for geometric edges and bimaterial outer edges

It is the aim of this section to discuss the edge singularities for geometric edges and bimaterial outer edges; precisely, we intend to show the following two theorems:

**Theorem 3.9.1.** For any geometric edge E the kernels of the associated operators $A_{\lambda}$ are trivial in each of the following two cases:

a) the opening angle $\theta_1 - \theta_0$ is not larger than $\pi$ and $\Re \lambda \in ]0,1[$.

b) $\theta_1 - \theta_0 \in ]\pi,2\pi[$ and $\Re \lambda \in ]0,1/2[$.

**Theorem 3.9.2.** Let $K_{\theta_0}^0, K_{\theta_1}^0$ be two neighbouring sectors in $\mathbb{R}^2$ with $\theta_1 - \theta_0, \theta_2 - \theta_1 \leq \pi$ and $\theta_2 - \theta_0 < 2\pi$. Let $p^0, \rho^2$ be two real, symmetric, positive definite $2 \times 2$ matrices corresponding to the sectors $K_{\theta_0}^0, K_{\theta_1}^0$. Let $t_{\lambda}$ be the form defined in (3.4.2) either on $H^1([\theta_0,\theta_2])$ or on $H^1([\theta_0,\theta_2])$. Then there is an $\epsilon > 0$ such that the kernel of the corresponding operator $A_{\lambda}$ (see Definition 3.4.1) is trivial if $\Re \lambda \in ]0,1/2 + \epsilon[$.

We will prove the theorems in several steps, starting with the following

**Lemma 3.9.3.** Let $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, and define for $\gamma \in ]-\pi,\pi]$ the number

$$
\sigma = \arg \frac{ae^{-2i\gamma} + 1}{\alpha + 1} \in ]-\pi,\pi].
$$

Then either $\gamma, \gamma + \sigma \in ]-\pi,0[$ or $\gamma = \sigma = 0$ or $\gamma, \gamma + \sigma \in ]0,\pi[$ or $\gamma = \gamma + \sigma = \pi$.

**Proof.** The cases $\gamma = 0$ and $\gamma = \pi$ are straightforward. In the remaining cases one has

$$
e^{i(\gamma + \sigma)} = e^{i\gamma} \frac{ae^{-2i\gamma} + 1}{\alpha + 1} \frac{|\alpha + 1|}{|ae^{-2i\gamma} + 1|} = \frac{(ae^{-i\gamma} + e^{i\gamma})(\pi + 1)}{|\alpha + 1||ae^{-2i\gamma} + 1|} = \frac{|\alpha|^2 e^{-i\gamma} + e^{i\gamma} + 2 \Re(\alpha e^{-i\gamma})}{|\alpha + 1||ae^{-2i\gamma} + 1|}.
$$

Thus, the imaginary part of $e^{i(\gamma + \sigma)}$ equals $\frac{1 - |\alpha|^2}{1 + \alpha^2} \sin \gamma$, and its sign depends in an obvious way only on $\gamma$. \hfill $\Box$

It follows the proof of Theorem 3.9.1; without loss of generality we may assume $\theta_1 = \pi$. Again exploiting the ansatz functions (3.4.8), the Dirichlet conditions in $\theta_0, \theta_1 \in ]-\pi,\pi]$ read

$$
c_+e^{-i\lambda\theta_0}(ae^{2i\theta_0} + 1)^\lambda + c_-e^{i\lambda\theta_0}(ae^{-2i\theta_0} + 1)^\lambda = 0 \quad (3.9.1)
$$

$$
c_+e^{-i\lambda\pi}(\alpha + 1)^\lambda + c_-e^{i\lambda\pi}(\alpha + 1)^\lambda = 0. \quad (3.9.2)
$$
These equations are nontrivially solvable in $c_+, c_-$ iff

$$
1 = e^{-2i\lambda\epsilon}e^{2i\lambda\theta_0} \frac{\bar{\alpha}e^{-2i\theta_0} + 1}{(\bar{\alpha} + 1)^\lambda} \frac{(\alpha + 1)^\lambda}{(\alpha e^{2i\theta_0} + 1)^\lambda} = e^{-2i\lambda\epsilon}e^{2i\lambda\theta_0} \frac{\bar{\alpha}e^{-2i\theta_0} + 1}{(\bar{\alpha} + 1)^\lambda} \frac{(\alpha + 1)^\lambda}{(\alpha e^{2i\theta_0} + 1)^\lambda},
$$

(compare the considerations in Subsection 3.6.3, in particular (3.6.32). Putting

we may write (3.9.3) as

in case a), where $\theta_0 \in [0,\pi]$, we obtain $\theta_0 + \nu \in [0,\pi]$ by Lemma 3.9.3, which excludes $\Re \lambda \in [0,1]$. If $\theta_0 \in ]-\pi,0[$, then, by Lemma 3.9.3, we have $\theta_0 + \nu \in ]-\pi,0[$, which shows the assertion in case b).

Concerning Theorem 3.9.2, we may apply a rotation (corresponding to a shift in the angle space) and thus reduce the general case to that one where $\theta_0 = -\gamma$, $\theta_1 = 0$ and $\theta_2 = \delta$.

Again using the ansatz functions (3.4.8) we are getting the following equations expressing the transmission conditions in 0, see Corollary 3.6.2

$$
c_+(\alpha + 1)^\lambda + c_-(\bar{\alpha} + 1)^\lambda = d_+(\beta + 1)^\lambda + d_- (\bar{\beta} + 1)^\lambda
$$

(3.9.4)

and

$$
D_m^{1/2}[c_+(\alpha + 1)^\lambda - c_-(\bar{\alpha} + 1)^\lambda] = D_o^{1/2}[d_+(\beta + 1)^\lambda - d_- (\bar{\beta} + 1)^\lambda].
$$

(3.9.5)

We define

$$
e_\gamma \overset{\text{def}}{=} \begin{cases} -1 & \text{if Dirichlet in } -\gamma \\ 1 & \text{if Neumann in } -\gamma \end{cases}
$$

and analogously for $\delta$. In this convention, (see Lemma 3.6.1) the boundary condition in $-\gamma$ yields

$$
c_+e^{i\lambda\gamma}(\alpha e^{-2i\gamma} + 1)^\lambda - e_\gamma c_- e^{-i\lambda\gamma}(\bar{\alpha} e^{2i\gamma} + 1)^\lambda = 0
$$

or, what is the same,

$$
c_- = e_\gamma c_+ e^{2i\lambda\gamma}(\alpha e^{-2i\gamma} + 1)^\lambda (\bar{\alpha} e^{2i\gamma} + 1)^\lambda = 0
$$

(3.9.6)

On the other hand, the corresponding boundary condition in $\delta$ implies

$$
d_+ e^{-i\lambda\delta}(\beta e^{2i\delta} + 1)^\lambda - e_\delta d_- e^{i\lambda\delta}(\bar{\beta} e^{-2i\delta} + 1)^\lambda = 0
$$

or, equivalently,

$$
d_+ = e_\delta d_- e^{2i\lambda\delta}(\bar{\beta} e^{-2i\delta} + 1)^\lambda (\beta e^{2i\delta} + 1)^\lambda.
$$

(3.9.7)

We insert (3.9.6) and (3.9.7) in (3.9.4) and (3.9.5) and obtain

$$
c[(\alpha + 1)^\lambda + e_\gamma e^{2i\lambda\gamma}(\alpha e^{-2i\gamma} + 1)^\lambda (\bar{\alpha} + 1)^\lambda] - d[(e_\delta e^{2i\lambda\delta}(\bar{\beta} e^{-2i\delta} + 1)^\lambda (\beta + 1)^\lambda + (\bar{\beta} + 1)^\lambda)] = 0
$$

(3.9.8)

and

$$
D_m^{1/2}[c[(\alpha + 1)^\lambda - e_\gamma e^{2i\lambda\gamma}(\alpha e^{-2i\gamma} + 1)^\lambda (\bar{\alpha} + 1)^\lambda] + D_o^{1/2}[d[(\beta + 1)^\lambda - e_\delta e^{2i\lambda\delta}(\bar{\beta} e^{-2i\delta} + 1)^\lambda (\beta + 1)^\lambda] = 0
$$

(3.9.9)
for \( c = c^+ \) and \( d = d^- \). (3.9.8), (3.9.9) are nontrivially solvable iff

\[
D_o^{1/2} \left[ 1 + \epsilon_\gamma e^{2i\lambda \gamma} \frac{(ae^{-2i\gamma} + 1)\lambda}{(\alpha + 1)\lambda} \right] \left[ 1 - \epsilon_\delta e^{2i\lambda \delta} \frac{(\beta e^{-2i\delta} + 1)\lambda}{(\beta + 1)\lambda} \right] + D_m^{1/2} \left[ 1 + \epsilon_\delta e^{2i\lambda \delta} \frac{(\beta e^{-2i\delta} + 1)\lambda}{(\beta + 1)\lambda} \right] \left[ 1 - \epsilon_\gamma e^{2i\lambda \gamma} \frac{(ae^{-2i\gamma} + 1)\lambda}{(\alpha + 1)\lambda} \right] = 0. \tag{3.9.10}
\]

Putting

\[
\sigma = \arg \frac{ae^{-2i\gamma} + 1}{\alpha + 1}, \quad \kappa = \arg \frac{\beta e^{-2i\delta} + 1}{\beta + 1},
\]

and arguing as in (3.6.32)–(3.6.34), this altogether enables us to rewrite (3.9.10) as

\[
D_o^{1/2} \left[ 1 + \epsilon_\gamma e^{2i\lambda (\gamma + \sigma)} \right] \left[ 1 - \epsilon_\delta e^{2i\lambda (\delta + \kappa)} \right] + D_m^{1/2} \left[ 1 + \epsilon_\delta e^{2i\lambda (\delta + \kappa)} \right] \left[ 1 - \epsilon_\gamma e^{2i\lambda (\gamma + \sigma)} \right] = 0,
\]

or, what is the same,

\[
D_o^{1/2} \left[ e^{-i\lambda (\gamma + \sigma)} + \epsilon_\gamma e^{i\lambda (\gamma + \sigma)} \right] \left[ e^{-i\lambda (\delta + \kappa)} - \epsilon_\delta e^{i\lambda (\delta + \kappa)} \right] + D_m^{1/2} \left[ e^{-i\lambda (\delta + \kappa)} + \epsilon_\delta e^{i\lambda (\delta + \kappa)} \right] \left[ e^{-i\lambda (\gamma + \sigma)} - \epsilon_\gamma e^{i\lambda (\gamma + \sigma)} \right] = 0. \tag{3.9.11}
\]

This means that in the pure Dirichlet case (with \( \epsilon_\gamma = \epsilon_\delta = -1 \)) (3.9.10) can be written equivalently as

\[
D_o^{1/2} \sin \lambda (\gamma + \sigma) \cos \lambda (\delta + \kappa) + D_m^{1/2} \cos \lambda (\gamma + \sigma) \sin \lambda (\delta + \kappa) = 0 \tag{3.9.12}
\]

and in the pure Neumann case (with \( \epsilon_\gamma = \epsilon_\delta = 1 \)) as

\[
D_o^{1/2} \cos \lambda (\gamma + \sigma) \sin \lambda (\delta + \kappa) + D_m^{1/2} \sin \lambda (\gamma + \sigma) \cos \lambda (\delta + \kappa) = 0. \tag{3.9.13}
\]

Because \( D_o \) and \( D_m \) are arbitrary positive constants it suffices to focus the following discussion on (3.9.12).

Lemma 3.9.4. If \( 0 < \gamma, \delta < \pi \) with \( \gamma + \delta < 2\pi \), then any solution \( \lambda \) of (3.9.12) satisfies \( \Re \lambda \notin [0, 1/2 + \epsilon] \) for any \( \epsilon > 0 \).

Proof. Since, by Lemma 3.9.3, \( \sin \lambda (\gamma + \sigma) \neq 0 \) and \( \sin \lambda (\delta + \kappa) \neq 0 \), if \( 0 < \Re \lambda < 1 \), we can rewrite (3.9.12) as

\[
D_o^{1/2} \cot \lambda (\delta + \kappa) + D_m^{1/2} \cot \lambda (\gamma + \sigma) = 0. \tag{3.9.14}
\]

Note that

\[
\Re \cot (\xi + i\eta) = \frac{(\cosh^2 \eta - \sinh^2 \eta) \sin \xi \cos \xi}{(\sin \xi \cosh \eta)^2 + (\cos \xi \sinh \eta)^2} = \frac{\sin 2\xi}{2(\sin^2 \xi + \sinh^2 \eta)},
\]

hence, with \( \lambda = \vartheta + i\nu \), the real part of (3.9.14) satisfies

\[
\frac{D_o^{1/2} \sin 2\vartheta (\delta + \kappa)}{\sin^2 \vartheta (\delta + \kappa) + \sinh^2 \nu (\delta + \kappa)} + \frac{D_m^{1/2} \sin 2\vartheta (\gamma + \sigma)}{\sin^2 \vartheta (\gamma + \sigma) + \sinh^2 \nu (\gamma + \sigma)} = 0. \tag{3.9.15}
\]

If \( 0 < \vartheta \leq 1/2 \), then Lemma 3.9.3 shows that \( 0 < 2\vartheta (\delta + \kappa), 2\vartheta (\gamma + \sigma) \leq \pi \), and therefore both terms on the left hand side of (3.9.15) are non-negative. Due to \( \gamma + \delta < 2\pi \), at most one of them may be zero. This proves the assertion.

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Remark 3.9.5. If one is confronted with a bimaterial outer edge supplemented by a Dirichlet condition in $-\gamma$ and a Neumann condition in $\delta$ (what means $-\epsilon_\gamma = \epsilon_\delta = 1$), then (3.9.11) reads as

$$-D_o^{1/2} \sin(\lambda(\gamma + \sigma)) \sin(\lambda(\delta + \kappa)) + D_m^{1/2} \cos(\lambda(\gamma + \sigma)) \cos(\lambda(\delta + \kappa)) = 0. \quad (3.9.16)$$

If we again suppose $\gamma, \delta \in ]0, \pi[, \text{ then we may divide (3.9.16) by } \sin(\lambda(\gamma + \sigma)) \sin(\lambda(\delta + \kappa))$ (provided $\text{Re} \lambda \in ]0, 1[$) and obtain the equivalent condition

$$\cot(\lambda(\gamma + \sigma)) \cot(\lambda(\delta + \kappa)) = \frac{D_o^{1/2}}{D_m^{1/2}}. \quad (3.9.17)$$

It is not hard to see that there are parameter configurations $\gamma, \delta, \alpha, \beta, D_o, D_m$ such that (3.9.17) is fulfilled for $\lambda$ with arbitrarily small (positive) real part; see also [Mer03], where the case of scalar multiples of the Laplacian already was treated.

Remark 3.9.6. In fact, the results of Theorem 3.9.1 and Theorem 3.9.2 are already proved in [EKRS07] (see Lemma 2.9 and Lemma 2.5) by completely different methods and based on the results of Il’yin [Ily73], [Ily74]. Our intention was here to give a proof which is straightforward and self-contained.

Acknowledgement. Part of the ideas from the appendix are due to our colleagues J. Elschner and G. Schmidt. We are grateful for being given the possibility to publish this here.
References (Article 3)


References (Article 3)


References (Article 3)


References (Article 3)


4. Irreducibility and Mixed Boundary Conditions

by Robert Haller-Dintelmann, Matthias Hieber and Joachim Rehberg

In memoriam Helmut H. Schaefer

Abstract

In this paper we consider positive semigroups on $L^p(\Omega)$ generated by elliptic operators $A$ subject to mixed Dirichlet-Neumann boundary conditions on non-smooth domains $\Omega$. We show in particular that these semigroups as well as those generated by multiplicative perturbations $bA$ of $A$ are irreducible, provided $b \in L^\infty(\Omega)$ is real and satisfies $b \geq \delta$ for some $\delta > 0$.

4.1. Introduction

The theory of positive operators and positive semigroups has many applications to evolutionary problems, see e.g. [Sch74], [Dav90], [EN00], [ABHN01]. In this note we consider operators related with diffusion equations in divergence form with mixed Dirichlet-Neumann boundary conditions on non-smooth domains $\Omega \subset \mathbb{R}^n$. It is known that the realizations $A_p$ of the associated elliptic second order differential operators in $L^p(\Omega)$ generate positive semigroups on $L^p(\Omega)$ for all $p \in (1, \infty)$; see e.g. [GKR01]. Note that formally the operators $A_p$ may be expressed as $(\nabla \cdot \mu \nabla)$, where $\mu \in L^\infty(\Omega; M_{n \times n})$ denotes the diffusion coefficient.

Motivated by certain quasilinear problems, see e.g. [HR06], we are interested in multiplicative perturbations of $A_p$ by real-valued $L^\infty$-functions $b$ bounded away from 0. These operators are formally given by $b(\nabla \cdot \mu \nabla)$.

It is an interesting question to ask whether the semigroup $e^{tA_p}$ is irreducible and further whether $bA_p$ generates a positive or even irreducible semigroup $e^{t bA_p}$ on $L^p(\Omega)$. It is the aim of this paper to give an affirmative answer to this question. Observe that the semigroup property in the case where the boundary of $\Omega$ consists of two separate components with either Dirichlet or Neumann conditions was already considered by Amann [Ama93] in the situation of smooth boundaries.

In our situation, the underlying domains are Lipschitz domains and the boundary condition is mixed. By mixed Dirichlet-Neumann boundary condition we mean roughly speaking the following: there is an open set $\Gamma_N \subset \partial \Omega$ such that the elements of the domain of the operator have vanishing trace on $\partial \Omega \setminus \Gamma_N$ and a vanishing conormal derivative on $\Gamma_N$ (in the distributional sense).
4. Irreducibility and Mixed Boundary Conditions

4.2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and denote by $\Gamma_N \subset \partial \Omega$ an open subset of $\partial \Omega$. For $1 < p < \infty$ we define $W^{1,p}_{\Gamma_N}(\Omega)$ as the closure of

$$\{ \psi|_\Omega : \psi \in C_\infty_c(\mathbb{R}^n), \text{supp} \psi \cap (\partial \Omega \setminus \Gamma_N) = \emptyset \}$$

in the Sobolev space $W^{1,p}(\Omega)$. If $p = 2$, we write $H^1(\Omega)$ or $H^1_{\Gamma_N}(\Omega)$ instead of $W^{1,2}_{\Gamma_N}(\Omega)$. Of course, if $\Gamma_N = \emptyset$, then $W^{1,p}_{\Gamma_N}(\Omega) = W^{1,p}_0(\Omega)$.

Consider $\mu \in L^\infty(\Omega; M_{n \times n})$, where $M_{n \times n}$ denotes the set of all real, symmetric $n \times n$ matrices. Suppose that additionally

$$\inf_{x \in \Omega} \inf_{|\varsigma| = 1} \mu(x) \varsigma \cdot \varsigma > 0.$$ 

For a closed subspace $V \subset H^1(\Omega)$ such that $H^1_0(\Omega) \subset V$ we define the form $a : V \times V \to \mathbb{R}$ by

$$a(u,v) := -\int_\Omega \mu \nabla u \cdot \nabla v \, dx, \quad u, v \in V.$$ 

The form induces a continuous mapping $A : V \to V'$ such that

$$a(u,v) = \langle Au, v \rangle, \quad u, v \in V.$$ 

Here, for $v \in L^2(\Omega)$, $f_v(u) := \langle v, u \rangle_{L^2}$ defines an element $f_v \in V'$ and $v \mapsto f_v : L^2(\Omega) \to V'$ defines a continuous injection. In the following, we identify $v$ with $f_v$. We then define the operator $A$ as

$$D(A) := \{ u \in V : \exists f \in L^2(\Omega), \ a(u, \phi) = \langle f, \phi \rangle \ \forall \phi \in V \},$$

$$Au := f.$$ 

It is well known that $A$ generates an analytic semigroup on $L^2(\Omega)$ which is positivity preserving.

The following result gives sufficient conditions on the subspace $V$, such that $e^{tA}$ satisfies an upper Gaussian bound.

**Proposition 4.2.1.** [Arendt, terElst [AtE97]] Assume that $V$ is a closed subspace of $H^1(\Omega)$ satisfying

1. $H^1_0(\Omega) \subseteq V$,
2. $V$ has the $L^1$-$H^1$ extension property,
3. $u \in V$ implies $|u|, \inf(|u|, 1) \in V$,
4. $u \in V$, $v \in H^1(\Omega)$, $|v| \leq u$ implies $v \in V$.

Then $e^{tA}$ satisfies an upper Gaussian estimate, i.e.

$$e^{tA} f(x) = \int_\Omega K_t(x,y) f(y) \, dy, \quad x \in \Omega, \ f \in L^2(\Omega)$$

for some measurable function $K_t : \Omega \times \Omega \to \mathbb{R}_+$ and there exist constants $M, a > 0$ and $\omega \in \mathbb{R}$ such that

$$0 \leq K_t(x,y) \leq \frac{M}{t^{n/2}} e^{-\frac{a|x-y|^2}{t}} e^{\omega t}, \quad t > 0, \ \text{a.a.} \ x, y \in \Omega.$$ 

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Let $H_{1}^{2} \Gamma_{N} (\Omega)$ be defined as above. It was shown in [HR06] that $V := H_{1}^{2} \Gamma_{N} (\Omega)$ satisfies the assumptions a) – d) of Proposition (4.2.1).

This means that $e^{tA}$ satisfies an upper Gaussian bound. Thus $e^{tA}$ extends to an analytic semigroup on $L^{p}(\Omega)$ for all $1 < p < \infty$ (see e.g. [Dav90] or [AtE97, Theorem 5.3]). We will denote the generator of the semigroup on $L^{p}(\Omega)$ by $A_{p}$. Furthermore, these semigroups are even contractive for all these $p$. This can be seen by combining Proposition 4.11 and Theorem 4.28 in [Ouh05].

We now turn to multiplicative perturbations of $A$. Let $b \in L^{\infty}(\Omega)$ be real-valued with $b \geq \delta$ for some $\delta > 0$. Then we define the operator $bA$ on $L^{2}(\Omega)$ with $D(bA) = D(A)$ and $bAu(x) = b(x)Au(x)$ for all $u \in D(A)$. It is remarkable that this multiplicative perturbation preserves the Gaussian bound. This essentially follows from a result due to Duong and Ouhabaz [DO99].

**Proposition 4.2.2.** Let $b \in L^{\infty}(\Omega; \mathbb{R})$ and $b \geq \delta > 0$. Then $bA$ generates an analytic $C_{0}$-semigroup on $L^{2}(\Omega)$ given by kernel operators. Moreover, the associated kernels $p_{t}(x, y)$ satisfy an upper Gaussian bound of the form

$$|p_{t}(x, y)| \leq \frac{C}{\mu^{2}} e^{-\frac{1}{2}|x-y|^{2}} e^{\gamma t}, \quad t > 0, \text{ a.a. } x, y \in \Omega,$$

for some constants $c, C > 0$ and $\gamma \in \mathbb{R}$.

**Proof.** The fact that $bA$ generates an analytic $C_{0}$-semigroup follows from [DO99, Proposition 2.1].

Let $\gamma \geq \omega |b|_{\infty}$. Then $\gamma/b - \omega \geq 0$, so $e^{t(A-\omega)}$ and $e^{-t(\gamma/b-\omega)}$ are contraction semigroups. Thus, by the Trotter product formula (see e.g. [EN00, Corollary III.5.8]) we have for every $f \in L^{2}(\Omega)$, using the positivity of $e^{t(A-\omega)}$,

$$|e^{t(A-\omega-(\gamma/b-\omega))}f| = \lim_{n \to \infty} \left| \left( e^{t/n(A-\omega)} - t/n(\gamma/b-\omega) \right) e^{-t/n(\gamma/b-\omega)} \right| \leq \lim_{n \to \infty} \left| e^{t/n(A-\omega)} \left( e^{t/n(A-\omega)} - t/n(\gamma/b-\omega) \right) e^{-t/n(\gamma/b-\omega)} \right| \leq \cdots \leq \lim_{n \to \infty} \left| e^{t/n(A-\omega)} \right| f| = e^{t(A-\omega)}|f|.$$ 

This implies that the semigroup $e^{t(A-\gamma/b)} = e^{t(A-\omega-(\gamma/b-\omega))}$ has the same Gaussian estimate as $e^{t(A-\omega)}$, that is (4.2.2) without the factor $e^{\omega t}$. So we may now apply [DO99, Theorem 2.2] to this semigroup and get a Gaussian bound without exponential factor for the semigroup $e^{t(bA-\gamma)}$, which yields the assertion.

Next, we remind the reader that the concept of irreducibility has undergone many developments in the last decades. In the following, we follow the approach and definition introduced by H. H. Schaefer [Sch74].

**Definition 4.2.3.** Let $X$ be a Banach lattice. A linear subspace $Y$ of $X$ is called an ideal, if for every choice of $f \in Y$ and $g \in X$ with $|g| \leq |f|$, we have $g \in Y$.

A $C_{0}$-semigroup $T(t)$ on $X$ is called irreducible, if there is no $T(t)$-invariant ideal except $\{0\}$ and $X$.

**Remark 4.2.4.** It is interesting to note that for positive semigroups $T(t)$ on Banach lattices of the form $X = L^{p}(\Omega), 1 \leq p < \infty$, which may be represented as kernel operators, the following characterization of irreducibility in terms of the underlying kernels is known:

Let $T(t)$ be a $C_{0}$-semigroup on $L^{p}(\Omega)$ and assume that $T(t)$ is given by kernels as in (4.2.1). Then $T(t)$ is irreducible if and only if for any two measurable subsets $M$ and $N$ of $\Omega$ with
positive and finite measure, such that \( M \cap N \) is a nullset, there exists \( t > 0 \) with
\[
\int_M \int_N K_t(x, y) \, dx \, dy > 0.
\]
For a proof, see [Nag86, Example C-III.3.4].

### 4.3. Main Result and Proof

We are now in the position to state the main result of this paper.

**Theorem 4.3.1.** Let \( 1 < p < \infty \) and let \( b \in L^\infty(\Omega; \mathbb{R}) \) such that \( b \geq \delta \) for some \( \delta > 0 \). Then the semigroup \( e^{tbA_p} \) is positive, irreducible and compact.

Note that in the case \( b = 1 \), irreducibility was proved in [Ouh05] by Ouhabaz; see also [Are04], Section 8.2.

The main idea of the proof of irreducibility of \( e^{tbA_p} \) is to show, that the resolvent of \( bA \) is positivity improving. This means, that for some \( \lambda > s(A) \) and all \( f \in L^p(\Omega) \setminus \{0\} \) with \( f \geq 0 \) almost everywhere the function \( R(\lambda, A)f \) is strictly positive, i.e. \( R(\lambda, A)f > 0 \) almost everywhere in \( \Omega \). Then irreducibility follows by a contradiction argument using Remark 4.2.4.

Due to the relation
\[
(\lambda - bA)^{-1} = (W_\lambda - A)^{-1} b^{-1}
\]
with \( W_\lambda := \frac{\lambda}{b} \), this is obtained whenever the resolvent of \( A - W_\lambda \) is positivity improving. Thus, it is our first aim to show this.

In order to do so, we cite the following minimum principle from [GT77, Section 8.7].

**Proposition 4.3.2.** Let \( V \in L^\infty(\Omega) \) be positive and let \( u \in H^1(\Omega) \) satisfy \( (A - V)u \leq 0 \) in \( \Omega \). If for some open ball \( B \) with \( B \subseteq \Omega \) we have \( \inf_B u = \inf_\Omega u \leq 0 \), then \( u \) is constant in \( \Omega \).

In the following, let \( W \in L^\infty(\Omega) \) satisfy \( W \geq \epsilon \) for some \( \epsilon > 0 \). Since \( A - W \) is self-adjoint and has compact resolvent, the spectrum of this operator consists only of eigenvalues and since \( s(A) \leq 0 \) and \( W \geq \epsilon \), we have \( \sigma(A - W) \subseteq (-\infty, \lambda_1] \), where \( \lambda_1 < 0 \) denotes the largest eigenvalue. Furthermore we have
\[
\lambda_1 = \sup \{ \langle (A - W)\eta, \eta \rangle : \eta \in H^1_G(\Omega), \|\eta\|_2 = 1 \}
\]
and the supremum is attained for every eigenfunction of \( A - W \) associated to \( \lambda_1 \).

**Lemma 4.3.3.** If \( \psi \neq 0 \) is a positive eigenfunction of \( A - W \) belonging to \( \lambda_1 \), then \( \psi > 0 \) in \( \Omega \).

**Proof.** If \( \inf_\Omega \psi > 0 \) there is nothing to prove, so we concentrate on the case where \( \inf_\Omega \psi = 0 \). Thus \( (A - W)\psi = \lambda_1 \psi \leq 0 \), so if we assume that \( \psi \) does not admit a strictly positive lower bound on a ball \( B \) with \( \overline{B} \subseteq \Omega \), then \( \psi \) has to be constant on \( \Omega \) in view of Proposition 4.3.2. But this can be true, since the infimum of \( \psi \) on \( \Omega \) is zero, only provided \( \psi = 0 \).

**Lemma 4.3.4.** If \( \psi \neq 0 \) is a real eigenfunction of \( A - W \) belonging to \( \lambda_1 \), then \( \psi > 0 \) on \( \Omega \) or \( \psi < 0 \) on \( \Omega \). Any other eigenfunction belonging to \( \lambda_1 \) is a (scalar) multiple of \( \psi \).
Proof. Without loss of generality we may normalize our eigenfunction to $\|\psi\|_2 = 1$. Let $\psi = \psi_+ - \psi_-$ be the usual decomposition of $\psi$ in a positive and negative part. If $\psi_- = 0$ one may directly apply Lemma 4.3.3 and obtains the first assertion. The same is true, if $\psi_+ = 0$, arguing via $-\psi$. Let us now suppose $\psi_+ \neq 0 \neq \psi_-$. First we will show that the function $\tilde{\psi}_+ := \frac{\psi_+}{\|\psi_+\|_2}$ also maximizes the expression in (4.3.2). In order to do so, consider functions $\eta = \alpha\psi_+ - \beta\psi_-$, with the additional conditions $\alpha \in \left[0, \frac{1}{\|\psi_+\|_2}\right]$ and $\|\eta\|_2 = 1$. In view of $\psi_- \neq 0$ it is clear that $\|\psi_+\|_2 < 1$.

Obviously, the normalizing condition $\|\eta\|_2 = 1$ is equivalent to

$$\beta^2 = 1 - \frac{\alpha^2\|\psi_+\|_2^2}{\|\psi_-\|_2^2}. \quad (4.3.3)$$

By an explicit calculation and taking into account (4.3.3) we obtain

$$\langle (A - W)\eta, \eta \rangle = -\alpha^2 \left(\int_{\Omega} \langle \mu \nabla \psi_+, \nabla \psi_+ \rangle + \int_{\Omega} W\psi_+^2 \right) - \beta^2 \left(\int_{\Omega} \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_{\Omega} W\psi_-^2 \right) = -\alpha^2 \left(\int_{\Omega} \langle \mu \nabla \psi_+, \nabla \psi_+ \rangle + \int_{\Omega} W\psi_+^2 \right) - \frac{\alpha^2\|\psi_+\|_2^2}{\|\psi_-\|_2^2} \left(\int_{\Omega} \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_{\Omega} W\psi_-^2 \right) \alpha^2$$

$$= -\frac{1}{\|\psi_-\|_2^2} \left(\int_{\Omega} \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_{\Omega} W\psi_-^2 \right).$$

This means that the prefactor of $\alpha^2$ must be zero, since otherwise the expression, seen as a function in $\alpha$, is strictly decreasing or increasing (depending on the sign of the prefactor) and $\alpha = 1$ cannot maximize (4.3.2) on the interval $\left[0, \frac{1}{\|\psi_+\|_2}\right]$. This implies

$$\frac{\int_{\Omega} \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_{\Omega} W\psi_-^2}{\|\psi_-\|_2^2} = \frac{\int_{\Omega} \langle \mu \nabla \psi_+, \nabla \psi_+ \rangle + \int_{\Omega} W\psi_+^2}{\|\psi_+\|_2^2}$$

and, setting $\alpha = \beta = 1$ (observe that $\eta = \psi$ in this case),

$$\langle (A - W)\psi, \psi \rangle = -\frac{1}{\|\psi_-\|_2^2} \left(\int_{\Omega} \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_{\Omega} W\psi_-^2 \right) = \langle (A - W)\tilde{\psi}_+, \tilde{\psi}_+ \rangle.$$

Thus, the normalized function $\tilde{\psi}_+$ also maximizes (4.3.2).

Since $\tilde{\psi}_+$ is an element of the form domain, and hence from the domain of $(A - W)^{1/2}$, the series

$$\sum_m \langle \tilde{\psi}_+, u_m \rangle^2 \lambda_m$$

converges. Here the $\lambda_m$’s denote the eigenvalues of $A - W$ and $\{u_m\}_{m \in \mathbb{N}}$ is an orthonormal eigenbasis of $A - W$. Hence, we have

$$\sum_m \langle \tilde{\psi}_+, u_m \rangle^2 \lambda_1 = \lambda_1 = \langle (A - W)\tilde{\psi}_+, \tilde{\psi}_+ \rangle = \sum_m \langle \tilde{\psi}_+, u_m \rangle^2 \lambda_m.$$
belong to the eigenspace of \( \lambda_1 \). Now Lemma 4.3.3 shows that \( \psi_+ > 0 \) in \( \Omega \) and \( \psi_- \) must vanish, in contradiction to our above supposition.

Assume now that \( \psi \) and \( \varphi \) are two linearly independent eigenfunctions belonging to \( \lambda_1 \). Then we have \((A - W - \lambda_1)\psi = (A - W - \lambda_1)\varphi = 0\), so by [GT77, Theorem 8.22] \( \psi \) and \( \varphi \) are locally Hölder continuous in \( \Omega \). Thus, by linear independence, there are points \( x, y \in \Omega \) such that the system

\[
\psi(x)r + \varphi(x)s = -1, \quad \psi(y)r + \varphi(y)s = 1
\]

has a solution \((r, s)\). But this implies that the function \( r\psi + s\varphi \) is also an eigenfunction to the eigenvalue \( \lambda_1 \) that takes positive and negative values. This, however, is impossible by the preceding considerations.

**Lemma 4.3.5.** \( R(\lambda, bA) \) is positivity improving for all \( \lambda > 0 \).

**Proof.** Putting \( W_\lambda := \frac{\lambda}{b} \), one first notices \( W_\lambda > \epsilon_\lambda > 0 \) almost everywhere. Hence, for every \( \lambda > 0 \) Lemma 4.3.3 and Lemma 4.3.4 apply for \( W = W_\lambda \). Thus the largest eigenvalue of \( A - W_\lambda \) is simple and there is a corresponding eigenfunction which is strictly positive. Furthermore, this operator is self-adjoint, bounded from above and the semigroup \( e^{t(A-W_\lambda)} \) is positive. To see this last point, observe that \( e^{tA} \) and \( e^{-tW_\lambda} \) are both positive contraction semigroups, so again by the Trotter product formula, we obtain

\[
e^{t(A-W_\lambda)}f = \lim_{n \to \infty} (e^{\frac{\lambda}{b}A}e^{-\frac{\lambda}{b}W_\lambda})^n f,
\]

which yields the claimed positivity.

Thus, we may apply [RS78, Theorem XIII.44], which states that \( R(0, A - W_\lambda) \) is positivity improving. According to (4.3.1) this also holds for \((\lambda - bA)^{-1}\).

**Proof of Theorem 4.3.1.** Since \( e^{tbA} \) is analytic, it maps \( L^2(\Omega) \) into \( D(A) \). The latter is contained in the form domain and thus in \( H^1(\Omega) \) and so \( e^{tbA} \) is compact by the compact embedding of \( H^1(\Omega) \) in \( L^2(\Omega) \). As \( e^{tbA\varphi}, 1 < p < \infty, \) is a consistent family of semigroups, the compactness in \( L^p(\Omega) \) for \( p \in [2, \infty) \) follows by complex interpolation via [Dav90, Theorem 1.6.1] and for \( p \in (1, 2) \) by duality. This proves the last assertion.

Regarding positivity, observe that it suffices to handle the case \( p = 2 \) due to consistency and to show that \( R(\lambda, bA) \) is a positive operator for large \( \lambda \). In order to do so, set again \( W_\lambda := \lambda/b \). Then, we already saw in the proof of Lemma 4.3.5, that \( e^{t(A-W_\lambda)} \) is positive. Hence, the same is true for \( R(0, (A - W_\lambda)) = (W_\lambda - A)^{-1} \). Using once more (4.3.1), we see that \( R(\lambda, bA) \) is positive.

We now turn to prove irreducibility. By Proposition 4.2.2, the semigroup \( e^{tbA\varphi} \) is given by kernels \( p_t \in L^p(\Omega \times \Omega) \) for \( t > 0 \). Thus in view of the criterion given in Remark 4.2.4, it suffices to argue in \( L^2(\Omega) \).

Also, the upper Gaussian bound implies by [Hie96, Theorem 2.2] that the resolvent of \( bA \) at the point 1, i.e. \((1 - bA)^{-1}\) is a regular integral operator with kernel

\[
K^R(x, y) = \int_0^\infty e^{-t}p_t(x, y) \, dt.
\]

Due to the positivity of \( e^{tbA} \) this kernel is positive. Furthermore, for every choice of measurable sets \( M, N \subset \Omega \) with positive measure it satisfies

\[
\int_M \int_N K^R(x, y) \, dx \, dy = \int_N \left(R(1,bA)1_M\right)(x) \, dx > 0,
\]

since the characteristic function of \( M \) is positive and not identically zero. Hence, \( R(1,bA)1_M \) is strictly positive by Lemma 4.3.5.
Finally, suppose that the semigroup $e^{tbA}$ is not irreducible. By Remark 4.2.4 this means that there exist measurable sets $M_0, N_0 \subset \Omega$ with positive measure such that for all $t > 0$ we have

$$\int_{M_0} \int_{N_0} p_t(x, y) \, dx \, dy = 0.$$ 

Note that this expression cannot be negative, since $p_t \geq 0$. But this implies immediately

$$\int_{M_0} \int_{N_0} K^R(x, y) \, dx \, dy = \int_0^\infty e^{-t} \left( \int_{M_0} \int_{N_0} p_t(x, y) \, dx \, dy \right) = 0,$$

in contradiction to (4.3.4). Hence $e^{tbA}$ must be irreducible. \hfill \Box

Remark 4.3.6. After submitting the paper, we were informed by Wolfgang Arendt that, combining techniques from Ouhabaz [Ouh05] and himself [Are04], one obtains a different proof of our main result, Theorem 4.3.1, which is based on form methods only.
References (Article 4)


References (Article 4)


5. Kolmogorov Kernel Estimates for the Ornstein-Uhlenbeck Operator

by Robert Haller-Dintelmann and Julian Wiedl

Abstract
Replacing the Gaussian semigroup in the heat kernel estimates by the Ornstein-Uhlenbeck semigroup on \( \mathbb{R}^d \), we define the notion of Kolmogorov kernel estimates. This allows us to show that under Dirichlet boundary conditions Ornstein-Uhlenbeck operators are generators of consistent, positive, (quasi-)contractive \( C_0 \)-semigroups on \( L^p(\Omega) \) for all \( 1 \leq p < \infty \) and for every domain \( \Omega \subseteq \mathbb{R}^d \). For exterior domains with sufficiently smooth boundary a result on the location of the spectrum of these operators is also given.

5.1. Introduction

Heat kernel estimates have proved to be a powerful tool for the analysis of elliptic differential operators. Beside many other things, they allow the extension of a given semigroup on \( L^{p_0} \) for some \( p_0 \) to the whole scale of \( L^p \)-spaces for \( 1 \leq p < \infty \) in a consistent way, transferring certain nice properties to all these semigroups, such as analyticity, and yielding \( p \)-invariance of the spectrum.

Dealing with Ornstein-Uhlenbeck operators on \( L^p(\Omega) \) for unbounded domains \( \Omega \) we evidently cannot expect to get heat kernel estimates, as the spectrum of these operators in \( L^p(\mathbb{R}^d) \) already depends heavily on \( p \). Nevertheless, the well-known representation for the Ornstein-Uhlenbeck semigroup on \( L^p(\mathbb{R}^d) \) that is due to A. N. Kolmogorov and given by

\[
T(t)f(x) = \int_{\mathbb{R}^d} k_t(e^{tB}x - y)f(y) \, dy = (k_t * f)(e^{tB}x),
\]

where the Kolmogorov kernel

\[
k_t(x) = \frac{1}{(4\pi)^{d/2}(\det Q_t)^{\frac{d}{2}}} \exp\left(-\frac{1}{4}Q_t^{-1}x \cdot x\right) \quad \text{with} \quad Q_t = \int_0^t e^{sB}Qe^{sB^*} \, ds
\]

looks very similar to the heat semigroup. So it is a natural idea to consider Kolmogorov kernel estimates, by replacing the Gaussian semigroup by the Kolmogorov semigroup \( (T(t))_{t \geq 0} \), thus getting a majorising semigroup that is well adapted to operators of the Ornstein-Uhlenbeck type. These estimates then allow us to extend semigroups on \( L^{p_0} \) to the whole scale of \( L^p \) for \( 1 \leq p < \infty \), analogously to the case of heat kernel bounds.

Having established this idea, in the sequel we apply it to Ornstein-Uhlenbeck operators with Dirichlet boundary conditions in \( L^p(\Omega) \), where \( 1 \leq p < \infty \) and \( \Omega \) is a domain in \( \mathbb{R}^d \). Ornstein-Uhlenbeck operators are differential operators, formally given by

\[
(Au)(x) = \sum_{i,j=1}^d q_{ij}D_iD_ju(x) + Bx \cdot \nabla u(x),
\]

(5.1.1)
where $Q = (q_{ij})_{j=1}^d \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix and $B = (b_{ij})_{j=1}^d \in \mathbb{R}^{d \times d}$ is an additional drift. In this context one usually works in spaces of continuous functions on $\mathbb{R}^d$ or in the spaces $L^p(\mathbb{R}^d)$, where $\mu$ is the invariant measure of the underlying process.

Recently, it became clear that an analytic treatment of these operators is of great interest. For instance, looking at the Stokes equation in the exterior of a rotating obstacle leads to operators of the Ornstein-Uhlenbeck type, see [His99], [HS05] and [GHH05]. Thus one is interested in their behaviour on $L^p(\Omega)$ with respect to the Lebesgue measure for domains and especially exterior domains.

Passing from the invariant measure to the Lebesgue measure changes the properties of the operator completely. The spectrum is no longer contained in the negative real axis, instead it contains a vertical line (cf. [Met01]), so the semigroup $(T(t))_{t \geq 0}$ on $L^p(\mathbb{R}^d)$ is not analytic, as it is in $L^p(\mathbb{R}^d, \mu)$ (cf. [MPP02]), and it even fails to be eventually norm-continuous.

Whereas Ornstein-Uhlenbeck operators are well understood in $L^p(\mathbb{R}^d)$ (cf. [MPV05], [MPRS02], [Met01]) and for bounded domains, where they can be viewed as a perturbation of lower order of the elliptic diffusion part, there are very few results for unbounded domains. M. Geissert, H. Heck, M. Hieber and I. Wood showed in [GHHW05] that in the case of an exterior domain $\Omega$ with sufficiently smooth boundary, a realisation of $\mathcal{A}$ on $L^p(\Omega)$ generates a $C_0$-semigroup and G. da Prato and A. Lunardi treat the case of $L^2$-spaces of convex sets with respect to Neumann boundary conditions and infinitesimally invariant measures in [DPL04]. The Dirichlet problem in spaces of bounded continuous functions on smooth domains is treated by S. Forano, G. Metafune and E. Priola in [FMP04]. An overview may be found in [BL].

In this paper we show that for arbitrary domains $\Omega \subset \mathbb{R}^d$ a realisation of $\mathcal{A}$ in $L^2(\Omega)$ generates a (quasi-)contractive, positive $C_0$-semigroup, that has a Kolmogorov kernel estimate, see Theorem 5.3.2. In the sequel this allows us to define consistent Ornstein-Uhlenbeck semigroups on $L^p(\Omega)$ for $1 \leq p < \infty$ that have the same contractivity, positivity and domination properties. This immediately gives an upper bound on the growth bound of the semigroups and implies that the Ornstein-Uhlenbeck operator admits a bounded $H^\infty$-calculus on $L^p(\Omega)$ (Proposition 5.4.5).

In the special case of an exterior domain with sufficiently smooth boundary, it turns out that the domain is

$$W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega) \cap \{ f \in L^p(\Omega) : Bx \cdot \nabla f \in L^p(\Omega) \}$$

and we even deduce in Theorem 5.5.2 that the same vertical line as in the case of the whole space is contained in the spectrum of the operator, so the spectral behaviour is the same as for the case $\Omega = \mathbb{R}^d$. This means that also in this case the semigroup is not eventually norm-continuous. Nevertheless we can show that its growth bound and the spectral bound of its generator coincide, which is no longer clear by standard spectral theory for semigroups.

The paper is organised as follows. In section 5.2 we introduce the notion of Kolmogorov kernel estimates and prove their main implications. The generation result for $\mathcal{A}$ in $L^2(\Omega)$ is contained in section 5.3 and in section 5.4 we show that this semigroup is positive and admits a Kolmogorov kernel estimate. Section 5.5 finally contains the results for exterior domains.

Notations

Throughout this paper we use the following notation.

For a closed operator $(\mathcal{A}, D(\mathcal{A}))$ on some Banach space $X$ we denote by $\sigma(\mathcal{A})$ the spectrum, by $\varrho(\mathcal{A})$ the resolvent set and by $R(\lambda, \mathcal{A}) = (\lambda - A)^{-1}$, $\lambda \in \varrho(\mathcal{A})$, the resolvent of $\mathcal{A}$. Furthermore, the space of all bounded linear operators on $X$ is denoted by $\mathcal{L}(X)$.

As usual, for $\Omega \subset \mathbb{R}^d$ open, $\| \cdot \|_p$ stands for the norm of the Lebesgue spaces $L^p(\Omega)$ whenever the set $\Omega$ is clear from the context. We write $W^{k,p}(\Omega)$, or $H^k(\Omega)$ in the case $p = 2$, for the
Sobolev spaces, $C_c^\infty(\Omega)$ for the space of all smooth functions having compact support in $\Omega$ and $W_0^{1,p}(\Omega)$, or $H_0^1(\Omega)$, is the closure of $C_c^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$ or $H^1(\Omega)$, respectively. Furthermore, if $X$ is a function space, $X_+$ stands for the cone of all positive functions in $X$.

Finally, $B_r(x_0)$ is the open ball of radius $r$ with centre $x_0$ and, given a matrix $B \in \mathbb{R}^{d \times d}$, we write $\text{tr}(B) = \sum_{j=1}^d b_{jj}$ for its trace.

### 5.2. Kolmogorov kernel estimates

Given a matrix $B \in \mathbb{R}^{d \times d} \setminus \{0\}$ and a positive definite matrix $Q \in \mathbb{R}^{d \times d}$, we define the Kolmogorov semigroup $(K_p(t))_{t \geq 0}$ on $L^p(\mathbb{R}^d)$ by

$$
(K_p(t)f)(x) = \int_{\mathbb{R}^d} k_t(e^{tB}x - y)f(y) \, dy, \quad f \in L^p(\mathbb{R}^d),
$$

where the Kolmogorov kernel $k_t$ is given by

$$
k_t(x) = \frac{1}{(4\pi)^{d/2}(\det Q_t)^{1/2}} \exp\left(-\frac{1}{4}Q_t^{-1}x \cdot x\right) \quad \text{with} \quad Q_t = \int_0^t e^{sB}Qe^{sB^*} \, ds.
$$

It is well known (cf. [Met01]) that $(K_p(t))_{t \geq 0}$ is a positive $C_0$-semigroup on $L^p(\mathbb{R}^d)$ for every $1 \leq p < \infty$ and it is straightforward by substitution and Young’s inequality that for every $f \in L^p(\mathbb{R}^d)$

$$
\|K_p(t)f\|_p = \left(\int_{\mathbb{R}^d} |(k_t \ast f)(e^{tB}x)|^p \, dx\right)^{1/p} = e^{\frac{\text{tr}(B)}{p}}\|k_t \ast f\|_p \leq e^{\frac{\text{tr}(B)}{p}}\|f\|_p,
$$

since $\|k_t\|_1 = 1$. The generator $A_{R^d,p}$ of this semigroup is the Ornstein-Uhlenbeck operator $\mathcal{A}$ given in (5.1.1) and G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt showed in [MPRS02] that its domain is

$$
D(A_{R^d,p}) = W^{2,p}(\mathbb{R}^d) \cap \{ f \in L^p(\mathbb{R}^d) : Bx \cdot \nabla f \in L^p(\mathbb{R}^d) \}
$$

for $1 < p < \infty$.

Now, let $\Omega \subseteq \mathbb{R}^d$ be an open set, $1 \leq p < \infty$ and let $(T(t))_{t \geq 0}$ be a $C_0$-semigroup on $L^p(\Omega)$ with generator $\mathcal{A}$.

**Definition 5.2.1.** We say, that the semigroup $(T(t))_{t \geq 0}$ on $L^p(\Omega)$ satisfies a Kolmogorov kernel estimate, if there exist a matrix $B \in \mathbb{R}^{d \times d} \setminus \{0\}$, a positive definite matrix $Q \in \mathbb{R}^{d \times d}$, $M \geq 0$ and $\omega \in \mathbb{R}$, such that for all $f \in L^p(\Omega)$ and all $t \geq 0$ we have the pointwise estimate

$$
|T(t)f| \leq Me^{\omega t}K_p(t)|\tilde{f}|,
$$

where $\tilde{f}$ denotes the trivial extension of $f$ to $\mathbb{R}^d$.

For such semigroups we have the following result.

**Proposition 5.2.2.** Let $\Omega \subseteq \mathbb{R}^d$ be open, $1 \leq p < \infty$ and let $(T(t))_{t \geq 0}$ be a $C_0$-semigroup on $L^p(\Omega)$ that satisfies a Kolmogorov kernel estimate for some matrix $B$, a positive definite matrix $Q$ and constants $M$ and $\omega$. Then for $1 \leq q < \infty$ there exist consistent $C_0$-semigroups $(T_q(t))_{t \geq 0}$ on $L^q(\Omega)$, such that $T_p = T$ and

$$
\|T_q(t)\|_{L^q(\Omega)} \leq Me^{(\omega - \frac{\text{tr}(B)}{p})t}
$$

for every $t \geq 0$. Furthermore, $T_q$ satisfies the same Kolmogorov kernel estimate and if $T$ is a positive semigroup, $T_q$ is also positive.
Proof. Let \( g \in L^p(\Omega) \cap L^q(\Omega) \) and \( t \geq 0 \). Then, by consistency of the semigroups \( K_q \) for \( 1 \leq q < \infty \) and (5.2.1), we have
\[
\|T(t)g\|_q = \left( \int_\Omega |T(t)g|^q \right)^{\frac{1}{q}} \leq Me^{ct} \left( \int_{\mathbb{R}^d} (K_p(t)|\tilde{g}|^q) \right)^{\frac{1}{q}} = Me^{ct} \|K_q(t)|\tilde{g}\|_q \leq Me^{(\omega - \frac{\text{tr}(B)}{q})t}\|g\|_q.
\]
Thus we can extend the operator \( T(t) \) continuously to an operator \( T_q(t) \) on \( L^q(\Omega) \), obtaining \( C_0 \)-semigroups that are consistent by construction and that obey the stated norm estimate by the above calculation.

For \( f \in L^q(\Omega) \cap L^p(\Omega) \) the inequality
\[
|T_q(t)f| \leq Me^{ct}K_q(t)|\tilde{f}|
\]
is immediate by consistency. Thus \( Me^{ct}K_q(t)|\tilde{f}| - |T_q(t)f| \) is positive for all these functions and the Kolmogorov kernel estimates follow for arbitrary \( f \in L^q(\Omega) \), since \( L^q(\Omega)_+ \) is closed in \( L^q(\Omega) \).

Finally, the same closedness argument yields the positivity of \( T_q \), whenever \( T \) is positive. \( \square \)

For every \( 1 \leq q < \infty \) we denote the generator of \((T_q(t))_{t \geq 0}\) by \( A_q \). The Kolmogorov kernel estimates also provide consistency results for these operators and their resolvents. We collect them in the next proposition.

**Proposition 5.2.3.** Let \( 1 \leq q < \infty \). Then

a) \( R(\lambda, A_q)f = R(\lambda, A_r)f \) for all \( f \in L^q(\Omega) \cap L^r(\Omega), \) all \( 1 \leq r < \infty \) and all \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > \lambda_0 := \max(\omega - \frac{\text{tr}(B)}{q}, \omega - \frac{\text{tr}(B)}{q}) \).

b) The set \( \{ f \in D(A) \cap L^q(\Omega) : Af \in L^q(\Omega) \} \) is contained in \( D(A_q) \) and \( A_qf = Af \) for all such \( f \).

c) \( |R(\lambda, A_q)f| \leq MR(\lambda, \omega + A_{R^d,q})|\tilde{f}| \) for all \( f \in L^q(\Omega) \) and all \( \lambda > \omega - \frac{\text{tr}(B)}{q} \).

**Proof.**

a) By Proposition 5.2.2 we know, that the growth bounds of \( T_q \) and \( T_r \) are at most \( \lambda_0 \). Taking \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > \lambda_0 \), this allows us to conclude with the help of the Laplace transform
\[
R(\lambda, A_q)f = \int_0^\infty e^{-\lambda t}T_q(t)f \, dt = \int_0^\infty e^{-\lambda t}T_r(t)f \, dt = R(\lambda, A_r)f
\]
for every \( f \in L^q(\Omega) \cap L^r(\Omega) \) by the consistency of the semigroups.

b) Let \( f \in D(A) \cap L^q(\Omega) \) with \( Af \in L^q(\Omega) \). Then, choosing \( \lambda > \omega + |\text{tr}(B)| \), we have \( \lambda \in \mathfrak{g}(A) \cap \mathfrak{g}(A_q) \) and since \( (\lambda - A)f \in L^p(\Omega) \cap L^q(\Omega) \), we get
\[
f = R(\lambda, A)(\lambda - A)f = R(\lambda, A_q)(\lambda - A)f \in D(A_q)
\]
by part a) of this proof. The above equality also yields \( A_qf = Af \).

c) Let \( \lambda > \omega - \frac{\text{tr}(B)}{q} \). Then we may again use Laplace transform for \( R(\lambda, A_q) \) and we obtain for \( f \in L^q(\Omega) \)
\[
|R(\lambda, A_q)f| = \left| \int_0^\infty e^{-\lambda t}T_q(t)f \, dt \right| \leq M \int_0^\infty e^{-\lambda t}K_q(t)|\tilde{f}| \, dt = MR(\lambda, \omega + A_{R^d,q})|\tilde{f}|,
\]
since \( T_q \) also has a Kolmogorov kernel estimate by Proposition 5.2.2. \( \square \)
In the special case that $M = 1$ in the kernel estimates, the operators $A_q - \omega + \text{tr}(B)/q$ generate contraction semigroups for all $q \in (1, \infty)$. Since $|\text{tr}(B)/q| \leq |\text{tr}(B)|$ for all $q \in (1, \infty)$, the amount of the shift is bounded. Thus $A_q - \omega - |\text{tr}(B)|$, $1 < q < \infty$, is a family of generators of contraction semigroups on $L^q(\Omega)$. If in addition the semigroup $(T(t))_{t \geq 0}$ is positive, by Proposition 5.2.2 all the semigroups are positive. This immediately yields a bounded $H^\infty$-calculus for their generators, see [HP98].

**Proposition 5.2.4.** The operators $A_q - \omega - |\text{tr}(B)|$ admit a bounded $H^\infty$-calculus for every $1 < q < \infty$, whenever $(T(t))_{t \geq 0}$ is a positive semigroup satisfying a Kolmogorov kernel estimate with $M = 1$.

### 5.3. The Ornstein-Uhlenbeck semigroup on $L^2(\Omega)$

In the following we want to use Kolmogorov kernel estimates to show that the Ornstein-Uhlenbeck operator $\mathcal{A}$ is the generator of a positive $C_0$-semigroup on $L^p(\Omega)$ for every open and connected subset $\Omega$ of $\mathbb{R}^d$. As we already mentioned in the introduction, the key is a generation result for the case $p = 2$ and Ornstein-Uhlenbeck kernel estimates. We will prove these two items in this and the following section.

We define the realisation of the Ornstein-Uhlenbeck operator in $L^2(\Omega)$ by

$$D(A_{\Omega,2}) = H^1_0(\Omega) \cap \{ u \in H^2_{\text{loc}}(\Omega) : \mathcal{A}u \in L^2(\Omega) \},$$

$$\begin{aligned}
(A_{\Omega,2}u)(x) &= \sum_{i,j=1}^{d} q_{ij} D_i D_j u(x) + Bx \cdot \nabla u(x) \\
&= \sum_{i,j=1}^{d} q_{ij} D_i D_j u(x) + \sum_{i,j=1}^{d} b_{ij} x_j D_i u(x), \quad x \in \Omega \subset \mathbb{R}^d,
\end{aligned}$$

where $Q = (q_{ij})_{i,j=1}^{d} \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix and $B = (b_{ij})_{i,j=1}^{d} \in \mathbb{R}^{d \times d}$ by dissipativity, the equality of the two domains follows by the following general observation.

**Remark 5.3.1.** Let $B$ be the generator of a $C_0$-semigroup on some Banach space $X$ and let $A \supseteq B$ be dissipative. Then we already have $A = B$. In fact, there exists $\lambda > 0$, such that $\lambda - B$ is surjective and $\lambda - A$ is injective by dissipativity. Thus the claim follows by [EN00, IV. 1.21. (5)].

Now we can formulate our result for the case $p = 2$.

**Theorem 5.3.2.** Let $\Omega \subseteq \mathbb{R}^d$ be a domain. Then the operator $A_{\Omega,2}$ generates a positive $C_0$-semigroup $(T_{\Omega,2}(t))_{t \geq 0}$ on $L^2(\Omega)$ with $\|T_{\Omega,2}(t)\|_{\mathcal{L}(L^2(\Omega))} \leq e^{-\frac{\omega + \text{tr}(B)}{2} t}$ for all $t \geq 0$. Moreover for every $\lambda > -\text{tr}(B)/2$ and every $t \geq 0$ we have the domination properties

$$\begin{aligned}
|R(\lambda, A_{\Omega,2}) f| &\leq R(\lambda, A_{\mathbb{R}^d,2}) \| f \|, \quad f \in L^2(\Omega), \\
|T_{\Omega,2}(t) f| &\leq T_{\mathbb{R}^d,2}(t) \| f \| = K_2(t) \| f \|, \quad f \in L^2(\Omega),
\end{aligned}$$

where $\tilde{f}$ denotes the extension of $f$ by $0$. In particular $T_{\Omega,2}$ fulfills a Kolmogorov kernel estimate with $M = 1$ and $\omega = 0$. 

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In the following we use the notation
\[(A_0 u)(x) = \sum_{i,j=1}^{d} q_{ij} D_i D_j u(x) \quad \text{and} \quad (Lu)(x) = Bx \cdot \nabla u(x)\]
for the diffusion and the drift part of \(A\), respectively. We start with a simple lemma, that will be useful for many proofs.

**Lemma 5.3.3.** Let \(G \subseteq \mathbb{R}^d\) be open and \(u \in H^1_0(G)\) be a real-valued function. For any \(\varphi \in C^\infty_c(\mathbb{R}^d)\) we have
\[
\int_G (Lu) u \varphi = -\frac{\text{tr}(B)}{2} \int_{\Omega} u^2 \varphi - \frac{1}{2} \int_{\Omega} u^2 L \varphi.
\]

**Proof.** Since \(u \in H^1_0(G)\), there is a sequence \((u_n)_{n \in \mathbb{N}} \subseteq C^\infty_c(\mathbb{R}^d)\) converging to \(u\) in \(H^1(G)\). Therefore, we have \(\int_{G} (Lu) u \varphi = \lim_{n \to \infty} \int_{G} Lu_n u_n \varphi\), since \(Bx\) is bounded on the support of \(\varphi\). We get
\[
\int_G Lu_n u_n \varphi = \int_G Bx \cdot \nabla u_n u_n \varphi = -\int_G \text{div}(u_n Bx) u_n
\]
\[= -\int_G Lu_n u_n \varphi - \text{tr}(B) \int_G u_n^2 \varphi - \int_G u_n^2 L \varphi.
\]
Letting \(n\) tend to \(\infty\), we derive
\[
\int_G (Lu) u \varphi = -\frac{\text{tr}(B)}{2} \int_G u^2 \varphi - \frac{1}{2} \int_G u^2 L \varphi.
\]

**Remark 5.3.4.** To be precise, one has to check that integration by parts is allowed in the proof of Lemma 5.3.3. This will be used again later on, so it might be useful to note the following generalisation. For any open set \(G\), integration by parts is possible if \(u \in C^\infty_c(\mathbb{R}^d)\) and \(v \in H^1_{\text{loc}}(G)\).
In this case, there is a compact set \(K \subseteq G\) with \(\text{supp}(u) \subseteq K^\circ\). Then \(v \in H^1(K)\) and by the definition of weak derivatives one gets for all \(1 \leq i \leq d\)
\[
\int_G u(D_i v) = \int_K u(D_i v) = -\int_K (D_i u)v = -\int_G (D_i u)v.
\]
In order to show that \(A_{\Omega,2}\) is a generator of a \(C_0\)-semigroup we will apply the Lumer-Phillips theorem. So we first need dissipativity.

**Proposition 5.3.5.** The operator
\[
A := A_{\Omega,2} + \frac{\text{tr}(B)}{2}
\]
is dissipative in \(L^2(\Omega)\).

**Proof.** Let \(f \in D(A_{\Omega,2}) = D(A)\). Writing \(f = u + iv\) for suitable real \(u, v \in L^2(\Omega)\), we get
\[
\text{Re} \int_{\Omega} (Af) = \int_{\Omega} (Au) + \int_{\Omega} (Av),
\]
so it suffices to show \( \int_\Omega (Au)u \leq 0 \) for real-valued functions \( u \). Note that \( u, v \in D(\mathcal{A}) \), since the coefficients of \( \mathcal{A} \) are real.

We choose \( \eta \in C^\infty(\mathbb{R}^d) \) vanishing outside of \( B_2(0) \) with \( \eta|_{B_1(0)} = 1 \) and define \( \eta_m(x) = \eta(\frac{x}{m}) \) for \( m \in \mathbb{N} \). Since \( (Au)u \in L^1(\Omega) \) we derive \( \lim_{m \to \infty} \int_\Omega (Au)\eta_m = \int_\Omega (Au)u \) by the pointwise convergence of \( \eta_m \) to \( 1 \) and Lebesgue’s Theorem.

Next we choose a sequence \( (u_n)_{n \in \mathbb{N}} \) of \( C^\infty \)-functions converging to \( u \) with respect to the \( H^1 \)-norm. Now, partial integration (as in Remark 5.3.4) yields

\[
\int_\Omega (A_{\Omega,2}u)(x)u_n(x)\eta_m(x) \, dx
\]

\[
= \int_\Omega \sum_{i,j=1}^d q_{ij} D_i D_j u(x)u_n(x)\eta_m(x) \, dx + \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u(x)u_n(x)\eta_m(x) \, dx
\]

\[
= - \int_\Omega \sum_{i,j=1}^d q_{ij} D_i D_j u(x)u_n(x)\eta_m(x) \, dx - \int_\Omega \sum_{i,j=1}^d q_{ij} D_i u(x)u_n(x)D_j \eta_m(x) \, dx
\]

\[
+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u(x)u_n(x)\eta_m(x) \, dx - \frac{1}{2} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u_n(x)\eta_m(x) \, dx
\]

\[
- \frac{1}{2} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j u_n(x)u(x)D_i \eta_m(x) \, dx - \frac{\text{tr}(B)}{2} \int_\Omega u(x)u_n(x)\eta_m(x) \, dx.
\]

Note that, thanks to the bounded supports of the functions \( \eta_m \) and \( u_n \), all integrals in the above calculations are well defined.

The \( H^1 \)-convergence of the sequence \( (u_n)_{n \in \mathbb{N}} \) yields

\[
\lim_{n \to \infty} \int_\Omega \sum_{i,j=1}^d q_{ij} D_i D_j u_n(x)\eta_m(x) \, dx = \int_\Omega \sum_{i,j=1}^d q_{ij} D_i D_j u(x)\eta_m(x) \, dx
\]

and

\[
\lim_{n \to \infty} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u_n(x)u(x)\eta_m(x) \, dx = \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u(x)u(x)\eta_m(x) \, dx,
\]

since \( x_j \eta_m(x) \) is bounded. The other summands can be treated analogously, so we derive

\[
\int_\Omega Au(x)u(x)\eta_m(x) \, dx = \lim_{n \to \infty} \int_\Omega (Au)(x)u_n(x)\eta_m(x) \, dx
\]

\[
= - \int_\Omega \sum_{i,j=1}^d q_{ij} D_i D_j u(x)u_n(x)\eta_m(x) \, dx - \int_\Omega \sum_{i,j=1}^d q_{ij} D_i u(x)u_n(x)D_j \eta_m(x) \, dx
\]

\[
+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u(x)u_n(x)\eta_m(x) \, dx - \frac{1}{2} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j D_i u_n(x)\eta_m(x) \, dx
\]

\[
- \frac{1}{2} \int_\Omega \sum_{i,j=1}^d b_{ij} x_j u_n(x)u(x)D_i \eta_m(x) \, dx.
\]

Next, we want to pass to the limit \( m \to \infty \), so we have to consider the terms containing derivatives of \( \eta_m \). The equality \( (D_i \eta_m)(x) = \frac{1}{m}(D_i \eta)(\frac{x}{m}) \) implies

\[
x_j D_i \eta_m(x) = \frac{x_j}{m} D_i \eta \left( \frac{x}{m} \right) = 0 \quad \text{for} \quad \left| \frac{x}{m} \right| > 2,
\]

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and therefore all functions \( x_j D_i \eta_m(x) \), \( 1 \leq j \leq d \), are bounded with
\[
| x_j D_i \eta_m(x) | = \left| \frac{x_j}{m} D_i \eta \left( \frac{x}{m} \right) \right| < 2 \| \nabla \eta \|_\infty \]
and have support inside \( \{ x \in \mathbb{R}^d : m \leq |x| \leq 2m \} \).

Now let \( \varepsilon > 0 \). Since \( u^2 \in L^1(\Omega) \), there is a compact subset \( K_\varepsilon \subseteq \Omega \) with
\[
\int_{\Omega \setminus K_\varepsilon} u^2 \leq \frac{\varepsilon}{2 \| \nabla \eta \|_\infty}.
\]
If we choose \( m_0 \) large enough, we have \( K_\varepsilon \cap \text{supp}(x_j D_i \eta_m) = \emptyset \) for all \( m \geq m_0 \) and therefore
\[
\left| \int_{\Omega} u^2(x) x_j D_i \eta_m(x) \, dx \right| = \left| \int_{\Omega \setminus K_\varepsilon} u^2(x) x_j D_i \eta_m(x) \, dx \right| \leq \varepsilon.
\]
This proves
\[
\lim_{m \to \infty} \int_{\Omega} \sum_{i,j=1}^d b_{ij} x_j u(x) u(x) D_j \eta_m(x) \, dx = 0.
\]
Since \( \| D_i \eta_m \|_\infty \leq \frac{1}{m} \| \nabla \eta \|_\infty \), we also have
\[
\lim_{m \to \infty} \int_{\Omega} \sum_{i,j=1}^d q_{ij} D_i u(x) u(x) D_j \eta_m(x) \, dx = 0,
\]
Finally we obtain dissipativity of \( A \) by
\[
\int_{\Omega} (Au) u = \lim_{m \to \infty} \int_{\Omega} (Au) u \eta_m = - \int_{\Omega} \sum_{i,j=1}^d q_{ij} D_i u(x) D_j u(x) \, dx \leq 0,
\]
as \( Q \) is positive definite.

In order to show that \( A_{\Omega,2} \) is a generator, it remains to be proven that \( \lambda - A \) is surjective for some fixed \( \lambda > 0 \). This will be done by approximating the solution \( u \) of the resolvent problem \( (\lambda - A)u = f \), \( f \in L^2(\Omega) \), with solutions of the same problem on bounded and regular subdomains of \( \Omega \).

By \([DL90, \text{II.4, Lemma 1}]\), there exists an increasing sequence \((\Omega_n)_{n \in \mathbb{N}}\) of bounded subdomains of \( \Omega \), that have a \( C^2 \)-boundary, such that \( \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \). Note that the specific choice of this sequence is not important, since by dissipativity of \( A \) a solution of the resolvent problem is unique, whenever it exists.

Since the coefficients of \( L \) are bounded on bounded sets, by standard perturbation theory, the operator \( A + \text{tr}(B)/2 \) generates a \( C_0 \)-semigroup on \( L^2(\Omega_n) \) for every \( n \in \mathbb{N} \), when we equip it with the domain \( D = H^1_0(\Omega_n) \cap H^2(\Omega_n) \). As \( D(A_{\Omega_n,2}) \) contains \( D \), we get \( D(A_{\Omega,2}) = D \) again by Remark 5.3.1.

This coincidence of the domains even gives us some more precious information. Since the generator \( A_{\Omega_n,2} + \text{tr}(B)/2 \) is dissipative it even generates a contraction semigroup on \( L^2(\Omega_n) \) for every \( n \in \mathbb{N} \). Thus the bounds on the resolvent do not depend on \( n \), which will be important in the following.

Fixing \( \lambda > 0 \) and \( f \in L^2(\Omega) \) that means we find a unique solution \( u_n \in H^1_0(\Omega_n) \cap H^2(\Omega_n) \) for the problem \( \lambda u_n - A_{\Omega_n,2} u_n - \frac{\text{tr}(B)}{2} u_n = f |_{\Omega_n} \) for every \( n \in \mathbb{N} \) with
\[
\| u_n \|_{L^2(\Omega_n)} = \left\| R \left( \lambda, A_{\Omega_n,2} + \frac{\text{tr}(B)}{2} \right) f |_{\Omega_n} \right\|_{L^2(\Omega_n)} \leq \frac{1}{\lambda} \| f \|_{L^2(\Omega)}, \quad (5.3.1)
\]
As a first step, we will prove that a solution exists. We consider the sequence $(u_n)_{n \in \mathbb{N}}$. It only remains to be shown that for a fixed $K > 0$, independently of the domain $\Omega$, the trivial extension of $u_n$ is an element of $H^1(\mathbb{R}^d)$ and hence of $H^1_0(\Omega)$, we may regard $(u_n)_{n \in \mathbb{N}}$ as a sequence in $H^1_0(\Omega)$. The next lemma will show that even the gradients of the functions $u_n$ are bounded independently of the chosen domain, which does not follow by elliptic regularity alone.

**Lemma 5.3.6.** Let $G \subseteq \mathbb{R}^d$ be a bounded domain with $C^2$-boundary and $u \in H^1_0(G) \cap H^2(G)$ be a solution of $\lambda u - A_{G,2}u - \frac{\mu(B)}{2}u = g$ for some $\lambda > 0$. Then

$$\|\nabla u\|_2 \leq \sqrt{\frac{2}{\lambda}}\|Q^{-\frac{1}{2}}\|g\|_2.$$  

**Proof.** Let $u = v + iw$ with real-valued $v, w \in H^1_0(G) \cap H^2(G)$. Then

$$
\|D^\alpha u\|_2^2 = \int_G D^\alpha u \overline{D^\alpha u} = \int_G (D^\alpha v + iD^\alpha w)(D^\alpha v - iD^\alpha w) = \|D^\alpha v\|_2^2 + \|D^\alpha w\|_2^2
$$

for any multiindex $\alpha$ with $|\alpha| \leq 2$. For the function $v$ we calculate by Lemma 5.3.3

$$
\int_G Q^\frac{1}{2}\nabla v Q^\frac{1}{2}\nabla v = \int_G Q\nabla v \nabla v = -\int_G (A_0 v)v
$$

and

$$
\int_G \left(\lambda v - A_{G,2}v - \frac{\mu(B)}{2}v\right) v = \lambda \int_G vv + \int_G (A_0^*)v + \frac{\mu(B)}{2} \int_G vv
$$

$$
= \int_G \left(\lambda v - A_{G,2}v - \frac{\mu(B)}{2}v\right) v
$$

$$
= \int_G \text{Re}(g) R \left(\lambda, A_{G,2} + \frac{\mu(B)}{2}\right) \text{Re}(g) = \lambda \left\| R \left(\lambda, A_{G,2} + \frac{\mu(B)}{2}\right) \text{Re}(g) \right\|_2^2.
$$

We conclude by (5.3.1)

$$
\|\nabla v\|_2^2 = \|Q^{-\frac{1}{2}}(Q^\frac{1}{2}\nabla v)\|_2^2 \leq \|Q^{-\frac{1}{2}}\|_2^2 \|\text{Re}(g)\|_2^2.
$$

Repeating the same calculations for $w$, we obtain

$$
\|\nabla u\|_2^2 \leq \|Q^{-\frac{1}{2}}\|_2^2 \|\text{Re}(g)\|_2^2 + \|Q^{-\frac{1}{2}}\|_2^2 \|\text{Im}(g)\|_2^2 = \|Q^{-\frac{1}{2}}\|_2^2 \|g\|_2^2.
$$

\[\square\]

**Proposition 5.3.7.** Let $\Omega \subseteq \mathbb{R}^d$ be a domain. Then $A_{\Omega,2}$ is the generator of a $C_0$-semigroup $(T_{\Omega,2}(t))_{t \geq 0}$ on $L^2(\Omega)$ with $\|T_{\Omega,2}(t)\| \leq e^{-\frac{\mu(B)}{2}t}$.

**Proof.** It only remains to be shown that for a fixed $\lambda > 0$ and for every $f \in L^2(\Omega)$ there exists a function $u \in H^1_0(\Omega) \cap \{v \in H^2_0(\Omega) : Av \in L^2(\Omega)\}$ with $\lambda u - Au = f$. We consider the sequence $(u_n)_{n \in \mathbb{N}}$ of trivial extensions of the solutions on $\Omega_n$ mentioned above. In view of Lemma 5.3.6, it is bounded in $H^1_0(\Omega)$, so there exists a weakly convergent subsequence $(u_{nk})_{k \in \mathbb{N}}$. We denote its limit by $u$ and show in the following that $u$ is the desired solution.

As a first step, we will prove that $u \in H^2_0(\Omega)$. Fix two compact sets $K_1, K_2 \subseteq \Omega$ with $K_1 \subseteq K_2$. Then, by construction, $K_2 \subseteq \Omega_{nk}$ for sufficiently large $k$. The coefficients of $A$ are bounded on $K_2$, so [GT83, Theorem 9.11], implies that there is a constant $C$ depending on

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$K_1, K_2$ and the bound of the coefficients on $K_2$, such that $\|u_n\|_{H^2(K_1)} \leq C(\|u_n\|_2 + \|f\|_2)$, so there exists a weakly convergent subsequence $(u_{n_k})_{k \in \mathbb{N}}$ in $H^2(K_1)$. Let $v$ denote its limit. Since the sequence $(u_{n_k} |_{K_1})_{k \in \mathbb{N}}$ also converges weakly to $u|_{K_1}$ in $L^2(K_1)$, we derive $v = u|_{K_1}$ from the uniqueness of weak limits in $L^2(K_1)$. This shows $u \in H^2_{loc}(\Omega)$. In order to finish the proof, it remains to show $\lambda u - Au = f$. Let $g \in C^\infty(\Omega)$ and fix a compactum $K$ with supp$(g) \subseteq K^\circ$. Then $K \subseteq \Omega_{n_k}$ for $k$ large enough. Let $(u_{n_k})_{k \in \mathbb{N}}$ be again a weakly convergent subsequence of $(u_{n_k})_{k \in \mathbb{N}}$ on $H^2(K)$ and $A^*$ be the formal adjoint of $A$. Then we conclude
\[
\int_{\Omega} (\lambda u - Au - f)g = \int_K (\lambda u - Au - f)g = \int_K (\lambda u g - uA^*g - fg) = \lim_{l \to \infty} \int_K (\lambda u_{n_k}g - u_{n_k}A^*g - fg) = \lim_{l \to \infty} \int_K (\lambda u_{n_k} - Au_{n_k} - f)g = 0.
\]
Thus the assertion follows by the fundamental theorem of variational calculus. 

5.4. Domination and positivity of the semigroup

The aim of this section is to prove Kolmogorov kernel estimates for the semigroup $(T_{\Omega,2}(t))_{t \geq 0}$ obtained in Proposition 5.3.7. While doing so, we also obtain positivity of the semigroup. Our method is inspired by the proof of the heat kernel estimates for the Dirichlet Laplacian, cf. [AB99].

Lemma 5.4.1. Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $\lambda > \text{tr}(B)/2$ and let $u \in D(A_{\Omega,2})$, $v \in H^1(\Omega) \cap \{ f \in H^2_{loc}(\Omega) : Af \in L^2(\Omega) \}$ be real-valued functions, such that $v \geq 0$. Then the inequality $(\lambda - A)u \leq (\lambda - A)v$ a.e. implies $u \leq v$ a.e.

Proof. As in the proof of Proposition 5.3.5, we choose a positive $\eta \in C^\infty_c(\mathbb{R}^d)$ vanishing outside of $B_2(0)$ with $\eta|_{B_1(0)} = 1$ and define $\eta_m(x) = \eta(\frac{x}{m})$ for $m \in \mathbb{N}$. By hypotheses, we have
\[
\lambda(u - v) - A_0(u - v) - L(u - v) \leq 0, \quad \text{a.e.}
\]
so we obtain
\[
\lambda \int_\Omega (u - v)\varphi \eta_m - \sum_{i,j=1}^d \int_\Omega q_{ij} D_i D_j (u - v) \varphi \eta_m - \int_\Omega Bx \cdot \nabla (u - v) \eta_m \varphi \leq 0
\]
for all $m \in \mathbb{N}$ and all positive $\varphi \in C^\infty_c(\Omega)$. By integration by parts, cf. Remark 5.3.4, we conclude that
\[
\lambda \int_\Omega (u - v)\varphi \eta_m + \sum_{i,j=1}^d \int_\Omega q_{ij} D_i (u - v) D_j \varphi \eta_m + \sum_{i,j=1}^d \int_\Omega q_{ij} D_i (u - v) \varphi D_j \eta_m
\]
\[
- \int_\Omega Bx \cdot \nabla (u - v) \eta_m \varphi \leq 0.
\]
for all $m \in \mathbb{N}$. Now, this last inequality is even valid for all $\varphi \in H^1_{0_+}(\Omega)$ by density. In the following we show, that $(u - v)^+ \in H^1_{0_+}(\Omega)$. In order to do so, we choose a sequence $(u_n) \subseteq C^\infty_c(\Omega)$, that converges to $u$ in $H^1(\Omega)$. Then the function $(u_n - v)^+$ is in $H^1(\Omega)_+$ for every $n \in \mathbb{N}$ and since $v \geq 0$, we get supp$(u_n - v)^+ \subseteq$ supp$(u_n)$. Thus $(u_n - v)^+$ has compact support in $\Omega$, which implies $(u_n - v)^+ \in H^1_{0_+}(\Omega)$. This finally yields $(u - v)^+ \in H^1_{0_+}(\Omega)$, as $H^1_{0_+}(\Omega)$ is a closed subspace of $H^1(\Omega)$. 

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Putting $\varphi = (u-v)^+$ in the above inequality and observing that then all the integrals vanish on the set \( \{ u \leq v \} \), we get

$$
\lambda \int_{\Omega} ((u-v)^+)^2 \eta_m + \int_{\Omega} \nabla (u-v)^+ Q \nabla (u-v)^+ \eta_m \\
- \int_{\Omega} Bx \cdot \nabla (u-v)^+ (u-v)^+ \eta_m + \sum_{i,j=1}^{n} \int_{\Omega} q_{ij} D_j (u-v)^+ (u-v)^+ D_i \eta_m \leq 0.
$$

Now Lemma 5.3.3 yields for the third integral

$$
\int_{\Omega} Bx \cdot \nabla (u-v)^+ (u-v)^+ \eta_m = -\frac{\text{tr}(B)}{2} \int_{\Omega} ((u-v)^+)^2 \eta_m - \int_{\Omega} Bx \cdot \nabla \eta_m ((u-v)^+)^2.
$$

As all the limits for \( m \to \infty \) on the left hand side exist (cf. proof of Proposition 5.3.5), this implies

$$
- \int_{\Omega} Bx \cdot \nabla (u-v)^+ (u-v)^+ = \frac{\text{tr}(B)}{2} \int_{\Omega} ((u-v)^+)^2.
$$

We derive the inequality

$$
\left( \lambda + \frac{\text{tr}(B)}{2} \right) \int_{\Omega} ((u-v)^+)^2 + \int_{\Omega} \nabla (u-v)^+ Q \nabla (u-v)^+ \leq 0.
$$

Since \( Q \) is positive definite and \( \lambda > -\text{tr}(B)/2 \), this can only be true, if both integrals are zero, in particular, we get \( \| (u-v)^+ \|_{L^2(\Omega)} = 0 \), which implies \( u \leq v \) a.e. \( \square \)

Setting \( v = 0 \), as an immediate corollary, we get the following positivity result for \( A \).

**Corollary 5.4.2.** Let \( \lambda > -\text{tr}(B)/2 \) and let \( u \in D(A_{\Omega,2}) \) be a real-valued function. Then

\[
(\lambda - A)u \leq 0 \text{ a.e. } \implies u \leq 0 \text{ a.e. and } \\
(\lambda - A)u \geq 0 \text{ a.e. } \implies u \geq 0 \text{ a.e.}
\]

The next step is to show that the resolvent and the semigroup react monotonely, when the domain \( \Omega \) is enlarged. As a byproduct of the proof, we also deduce the positivity of the semigroup and the resolvent.

**Proposition 5.4.3.** Let \( \Omega_1, \Omega_2 \subseteq \mathbb{R}^d \) be domains and let \( \Omega_1 \subseteq \Omega_2 \). Then for every \( f \in L^2(\Omega_1)_+ \), for every \( \lambda > -\text{tr}(B)/2 \) and every \( t \geq 0 \), we have

\[
a) \; 0 \leq R(\lambda, A_{\Omega_1,2})f \leq R(\lambda, A_{\Omega_2,2})\tilde{f}, \\
b) \; 0 \leq T_{\Omega_1,2}(t)f \leq T_{\Omega_2,2}(t)\tilde{f}
\]

almost everywhere. Here \( \tilde{f} \) denotes the extension of \( f \) by 0.

**Proof.** a) Put \( u := R(\lambda, A_{\Omega_1,2})f \) and \( v := R(\lambda, A_{\Omega_2,2})\tilde{f} \). Then, both \( u \) and \( v \) are real-valued. Indeed, for \( u = u_1 + iu_2 \), we have \( f = (\lambda - A_{\Omega_1,2})u = (\lambda - A_{\Omega_1,2})u_1 + i(\lambda - A_{\Omega_1,2})u_2 \), where \( (\lambda - A_{\Omega_1,2})u_1, f = 1,2 \), are real-valued functions. This implies \( (\lambda - A_{\Omega_1,2})u_2 = 0 \), and, by injectivity of \( (\lambda - A_{\Omega_1,2}) \), even \( u_2 = 0 \). The argument for \( v \) is the same.

Since \( u \in D(A_{\Omega_1,2}) \), \( v \in D(A_{\Omega_2,2}) \) and \( f \) and \( \tilde{f} \) are positive we get \( u, v \geq 0 \) a.e. by Corollary 5.4.2. It remains to show \( u \leq v \). As \( (\lambda - A)u = f = (\lambda - A)v \) a.e. in \( \Omega_1 \), we have in particular \( (\lambda - A)u \leq (\lambda - A)v \). Furthermore, we know that \( u \in D(A_{\Omega_1,2}) \) and the restriction of \( v \) to \( \Omega_1 \) is in \( H^1(\Omega_1) \cap \{ f \in H^2_{\text{loc}}(\Omega_1) : Af \in L^2(\Omega_1) \} \). Thus we may apply Lemma 5.4.1, in order to get \( u \leq v \) a.e. in \( \Omega_1 \).
Corollary 5.4.4. If $\Omega_1$ and $\Omega_2$ are as in Proposition 5.4.3, we have for every $f \in L^2(\Omega_1)$, for every $\lambda > -\text{tr}(B)/2$ and every $t \geq 0$

\begin{itemize}
  \item [\textbf{a)}] $|T_{\Omega_2}(t)f| \leq T_{\Omega_2}(2t)|\hat{f}|$
  \item [\textbf{b)}] $|R(\lambda, A_{\Omega_2})f| \leq R(\lambda, A_{\Omega_2})|\hat{f}|$
\end{itemize}

Proof. Looking at the special case $\Omega_2 = \mathbb{R}^d$ we see that the semigroup $(T_{\Omega_2}(t))_{t \geq 0}$ is dominated by the semigroup $(T_{\mathbb{R}^d}(t))_{t \geq 0}$, which is given by positive, regular integral kernel operators. It is a classical result (cf. \cite{Sch74, IV, §9}), that in this situation also the operators $T_{\Omega_2}(t), t > 0,$ are given by regular integral kernels $K_t$, that are positive, since $T_{\Omega_2}(t)$ are positive operators. Thus

\[
|T_{\Omega_2}(t)f| = \left| \int_{\Omega_2} K_t(\cdot, y)f(y) \, dy \right| \leq \int_{\Omega_2} K_t(\cdot, y)|f(y)| \, dy = T_{\Omega_2}(2t)|\hat{f}|
\]

by Proposition 5.4.3. Now, the resolvent estimate follows as in the proof of Proposition 5.2.3 \(c\).

\[\square\]

For arbitrary $f \in L^2(\Omega_1)$, we get:

**Corollary 5.4.4.** If $\Omega_1$ and $\Omega_2$ are as in Proposition 5.4.3, we have for every $f \in L^2(\Omega_1)$, for every $\lambda > -\text{tr}(B)/2$ and every $t \geq 0$

\begin{itemize}
  \item [\textbf{a)}] $|T_{\Omega_2}(t)f| \leq T_{\Omega_2}(2t)|\hat{f}|$
  \item [\textbf{b)}] $|\lambda(A_{\Omega_2})f| \leq |\lambda(A_{\Omega_2})|\hat{f}|$
\end{itemize}

Proof. Looking at the special case $\Omega_2 = \mathbb{R}^d$ we see that the semigroup $(T_{\Omega_2}(t))_{t \geq 0}$ is dominated by the semigroup $(T_{\mathbb{R}^d}(t))_{t \geq 0}$, which is given by positive, regular integral kernel operators. It is a classical result (cf. \cite{Sch74, IV, §9}), that in this situation also the operators $T_{\Omega_2}(t), t > 0,$ are given by regular integral kernels $K_t$, that are positive, since $T_{\Omega_2}(t)$ are positive operators. Thus

\[
|\lambda(A_{\Omega_2})f| = \left| \int_{\Omega_2} \lambda(A_{\Omega_2})f(y) \, dy \right| \leq \int_{\Omega_2} \lambda(A_{\Omega_2})|f(y)| \, dy = T_{\Omega_2}(2t)|\hat{f}|
\]

by Proposition 5.4.3. Now, the resolvent estimate follows as in the proof of Proposition 5.2.3 \(c\).

\[\square\]

Setting $\Omega_2 = \mathbb{R}^d$ in Corollary 5.4.4, we have finally completed the proof of Theorem 5.3.2.

Now we have shown that $A_{\Omega_2}$ satisfies a Kolmogorov kernel estimate with $M = 1$ and $\omega = 0$, so the results stated in Chapter 2 hold true. We collect them in the following theorem.

**Theorem 5.4.5.** Let $\Omega \subseteq \mathbb{R}^d$ be a domain. Then $(T_{\Omega,p}(t))_{t \geq 0}, 1 \leq p < \infty,$ is a family of consistent, positive $C_0$-semigroups on $L^p(\Omega)$ with $\|T_{\Omega,p}(t)\|_{L^p(\Omega)} \leq e^{-\frac{\text{tr}(B)}{p}t}$ for all $1 \leq p < \infty$ and all $t \geq 0$. For every $\lambda > -\text{tr}(B)/p$ and every $t > 0$ we have the domination properties

\[
|T_{\Omega,p}(t)f| \leq T_{\mathbb{R}^d,p}(t)|\hat{f}| = (k_t * |\hat{f}|)(e^{|B|t}), f \in L^p(\Omega),
\]

where $\hat{f}$ denotes the extension of $f$ by 0. Moreover, the operators $A_{\Omega,p} - |\text{tr}(B)|$ admit a bounded $H^\infty$-calculus for every $1 < p < \infty$ and we have

\begin{itemize}
  \item [\textbf{a)}] $R(\lambda, A_{\Omega,p})f = R(\lambda, A_{\Omega,q})f$ for all $f \in L^p(\Omega) \cap L^q(\Omega), 1 \leq p, q < \infty$ and all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \max(-\frac{\text{tr}(B)}{p}, -\frac{\text{tr}(B)}{q}).$
  \item [\textbf{b)}] The set $\{f \in D(A_{\Omega,2}) \cap L^p(\Omega) : A_{\Omega,2}f \in L^p(\Omega)\}$ is contained in $D(A_{\Omega,p})$ and $A_{\Omega,p}f = A_{\Omega,2}f$ for all such $f$ and $1 \leq p < \infty.$
\end{itemize}
5.5. The spectrum of $A_{\Omega,p}$ on exterior domains

We finally turn our attention to the special case of an exterior domain, i.e. $\Omega = \mathbb{R}^d \setminus K$ for some compact set $K \subseteq \mathbb{R}^d$ with $C^{1,1}$-boundary. Scaling the set $K$ down to $\{0\}$, we show that the spectral behaviour of $A_{\Omega,p}$ is the same as in the case $\Omega = \mathbb{R}^d$. That means that the spectrum of the drift operator $L$ on $\mathbb{R}^d$, i.e. the whole vertical line $-\text{tr}(B)/p + i\mathbb{R}$ (or in the case $\text{tr}(B) = 0$ at least an unbounded subgroup of it) is contained in $\sigma(A_{\Omega,p})$. This implies that the semigroup $(T_{\Omega,p}(t))_{t \geq 0}$ is not eventually norm-continuous. Nevertheless, we have that the spectral bound of $A_{\Omega,p}$ and the growth bound of $T_{\Omega,p}$ coincide and $s(A_{\Omega,p}) = \omega_0(T_{\Omega,p}) = -\text{tr}(B)/p$. Before we can formulate this theorem, we have to introduce the realisation $L_{\mathbb{R}^d,p}$ of $L$ in $L^p(\mathbb{R}^d)$:

$$D(L_{\mathbb{R}^d,p}) = \{u \in L^p(\mathbb{R}^d) : Lu \in L^p(\mathbb{R}^d)\}, \quad L_{\mathbb{R}^d,p}u = Lu,$$

where $Lu$ is understood in the sense of distributions. We collect the information on the operators $L_{\mathbb{R}^d,p}$ that we need in the following in the next proposition. For proofs, see [Met01].

**Proposition 5.5.1.** Let $1 \leq p < \infty$. Then the following holds:

a) The operator $L_{\mathbb{R}^d,p}$ is the generator of a $C_0$-semigroup $(S(t))_{t \geq 0}$ on $L^p(\mathbb{R}^d)$, given by $(S(t)f)(x) = f(e^{tB}x)$, $x \in \mathbb{R}^d$, for every $f \in L^p(\mathbb{R}^d)$.

b) If $\text{tr}(B) \neq 0$, then $\sigma(L_{\mathbb{R}^d,p}) = -\text{tr}(B)/p + i\mathbb{R}$.

c) If $\text{tr}(B) = 0$, then $\sigma(L_{\mathbb{R}^d,p})$ is an additive subgroup of $i\mathbb{R}$, that is not $\{0\}$.

Now the result that we want to prove in this section can be formulated as follows.

**Theorem 5.5.2.** Let $K \subset \mathbb{R}^d$ be compact with a $C^{1,1}$-boundary and $\Omega = \mathbb{R}^d \setminus K$. Then for every $1 < p < \infty$ we have

$$D(A_{\Omega,p}) = W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega) \cap \{ f \in L^p(\Omega) : Lf \in L^p(\Omega) \}.$$  \hspace{1cm} (5.5.1)

Furthermore we have the inclusion $\sigma(L_{\mathbb{R}^d,p}) \subseteq \sigma(A_{\Omega,p})$ and $s(A_{\Omega,p}) = \omega_0(T_{\Omega,p}) = -\frac{\text{tr}(B)}{p}$.

Note, that the whole picture changes completely for bounded domains as the semigroup then becomes analytic. It remains however an open question, whether Theorem 5.5.2 is still valid for more general unbounded domains.

**Remark 5.5.3.** If $\Omega$ is an exterior domain with $C^{1,1}$-boundary, we already know from [GHHW05] that the Ornstein-Uhlenbeck operator equipped with the domain $W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega) \cap \{ f \in L^p(\Omega) : Lf \in L^p(\Omega) \}$ generates a $C_0$-semigroup for every $1 < p < \infty$. For the following results, it is necessary to check that $A_{\Omega,p}$ coincides with this operator, which we will denote by $\hat{A}_p$. For $p = 2$, the inclusion $D(\hat{A}_2) \subseteq D(A_{\Omega,2})$ is clear, so we get $D(\hat{A}_2) = D(A_{\Omega,2})$ by Remark 5.3.1.

The construction of the semigroups in [GHHW05] now immediately yields consistency. From this we deduce that the semigroups generated by $\hat{A}_p$ and $A_{\Omega,p}$ and hence their generators coincide for $1 < p < \infty$.

In the following we often identify functions $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ with their extension by $f(0) = 0$ and thus view $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ as a subspace of $C_c^\infty(\mathbb{R}^d)$. The importance of the space $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ for the following is due to the following lemma.

**Lemma 5.5.4.** The subspace $C_c^\infty(\mathbb{R}^d \setminus \{0\}) \subseteq D(L_{\mathbb{R}^d,p})$ is a core for $L_{\mathbb{R}^d,p}$. 

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Proof. Let \( f \in C^\infty_c(\mathbb{R}^d \setminus \{0\}) \) and \( t > 0 \). Then \( 0 \notin \text{supp}(f) \). In view of the linearity, bijectivity and continuity of the map \( x \mapsto e^{-tB}x \), we conclude that \( \text{supp}(S(t)f) = e^{-tB} \text{supp}(f) \) is a compact set not containing 0, hence \( S(t)f \in C^\infty_c(\mathbb{R}^d \setminus \{0\}) \) for every \( t \geq 0 \). Moreover, \( C^\infty_c(\mathbb{R}^d \setminus \{0\}) \) is dense in \( L^p(\mathbb{R}^d) \), so the lemma follows from \([En00, \text{Proposition I.1.7}]\). □

Now, if \( \Omega = \mathbb{R}^d \setminus K \) for some compact set \( K \subseteq \mathbb{R}^d \) with \( C^{1,1} \)-boundary, for every \( k \in \mathbb{N} \) we set

\[
\Omega_k := \{ x \in \mathbb{R}^d : kx \in \Omega \} = \mathbb{R}^d \setminus \frac{1}{k}K
\]

and consider the operator

\[
A_k = \frac{1}{k^2} \sum_{i,j=1}^d q_{ij} D_i D_j + \mathcal{L},
\]

with domain

\[
D(A_k) = W^{1,p}_0(\Omega_k) \cap W^{2,p}(\Omega_k) \cap \{ f \in L^p(\Omega_k) : \mathcal{L} f \in L^p(\Omega_k) \}
\]

for some given \( p \in (1, \infty) \).

By Remark 5.5.3 we have \( D(A_k) = D(A_{\Omega_k,p}) \), so Theorem 5.4.5 implies that \( \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) > -\frac{\vartheta(\{B\})}{p} \} \subseteq \sigma(A_k) \) for all \( k \in \mathbb{N} \). Thus we may compare the resolvents of \( A_k \) with the resolvent of \( A_{\Omega,p} = A_k \) for all these \( \lambda \). We adapt a technique introduced in \([D'S86]\) to obtain the following result.

Lemma 5.5.5. We have \( \|R(\lambda, A_k)\|_{L^p(\Omega_k)} = \|R(\lambda, A_{\Omega,p})\|_{L^p(\Omega)} \) for every \( k \in \mathbb{N} \) and every \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > -\frac{\vartheta(\{B\})}{p} \).

Proof. We consider the map

\[
V_k : L^p(\Omega) \to L^p(\Omega_k), \quad V_k(f)(x) = k^{\frac{d}{p}} f(kx).
\]

The transformation formula yields that \( V_k \) is an isometry, whose inverse is given by

\[
V_k^{-1}(f)(x) = k^{-\frac{d}{p}} f \left( \frac{1}{k}x \right).
\]

For \( f \in W^{2,p}(\Omega) \) we get by the chain rule \( V_k(f) \in W^{2,p}(\Omega_k) \) and

\[
V_k^{-1} A_k V_k f = k^{\frac{d}{p}} V_k^{-1} A_k f(kx)
\]

\[
= V_k^{-1} \left[ k^{\frac{d}{p}} \left( \frac{1}{k^2} \sum_{i,j=1}^d k^2 q_{ij} D_i D_j f(kx) + \sum_{i,j=1}^d b_{ij} k x_j D_i f(kx) \right) \right]
\]

\[
= V_k^{-1} \left[ k^{\frac{d}{p}} \left( \sum_{i,j=1}^d q_{ij} D_i D_j f(kx) + \sum_{i,j=1}^d b_{ij} k x_j D_i f(kx) \right) \right] = Af.
\]

If \( f \in W^{1,p}_0(\Omega) \), there is a sequence \( f_n \in C^\infty_c(\Omega) \), which converges to \( f \) in \( W^{1,p}(\Omega) \). Then the sequence \( (V_k f_n)_{n \in \mathbb{N}} \) is in \( C^\infty_c(\Omega_k) \) with \( \text{supp}(V_k f_n) = \frac{1}{k} \text{supp}(f_n) \) and it converges to \( V_k f \) in \( W^{1,p}(\Omega_k) \). Again, the transformation formula implies \( \mathcal{L} V_k f \in L^p(\Omega_k) \). For \( V_k^{-1} \) one can argue analogously. So \( V_k \) induces a bijection from \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap \{ f \in L^p(\Omega) : \mathcal{L} f \in L^p(\Omega) \} \) to \( W^{2,p}(\Omega_k) \cap W^{1,p}_0(\Omega_k) \cap \{ f \in L^p(\Omega_k) : \mathcal{L} f \in L^p(\Omega_k) \} \). In fact, we have \( A_{\Omega,p} f = V_k^{-1} A_k V_k f \).
for all \( f \in D(A_{\Omega,p}) \), and \((\lambda - A_{\Omega,p})f = (\lambda - V_k^{-1}A_k V_k)f = V_k^{-1}(\lambda - A_k)V_k f\). Now we get \( R(\lambda, A_{\Omega,p}) = V_k^{-1}(\lambda - A_k)^{-1}V_k \). We conclude
\[
\| R(\lambda, A_{\Omega,p})f \|_p = \| V_k^{-1}(\lambda - A_k)^{-1}V_k f \|_p = \| V_k^{-1}(\lambda - A_k)^{-1}V_k f \|_p
\leq \| V_k^{-1}\| \| (\lambda - A_k)^{-1} \| \| V_k f \|_p
\leq \| R(\lambda, A_k) \| \| (\lambda - A_k) \| \| f \|_p.
\]
The reverse inequality follows analogously.

Having this equality in hand, we may finally show that the resolvents of \( A_{\Omega,p} \) obey the inequality \( \| R(\lambda, L_{\mathbb{R}^d,p}) \|_{L(\mathbb{R}^d)} \leq \| R(\lambda, A_{\Omega,p}) \|_{L(\Omega)} \) for \( \Re(\lambda) > -\frac{\text{tr}(B)}{p} \). Since \( \sigma(L_{\mathbb{R}^d,p}) \subseteq -\text{tr}(B)/p + i\mathbb{R} \) by Proposition 5.5.1, we can approximate every \( \mu \in \sigma(L_{\mathbb{R}^d,p}) \) by \( \lambda \in \sigma(A_{\Omega,p}) \). The above inequality then implies the divergence of \( \| R(\lambda, A_{\Omega,p}) \|_{L(\Omega)} \) for \( \lambda \to \mu \). This yields \( \sigma(L_{\mathbb{R}^d,p}) \subseteq \sigma(A_{\Omega,p}) \). So we have shown
\[
-\frac{\text{tr}(B)}{p} \leq \omega(A_{\Omega,p}) \leq \omega_0(T_{\Omega,p}) \leq -\frac{\text{tr}(B)}{p},
\]
which implies the equality stated in Theorem 5.5.2.

**Proposition 5.5.6.** For \( \Re(\lambda) > -\frac{\text{tr}(B)}{p} \) we have
\[
\| R(\lambda, L_{\mathbb{R}^d,p}) \|_{L(\mathbb{R}^d)} \leq \| R(\lambda, A_{\Omega,p}) \|_{L(\Omega)}.\]

**Proof.** Let \( g \in C_c^\infty(\mathbb{R}^d \setminus \{0\}) \). Then \( g \in D(A_k) \) for \( k \) large enough. It follows
\[
\| A_k g - L_{\mathbb{R}^d,p} g \|_p = \left\| \frac{1}{k^2} \sum_{i,j=1}^d q_{ij} D_i D_j g \right\|_p \longrightarrow 0
\]
for \( k \to \infty \). Now we consider \( f = (\lambda - L)g \), \( g \in C_c^\infty(\mathbb{R}^d \setminus \{0\}) \). For \( k \) large enough we have \( f \in D(A_k) \). Then for \( n \geq k \) we get with the previous Lemma
\[
\| R(\lambda, A) f - R(\lambda, L_{\mathbb{R}^d,p}) f \|_p = \| R(\lambda, A)(\lambda - L_{\mathbb{R}^d,p}) g - g \|_p
= \| R(\lambda, A)((\lambda - A_n)g - (\lambda - A_n)g + (\lambda - L_{\mathbb{R}^d,p}) g - g \|_p
= \| R(\lambda, A_n)(A_n - L_{\mathbb{R}^d,p}) g \|_p \leq \| R(\lambda, A_{\Omega,p}) \|_{L(\Omega)} \| (A_n - L_{\mathbb{R}^d,p}) g \|_p,
\]
so \( R(\lambda, A_n)f - R(\lambda, L_{\mathbb{R}^d,p})f \) converges to 0 in \( L^p(\Omega) \). Then for each \( \varepsilon > 0 \) and \( f \in (\lambda - L_{\mathbb{R}^d,p})C_c^\infty(\mathbb{R}^d \setminus \{0\}) \) there exists a \( k \) with
\[
\| R(\lambda, L_{\mathbb{R}^d,p}) f \|_p \leq \| R(\lambda, A_k) f \|_p + \varepsilon \leq \| R(\lambda, A_{\Omega,p}) \|_{L(\Omega)} \| f \|_p + \varepsilon.
\]
Thus we have
\[
\| R(\lambda, L_{\mathbb{R}^d,p}) f \|_p \leq \| R(\lambda, A_{\Omega,p}) \|_{L(\Omega)} \| f \|_p
\]
for all \( f \in (\lambda - L_{\mathbb{R}^d,p})C_c^\infty(\mathbb{R}^d \setminus \{0\}) \). Since \( C_c^\infty(\mathbb{R}^d \setminus \{0\}) \) is a core for \( L_{\mathbb{R}^d,p} \) by Lemma 5.5.4, the assertion follows. \( \square \)
References (Article 5)


References (Article 5)


6. Elliptic Operators with Unbounded Drift Coefficients on Domains

by Robert Haller-Dintelmann and Julian Wiedl

Abstract
We show that second-order elliptic differential operators, whose coefficients of the first order terms are growing not more than $|x| \log(|x|)$ at infinity, equipped with Dirichlet boundary conditions, are generators of consistent, positive $C_0$-semigroups on $L^p(\Omega)$, $1 < p < \infty$, for every open set $\Omega \subseteq \mathbb{R}^d$. Furthermore, when the domain is enlarged, the semigroups increase monotonously in the sense of positive operators, as in the case of the Dirichlet Laplacian.

6.1. Introduction
In this paper we study the differential operator
$$A u(x) = A_0 u(x) + L u(x) = \text{div}(a(x)\nabla u(x)) + b(x) \cdot \nabla u(x), \quad x \in \Omega.$$  
Here $\Omega \subseteq \mathbb{R}^d$ is an open set and the functions $a \in C_b^1(\Omega; \mathbb{R}^{d \times d})$ and $b \in W^{1,\infty}\text{loc}(\Omega; \mathbb{R}^d)$ satisfy the following hypotheses for some constants $\alpha > 0$ and $c \geq 0$.

(A1) $\xi^T a(x) \xi \geq \alpha |\xi|^2$ for all $\xi \in \mathbb{R}^d$ and almost all $x \in \Omega$,

(B1) $|b(x)| \leq c(1 \vee |x| \log(|x|))$ for almost all $x \in \Omega$,

(B2) $\text{div} b$ is bounded from below.

We show that suitable realisations of these operators generate positive, consistent $C_0$-semigroups on $L^p(\Omega)$ for $1 < p < \infty$ and that all these semigroups are dominated in the sense of positive operators by the corresponding semigroups on $\mathbb{R}^d$. Furthermore, a gradient estimate for the resolvent in $L^2(\Omega)$ is deduced.

In [HW05] we achieved this in the special case of Ornstein-Uhlenbeck operators where $A_0$ has constant coefficients and $b(x) = Bx$ for some matrix $B \in \mathbb{R}^{d \times d}$. In this note we show how these results can be generalised to the above setting. Since part of the arguments stay the same, we do not always give self-contained proofs, but sometimes refer to [HW05]. Also for references on earlier results on Ornstein-Uhlenbeck operators we refer to this article.

In the recent article [AMP06] W. Arendt, G. Metafune and D. Pallara study Schrödinger operators with unbounded drift coefficients. In contrast to this note, their focus lies on the interplay between the potential and the drift coefficient and on conditions that guarantee Gaussian estimates for the semigroups. Setting the potential equal to zero, some of our results can, at least implicitly, also be found in this article. Note, that although it is only formulated for the case of $\Omega = \mathbb{R}^d$, their proofs of the generation results carry over to general open sets $\Omega$, cf. [AMP06, Remark 5.8].

An alternative approach to operators with superlinearly growing drift coefficients on the whole space $\mathbb{R}^d$ was used by J. Prüss, A. Rhandi and R. Schnaubelt in [PRS05]. There a generation
result is shown and the domain of the operator is determined. Furthermore, the authors show by an example for \( d = 1 \) that, imposing a domain that contains at least \( C_b(\mathbb{R}) \cap C^2(\mathbb{R}) \), no generation result can be expected for some drift coefficient \( b \) that grows like \( |x|^{1+\varepsilon} \) at infinity even for arbitrarily small \( \varepsilon > 0 \).

Thus our growth assumption (B1) is a somehow natural restriction, even if it may be slightly weakened (cf. Remark 6.3.2).

Comparing our results in the special case \( \Omega = \mathbb{R}^d \) with the results in [PRS05], we weaken the hypotheses on the coefficients, allowing non-bounded divergence of \( b \) and eliminating a condition that links \( b \) to the first derivatives of \( a \), for the price of a less exact description of the domain of the operator.

If \( \Omega \) is not the whole space, the description of the domain of the operator seems to be an open problem, even for regular boundaries.

### 6.2. Main result

We define the realisation of \( A \) on \( L^2(\Omega) \) by

\[
D(A_{\Omega,2}) = H^1_0(\Omega) \cap \{ u \in H^2_{\text{loc}}(\Omega) : Au \in L^2(\Omega) \}, \quad A_{\Omega,2}u = Au
\]

and we set for \( 1 < p < \infty \)

\[
\omega_p := \text{essinf}_{x \in \Omega} \frac{\text{div}(b(x))}{p}
\]

which exists by (B2).

Now we can formulate our main result.

**Theorem 6.2.1.** Let \( \Omega \subseteq \mathbb{R}^d \) be open and (A1), (B1) and (B2) be satisfied. Then the operator \( A_{\Omega,2} \) generates a positive \( C_0 \)-semigroup \( (T_{\Omega,2}(t))_{t \geq 0} \) on \( L^2(\Omega) \) with \( \|T_{\Omega,2}(t)\| \leq e^{-\omega_p t} \) and for every \( \lambda > 0 \) we have the estimate

\[
\|\nabla R(\lambda, A_{\Omega,2} + \omega_2)\|_{L^1(\Omega)} \leq \frac{C}{\sqrt{\lambda}}.
\]

Moreover, there exists a family of consistent positive \( C_0 \)-semigroups \( (T_{\Omega,p}(t))_{t \geq 0} \) on \( L^p(\Omega) \) for \( 1 < p < \infty \) with \( \|T_{\Omega,p}(t)\| \leq e^{-\omega_p t} \) and for every \( \lambda > -\omega_p \) and every \( t \geq 0 \) we have the domination properties

\[
|T_{\Omega,p}(t)f| \leq T_{\mathbb{R}^d,p}(t)|\hat{f}|, \quad f \in L^p(\Omega),
\]

\[
|R(\lambda, A_{\Omega,p})f| \leq R(\lambda, A_{\mathbb{R}^d,p})|\hat{f}|, \quad f \in L^p(\Omega),
\]

where \( \hat{f} \) denotes the extension of \( f \) by 0 and \( A_{\Omega,p} \) is the generator of \( (T_{\Omega,p}(t))_{t \geq 0} \).

We give some comments on the strategy of the proof. Proceeding as in [HW05], we first show that \( A_{\Omega,2} \) is dissipative for arbitrary open sets \( \Omega \) (Proposition 6.3.1) and show surjectivity of \( \lambda - A_{\Omega,2} \) for some positive \( \lambda \) by approximating the domain from the interior with bounded regular open subsets \( \Omega_n \). On all these subsets the coefficients of our operator are bounded, so we get a sequence of solutions \( u_n \) on \( \Omega_n \), which can be shown to contain a subsequence that converges weakly to a solution of the elliptic problem on \( \Omega \) (Proposition 6.4.1). In fact, the whole sequence converges even strongly, as can be seen from the proof of [AMP06, Theorem 3.1], but this is not needed in the following.

In a second step we show, also analogously to [HW05], that the semigroups \( (T_{\Omega,2}(t))_{t \geq 0} \) generated by \( A_{\Omega,2} \) are positive and growing in the sense of positive operators when the domain is enlarged, thus giving us the domination result of Theorem 6.2.1 for \( p = 2 \) (Proposition 6.4.2).
6.3. Dissipativity of $A_{\Omega,p}$

In order to show dissipativity for all $1 < p < \infty$, we need the integration by parts formula

$$
\int_{\Omega} \varphi \overline{u} |u|^{p-2} \nabla u = - \int_{\Omega} \varphi |u|^{p-2} u \nabla \overline{u} - \frac{p-2}{2} \int_{\Omega} \varphi |u|^{p-2}(\overline{\nabla u} + u \nabla \overline{u}) - \int_{\Omega} \nabla \varphi |u|^p, \tag{6.3.1}
$$

that is valid for every Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ and for every $1 < p < \infty$, every $u \in W^{1,p}_0(\Omega)$ and every $\varphi \in W^{1,\infty}(\Omega)$. The proof follows the lines of [MS04, Theorem 3.1] and can be found in [Woo05, Proposition 5.2.2.], where it is formulated for $\varphi \in C_\infty(\mathbb{R}^d)$. It easily carries over to our more general functions $\varphi$.

Note, that for $p = 2$, this formula reduces to

$$
\int_{\Omega} \varphi \nabla \overline{u} = - \int_{\Omega} \varphi u \nabla \overline{u} - \int_{\Omega} \nabla \varphi |u|^2, \tag{6.3.2}
$$

which is easily verified by partial integration and valid for arbitrary open sets $\Omega \subseteq \mathbb{R}^d$.

**Proposition 6.3.1.** Let $p = 2$ and $\Omega \subseteq \mathbb{R}^d$ be open or let $p \in (1, \infty)$ and $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain. Then the operator $A$ given by

$$
D(A) = W^{1,p}_0(\Omega) \cap \{ f \in W^{2,p}_{\text{loc}}(\Omega) : Af \in L^p(\Omega) \}, \quad Au = Au + \omega_p u
$$

is dissipative.

**Proof.** Choose a cut-off function $\eta \in C_\infty(\mathbb{R})$ with $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$, $\eta(t) = 1$ for $t \in [-1, 1]$ and $\eta(t) = 0$ for $|t| \geq 2$. In order to balance the superlinear growth of $b$ we now put in contrast to [HW05] $\eta_m(x) = \eta(\log(|x|)/m)$ for $m \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Then for all $m \in \mathbb{N}$ and $u \in D(A)$ we have by (6.3.1) or (6.3.2), respectively

$$
\text{Re} \left( \int_{\Omega} L u \overline{u} |u|^{p-2} \eta_m \right) = \text{Re} \left( - \int_{\Omega} u b \nabla \overline{u} |u|^{p-2} \eta_m - \frac{p-2}{2} \int_{\Omega} |u|^{p-2}(\overline{\nabla u} + u \nabla \overline{u}) \eta_m - \int_{\Omega} \text{div}(\eta_m b) |u|^p \right).
$$

Since $b$ is real-valued we have $\text{Re}(u b \nabla \overline{u}) = \text{Re}(\overline{b} \nabla u)$, so we get

$$
\text{Re} \left( -(p-1) \int_{\Omega} L u \overline{u} |u|^{p-2} \eta_m \right) = - \int_{\Omega} \text{div} b |u|^p \eta_m - \int_{\Omega} b \nabla \eta_m |u|^p,
$$

which finally yields

$$
\text{Re} \left( \int_{\Omega} L u \overline{u} |u|^{p-2} \eta_m \right) = \frac{1}{p} \left( - \int_{\Omega} \text{div} b |u|^p \eta_m - \int_{\Omega} b \nabla \eta_m |u|^p \right).
$$
By dissipativity of \( A_0 \) and the definition of \( \omega_p \), this implies
\[
\text{Re} \left( \int_{\Omega} A u \overline{u} |u|^{p-2} \eta_m \right) \leq \int_{\Omega} \left( \frac{\omega_p - \text{div} b}{p} \right) |u|^p \eta_m - \frac{1}{p} \int_{\Omega} b \nabla \eta_m |u|^p \\
\leq -\frac{1}{p} \int_{\Omega} b \nabla \eta_m |u|^p.
\]

Now, we want to let \( m \to \infty \). For the left hand side this is easy due to \( A u \in L^p(\Omega) \) and the boundedness of \( \eta_m \). Concerning the integral on the right hand side, we note that
\[
\nabla \eta_m(x) = \eta' \left( \frac{\log(|x|)}{m} \right) \frac{1}{m|x|} \frac{x}{|x|},
\]
Since \( \eta' (\log(|x|)/m) = 0 \), whenever \( \log(|x|) \notin [m, 2m] \), the integrand converges pointwise to 0 as \( m \to \infty \). Furthermore,
\[
|\nabla \eta_m(x)| \leq \| \eta' \|_{\infty} \frac{1}{m|x|} 1_{\{\log(|x|)/2 \leq m \leq \log(|x|)\}} \leq C \frac{1}{1 \vee |x| \log(|x|)/2}
\]
and (B1) imply
\[
|b(x) \cdot \nabla \eta_m(x)| u(x)|^p \leq C \frac{1 \vee |x| \log(|x|) \log(|x|)/2}{1 \vee |x| \log(|x|)/2} |u(x)|^p \leq C |u(x)|^p.
\]
So by Lebesgue’s Theorem we end up with
\[
\int_{\Omega} A u \overline{u} |u|^{p-2} \leq -\frac{1}{p} \lim_{m \to \infty} \int_{\Omega} b \nabla \eta_m |u|^p = 0.
\]
\( \square \)

**Remark 6.3.2.** By putting \( \log[1](|x|) = \log(|x|) \) and \( \log[n+1](|x|) = \log(\log[n](|x|)) \) for \( n \geq 1 \) and setting
\[
\eta_m(x) = \eta \left( \frac{\log[n](|x|)}{m} \right)
\]
for some \( n \in \mathbb{N} \) in the above proof, we may also deal with a slightly stronger growth of the coefficient \( b \). In fact, for this choice of \( \eta_m \), we have
\[
\nabla \eta_m(x) = \eta' \left( \frac{\log[n](|x|)}{m} \right) \frac{1}{|x| \log(|x|) \log^{[2]}(|x|) \ldots \log^{[n-1]}(|x|) m|x|} \frac{x}{m|x|}.
\]
As above \( \nabla \eta_m(x) = 0 \) whenever \( \log[n](|x|) \notin [m, 2m] \), consequently
\[
|\nabla \eta_m(x)| \leq C \frac{1 \vee |x| \log(|x|) \log^{[2]}(|x|) \ldots \log^{[n]}(|x|)}{1 \vee |x| \log(|x|) \log^{[2]}(|x|) \ldots \log^{[n]}(|x|)}
\]
in this case, and hypotheses (B1) can be weakened to
(B1’) \( |b(x)| \leq c(1 \vee |x| \log(|x|) \log^{[2]}(|x|) \ldots \log^{[n]}(|x|)) \) for some \( n \in \mathbb{N} \) and for almost all \( x \in \Omega \).
6.4. The case $p = 2$

The next goal is a generation result for the operator $A_{\Omega,2}$. In view of Proposition 6.3.1, it suffices to show the following statement.

**Proposition 6.4.1.** For every $\lambda > 0$ the operator $\lambda - \omega_2 - A_{\Omega,2}$ is surjective.

**Proof.** Let $\lambda > 0$ and $f \in L^2(\Omega)$ be given. Proceeding as in [HW05], by [DL90, II.4, Lemma 1] we choose a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of bounded open subsets of $\Omega$ with smooth boundaries, such that $\Omega_n \subseteq \Omega_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. Note, that in [DL90] the sets $\Omega_n$ are constructed in such a way that every $\Omega_n$ has only a finite number of connected components, so we get unique solutions $u_n \in H^2(\Omega_n) \cap H_0^1(\Omega_n)$ of

$$\begin{cases}
\lambda u_n - \omega_2 u_n - A_{\Omega_n,2} u_n = f|_{\Omega_n} & \text{in } \Omega_n, \\
u_n = 0 & \text{on } \partial \Omega_n.
\end{cases}$$

Since $A_{\Omega_n,2} + \omega_2$ is dissipative by Proposition 6.3.1 we get the estimate

$$\|u_n\|_{L^2(\Omega_n)} \leq \frac{1}{\lambda} \|f\|_{L^2(\Omega)} \quad (6.4.1)$$

independently from $n \in \mathbb{N}$.

In order to bound $\|\nabla u_n\|_{L^2(\Omega_n)}$ analogously to [HW05, Lemma 3.6], we suppose by the same argument as therein, $u_n$ to be real-valued and we estimate by hypotheses (A1)

$$\|\nabla u_n\|_{L^2(\Omega_n)}^2 \leq C \int_{\Omega_n} \nabla u_n \cdot \nabla u_n = -C \int_{\Omega_n} A_0 u_n u_n.$$

Following the proof of [HW05, Lemma 3.6] this implies

$$\|\nabla u_n\|_{L^2(\Omega_n)}^2 \leq C \left( \int_{\Omega_n} (\lambda - \omega_2 - A_{\Omega_n,2}) u_n u_n - \lambda \int_{\Omega_n} u_n^2 - \int_{\Omega_n} \left( \frac{\text{div} b}{2} - \omega_2 \right) u_n^2 \right)$$

$$\leq C (\|f\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega_n)} + \lambda \|u_n\|_{L^2(\Omega_n)}^2) \leq \frac{C}{\lambda} \|f\|_{L^2(\Omega)}^2$$

by (6.4.1). Thus we have shown

$$\|\nabla u_n\|_{L^2(\Omega_n)} \leq \frac{C}{\sqrt{\lambda}} \|f\|_{L^2(\Omega)} \quad (6.4.2)$$

independently from $n \in \mathbb{N}$. Consequently, there is a subsequence of $(u_n)_{n \in \mathbb{N}}$ in $H_0^1(\Omega)$ that converges weakly to some $u$. As in the proof of [HW05, Proposition 3.7] one sees, that $u \in D(A_{\Omega,2})$ and $(\lambda - \omega_2 - A_{\Omega,2}) u = f$.

Therefore, by the Lumer-Phillips Theorem $A_{\Omega,2}$ generates a $C_0$-semigroup on $L^2(\Omega)$, that will be denoted by $(T_{\Omega,2}(t))_{t \geq 0}$ in the following.

**Proposition 6.4.2.** The semigroup $(T_{\Omega,2}(t))_{t \geq 0}$ is positive and if $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ are domains with $\Omega_1 \subseteq \Omega_2$, then for every $\lambda > -\omega_2$, every $t \geq 0$ and every function $f \in L^2(\Omega_1)$

(a) $|T_{\Omega_1,2}(t)f| \leq T_{\Omega_2,2}(t)|f|$,

(b) $|R(\lambda, A_{\Omega_1,2})f| \leq R(\lambda, A_{\Omega_2,2})|f|$,

where $\tilde{f}$ denotes the trivial extension of $f$ to $\Omega_2$. 

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Proof. The proofs in [HW05, Section 4] can be copied almost completely, replacing again, as we did in the proof of dissipativity, the linear scaling in the cut-off functions \( \eta_m \) by a logarithmic one. Only at the end of Lemma 6.1 we have to give an extra argument:

Let \( u \in D(A_{\Omega,2}) \) and \( v \in H^1(\Omega) \cap \{ f \in H^2_{\text{loc}}(\Omega) : Af \in L^2(\Omega) \} \) be real-valued functions with \( v \geq 0 \), that fulfill \( (\lambda - A)u \leq (\lambda - A)v \) a.e. for some \( \lambda > -\omega_2 \). Then we get as in the proof of [HW05, Lemma 4.1] the inequality

\[
\int_\Omega [\nabla (u - v)^+]^T a \nabla (u - v)^+ \eta_m + \int_\Omega [\nabla \eta_m]^T a \nabla (u - v)^+ (u - v)^+ \\
+ \int_\Omega \left( \frac{\text{div} b}{2} + \lambda \right) [(u - v)^+]^2 \eta_m + \frac{1}{2} \int_\Omega b \nabla \eta_m [(u - v)^+]^2 \leq 0,
\]

where \( \eta_m \) is defined as in the proof of Proposition 6.3.1. By positivity of the third integral and letting \( m \to \infty \), this yields

\[
\int_\Omega [\nabla (u - v)^+]^T a \nabla (u - v)^+ \leq 0.
\]

Thus by the ellipticity condition (A1) we have \( \nabla (u - v)^+ = 0 \). As \( (u - v)^+ \) is known to be in \( H^0_0(\Omega) \), this yields \( u \leq v \) a.e. \( \square \)

### 6.5. Proof of Theorem 6.2.1

For \( p = 2 \) the proof of Theorem 6.2.1 is contained in the last section. Note that the estimate for the gradient of the resolvent follows directly from (6.4.2), since the resolvent is the weak limit of a subsequence of \((u_n)_{n \in \mathbb{N}}\). In order to prove the second part of Theorem 6.2.1 we begin with the case \( \Omega = \mathbb{R}^d \). For \( 1 < p < \infty \) we define the realisation

\[
D(A_{\mathbb{R}^d,p}) = \{ f \in W_{\text{loc}}^{2,p}(\mathbb{R}^d) : Au \in L^p(\mathbb{R}^d) \}, \quad A_{\mathbb{R}^d,p}u = Au.
\]

**Proposition 6.5.1.** The operator \( A_{\mathbb{R}^d,p} \) generates a \( C_0 \)-semigroup \((T_{\mathbb{R}^d,p}(t))_{t \geq 0}\) on \( L^p(\mathbb{R}^d) \) with \( \|T_{\mathbb{R}^d,p}(t)\| \leq e^{-\omega t} \) that is consistent with \((T_{\mathbb{R}^d,2}(t))_{t \geq 0}\).

**Proof.** By Proposition 6.3.1 the operator \( A := A_{\mathbb{R}^d,p} + \omega \) is dissipative. In order to show surjectivity of \( \lambda - A \) for some \( \lambda > 0 \), we approximate solutions in \( \mathbb{R}^d \) again by solutions in bounded, regular domains \( \Omega_n \) (e.g. balls) as in the proof of Proposition 6.4.1. Since the coefficients of \( A \) are bounded on \( \Omega_n \) for every \( n \in \mathbb{N} \) we again get unique solutions \( u_n \in W^{2,p}(\Omega_n) \cap W^{1,p}_0(\Omega_n) \) of

\[
\begin{cases}
\lambda u_n - Au_n = f|_{\Omega_n} \text{ in } \Omega_n, \\
u_n = 0 \text{ on } \partial \Omega_n,
\end{cases}
\]

that by Proposition 6.3.1 obey the estimate

\[
\|u_n\|_{L^p(\Omega_n)} \leq \frac{1}{\lambda} \|f\|_{L^p(\Omega)}
\]

independently from \( n \in \mathbb{N} \). Extending all these solutions to \( \mathbb{R}^d \) by zero, we get a bounded sequence \((u_n)_{n \in \mathbb{N}}\) in \( L^p(\mathbb{R}^d) \), from which we may extract a weakly convergent subsequence with weak limit \( u \in L^p(\mathbb{R}^d) \). Redoing the proof of [HW05, Proposition 3.7] in the \( L^p \)-setting we see, that \( u \in D(A_{\mathbb{R}^d,p}) \) and \( (\lambda - A)u = f \). Thus \( A \) generates a contraction semigroup and the rescaled semigroup \((T_{\mathbb{R}^d,p}(t))_{t \geq 0}\) satisfies the stated norm estimate.
Finally we have to prove consistency. We will do this for the resolvents of \( A_{\mathbb{R}^d, p} \) and \( A_{\mathbb{R}^d, 2} \) in the case of large \( \lambda \). Then consistency of the semigroups follows by

\[
T_{\mathbb{R}^d, 2}(t)f = \lim_{n \to \infty} \left[ \frac{n}{t} R \left( \frac{n}{t}, A_{\mathbb{R}^d, 2} \right) f \right]^n = \lim_{n \to \infty} \left[ \frac{n}{t} R \left( \frac{n}{t}, A_{\mathbb{R}^d, p} \right) f \right]^n = T_{\mathbb{R}^d, p}(t)f
\]

for all \( f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \). Now let \( \lambda > \max\{-\omega_2, -\omega_1\} \) and \( f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \) be given. Since the coefficients of \( A \) are bounded on \( \Omega_n \) for every \( n \in \mathbb{N} \), we have consistency of \( R(\lambda, A_{\Omega_n, p}) \) and \( R(\lambda, A_{\Omega_n, 2}) \). Thus

\[
u_n = R(\lambda, A_{\Omega_n, p})f|_{\Omega_n} = R(\lambda, A_{\Omega_n, 2})f|_{\Omega_n} \in L^2(\Omega_n) \cap L^p(\Omega_n)
\]

for every \( n \in \mathbb{N} \). By the above construction, there is a weakly convergent subsequence of \((u_n)_{n \in \mathbb{N}}\) in \( L^p(\mathbb{R}^d) \) with weak limit \( u = R(\lambda, A_{\mathbb{R}^d, p})f \). Note that by dissipativity this weak limit is independent of the chosen subsequence. On the other hand, by the proof of Proposition 6.4.1, we may extract from this subsequence another subsequence, that converges weakly in \( L^2(\mathbb{R}^d) \) to \( R(\lambda, A_{\mathbb{R}^d, 2})f \). Denoting this final subsequence again by \((u_n)_{n \in \mathbb{N}}\) we obtain for every \( \varphi \in C^\infty_c(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} R(\lambda, A_{\mathbb{R}^d, p})f \varphi = \lim_{n \to \infty} \int_{\mathbb{R}^d} u_n \varphi = \int_{\mathbb{R}^d} R(\lambda, A_{\mathbb{R}^d, 2})f \varphi.
\]

The desired consistency of the resolvents follows. \( \square \)

Now, let \( \Omega \subseteq \mathbb{R}^d \) be open and \( f \in L^p(\Omega) \cap L^2(\Omega) \). Then for every \( t \geq 0 \) we have by Proposition 6.4.2 and consistency

\[
\|T_{\Omega, 2}(t)f\|_{L^p(\Omega)} \leq \|T_{\mathbb{R}^d, 2}(t)\tilde{f}\|_{L^p(\mathbb{R}^d)} = \|T_{\mathbb{R}^d, p}(t)\tilde{f}\|_{L^p(\mathbb{R}^d)} \leq e^{-\omega_p t} \|f\|_{L^p(\Omega)}.
\]

Thus as in [HW05, Section 2] we may extend the operators \( T_{\Omega, 2}(t) \) continuously to operators \( T_{\Omega, p}(t) \) on \( L^p(\Omega) \), obtaining \( C_0 \)-semigroups, that are consistent with \( (T_{\Omega, 2}(t))_{t \geq 0} \) by construction and fulfill \( \|T_{\Omega, p}(t)\| \leq e^{-\omega_p t} \) by the above calculation.

Furthermore, we have for \( t \geq 0 \) and \( f \in L^2(\Omega) \cap L^p(\Omega) \)

\[
T_{\mathbb{R}^d, p}(t)\tilde{f} - |T_{\Omega, p}(t)f| = T_{\mathbb{R}^d, 2}(t)\tilde{f} - |T_{\Omega, 2}(t)f| \geq 0
\]

and since the cone of all non-negative functions is closed in \( L^p(\Omega) \) we have

\[
|T_{\Omega, p}(t)f| \leq T_{\mathbb{R}^d, p}(t)\tilde{f}
\]

for every \( f \in L^p(\Omega) \) and \( t \geq 0 \). By the same closedness argument we get positivity of \( (T_{\Omega, p}(t))_{t \geq 0} \) from

\[
T_{\Omega, p}(t)f = T_{\Omega, 2}(t)f \geq 0
\]

for positive functions \( f \in L^p(\Omega) \cap L^2(\Omega) \).

Finally, for every \( \lambda > -\omega_p \) and every \( f \in L^p(\Omega) \) we have

\[
|R(\lambda, A_{\Omega, p})f| = \left| \int_0^\infty e^{-\lambda t} T_{\Omega, p}(t)f \, dt \right| \leq \int_0^\infty e^{-\lambda t} T_{\mathbb{R}^d, p}(t)\tilde{f} \, dt = R(\lambda, A_{\mathbb{R}^d, p})\tilde{f},
\]

completing the proof of Theorem 6.2.1.

**Remark 6.5.2.** It is worthwhile to observe that the semigroups generated by \( A_{\Omega, p} \) constructed in the proof of Theorem 6.2.1 coincide with the corresponding semigroups from [AMP06] (whose generators will be denoted by \( A_{\Omega, p} \) in the following). Since both families are consistent it is enough to show this for the resolvents in the case \( p = 2 \).
Let $f \in L^2(\Omega)$ be given. In [AMP06] $R(\lambda, \tilde{A}_{\Omega,2})f$ is obtained as the strong limit of $R(\lambda, \hat{A}_{\Omega,2})f$. Now $\Omega_n$ is bounded and we have $R(\lambda, A_{\Omega_n,2}) = R(\lambda, \tilde{A}_{\Omega_n,2})$. So every subsequence of this sequence converges weakly to the strong limit, which implies $R(\lambda, A_{\Omega,2})f = R(\lambda, \tilde{A}_{\Omega,2})f$.

It is also of interest, that the strong convergence of the resolvents of $A_{\Omega_n,2}$ implies a convergence result for the corresponding semigroups, see [AMP06, Proposition 3.6].
References (Article 6)


7. Expansions in Generalized Eigenfunctions of the Weighted Laplacian on Star-Shaped Networks

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Abstract

We are interested in evolution phenomena on star-shaped networks composed of \( n \) semi-infinite branches which are connected at their origins. Using spectral theory we construct the equivalent of the Fourier transform, which diagonalizes the weighted Laplacian on the \( n \)-star. It is designed for the construction of explicit solution formulas to various evolution equations such as the heat, wave or the Klein-Gordon equation with different leading coefficients on the branches.

7.1. Introduction

We study the foundations for the understanding of evolution phenomena on star-shaped networks composed of \( n \) semi-infinite branches which are connected at their origins. To this end, we construct the equivalent of the Fourier transform which diagonalizes the weighted Laplacian on the \( n \)-star, using spectral theory. This allows us to formulate a functional calculus for the weighted Laplacian, designed to construct explicit solution formulas to various evolution equations such as the heat, wave or the Klein-Gordon equation with different leading coefficients on the branches. The model of the \( n \)-star should lead to a comprehension of the phenomena happening locally in time and space near the ramification nodes of more complicated networks. The investigation of evolution equations on networks starts with G. Lumer [Lum80] and subsequent papers. See [AM94, AMvBN01, Are06] and the references mentioned therein.

Let \( N_1, \ldots, N_n \) be \( n \) disjoint copies of \((0; +\infty)\) \((n \in \mathbb{N}, n \geq 2)\) and \( c_k > 0 \), for \( k \in \{1, \ldots, n\} \).

A vector \((u_1, \ldots, u_n)\) of functions \( u_k : \overline{N_k} \to \mathbb{C} \) is said to satisfy the transmission conditions

\[
\begin{align*}
(T_0), \quad & u_i(0) = u_k(0) \text{ for all } (i, k) \in \{1, \ldots, n\}^2, \\
(T_1), \quad & \sum_{k=1}^{n} c_k^2 \partial_x u_k(0^+) = 0.
\end{align*}
\]

A vector \((u_k)_{k=1}^{n} \) satisfying \((T_0)\) can also be viewed as a function on \( N := \bigcup_{k=1}^{n} \overline{N_k} \), where the \( n \) boundary points corresponding to \( 0 \in \overline{N_k} \) are identified. This domain is called a star-shaped network or \( n \)-star with the branches \( N_1, \ldots, N_n \).
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In this paper, we study the weighted Laplacian submitted to \((T_0)\) and \((T_1)\):

\[
D(A) := \left\{ (u_k) \in \prod_{k=1}^{n} H^2(N_k) \mid (u_k) \text{ satisfies } (T_0) \text{ and } (T_1) \right\},
\]

\[
A(u_k) := (-c_k^2 \cdot \partial_x^2 u_k)_{k=1,...,n}.
\]

This operator can be inserted for example in the abstract wave equation

\[
\begin{align*}
\ddot{u}(t) + Au(t) &= 0, \\
u(0) &= u_0, \quad \dot{u}(0) = v_0,
\end{align*}
\]

which means in concrete terms:

\[
\begin{align*}
&\frac{\partial^2}{\partial t^2} - c_k^2 \frac{\partial^2}{\partial x^2} u_k(t, x) = 0, \quad \forall \ k \in \{1, \ldots, n\}, \\
u_i(t, 0) = u_k(t, 0), \quad \forall \ (i, k) \in \{1, \ldots, n\}^2, \\
&\sum_{k=1}^{n} c_k^2 \partial_x u_k(t, 0^+) = 0, \\
u_k(0, x) &= u_k^0(x), \quad \forall \ k \in \{1, \ldots, n\}, \\
\partial_t u_k(0, x) &= v_k^0(x), \quad \forall \ k \in \{1, \ldots, n\}
\end{align*}
\]

for \(x, t \geq 0\), where \(u_0 = (u_k^0)_{k=1,\ldots,n}, v_0 = (v_k^0)_{k=1,\ldots,n}\) and \(u(t) = (u_k(t, \cdot))_{k=1,\ldots,n}\).

The operator \(A\) is self-adjoint, its spectrum is \([0; +\infty)\) and has multiplicity \(n\) (in the sense of ordered spectral representations, see Definition XII.3.15, p. 1216 of [DS63]). The analytical core of this paper is a representation of the kernel of the resolvent of \(A\) in terms of a special choice of a family of \(n\) generalized eigenfunctions parameterized by \(\lambda \in [0; +\infty)\).

After having proved a limiting absorption principle for the resolvent, we insert \(A\) in Stone’s formula to obtain a representation of the resolution of the identity of \(A\) in terms of the generalized eigenfunctions. This classical procedure (see for example [AM96]) should lead to an expansion formula for functions in \(H = \prod_{k=1}^{n} L^2(N_k)\) in terms of the family of generalized eigenfunctions.

We observe that the transition from the formula for the resolution of the identity to an expansion formula involving a generalized Fourier transform, which diagonalizes \(A\), is not straightforward in the case of the \(n\)-star. This comes from the fact that the resolvent kernel, which is defined on \(N \times N\), changes its structure when crossing the \(n\) diagonals of \(N_k \times N_k\), \(k = 1, \ldots, n\). These diagonals cut \(N \times N\) into \(n\) connected pieces in accordance with the structure of the resolvent. Our special choice of the generalized eigenfunctions allows us to recombine the inner integral of the formula for the resolution of the identity across the diagonals of \(N_k \times N_k\) to an integral over all of \(N\), furnishing the desired generalized Fourier transformation \(V\) as well as its left inverse \(Z\). It is not obvious, whether this recombination is possible for all choices of generalized eigenfunctions, although theoretical results imply that an expansion in generalized eigenfunctions always exists [Ber68, PSW89]. Now, \(V\) can be extended to an isometry on \(H\), which diagonalizes \(A\), and an explicit functional calculus for \(A\) can be given. We plan to give explicit expressions for the solutions of evolution equations like the weighted wave, heat and Klein-Gordon equations on the \(n\)-star and to derive results on their qualitative behaviour in a subsequent paper.

Such expressions can be obtained (at least formally) also from representations of the resolution of the identity which are not recombined to Fourier-type transformations. But these expressions would be sums of terms with very poor regularity although their sum, representing the solution, is regular (like a decomposition of a \(C^\infty\)-function by multiplying it with characteristic functions on sub-domains). These artificial singularities are totally undesirable for
any kind of investigations. They occur for example in [DL93], a pioneering paper of theoretical physics explaining the phenomenon of advanced transmission of dispersive wave packets crossing a potential barrier. The authors obtain a solution formula using Laplace transform in time, but which splits up into irregular terms. They do not attempt to prove that their formula represents a solution of the original problem, which should be possible only in some very weak sense. But this (artificial) lack of regularity permits only to study the advanced transmission phenomenon for gaussian wave packets using a highly special method.

In [AMR04], the authors study the similar phenomenon of delayed reflection occurring at semi-infinite barriers. They construct an expansion in generalized eigenfunctions and thus avoid those artificial singularities. This expansion is used to define wave packets in frequency bands adapted to the transmission conditions. Thus it is possible to study the dependence of propagation patterns, in particular the delayed reflection, on the main frequency of the wave packets. In [AMR06] it is pointed out using similar methods, that classical causality is valid for nonlinear dispersive waves hitting a semi-infinite barrier. In [AMR03] a solution formula for the Klein-Gordon equation on the $n$-star but with one finite branch with an end with prescribed excitation is presented using Laplace transform in time. This result is not comparable with the present paper, because it does not concern an initial value problem.

There remains an unsatisfactory point in the present paper: our Fourier-type transformation $V$ is not a spectral representation of $A$ in the classical sense although it diagonalizes this operator: the natural norm on the range of $V$ making $V$ an isometry, as in the theorem of Plancherel, is not just a weighted $L^2$-norm on some measure space. This is due to the fact that the back transformation $Z$ has a different expression on each branch, and this is caused by the ramification of the domain.

It is not clear to us how one could find a family of generalized eigenfunctions leading to a spectral representation of $A$. The existing general literature on expansions in generalized eigenfunctions ([Ber68, PSW89, Wei80] for example) does not seem to be helpful for this kind of problem: their constructions start from an abstractly given spectral representation. But in concrete cases you do not have an explicit formula for it at the beginning.

In [vBL05] the relation of the eigenvalues of the Laplacian in a $L^\infty$—setting on infinite, locally finite networks to the adjacency operator of the network is studied. The question of the completeness of the corresponding eigenfunctions, viewed as generalized eigenfunctions in an $L^2$—setting, could be asked. The $n$-star we consider is a particular case of the geometry studied by J. von Below and the completeness of the eigenfunctions is established in a way. In a recent paper ([KS06]), the authors consider general networks with semi-infinite ends. They give a construction to compute some generalized eigenfunctions from the coefficients of the transmission conditions (scattering matrix). The eigenvalues of the associated Laplacian are the poles of the scattering matrix and their asymptotic behaviour is studied. But no attempt is made to show the completeness of a given family of generalized eigenfunctions. Spectral theory for the Laplacian on finite networks has been studied since the 1980ies for example by J.P. Roth, J.v. Below, S. Nicaise, F. Ali Mehmeti (see [AM94]).

Natural perspectives for our expansion result are investigations on the qualitative behaviour of solutions of evolution equations on the $n$-star. For the weighted heat equation on the $n$-star, our expansion permits to prove Gaussian estimates (this feature shall be treated in a subsequent paper). For bounded networks and variable coefficients this has already been proved by D. Mugnolo ([Mug05]) using different methods. In [KS05] the transport operator is considered on finite networks. The connection between the spectrum of the adjacency matrix of the network and the (discrete) spectrum of the transport operator is established. By adding semi-infinite branches to the finite network, continuous parts of the spectrum and generalized eigenfunctions might appear.
Many results have been obtained in spectral theory for elliptic operators on various types of unbounded domains in \( \mathbb{R}^n \). Using the existing results on stratified bands [CD95] for example, one could reduce the spectral analysis of the Laplacian on networks of bands locally near the nodes to the case of the \( n \)-star. Time asymptotics for the associated evolution equations have also been studied extensively. For the Klein-Gordon equation on the \( n \)-star we conjecture that the maximum of the absolute value of the solutions decays as \( t^{-1/2} \) when \( t \) tends to infinity as on the real line. For two branches with potential step this has been already proved using generalized eigenfunctions in [AM04]. An example for a three-dimensional coupled domain with singularities is treated in [AMMM97]. See also the other literature mentioned therein and in [AM96].

\section*{7.2. Data and functional analytic framework}

Let us introduce some notation which will be used throughout the rest of the paper:

- **Domain and functions:** Let \( N_1, \ldots, N_n \) be \( n \) disjoint sets identified with \((0; +\infty)\) \((n \in \mathbb{N}, n \geq 2)\) and put \( N := \bigcup_{k=1}^{n} N_k \). Furthermore, we write \([a, b]_{N_k}\) for the interval \([a, b]\) in the branch \( N_k \). For the notation of functions two viewpoints are used:
  - functions \( f \) on the object \( N \) taking their values in \( \mathbb{R} \) and \( f_k \) is then the restriction of \( f \) to \( N_k \).
  - \( n \)-tuples of functions on the branches \( N_k, k = 1, \ldots, n \); then sometimes we write \( f = (f_1, \ldots, f_n) \).

- **Transmission conditions:**
  \((T_0)\): \( (u_k)_{k=1, \ldots, n} \in \prod_{k=1}^{n} C^0(N_k) \) satisfies \( u_i(0) = u_k(0), \ \forall \ (i, k) \in \{1, \ldots, n\}^2 \).
  \((T_1)\): \( (u_k)_{k=1, \ldots, n} \in \prod_{k=1}^{n} C^1(N_k) \) satisfies \( \sum_{k=1}^{n} c_k^2 \cdot \partial_x u_k(0^+) = 0 \).

- **Definition of the operator:** Define the real Hilbert space \( H = \prod_{k=1}^{n} L^2(N_k) \) with scalar product \((\langle u_k, v_k \rangle)_H = \sum_{k=1}^{n} \langle u_k, v_k \rangle_{L^2(N_k)}\)
  and the operator \( A : D(A) \to H \) by

\[
D(A) = \left\{ (u_k) \in \prod_{k=1}^{n} H^2(N_k) \mid (u_k) \text{ satisfies } (T_0) \text{ and } (T_1) \right\},
\]

\[
A(u_k) = (A_k u_k)_{k=1, \ldots, n} = (-c_k^2 \cdot \partial_x^2 u_k)_{k=1, \ldots, n}.
\]

Note that, if \( c_k = 1 \) for every \( k \in \{1, \ldots, n\} \), \( A \) is the Laplacian in the sense of the existing literature.

- **Notation for the resolvent:** The resolvent of an operator \( T \) is denoted by \( R, \) i.e. \( R(z, T) = (zI - T)^{-1} \) for \( z \in \rho(T) \).

\textbf{Proposition 7.2.1} (spectrum of \( A \)). \textit{The operator } \( A : D(A) \to H \) \textit{defined above is self-adjoint and satisfies } \( \sigma(A) = [0; +\infty) \).

\textit{Proof.} Simple adaptation of the proof of Lemma 1.1.5 in [AM96]. \qed
7.3. Expansion in generalized eigenfunctions

The aim of this section is to find an explicit expression for the kernel of the resolvent of the operator \( A \) on the star-shaped network defined in the previous section.

**Definition 7.3.1** (generalized eigenfunction). Let \( \lambda \in \mathbb{C} \) be fixed. An element \( f \in \prod_{k=1}^{n} C^\infty(\overline{N_k}) \) is called generalized eigenfunction of \( A \) if it satisfies \((T_0), (T_1)\) and the formal differential expression \( Af = \lambda f \).

**Proposition 7.3.2** (an expression of the resolvent). Let \( \lambda \in \mathbb{C} \) be fixed. Let \( \text{Im}(\lambda) \neq 0 \) and \( e_1^\lambda, e_2^\lambda \) be generalized eigenfunctions of \( A \) such that the Wronskian \( w_1^\lambda(x) \) satisfies for every \( x \in N \)

\[
u_1^\lambda(x) = \det W(e_1^\lambda(x), e_2^\lambda(x)) = e_1^\lambda(x) \cdot (e_2^\lambda)'(x) - (e_1^\lambda)'(x) \cdot e_2^\lambda(x) \neq 0.
\]

If for some \( k \in \{1, \ldots, n\} \) we have \( e_1^\lambda|_{N_m} \in H^2(N_m) \) for all \( m \neq k \) and \( e_2^\lambda|_{N_k} \in H^2(N_k) \), then we have for any \( f \in H, \lambda \in \rho(A) \) and \( x \in N_k \)

\[
[R(\lambda, A)f](x) = \frac{1}{\nu_1^\lambda(w_1^\lambda, w_2^\lambda)} \left[ \int_{[x; +\infty)_{N_k}} e_1^\lambda(x)e_2^\lambda(x') f(x') \, dx' \right. \\
- \left. \int_{N\setminus[x; +\infty)_{N_k}} e_2^\lambda(x)e_1^\lambda(x') f(x') \, dx' \right].
\]

Note that by integral over \( N \), we mean the sum of the integrals over \( N_k, k = 1, \ldots, n \).

**Proof.** The arguments are the same as in the proof of Theorem 1.3.4 of [AM96] (see also [AM04]) and the calculations are analogous. The integration by parts is replaced here by the Green formula for the star-shaped network that is given in the next lemma.

**Lemma 7.3.3** (Green’s formula on the star-shaped network with \( n \) semi-infinite branches). Denote by \( V_{a_1, \ldots, a_n} \) the subset of the network \( N \) defined by

\[ V_{a_1, \ldots, a_n} = \{ x \in N \mid x \in [0; a_k), \text{ where } k \text{ is the index such that } x \in \overline{N_k} \}. \]

Then \( u, v \in D(A) \) implies

\[
\int_{V_{a_1, \ldots, a_n}} u''(x)v(x) \, dx = \int_{V_{a_1, \ldots, a_n}} u(x)v''(x) \, dx - \sum_{k=1}^{n} u(a_k)v'(a_k) + \sum_{k=1}^{n} u'(a_k)v(a_k).
\]

**Proof.** Two successive integrations by parts are used and since both \( u \) and \( v \) belong to \( D(A) \), they both satisfy the transmission conditions \((T_0)\) and \((T_1)\). So

\[
\sum_{k=1}^{n} u_k(0)v_k'(0) = u_1(0)\sum_{k=1}^{n} v_k'(0) = 0.
\]

Idem for \( \sum_{k=1}^{n} u_k'(0)v_k(0) \).

**Definition 7.3.4** (generalized eigenfunctions of \( A \)). For \( j \in \{1, \ldots, n\} \) let

\[
s_j := -c_j^{-1} \sum_{l \neq j} c_l, \quad d_{1,j} := (1 + s_j)/2 \quad \text{and} \quad d_{2,j} := (1 - s_j)/2.
\]
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The complex square root is chosen in such a way that $\sqrt{r\cdot e^{i\phi}} = \sqrt{r} e^{i\phi/2}$ with $r > 0$ and $
phi \in [-\pi; \pi]$. For $\lambda \in \mathbb{C}$ and $j, k \in \{1, \ldots, n\}$, $F_{\lambda}^{\pm,j} : N \to \mathbb{C}$ is defined for $x \in \overline{N}_j$ by $F_{\lambda}^{\pm,j}(x) := F_{\lambda,k}^{\pm,j}(x)$ with

$$F_{\lambda,k}^{\pm,j}(x) = d_{1,j} \cdot \exp(\pm i c_j^{-1} \sqrt{\lambda}x) + d_{2,j} \cdot \exp(\mp i c_j^{-1} \sqrt{\lambda}x),$$

$$F_{\lambda,k}^{\pm,j}(x) = \exp(\pm i c_j^{-1} \sqrt{\lambda}x),$$

for $k \neq j$.

**Remark 7.3.5.**
- $F_{\lambda}^{\pm,j}$ satisfies the transmission conditions $(T_0)$ and $(T_1)$.
- Formally it holds $AF_{\lambda}^{\pm,j} = \lambda F_{\lambda}^{\pm,j}$.
- Clearly $F_{\lambda}^{\pm,j}$ does not belong to $H$, thus it is not a classical eigenfunction.
- For $\text{Im}(\lambda) \neq 0$, the function $F_{\lambda,j}^{\pm,k}$, where the $+$-sign (respectively $-$-sign) is chosen if $\text{Im}(\lambda) > 0$ (respectively $\text{Im}(\lambda) < 0$), belongs to $H^2(N_k)$ for $k \neq j$. This feature is used in the formula for the resolvent of $A$.

**Definition 7.3.6** (kernel of the resolvent). For any $\lambda \in \mathbb{C}$, $j \in \{1, \ldots, n\}$ and $x \in \overline{N}_j$ we define

$$K(x, x', \lambda) = \begin{cases} \frac{1}{w(\lambda)} F_{\lambda,j}^{\pm,j}(x) F_{\lambda,j}^{\pm,j+1}(x'), & \text{for } x' \in \overline{N}_j, x' > x, \\ \frac{1}{w(\lambda)} F_{\lambda,j}^{\pm,j+1}(x) F_{\lambda,j}^{\pm,j}(x'), & \text{for } x' \in \overline{N}_k, k \neq j \text{ or } x' \in \overline{N}_j, x' < x, \end{cases}$$

where $w(\lambda) = \pm i \sqrt{\lambda} \cdot \sum_{j=1}^{n} c_j$. In the whole formula $+$ (respectively $-$) is chosen if $\text{Im}(\lambda) > 0$ (respectively $\text{Im}(\lambda) < 0$).

Here the index $j$ is to be understood modulo $n$, that is to say, if $j = n$, then $j + 1 = 1$.

Note that in particular, if $c_j = c$ for all $j \in \{1, \ldots, n\}$, then $w(\lambda) = \pm inc \sqrt{\lambda}$, for all $j \in \{1, \ldots, n\}$.

**Theorem 7.3.7** (expansion of the resolvent in the family $\{F_{\lambda,j}^{\pm,j}, j = 1, \ldots, n\}$). Let $f \in H$. Then, for $x \in N$ and $\lambda \in \rho(A)$

$$[R(\lambda, A)f](x) = \int_N K(x, x', \lambda)f(x') \, dx'.$$

**Proof.** In (7.3.1), the generalized eigenfunction $e_{\lambda}^{\pm}$ can be chosen to be $F_{\lambda}^{\pm,j}$. Then $e_{\lambda}^{\pm}$ can be $F_{\lambda}^{\pm,j}$ with any $l \neq j$ so we have chosen $j + 1$ to fix the formula. The choice has been done so that the integrands lie in $L^1(0, +\infty)$ (cf. the last item in Remark 7.3.5).

7.4. Application of Stone’s formula and limiting absorption principle

Let us first recall Stone’s formula (see Theorem XII.2.11 in [DS63]).

**Theorem 7.4.1** (Stone’s formula). Let $E$ be the resolution of the identity of a linear unbounded self-adjoint operator $T : D(T) \to H$ in a Hilbert space $H$ (i.e. $E(a,b) = 1_{(a,b)}(A)$ for $(a,b) \in \mathbb{R}^2$, $a < b$). Then, in the strong operator topology

$$h(T)E(a,b) = \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\alpha+\delta}^{\beta-\delta} \lambda h(\lambda)[R(\lambda - \epsilon i, T) - R(\lambda + \epsilon i, T)] \, d\lambda$$

for all $(a,b) \in \mathbb{R}^2$, $a < b$ and for any continuous scalar function $h$ defined on the real line.
To apply this formula we need to study the behaviour of the resolvent \( R(\lambda, A) \) for \( \lambda \) approaching the spectrum of \( A \).

**Theorem 7.4.2** (limiting absorption principle for \( A \)). For any \( (x, x') \in N^2 \) and \( (\lambda, \epsilon) \in (\mathbb{R}^+)^2 \), it holds with \( s_j, d_j \) as defined in Definition 7.3.4:

a) \( \lim_{\epsilon \to 0} K(x, x', \lambda - i \epsilon) = K(x, x', \lambda) \),

b) \( |K(x, x', \lambda - i \epsilon)| \leq M \cdot (\sqrt{\lambda})^{-1} \) with

\[
M = \max_{j \in \{1, \ldots, n\}} \left[ \max(1; |d_{1,j}| + |d_{2,j}|) \cdot \left( \sum_{j=1}^{n} c_j \right)^{-1} \right].
\]

**Proof.**  

a) The complex square root is, by definition, continuous on \( \{z \in \mathbb{C} \mid \text{Im}(z) \leq 0\} \) (cf. Definition 7.3.4), hence the continuity of \( K(x, x', \lambda) \) at real positive numbers \( \lambda \). (Note that \( x, x' \) are fixed parameters in this context.)

b) In concrete terms, the kernel is for \( \text{Im}(\mu) \leq 0 \) and \( x \in N_j \)

\[
K(x, x', \mu) = \frac{1}{w(\mu)} \begin{cases} 
 e^{-i \sqrt{\lambda} c_j^{-1} (x + c_j^{-1} x')}, & x' \in N_k, \ k \neq j, \\
 d_{2,j} e^{-i \sqrt{\lambda} c_j^{-1} (x - x')} + d_{1,j} e^{-i \sqrt{\lambda} c_j^{-1} (x + x')}, & x' \in N_j, \ x' < x, \\
 d_{2,j} e^{-i \sqrt{\lambda} c_j^{-1} (x - x')} + d_{1,j} e^{-i \sqrt{\lambda} c_j^{-1} (x + x')}, & x' \in N_j, \ x' > x.
\end{cases}
\]

Now

\[
\frac{1}{w(\mu)} = \left( \sum_{j=1}^{n} c_j \sqrt{|\lambda - i \epsilon|} \right)^{-1} = \left( \sum_{j=1}^{n} c_j \right)^{-1} (\lambda^2 + \epsilon^2)^{-1/4}
\]

\[\leq \left( \sum_{j=1}^{n} c_j \right)^{-1} \lambda^{-1/2}\]

for \( \mu = \lambda - i \epsilon, \lambda > 0, \epsilon \geq 0 \). Moreover, if \( x' < x \),

\[|e^{-i(\sqrt{\lambda}\epsilon)c_j^{-1}(x-x')}| = e^{\text{sgn}(\text{Im}(\lambda - i \epsilon))c_j^{-1}(x-x')} \leq 1,
\]

since \( \text{sgn}(\text{Im}(\sqrt{\lambda} - i \epsilon)) = \text{sgn}(\lambda - i \epsilon) \) (cf. Lemma 2.5.1 of [AM96], see also [AM04]). Idem for the other exponential terms. Hence the above estimate.

**Remark 7.4.3.** Note that, in particular, if \( c_j = c, j = 1, \ldots, n \), then \( M = c(n-1)/n \).

**Lemma 7.4.4.** For \( (x, x') \in N^2 \) and \( \lambda \in \mathbb{C} \), it holds \( K(x, x', \lambda) = K(x, x', \lambda) \).

**Proof.** The choice of the branch cut of the complex square root has been made such that \( \sqrt{\lambda} = \sqrt{\lambda} \) for all \( \lambda \in \mathbb{C} \).

This implies \( e^{i \sqrt{\lambda} x} = e^{i \sqrt{\lambda} x} = e^{-i \sqrt{\lambda} x} \) for all \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{R} \). Thus it holds

\[
F^{\lambda,j}_\chi(x) = F^{-\lambda,j}_\chi(x) \quad \text{and} \quad F^{\lambda,j}_\chi(x) = F^{\lambda,j}_\chi(x)
\]

for all \( \lambda \in \mathbb{C}, x \in N \) and \( j \in \{1, \ldots, n\} \). In the same way we have \( w(\lambda) = -w(\lambda) \). Observe, that switching from \( \lambda \) to \( \lambda \) the sign of the imaginary part is changing, so in the definition of \( K(x, x', \lambda) \) we have to take the other sign whenever there is a ±-sign in the formula. This gives the assertion.

\[\square\]
Proposition 7.4.5 (rewriting of the resolution of the identity of $A$).

Take $f \in H = \prod_{j=1}^{n} L^2(N_j)$, vanishing almost everywhere outside a compact set $B \subset N$ and let $-\infty < a < b < +\infty$. Then, for $x \in N$

$$(E(a, b)f)(x) = \text{Re} \left\{ \frac{1}{\pi} \int_{a}^{b} \prod_{j=1}^{n} \sum_{j=1}^{n} \sigma_j(\lambda, x) \cdot F^{\varepsilon, j+1}(x) \left( \int_{N} f(x') \cdot F^{\varepsilon, j}(x') \, dx' \right) \, d\lambda \right\},$$

where $E$ is the resolution of the identity of $A$ (cf. Theorem 7.4.1) and

$$\sigma_j(\lambda, x) := \frac{1}{\sqrt{\lambda}} \sigma_j(x), \text{ where } \sigma_j(x) := \frac{1}{N_j} (x) \cdot \frac{1}{C} \text{ for } j \in \{1, \ldots, n\}.$$

Here $C = (\sum_k c_k)$ and the index $j$ is to be understood modulo $n$, that is to say, if $j = n$, then $j + 1 = 1$.

Note that in particular if $c_j = c$ for all $j \in \{1, \ldots, n\}$, then $C = nc$, for all $j \in \{1, \ldots, n\}$.

Proof. The proof is analogous to that of Lemma 1.3.13 of [AM96] (see also [AM04]). Let in addition $g \in H$ be vanishing outside $B$. Then

$$(E(a, b)f, g)_{H}$$

$$= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [R(\lambda - \varepsilon i, A) - R(\lambda + \varepsilon i, A)] \, d\lambda f, g \right)_{H}$$

(7.4.1)

$$= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left( \int_{N} f(x') [K(\cdot, x', \lambda - i\varepsilon) - \overline{K(\cdot, x', \lambda - i\varepsilon)}] \, dx', g(\cdot) \right) \, d\lambda$$

(7.4.2)

$$= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left( \int_{N} f(x') 2i \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda$$

(7.4.3)

$$= \lim_{\delta \to 0^+} \frac{1}{\pi} \int_{a+\delta}^{b-\delta} \left( \int_{N} f(x') \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda$$

(7.4.4)

$$= \left( \frac{1}{\pi} \int_{a}^{b} \left( \int_{N} f(x') \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda \right)_{H}$$

(7.4.5)

$$= \left( \frac{1}{\pi} \int_{a}^{b} \left( \int_{N} f(x') \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda \right)_{H}$$

(7.4.6)

$$= \left( \frac{1}{\pi} \int_{a}^{b} \left( \int_{N} f(x') \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda \right)_{H}$$

(7.4.7)

$$= \left( \frac{1}{\pi} \int_{a}^{b} \left( \int_{N} f(x') \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda \right)_{H}$$

(7.4.8)

$$= \left( \frac{1}{\pi} \int_{a}^{b} \left( \int_{N} f(x') \text{ Im}(K(\cdot, x', \lambda - i\varepsilon)) \, dx', g(\cdot) \right) \, d\lambda \right)_{H}$$

(7.4.9)
7.4. Application of Stone’s formula and limiting absorption principle

\[ \int_{\mathcal{N}} \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{n} 1_{\mathcal{N}_{j}}(x) \Re \left[ \frac{1}{C} \left( 1_{\{x' \in \mathcal{N}_{j}, x' > x\}}(x') F_{\lambda}^{-j}(x) F_{\lambda}^{-j+1}(x') \right) \right] \, d\lambda g(x) \, dx \]

(7.4.10)

Here, the justifications for the equalities are the following:

(7.4.1): Stone’s formula (Theorem 7.4.1) applied with \( h(\lambda) \equiv 1 \).

(7.4.2): After applying the operator valued integral to \( f \), the two limits are in \( H \). So they commute with the scalar product in \( H \).

(7.4.3): \( (f, g)_{\mathcal{L}} \) is a continuous linear form on \( \mathcal{L}(H) \), and can therefore be commuted with the vector-valued integration.

(7.4.4): Theorem 7.3.7.

(7.4.5): Lemma 7.4.4.

(7.4.6): \( z - \overline{z} = 2i \cdot \text{Im} \, z \ \forall z \in \mathbb{C} \).

(7.4.7): Dominated convergence. Note that \( \text{supp } f, \text{supp } g \) and \( [a, b] \) are compact and use the limiting absorption principle (Theorem 7.4.2).

(7.4.8): Fubini.

(7.4.9): Definition 7.3.6.

(7.4.10): \( \text{Im}(z) = \text{Re}(z/i) \) for all \( z \in \mathbb{C} \). Note that, if \( \lambda \in \mathbb{R}^{-} \), then \( \lambda \in \rho(A) \) and thus the integrand in Stone’s formula is zero.

(7.4.11): Note that

\[
\begin{align*}
& \left( F_{\lambda,j}^{-j}(x) \right) \left( F_{\lambda,j}^{-j+1}(x') \right) = d_{2,j} e^{-ic_{j}^{-1} \sqrt{\lambda}(x-x')} + d_{1,j} e^{-ic_{j}^{-1} \sqrt{\lambda}(x+x')}, \\
& \left( F_{\lambda,j}^{-j+1}(x) \right) \left( F_{\lambda,j}^{-j}(x') \right) = d_{2,j} e^{-ic_{j}^{-1} \sqrt{\lambda}(x'-x)} + d_{1,j} e^{-ic_{j}^{-1} \sqrt{\lambda}(x+x')},
\end{align*}
\]

Since \( e^{-ic_{j}^{-1} \sqrt{\lambda}(x-x')} \) and \( e^{-ic_{j}^{-1} \sqrt{\lambda}(x'-x)} \) are conjugated for real \( \lambda \), both expressions have the same real part. Thus the integrals on \( \{x' \in \mathcal{N}_{j}, x' > x\} \) and its complement \( \mathcal{N} \setminus \{x' \in \mathcal{N}_{j}, x' > x\} \) recombine to a single integral on \( \mathcal{N} \). The formula of the theorem follows.

The assertion follows, because \( g \) was arbitrary with compact support. \( \square \)
7. Generalized Eigenfunctions of the Weighted Laplacian on Star-Shaped Networks

7.5. A Plancherel-type formula and a functional calculus for the operator

Now we use the explicit formula for the resolution of the identity of the operator $A$ obtained in Proposition 7.4.5 to prove a Plancherel-type formula. As in [AM96] (see also [AM04]), we define the Fourier-type transformation $V$ associated with the system of generalized eigenfunctions $\{F_\lambda^{-j} \mid \lambda \in [0; +\infty), j \in \{1, \ldots, n\}\}$ on regular functions using Proposition 7.4.5.

The main difficulty here is that the coefficient $\sigma_j(x)$ appearing in Proposition 7.4.5 depends on $x \in \mathbb{N}$: it is different on each branch of the star, unlike the situation in [AM04] and [AM96]. Thus $\sigma_j(x)$ does not commute with $V$ and therefore the scalar product making the range of $V$ a Hilbert space and $V$ an isometry cannot be directly defined as in [AM04] and [AM96], but must be transferred from $H$ via $V$. This introduces some additional technicalities. Apart from this we follow the lines of [AM04] and [AM96].

**Definition 7.5.1.**

a) For $f \in L^1(\mathbb{N})$ define $V_j f : [0; +\infty) \to \mathbb{R}$ by

$$V_j f(\lambda) = \int_\mathbb{N} f(x) \cdot F_\lambda^{-j}(x) \, dx, \quad j = 1, \ldots, n$$

and $V f : [0; +\infty) \to \mathbb{C}^n$ by $V f = (V_j f)_{j \in \{1, \ldots, n\}}$.

b) Let $\sigma$ be defined as in Proposition 7.4.5 and $\chi \in C^\infty(\mathbb{R})$ be such that $\chi \equiv 0$ on $(-\infty, 1)$ and $\chi \equiv 1$ on $(2, +\infty)$. For $K_j \in C^\infty((0, +\infty), \mathbb{C})$ such that $\chi K_j \in \mathcal{S}(\mathbb{R})$, for $j \in \{1, \ldots, n\}$ define $Z(K) : \mathbb{N} \to \mathbb{R}$ by

$$Z(K_1, \ldots, K_n)(x) = \frac{1}{\pi} \text{Re} \left\{ \int_0^{+\infty} \frac{1}{\sqrt{\lambda}} \sum_{j=1}^n \sigma_j(x) K_j(\lambda) F_\lambda^{-j+1}(x) \, d\lambda \right\}, \quad x \in \mathbb{N}.$$ 

Note that the integral on the right-hand side is absolutely convergent because $\lambda \mapsto 1/\sqrt{\lambda}$ is $L^1_{\text{loc}}$, $K_j$ is continuous and rapidly decreasing at $+\infty$ and $|F_\lambda^{-k}(x)| \leq \text{Const}$, for all $\lambda \in (0; +\infty)$, $x \in \mathbb{N}$, $k \in \{1, \ldots, n\}$.

**Remark 7.5.2.** Unlike $W$ in [AM96], $Z$ is not injective: an easy computation shows that $Z(K) = 0$ is equivalent to

$$\text{Re} \left[ \mathcal{F}(K_j(-\cdot)) \cdot 1_{[0; +\infty)}(\cdot) \right](x) = 0, \quad \forall \ x \in \mathbb{N}_j, \ \forall \ j \in \{1, \ldots, n\},$$

where $\mathcal{F}$ denotes the Fourier transform. And there exist non-vanishing functions $K_j$ satisfying this equation.

**Lemma 7.5.3** (asymptotic behaviour of $V_j f$). Consider $f \in \prod_{k=1}^n \mathcal{D}(N_k)$. Then $V_j f \in C^0([0; +\infty)) \cap C^\infty((0; +\infty))$ and $\chi V_j f \in \mathcal{S}(\mathbb{R})$ for any $j \in \{1, \ldots, n\}$ with $\chi$ as in Definition 7.5.1.

**Proof.** For $\lambda \in [0; +\infty)$ and $j \in \{1, \ldots, n\}$, it holds

$$V_j f(\lambda) = \int_\mathbb{N} f(x) F_\lambda^{-j}(x) \, dx = \sum_{k=1}^n \int_{N_k} f_k(x) F_{\lambda_k}^{-j}(x) \, dx.$$ 

Due to the definition of $F_{\lambda_k}^{-j}$ and due to the fact that $f_k$ is a test function having its support in $(0; +\infty)$, each term of the right-hand side is the Fourier transform of a test function and thus $C^\infty$ and rapidly decreasing in $\lambda$. \[\square\]
Theorem 7.5.5 (Plancherel-type formula and a functional calculus for the operator \( V \)). Let \( \sigma \) be defined as in the end of Proposition 7.4.5 and \( \chi \) as in Definition 7.5.1. Let \( f \in \prod_{k=1}^{n} D(N_k) \) and \( G = (G_1, \ldots, G_n) \in (C^\infty(0; +\infty))^n \cap (C^0[0; +\infty])^n \) such that \( \chi G_l \in S(\mathbb{R}) \) for \( l \in \{1, \ldots, n\} \). Define

\[
< Vf, G >_{\sigma,V} = \frac{1}{\pi} \operatorname{Re} \left\{ \sum_{j=1}^{n} \int_{0}^{+\infty} \frac{1}{\sqrt{\lambda}} V_{j+1} \left( \sigma_j(\cdot)f(\cdot) \right)(\lambda)G_j(\lambda) \, d\lambda \right\}.
\]

Then the integrals on the right-hand side are absolutely convergent and \( < Vf, G >_{\sigma,V} = (f, Z(G))_H \).

Proof. For \( \lambda \in (0; +\infty) \), it holds

\[
\left| \frac{1}{\sqrt{\lambda}} V_{j+1} \left( \sigma_j(\cdot)f(\cdot) \right)(\lambda) \right| = \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} \sigma_j(x)f(x) F_{j+1}^{-\lambda}(x) \, dx \leq C \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} |f(x)| \, dx. \tag{7.5.1}
\]

Together with the fact that \( G_j \) is rapidly decreasing and continuous for any \( j \in \{1, \ldots, n\} \), the latter estimate ensures the absolute convergence of the integrals.

\footnote{This formula is well defined using the expression for \( Z \) as defined in 7.5.1 in spite of the discontinuities introduced by the characteristic function.}

b) Let us fix \( f \in \prod_{k=1}^{n} D(N_k) \). Lemma 7.5.3 implies that there exists \( M_1(f) \geq 0 \), such that

\[
|V_j f(\lambda)| \leq \frac{M_1(f)}{1 + \lambda^2}, \quad \forall \lambda > 0, \ j \in \{1, \ldots, n\}.
\]

Clearly there exists \( M_2 \geq 0 \), such that

\[
\left| \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{n} \sigma_j(x) F_{j+1}^{-\lambda}(x) \right| \leq \frac{M_2}{\sqrt{\lambda}} \quad \forall \lambda > 0, \ x \in N.
\]

Thus the Theorem of Lebesgue implies that

\[
\frac{1}{\pi} \operatorname{Re} \left\{ \int_{0}^{+\infty} \frac{1}{\sqrt{\lambda}} 1_{(a,b)}(\lambda) \sum_{j=1}^{n} \sigma_j(x) V_j f(\lambda) F_{j+1}^{-\lambda}(x) \, d\lambda \right\}
\]

converges for \( a \to -\infty \) and \( b \to +\infty \) and almost every \( x \in N \) towards the same expression with \( 1_{(a,b)} \) replaced by 1.

c) Direct consequence of 2. \( \square \)

Now we shall introduce a structure on the range of \( V \) which shall be later on identified as a scalar product.
Estimate (7.5.1) also allows the application of the theorem of Fubini:

\[
<Vf, G>_{\sigma,V} = \frac{1}{\pi} \operatorname{Re} \left\{ \sum_{j=1}^{n} \int_{0}^{+\infty} \frac{1}{\sqrt{\lambda}} \left( \int_{N} \sigma_j(x) F_{\lambda}^{-j+1}(x) \, dx \right) G_j(\lambda) \, d\lambda \right\}
\]

\[
= \frac{1}{\pi} \operatorname{Re} \left\{ \sum_{j=1}^{n} \int_{N} \frac{1}{\sqrt{\lambda}} \left( \int_{0}^{+\infty} \sigma_j(x) F_{\lambda}^{-j+1}(x) G_j(\lambda) \, d\lambda \right) f(x) \, dx \right\}
\]

\[
= \int_{N} Z(G)(x) f(x) \, dx = (f, Z(G))_H.
\]

This Plancherel formula can now be combined with the fact that \( Z \) is the left inverse of \( V \) to prove that \( <\cdot, \cdot>_{\sigma,V} \) is a scalar product and that \( V \) is an isometry.

**Corollary 7.5.6.**

a) Let \((F, G) \in (V(\prod_{k=1}^{n} D(N_k)))^2\) and \((f, g) \in (D(N_k))^2\), such that \( F = Vf \) and \( G = Vg \). Then \( <F, G>_{\sigma,V} = <Vf, Vg>_{\sigma,V} = (f, g)_H \).

b) \( <\cdot, \cdot>_{\sigma,V} \) is a scalar product on \( V(\prod_{k=1}^{n} D(N_k)) \).

c) Let \( L_{\sigma,V}^2 \) be the completion of \( V(\prod_{k=1}^{n} D(N_k)) \) with respect to \( <\cdot, \cdot>_{\sigma,V} \). We denote the extended scalar product by the latter bracket as well. Thus \( (L_{\sigma,V}^2, <\cdot, \cdot>_{\sigma,V}) \) is a Hilbert space.

d) \( V : \prod_{k=1}^{n} D(N_k) \to V(\prod_{k=1}^{n} D(N_k)) \) extends to a surjective isometry \( \hat{V} : H \to L_{\sigma,V}^2 \).

e) \( Z = V^{-1} : V(\prod_{k=1}^{n} D(N_k)) \to \prod_{k=1}^{n} D(N_k) \) extends to a surjective isometry \( \hat{Z} : L_{\sigma,V}^2 \to H \). Thus \( \hat{Z} = \hat{V}^{-1} \).

**Proof.**

a) Lemma 7.5.3 implies that \( Vg \) is rapidly decreasing and thus Theorem 7.5.5 is applicable:

\[
<F, G>_{\sigma,V} = <Vf, Vg>_{\sigma,V} = (f, Z(Vg))_H = (f, g)_H.
\]

The last equality comes from Proposition 7.5.4.

b) \( V : \prod_{k=1}^{n} D(N_k) \to \text{Ran}V \) is linear and bijective (for the injectivity see Part 3 of Proposition 7.5.4). Thus \( <\cdot, \cdot>_{\sigma,V} \) inherits the property of being a scalar product from \( (\cdot, \cdot)_H \).

c) Clear by construction.

d) Clear by construction.

e) Theorem 7.5.5 implies \( <Vf, G>_{\sigma,V} = (f, Z(G))_H \) for all \( f \in \prod_{k=1}^{n} D(N_k) \) and \( G \in V(\prod_{k=1}^{n} D(N_k)) \). Thus it follows from 1.

\[
|(f, Z(G))_H| = |<Vf, G>_{\sigma,V}| \leq \|G\|_{\sigma,V} \|Vf\|_{\sigma,V} = \|G\|_{\sigma,V} \|f\|_H.
\]

Due to the denseness of \( \prod_{k=1}^{n} D(N_k) \) in \( H \), inequality (7.5.2) is valid for all \( f \in H \). Thus

\[
\|Z(G)\|_H \leq \|G\|_{\sigma,V}.
\]

Therefore \( Z \) extends by density-continuity to a continuous operator \( \hat{Z} \) on \( L_{\sigma,V}^2 \).
Theorem 7.5.7. Let \( h \in C(\mathbb{R}) \) and \( f \in H \), such that \( \lambda \mapsto (h(\lambda)/\sqrt{\lambda})\hat{V}f(\lambda) \) is absolutely integrable on \([0; +\infty)\). Then we have for \( x \in N \)

\[
h(A)f(x) = \frac{1}{\pi} \text{Re} \left\{ \int_0^{+\infty} \frac{h(\lambda)}{\sqrt{\lambda}} \sum_{j=1}^n \sigma_j(x)V_jf(\lambda)F_{-j+1}^\lambda(x) \, d\lambda \right\}. \tag{7.5.3}
\]

Proof. The same proof as in Proposition 7.4.5, but this time using Stone’s formula (Theorem 7.4.1) with arbitrary \( h \in C(\mathbb{R}) \), yields

\[
h(A)E(a, b)f(x) = \frac{1}{\pi} \text{Re} \left\{ \int_0^{+\infty} \frac{h(\lambda)}{\sqrt{\lambda}} 1_{(a,b)}(\lambda) \sum_{j=1}^n \sigma_j(x)V_jf(\lambda)F_{-j+1}^\lambda(x) \, d\lambda \right\}.
\]

Now, the assertion follows from dominated convergence and the fact that \( E(a, b) \) commutes with \( h(A) \) and tends to the identity if \( a \to -\infty \) and \( b \to \infty \). \( \square \)

Remark 7.5.8. a) Formally (7.5.3) reads like

\[
h(A)f = \tilde{Z}M_h\tilde{V}f, \tag{7.5.4}
\]

where \((M_h K)(\lambda) := h(\lambda)K(\lambda)\). It should be investigated, if under the hypotheses of Theorem 7.5.7 we have \( M_h\tilde{V}f \in L^2_{\sigma, V} \), and thus (7.5.4) is rigorously valid.

b) Using Theorem 7.5.7, we can represent solutions of evolution equations involving \( A \) (heat, wave, Klein-Gordon, . . . ) in view of obtaining qualitative informations like decay properties in time on the \( n \)-star. It remains the open problem of describing the relation of the belonging of \( f \) to \( D(A^s) \) and the decay of \( \tilde{V}f \) at infinity. This is important, because for example \( f \in D(A) \) ensures the twice differentiability of \( u(t) = \cos(\sqrt{A}t)f \) and thus the validity of the abstract wave equation \( \ddot{u}(t) + Au(t) = 0 \).
References (Article 7)


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References (Article 7)


8. $H^\infty$-Calculus for Products of Non-Commuting Operators

by Robert Haller-Dintelmann and Matthias Hieber

Abstract
It is shown that the product of two sectorial operators $A$ and $B$ admits a bounded $H^\infty$-calculus on a Banach space $X$ provided suitable commutator estimates and Kalton-Weis type assumptions on $A$ and $B$ are satisfied.

8.1. Introduction

The question of maximal $L^p$-regularity for partial differential equations has attracted much attention in the last decade. One reason for this is that, by linearization techniques, one obtains a powerful approach to many nonlinear parabolic problems.

Starting from the fundamental paper of Da Prato and Grisvard [DPG75], the so-called method of operator sums was further developed by Dore and Venni [DV87] and more recently by Kalton and Weis [KW01]. They proved, roughly speaking, that the sum $A + B$ of two commuting operators $A$ and $B$, equipped with its natural domain, has similar properties as $A$ and $B$. The most important examples fitting in this framework are of course the time derivative and differential operators with respect to the space variable.

Whereas problems of this kind for commuting operators may be regarded as fairly well understood, the situation is less clear in the non-commuting context. A first result in this direction was given by Monniaux and Prüss [MP97], who proved a theorem of Dore-Venni type, assuming the Labbas-Terreni commutator condition (see [LT87]). Very recently, Prüss and Simonett were able to prove a non-commutative version of the Kalton-Weis theorem for both Da Prato-Grisvard and Labbas-Terreni commutator conditions, see [PS04]. Applications of this result to parabolic equations on wedges and cones yield optimal regularity results for the solution of these equations.

For many applications it is essential to have results of this kind not only for the sum of $A$ and $B$ but also for products $AB$. Indeed, recent developments in free boundary value problems with moving contact lines show that regularity results on products of non commuting operators are very helpful in this context.

First results on products of non commuting sectorial operators under certain commutator estimates were obtained by Weber and Štrkalj. Indeed, a Dore-Venni type result for products was first obtained by Weber [Web98]. Štrkalj [Str01] proved that the product $AB$ of $A$ and $B$ is sectorial provided the underlying space is $B$-convex and assumptions of Kalton-Weis type are satisfied.

It is the aim of this paper to study the remaining question in this context: existence of an $H^\infty$-calculus for the product $AB$ of non commuting operators $A$ and $B$ under suitable commutator and Kalton-Weis type assumptions. In the following Theorems 8.3.1 and 8.3.2 we give an affirmative answer to this question.
8. **H∞-Calculus for Products of Non-Commuting Operators**

### 8.2. Preliminaries

In this section we introduce the notation being used throughout this article and collect certain properties of sectorial operators and operators with a bounded $H^\infty$-calculus.

If $X$ and $Y$ are Banach spaces, $\mathcal{L}(X,Y)$ denotes the space of all bounded, linear operators from $X$ to $Y$; moreover, $\mathcal{L}(X) := \mathcal{L}(X,X)$. The spectrum of a linear operator $A$ in $X$ is denoted by $\sigma(A)$, its resolvent set by $\varrho(A)$. As usual domain and range of an operator $A$ are denoted by $D(A)$ and $R(A)$, respectively.

Let $X$ be a complex Banach space, and $A$ be a closed linear operator in $X$. Then $A$ is called **sectorial** if $D(A) = X$, $R(A) = X$, $(-\infty,0] \subseteq \varrho(A)$ and

$$
\|t(t+A)^{-1}\| \leq M, \quad t > 0,
$$

for some $M < \infty$. We denote the class of sectorial operators in $X$ by $S(X)$. $\Sigma_\theta \subseteq \mathbb{C}$ means the open sector

$$
\Sigma_\theta = \{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta \}.
$$

If $A \in S(X)$ then $\varrho(-A) \supseteq \Sigma_\theta$ and $\sup \{ \|\lambda(\lambda + A)^{-1}\| : |\arg \lambda| < \theta \} < \infty$ for some $\theta > 0$. We thus define the spectral angle $\varphi_A$ of $A \in S(X)$ by

$$
\varphi_A = \inf \{ \phi : \varrho(-A) \supseteq \Sigma_{\pi - \phi}, \sup_{\lambda \in \Sigma_{\pi - \phi}} \|\lambda(\lambda + A)^{-1}\| < \infty \}.
$$

Evidently, we have $\varphi_A \in [0,\pi)$ and $\varphi_A \geq \sup \{ |\arg \lambda| : \lambda \in \sigma(A) \}$. For $\phi \in (0,\pi]$ we define the space of holomorphic functions on $\Sigma_\phi$ by $H(\Sigma_\phi) = \{ f : \Sigma_\phi \rightarrow \mathbb{C} \text{ holomorphic} \}$, and

$$
H^\infty(\Sigma_\phi) = \{ f : \Sigma_\phi \rightarrow \mathbb{C} \text{ holomorphic and bounded} \}.
$$

The space $H^\infty(\Sigma_\phi)$ with norm $\|f\|_\infty^\phi = \sup \{ |f(\lambda)| : |\arg \lambda| < \phi \}$ forms a Banach algebra. We also set

$$
H_0^\infty(\Sigma_\phi) := \bigcup_{\alpha,\beta<0} H_{\alpha,\beta}(\Sigma_\phi),
$$

where $H_{\alpha,\beta}(\Sigma_\phi) := \{ f \in H(\Sigma_\phi) : \|f\|_{\alpha,\beta}^\phi < \infty \}$, and $\|f\|_{\alpha,\beta}^\phi := \sup_{|\lambda| \leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda| \geq 1} |\lambda^{-\beta} f(\lambda)|$. Given $A \in S(X)$, fix any $\phi \in (\varphi_A,\pi]$ and let $\Gamma_\psi = (-\infty,0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ with $\varphi_A < \psi < \phi$. Then

$$
f(A) = \frac{1}{2\pi i} \int_{\Gamma_\psi} f(\lambda)(\lambda - A)^{-1} \, d\lambda, \quad f \in H_0^\infty(\Sigma_\phi),
$$

defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : H_0^\infty(\Sigma_\phi) \rightarrow \mathcal{L}(X)$ which is an algebra homomorphism. Following McIntosh [McI86], we say that a sectorial operator $A$ admits a bounded $H^\infty$-calculus if there are $\phi > \varphi_A$ and a constant $K_\phi < \infty$ such that

$$
\|f(A)\| \leq K_\phi \|f\|_\infty^\phi, \quad \text{for all } f \in H_0^\infty(\Sigma_\phi). \quad (8.2.1)
$$

The class of sectorial operators $A$ which admit a bounded $H^\infty$-calculus will be denoted by $\mathcal{H}(X)$ and the $H^\infty$-angle of $A$ is defined by

$$
\varphi_A^\infty = \inf \{ \phi > \varphi_A : (8.2.1) \text{ is valid} \}.
$$

If this is the case, the functional calculus for $A$ on $H_0^\infty(\Sigma_\phi)$ extends uniquely to $H^\infty(\Sigma_\phi)$.

We consider next another subclass of $S(X)$, namely operators with bounded imaginary powers. More precisely, a sectorial operator $A$ in $X$ is said to admit bounded imaginary powers if $A^{is} \in \mathcal{L}(X)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $\|A^{is}\| \leq C$ for $|s| \leq 1$. The class of such operators will be denoted by $BTP(X)$. We call

$$
\varphi_A^{BIP} = \lim_{|s| \rightarrow \infty} \frac{1}{|s|} \log \|A^{is}\|
$$

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the power angle of $A$. Since the functions $f_s$ defined by $f_s(z) = z^{is}$ belong to $H^\infty(\Sigma_{\phi})$, for any $s \in \mathbb{R}$ and $\phi \in (0, \pi)$, we obviously have the inclusions

$$\mathcal{H}^\infty(X) \subseteq B\mathcal{I}\mathcal{P}(X) \subseteq \mathcal{S}(X),$$

and the inequalities

$$\varphi_A^\infty \geq \varphi_A^{BIP} \geq \varphi_A \geq \sup\{|\arg \lambda| : \lambda \in \sigma(A)\}.$$

Let $Y$ be another Banach space. A family of operators $T \subseteq \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded, if there is a constant $C > 0$ and $p \in [1, \infty]$, such that for each $N \in \mathbb{N}$, $T_j \in T$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables $\varepsilon_j$ on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{L_p(\Omega; Y)} \leq C \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{L_p(\Omega; X)}$$

is valid. The smallest such $C$ is called $\mathcal{R}$-bound of $T$, we denote it by $\mathcal{R}(T)$. Observe that the concept of $\mathcal{R}$-boundedness does not depend on $p$, however $\mathcal{R}(T)$ does, see [CdPSW00], [Wei01], [KW01], [DHP03].

The concept of $\mathcal{R}$-bounded families of operators leads immediately to the notion of $\mathcal{R}$-sectorial operators. Indeed, a sectorial operator is called $\mathcal{R}$-sectorial (see [CP01]) if

$$\mathcal{R}_A(0) := \mathcal{R}(\{t(t + A)^{-1} : t > 0\}) < \infty.$$  

The $\mathcal{R}$-angle $\varphi_A^\mathcal{R}$ of $A$ is defined by means of

$$\varphi_A^\mathcal{R} := \inf\{\theta \in (0, \pi) : \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}(\{\lambda(\lambda + A)^{-1} : |\arg \lambda| \leq \theta\}).$$

Finally, we say that $A \in \mathcal{H}^\infty(X)$ admits an $\mathcal{R}$-bounded $H^\infty$-calculus, if the set

$$\{f(A) : f \in H^\infty(\Sigma_{\phi}), \|f\|_{\infty}^\phi \leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi \in (0, \pi)$. As above, the infimum $\varphi_A^{\mathcal{R} \infty}$ of such $\phi$ is called the $\mathcal{R}\mathcal{H}^\infty$-angle of $A$. The class of such operators is denoted by $\mathcal{R}\mathcal{H}^\infty(X)$.

Assume that the underlying space $X$ satisfies the so-called property $(\alpha)$, see [CdPSW00, Definition 3.11]. Then Kalton and Weis [KW01, Theorem 5.3] proved that every operator $A \in \mathcal{H}^\infty(X)$ already admits an $\mathcal{R}$-bounded $H^\infty$-calculus. More precisely, we have

$$\mathcal{H}^\infty(X) = \mathcal{R}\mathcal{H}^\infty(X) \quad \text{with} \quad \varphi_A^{\mathcal{R} \infty} = \varphi_A^\infty. \quad (8.2.2)$$

It is well known that $L^p$-spaces with $1 < p < \infty$ possess the property $(\alpha)$.

We are now able to state the Kalton-Weis theorem which gives a sufficient condition for the existence of an operator-valued $H^\infty$-calculus.

**Theorem 8.2.1 ([KW01]).** Let $X$ be a Banach space. Assume that $A \in \mathcal{H}^\infty(X)$, $F \in H^\infty(\Sigma_{\phi}; \mathcal{L}(X))$ such that

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda), \quad \mu \in \rho(A), \lambda \in \Sigma_{\phi},$$

and that $\phi > \varphi_A^\infty$ and $\mathcal{R}(F(\Sigma_{\phi})) < \infty$. Then there exists a constant $C$ independent of $F$ such that $F(A) \in \mathcal{L}(X)$ and

$$\|F(A)\|_{\mathcal{L}(X)} \leq C\mathcal{R}(F(\Sigma_{\phi})).$$
Consider for two sectorial operators $A$ and $B$ in $X$ their product $AB$ defined by

$$(AB)x := ABx, \quad D(AB) := \{ x \in D(B) : Bx \in D(A) \}.$$ 

We then observe that $AB$ is closed as soon as $A$ is invertible or $B$ is bounded. The Kalton-Weis theorem moreover implies for commuting operators $A$ and $B$ the following result.

**Corollary 8.2.2.** Let $X$ be a Banach space and assume that $A$ and $B$ are sectorial operators in $X$ which commute in the sense of resolvents. Suppose that $0 \in \rho(A)$, $A \in \mathcal{H}^{\infty}(X)$, $B \in \mathcal{RS}(X)$ and that $\varphi_A^\infty + \varphi_B^\infty < \pi$.

a) Then $AB$ is sectorial and $\varphi_{AB} \leq \varphi_A^\infty + \varphi_B^\infty$.

b) If in addition $B \in \mathcal{RH}(X)$ with $\varphi_A^\infty + \varphi_B^\infty < \pi$, then $AB \in \mathcal{H}^{\infty}(X)$ and $\varphi_{AB} \leq \varphi_A^\infty + \varphi_B^\infty$.

We say that a Banach space $X$ belongs to the class $\mathcal{H}$, if the Hilbert transform is bounded on $L^p(\mathbb{R}; X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform $H$ of a function $f \in \mathcal{S}(\mathbb{R}; X)$, the Schwartz space of rapidly decreasing $X$-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV \left( \frac{1}{t} \right) \ast f$$

These spaces are also called UMD Banach spaces, where the UMD stands for *unconditional martingale difference property*. It is a well known theorem that the set of Banach spaces of class $\mathcal{H}$ coincides with the class of UMD spaces; see e.g. [Bur86].

Throughout this paper, for $\psi \in (0, \pi)$ and $r \geq 0$ we denote by $\Gamma^r_\psi$ the path given by

$$\Gamma^r_\psi := -(-\infty, -r]e^{i\psi} \cup re^{-i[-\psi, \psi]} \cup [r, \infty)e^{-i\psi},$$

and we write $\Gamma_\psi := \Gamma^0_\psi$.

We remark that by $C$, $M$ and $c$ we denote various constants which may differ from line to line but which are always independent of the free variables.

### 8.3. The main result

Is the product of sectorial operators again sectorial? Let us recall that the first result in this direction for non-commuting operators was proved by Weber [Web98]. He showed that in UMD spaces, $\nu + AB$ with natural domain $D(AB) = \{ x \in D(B) : Bx \in D(A) \}$ is sectorial provided $A$ and $B$ have bounded imaginary powers of suitable power angles and certain commutator estimates are fulfilled. This result was later on extended by Strkalj [Str01] to Kalton-Weis type assumptions for operators defined in $B$-convex Banach lattices.

We start with a generalization of the latter result to arbitrary Banach spaces.

**Theorem 8.3.1.** Let $X$ be a Banach space. Assume that $A \in \mathcal{H}^{\infty}(X)$ and $B \in \mathcal{RS}(X)$ with $0 \in \rho(A)$ fulfill the following properties:

a) $(\mu - B)^{-1}D(A) \subseteq D(A)$ for some (all) $\mu \in \rho(B)$,

b) there are $\theta_A > \varphi_A^\infty$ and $\theta_B > \varphi_B^\infty$, such that $\theta_A + \theta_B < \pi$ and there exist constants $c, \alpha \geq 0$ and $\beta > 0$ with $\alpha + \beta < 1$, such that

$$\| [A, (\mu + B)^{-1}] (\lambda + A)^{-1} \| \leq \frac{c}{(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}}$$

for all $\lambda \in \Sigma_{\pi-\theta_A}$ and $\mu \in \Sigma_{\pi-\theta_B}$. 

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Then there exists $\nu \geq 0$ such that the operator $\nu + AB$ with domain $D(AB)$ is sectorial with $\varphi_{\nu+AB} \leq \theta_A + \theta_B$.

The following main result of the paper states that in the above situation we even have $AB \in \mathcal{H}^\infty(X)$ provided $B \in \mathcal{RH}^\infty(X)$.

**Theorem 8.3.2.** Let $X$ be a Banach space. Assume that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RH}^\infty(X)$ with $0 \in \varrho(A)$ fulfill the following properties:

a) $(\mu - B)^{-1}D(A) \subseteq D(A)$ for some (all) $\mu \in \varrho(B)$,

b) there are $\theta_A > \varphi_A^\infty$ and $\theta_B > \varphi_B^\infty$, such that $\theta_A + \theta_B < \pi$ and there exist constants $c, \alpha \geq 0$ and $\beta > 0$ with $\alpha + \beta < 1$, such that

$$\| [A, (\mu + B)^{-1}](\lambda + A)^{-1} \| \leq \frac{c}{(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}}$$

for all $\lambda \in \Sigma_{\pi-\theta_A}$ and $\mu \in \Sigma_{\pi-\theta_B}$.

Then there exists $\nu \geq 0$, such that the operator $\nu + AB$ with domain $D(AB)$ has a bounded $\mathcal{H}^\infty$-calculus with $\varphi_{\nu+AB}^\infty \leq \theta_A + \theta_B$.

**Remark 8.3.3.** a) As both theorems are not symmetric in the roles of $A$ and $B$, it is worthwhile to note that the same results hold true if the properties of $A$ and $B$ are interchanged. The proofs stay the same.

b) If $X$ has property (a), the classes $\mathcal{H}^\infty(X)$ and $\mathcal{RH}^\infty(X)$ coincide; see (8.2.2). Thus Theorem 8.3.2 may be formulated in this case with $B \in \mathcal{H}^\infty(X)$.

c) It will become apparent in the proof of the two theorems, that the amount of the shift $\nu$ is determined mainly by the constant $c$ in the commutator estimate.

Finally we observe, that the invertibility of $A$ implies $\lambda \in \varrho(A)$ for all $|\lambda| < \|A^{-1}\|^{-1}$ and $\|(\lambda + A)^{-1}\| \leq \|A^{-1}\|(1 - |\lambda||A^{-1}|)^{-1}$ for all these $\lambda$. Thus the commutator estimate can be extended to all $\lambda \in \Sigma_{\pi-\theta_A} \cup \{z \in \mathbb{C} : |z| < r\}$ whenever $r < \|A^{-1}\|^{-1}$.

### 8.4. Sectoriality of $\nu + AB$

In this section we give a proof of Theorem 8.3.1. Our method is inspired by the work of Weber [Web98] and Prüss and Simonett [PS04]. It is heavily based on properties of the families of operators $S_\mu$ and $T_\mu$ which are defined as follows.

We fix angles $\gamma \in (0, \pi - \theta_A - \theta_B)$ and $\varphi \in (\theta_A, \pi - \gamma - \theta_B)$, as well as a number $r \in (0, \|A^{-1}\|^{-1})$. Now let $\mu \in \Sigma_\gamma$. Then for all $z \in \Gamma_{\varphi, \gamma}$, we have $\mu/z \in \varrho(-B)$. Furthermore, by the choices of $\varphi$ and $r$, the inclusion $\Gamma_{\varphi, \gamma} \subseteq \varrho(A)$ holds true. We then define for $x \in D(A)$

$$S_\mu x := \frac{1}{2\pi i} \int_{\Gamma_{\varphi, \gamma}} \frac{1}{z^2} \left(\frac{\mu}{z} + B\right)^{-1} A(z - A)^{-1} x \, dz$$

$$T_\mu x := \frac{1}{2\pi i} \int_{\Gamma_{\varphi, \gamma}} \frac{1}{z^2} A(z - A)^{-1} \left(\frac{\mu}{z} + B\right)^{-1} x \, dz.$$
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The operators \(S_\mu\) are clearly bounded from \(D(A)\) to \(X\). For \(T_\mu\) this follows by the commutator estimate. Indeed for \(x \in D(A)\), we have \((\mu/z + B)^{-1}x \in D(A)\) and

\[
\left\| A \left( \frac{\mu}{z} + B \right)^{-1} x \right\| = \left\| \left( \frac{\mu}{z} + B \right)^{-1} + A \left( \frac{\mu}{z} + B \right)^{-1} \right\| A^{-1} \|Ax\| \\
\leq \left( \frac{|z|}{|\mu|} + \frac{C|z|^{1+\beta}}{|\mu|^{1+\beta}} \right) \|Ax\|.
\]

The operators \(S_\mu\) and \(T_\mu\) are even bounded on \(X\). In order to show this, we introduce the following lemma due to Kalton and Weis [KW01, Lemma 4.1]. Further proofs may be found also in [DDHPV04].

**Lemma 8.4.1.** Suppose \(A \in \mathcal{H}^\infty(X)\), \(\phi > \varphi_A^\infty\) and \(h \in H^\infty_0(\Sigma_\phi)\). Then there is a constant \(C > 0\), such that

\[
\left\| \sum_{k \in \mathbb{Z}} \alpha_k h(2^k tA) \right\|_{\mathcal{L}(X)} \leq C \sup_{k \in \mathbb{Z}} |\alpha_k|
\]

for all \(\alpha_k \in \mathbb{C}\) and \(t > 0\).

The above Lemma 8.4.1 enables us to prove that \(S_\mu\) and \(T_\mu\) are bounded on \(X\). More precisely, we have the following.

**Lemma 8.4.2.** Let \(\gamma \in (0, \pi - \theta_A - \theta_B)\). The operators \(S_\mu\) and \(T_\mu\) have unique bounded extensions on \(X\) for every \(\mu \in \Sigma_\gamma\) and there is a constant \(C_\gamma\), such that

\[
\|S_\mu\|_{\mathcal{L}(X)} + \|T_\mu\|_{\mathcal{L}(X)} \leq \frac{C_\gamma}{|\mu|} \left(1 + \frac{1}{|\mu|^\beta}\right).
\]

**Proof.** By Cauchy’s Theorem we may rewrite \(S_\mu\) for an arbitrary number \(a \in (0, 1)\) as

\[
S_\mu x = \frac{1}{2\pi i} \int_{\Gamma_\phi} \frac{1}{z^{1+a}} \left( \frac{\mu}{z} + B \right)^{-1} A^a(z-A)^{-1} x \, dz.
\]

In a second step we commute some part of \(A^a(z-A)^{-1}\) with the resolvent of \(B\) and get for every \(b \in (0, 1)\)

\[
S_\mu x = \frac{1}{2\pi i} \int_{\Gamma_\phi} \frac{1}{z^a} \left( A^a(z-A)^{-1} \right)^b (\mu + zB)^{-1} (A^a(z-A)^{-1})^{1-b} x \, dz \\
+ \frac{1}{2\pi i} \int_{\Gamma_\phi} \frac{1}{z^a} \left[ (\mu + zB)^{-1}, (A^a(z-A)^{-1})^b \right] (A^a(z-A)^{-1})^{1-b} x \, dz \\
=: I_1 + I_2.
\]

The right values of \(a\) and \(b\) will be chosen later on.

Analogously we get

\[
T_\mu x = \frac{1}{2\pi i} \int_{\Gamma_\phi} \frac{1}{z^a} \left( A^a(z-A)^{-1} \right)^{1-b} (\mu + zB)^{-1} (A^a(z-A)^{-1})^b x \, dz \\
- \frac{1}{2\pi i} \int_{\Gamma_\phi} \frac{1}{z^a} \left( A^a(z-A)^{-1} \right)^{1-b} \left[ (\mu + zB)^{-1}, (A^a(z-A)^{-1})^b \right] x \, dz,
\]

so that the proofs for \(S_\mu\) and \(T_\mu\) are basically the same and we will concentrate on \(S_\mu\).
The main step to estimate $I_2$ is to use the bounded $H^\infty$-calculus of $A$ and write \((A^a(z - A)^{-1})^b\) as a contour integral. To this end, we define the function \(g_z(\zeta) := (\zeta^a(z - \zeta)^{-1})^b\) and choose angles $\omega$ and $\vartheta$ with $\theta_A < \omega < \vartheta < \varphi$ as well as numbers $\tilde{r}$ and $\tilde{r}'$ satisfying $\tilde{r} < \tilde{r}' < \|A^{-1}\|^{-1}$. Then $g_z \in H^\infty_0(\Sigma_\omega)$ and we have

\[
(A^a(z - A)^{-1})^b = \frac{1}{2\pi i} \int_{\Gamma_\omega} g_z(\zeta)(\zeta - A)^{-1} \, d\zeta.
\]

Using this, we can rewrite the commutator in $I_2$ as

\[
\left[ (\mu + zB)^{-1}, (A^a(z - A)^{-1})^b \right] = \frac{1}{2\pi i} \int_{\Gamma_\omega} \frac{\zeta^{ab}}{(z - \zeta)^b} \left[ (\mu + zB)^{-1}, (\zeta - A)^{-1} \right] \, d\zeta.
\]

A closer look at the remaining commutator yields

\[
\left[ (\mu + zB)^{-1}, (\zeta - A)^{-1} \right] = \frac{1}{z} (\zeta - A)^{-1} \left[ \left( \frac{\mu}{z} + B \right)^{-1}, A \right] (\zeta - A)^{-1}
\]

and, using the commutator estimates, that leads to

\[
\begin{align*}
\left\| \left[ (\mu + zB)^{-1}, (A^a(z - A)^{-1})^b \right] \right\| &\leq C \frac{1}{|z|} \int_{\Gamma_\omega} \frac{|\zeta|^{ab}}{|z - \zeta|^b} \left\| (\zeta - A)^{-1} \right\| \left\| \left[ \left( \frac{\mu}{z} + B \right)^{-1}, A \right] (\zeta - A)^{-1} \right\| \, d|\zeta| \\
&\leq C |z|^{\beta - b} \int_{\Gamma_\omega} \frac{|\zeta|^{ab}}{|z - \zeta|^b} \left( 1 + |\zeta| \right)^{2 - \alpha} |\mu|^{1 + \beta} d|\zeta|.
\end{align*}
\]

(8.4.1)

In order to estimate this integral, we first observe, that there is a constant $C > 0$, such that for all $z \in \Gamma_\varphi$ and all $\zeta \in \Gamma_\omega$ we have

\[
|z - \zeta| \geq C(|z| + |\zeta|),
\]

(8.4.2)

at least if we have $\varphi - \omega < \pi/2$, but this can be guaranteed by a suitable choice of $\omega$ above. Thus (8.4.1) may be estimated further by

\[
C \frac{|z|^{\beta - b}}{|\mu|^{1 + \beta}} \int_{\Gamma_\omega} \left( \frac{|z|}{|z| + |\zeta|} \right)^{b} \frac{|\zeta|^{ab}}{(1 + |\zeta|)^{2 - \alpha}} \, d|\zeta| \leq C \frac{|z|^{\beta - b}}{|\mu|^{1 + \beta}} \int_{\Gamma_\omega} \frac{|\zeta|^{ab}}{(1 + |\zeta|)^{2 - \alpha}} \, d|\zeta|.
\]

This last integral is convergent if $a$ and $b$ satisfy $\alpha < 1 - ab$.

Using this result, we estimate $I_2$ as

\[
\|I_2\| \leq C \frac{1}{|\mu|^{1 + \beta}} \int_{\Gamma_\varphi} |z|^{\beta - a - b} \left\| (A^a(z - A)^{-1})^b \right\| \|x\| \, d|z|.
\]

As the function $x \mapsto x^a/(c + x)$, $x \in [0, \infty)$ takes its maximum at $x = ca/(1 - a)$, we see by the bounded $H^\infty$-calculus of $A$ and (8.4.2), that

\[
\left\| (A^a(z - A)^{-1})^b \right\| \leq C \sup_{x \in [0, \infty)} \left( \frac{x^a}{|z| + x} \right)^{1 - b} = C |z|^{(a - 1)(1 - b)}.
\]

This finally yields the estimate

\[
\|I_2\| \leq C \frac{1}{|\mu|^{1 + \beta}} \int_{\Gamma_\varphi} |z|^{\beta - ab - 1} \, d|z| \|x\|,
\]

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that gives us a second condition for \(a\) and \(b\) to make this integral converge: \(\beta < ab\). If we choose \(a, b \in (0, 1)\), such that \(\alpha + \beta < \alpha + ab < 1\), which is possible thanks to \(\alpha + \beta < 1\), both conditions are satisfied simultaneously and we get \(\|I_2\| \leq C/|\mu|^{1+\beta}\). We now turn our attention to \(I_1\). As we have seen, the integrals defining \(I_2\) and \(S_\mu x\) converge absolutely. Thus the integral defining \(I_1\) also converges absolutely. Since there is no singularity of the integrand for small \(|z|\), we only look at \(|z| \geq 2^{n_0}\), where \(n_0 \in \mathbb{Z}\) is a fixed number, such that \(2^{n_0} > r\). For this part of \(I_1\) we may write thanks to the absolute convergence

\[
\lim_{N \to \infty} \int_{\Gamma_{\sigma}^N} \frac{1}{z^a} \left(A^a(z-A)^{-1}\right)^b(\mu + zB)^{-1} \left(A^a(z-A)^{-1}\right)^{-b} x \, dz,
\]

where \(\Gamma_{\sigma}^N := \{ \lambda \in \Gamma_{\sigma} : 2^{n_0} \leq |\lambda| \leq 2^N \}\) for \(N > n_0\).

For one of these integrals we have

\[
K_N := \int_{2^{n_0}}^{2^N} \frac{1}{(te^{i\varphi})^a} \left(A^a(te^{i\varphi} - A)^{-1}\right)^b(\mu + te^{i\varphi} B)^{-1} \left(A^a(te^{i\varphi} - A)^{-1}\right)^{-b} x e^{i\varphi} \, dt.
\]

Substituting \(t = 2^k s\) and setting \(\tilde{g}_2 := g_2^{(1-b)/b}\), we obtain

\[
K_N = e^{i\varphi(1-a)} \int_1^{2^N} g_{e^{i\varphi}} \left(\frac{A}{2^k s}\right) (\mu + 2^k se^{i\varphi} B)^{-1} \left(\frac{A}{2^k s}\right) x \, ds.
\]

In order to estimate the norm of this expression, we choose by the Hahn-Banach theorem \(x^* \in X'\) with \(\|x^*\| = 1\) and

\[
\|K_N\| = \left\| \sum_{k=n_0}^{N-1} \int_1^{2^k} g_{e^{i\varphi}} \left(\frac{A}{2^k s}\right) (\mu + 2^k se^{i\varphi} B)^{-1} \left(\frac{A}{2^k s}\right) x, x^* \right\| \, ds.
\]

Having in mind the \(R\)-sectoriality of \(B\), we plug in independent, symmetric, \([-1, 1]\)-valued random variables \(\varepsilon_{n_0}, \ldots, \varepsilon_{N-1}\) on some probability space \(\Omega\).

\[
= \int_{\Omega} \left( \sum_{k=n_0}^{N-1} \varepsilon_k(\omega) g_{e^{i\varphi}} \left(\frac{A}{2^k s}\right) (\mu + 2^k se^{i\varphi} B)^{-1} \left(\frac{A}{2^k s}\right) x, x^* \right) \, d\omega \, ds.
\]

As these random variables are independent, we may write

\[
= \int_{\Omega} \left( \sum_{k=n_0}^{N-1} \varepsilon_k(\omega)(\mu + 2^k se^{i\varphi} B)^{-1} \left(\frac{A}{2^k s}\right) x, \sum_{k=n_0}^{N-1} \varepsilon_k(\omega) \left(\frac{A^*}{2^k s}\right) x^* \right) \, d\omega \, ds.
\]

Note that \(A^* \in \mathcal{H}^\infty(X')\) with \(\varphi^\infty_A = \varphi^\infty_A\) in case \(D(A^*)\) is dense in \(X'\). If this is not the case, we may use the sun-dual \(A^\odot\) on \(X^\odot\) instead (c.f. [vN92, Chapter 1.3]). By the
Cauchy-Schwarz inequality we see

\[
\leq \int_1^2 \left\| \sum_{k=n_0}^{N-1} \varepsilon_k (\mu + 2^k s e^{i \varphi}) B^{-1} \hat{g}_{e^{i \varphi}} \left( \frac{A}{2^k s} \right) x \right\|_{L^2(\Omega; X)} \cdot \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_{e^{i \varphi}} \left( \frac{A^*}{2^k s} \right) x^* \right\|_{L^2(\Omega; X')} ds \\
= \int_1^2 \frac{1}{|\mu|} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \frac{\mu}{2^k s e^{i \varphi}} \left( \frac{\mu}{2^k s e^{i \varphi}} + B \right)^{-1} \hat{g}_{e^{i \varphi}} \left( \frac{A}{2^k s} \right) x \right\|_{L^2(\Omega; X)} \cdot \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_{e^{i \varphi}} \left( \frac{A^*}{2^k s} \right) x^* \right\|_{L^2(\Omega; X')} ds,
\]

As \( B \) is \( \mathcal{R} \)-sectorial, and \( | \arg(\mu/(2^k s e^{i \varphi}))| < \pi - \theta_B \) we can estimate the latter term by

\[
\leq C_R \int_1^2 \frac{1}{|\mu|} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_{e^{i \varphi}} \left( \frac{A}{2^k s} \right) x \right\|_{L^2(\Omega; X)} \cdot \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_{e^{i \varphi}} \left( \frac{A^*}{2^k s} \right) x^* \right\|_{L^2(\Omega; X')} ds,
\]

where \( C_R := \mathcal{R} \left( \{ \lambda (\lambda + B)^{-1} : \lambda \in \Sigma_{\pi - \theta_B} \} \right) \).

Finally, we apply Lemma 8.4.1 to the two remaining norms. This yields

\[
\| K_N \| \leq \frac{C}{|\mu|} \int_1^2 \left( \sup_{k=n_0}^{N-1} |\varepsilon_k| \right)^2 \| x \| ds = \frac{C}{|\mu|} \| x \|.
\]

Summing up our considerations, we finally get for all \( x \in D(A) \)

\[
\| S_\mu x \| \leq \| I_1 \| + \| I_2 \| \leq C\frac{1}{|\mu|} \left( 1 + \frac{1}{|\mu|^2} \right) \| x \|.
\]

Thus, by density, we have the same estimate on all of \( X \), finishing the proof. \( \square \)

The above lemma is now the basis for the construction of a right inverse for \( \mu + AB \). Also, we show that \( \mu + AB \) is injective.

To this end, let \( x \in D(A) \cap R(A) \). Then, by the closedness of \( B \), we even have \( S_\mu x \in D(B) \) and we can calculate with the help of Cauchy’s Theorem

\[
BS_\mu x = \frac{1}{2\pi i} \int_{\Gamma_{\varphi}} \frac{1}{z^2} B \left( \frac{\mu}{z} + B \right)^{-1} A(z - A)^{-1} x dz \tag{8.4.3}
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_{\varphi}} \frac{1}{z} A(z - A)^{-1} x dz - \frac{\mu}{2\pi i} \int_{\Gamma_{\varphi}} \frac{1}{z^2} \left( \frac{\mu}{z} + B \right)^{-1} A(z - A)^{-1} x dz
\]

\[
= A^{-1} x - \frac{\mu}{2\pi i} \int_{\Gamma_{\varphi}} \frac{1}{z^2} \left( \frac{\mu}{z} + B \right)^{-1} (z - A)^{-1} x dz.
\]

This yields

\[
\| BS_\mu x \| \leq \| A^{-1} \| \| x \| + C|\mu| \int_{\Gamma_{\varphi}} \frac{1}{|z|^2} \frac{1}{|\mu|} \frac{1}{1+|z|} d|z| \| x \| \leq C\| x \|
\]

for every \( x \in D(A) \cap R(A) \). By density of \( D(A) \cap R(A) \) and the closedness of \( B \) this shows that we even have \( S_\mu \in \mathcal{L}(X, D(B)) \).
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Looking again at (8.4.3), we see, that for every $x \in D(A) \cap R(A)$ the integral on the right hand side is in $D(A)$. Thus we get

\[
BS_\mu x = A^{-1} \left( x - \frac{\mu}{2\pi i} \int_{\Gamma_\nu} \frac{1}{z^2} A \left( \frac{\mu}{z} + B \right)^{-1} (z - A)^{-1} x \, dz \right) = A^{-1} (x - \mu S_\mu x + Q_\mu x),
\]

where

\[
Q_\mu := \frac{\mu}{2\pi i} \int_{\Gamma_\nu} \frac{1}{z^2} \left[ \left( \frac{\mu}{z} + B \right)^{-1}, A \right] (z - A)^{-1} \, dz
\]
is in $L(X)$ with

\[
\|Q_\mu\|_{L(X)} \leq C \frac{c}{|\mu|^\gamma}, \quad (8.4.4)
\]
thanks to the commutator estimate. As $BS_\mu$ is a bounded operator on $X$ and $A$ is closed, this means, that even for every $x \in X$ we have $BS_\mu x \in D(A)$ and $ABS_\mu x = (1 + Q_\mu)x - \mu S_\mu x$. This implies $S_\mu x \in D(AB)$ for every $x \in X$ and

\[
(\mu + AB)S_\mu = 1 + Q_\mu
\]
in $L(X)$. Choosing $\nu \geq 0$, such that $\|Q_\mu\| \leq 1 < 1$ whenever $|\mu| \geq \nu$, we get the right inverse $R_\mu := S_\mu(1 + Q_\mu)^{-1}$ of $\mu + AB$. Summarizing we proved the following result.

**Lemma 8.4.3.** There exists $\nu \geq 0$ and for every $\gamma \in (0, \pi - \theta_A - \theta_B)$ there is a constant $C_\gamma \geq 0$ such that for every $\mu \in \Sigma_\gamma$ with $|\mu| \geq \nu$ the operator $\mu + AB$ is surjective with right inverse $R_\mu$ and

\[
\|R_\mu\| \leq C_{\gamma} \frac{1}{|\mu|} \left( 1 + \frac{1}{|\mu|^\gamma} \right).
\]

Before we start to prove that $\mu + AB$ is injective, we show that $D(AB) \cap D(A)$ is dense in $X$. We will need this later on, but on the other hand this also implies the density of $D(AB)$ in $X$, that we need to prove sectoriality of $\nu + AB$. In order to do so, fix $\lambda \in \Theta(B)$ and note that, thanks to $(\lambda - B)^{-1} D(A) \subseteq D(A)$ we have $(\lambda - B)^{-1} D(A) \subseteq D(AB) \cap D(A)$. Let $x \in X$. By density of $D(B)$, we approximate $x$ in $X$ by $(w_n) \subseteq D(B)$ and set $y_n := (\lambda - B)w_n$. Since $D(A)$ is also dense in $X$, for every $n \in \mathbb{N}$, there is a $z_n \in D(A)$, such that $\|z_n - y_n\| \leq 1/n$. Now, $(\lambda - B)^{-1}z_n \in D(AB) \cap D(A)$ for every $n \in \mathbb{N}$ and

\[
\|(\lambda - B)^{-1}z_n - x\| \leq \|(\lambda - B)^{-1}\|\|z_n - y_n\| + \|w_n - x\| \leq \frac{C}{n} + \|w_n - x\| \longrightarrow 0 \quad (n \to \infty).
\]

**Lemma 8.4.4.** There is a constant $\nu \geq 0$ such that the operator $\mu + AB$ is injective for all $\mu \in \Sigma_{\pi - \theta_A - \theta_B}$ with $|\mu| \geq \nu$.

**Proof.** The beginning of the proof is very similar to the construction of the right inverse, we just look at $T_\mu B$ instead of $BS_\mu$. Let $x \in D(AB) \cap D(A)$. Then we have $Bx \in D(A)$ and

\[
T_\mu Bx = \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{1}{z^2} A(z - A)^{-1} \left( \frac{\mu}{z} + B \right)^{-1} Bx \, dz
\]

\[
= A^{-1} x - \frac{\mu}{2\pi i} \int_{\Gamma_\nu} \frac{1}{z^2} (z - A)^{-1} \left( \frac{\mu}{z} + B \right)^{-1} x \, dz,
\]

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analogously to $(8.4.3)$. As $x$ is supposed to be in $D(A)$, this again implies $T_\mu Bx = A^{-1}(x - \mu T_\mu x)$. We deduce as before, that $AT_\mu B \in \mathcal{L}(X)$ and $AT_\mu B = I - \mu T_\mu$ thanks to the density of $D(AB) \cap D(A)$ in $X$.

Since $x \in D(AB)$ we may commute $A$ and $T_\mu$ and we see, that

$$T_\mu (\mu + AB)x = x - [T_\mu, A]Bx, \quad x \in D(AB).$$

In order to prove injectivity of $\mu + AB$, choose $x \in D(AB)$ with $(\mu + AB)x = 0$. Then our considerations above yield $x = [T_\mu, A]Bx = \mu[T_\mu, A]A^{-1}x$. Choosing $a \in (\beta, 1 - \alpha)$ and applying $A^{-a}$ to this identity, we obtain

$$A^{-a}x = \frac{\mu}{2\pi i} \int_{\Gamma_\mu} \frac{1}{z^2} A^{1-a}(z-A)^{-1} \left[ \left( \frac{\mu}{z} + B \right)^{-1}, A \right] A^{-1+a}A^{-a}x \, dz$$

$$= -\frac{\mu}{4\pi} \int_{\Gamma_\mu} \frac{A^{1-a}(z-A)^{-1}}{z^2\xi^{1-a}} \left[ \left( \frac{\mu}{z} + B \right)^{-1}, A \right] \beta, (\zeta - A)^{-1} A^{-a}x \, d\zeta \, dz,$n

by the Dunford calculus. The commutator estimate and $\|A^{1-a}(z-A)^{-1}\| \leq C|z|^{-a}$ then yields

$$\|A^{-a}x\| \leq C \frac{c}{|\mu|^\beta} \int_{\Gamma_\mu} \frac{1}{1 + |z|} \, d|z| \int_{\Gamma_\mu} \frac{1}{|\zeta|^{1-a}(1 + |\zeta|)^{1-\alpha}} \, d|\zeta| \|A^{-a}x\|.$$n

By the choice of $a$, these two integrals converge, so we end up with

$$\|A^{-a}x\| \leq C \frac{c}{|\mu|^\beta} \|A^{-a}x\|.$$n

If we choose $\nu$ so big, that $Cc/|\mu|^\beta < 1$, this implies $A^{-a}x = 0$, and hence $x = 0$, finishing the proof.

Summarizing, we have proved Theorem 8.3.1: In fact, if we choose $\nu \geq 0$ big enough, then, for all $\mu \in \Sigma - \theta_A - \theta_B \setminus B(0, \nu)$ the operator $\mu + AB$ is surjective by Lemma 8.4.3 and injective by Lemma 8.4.4. Furthermore $D(AB)$ and $R(AB)$ are dense in $X$ and for every $\gamma \in (0, \pi - \theta_A - \theta_B)$ we have the resolvent estimate

$$\|(\mu + AB)^{-1}\| = \|R_\mu\| \leq \frac{C_\gamma}{|\mu|} \left( 1 + \frac{1}{|\mu|^\beta} \right), \quad \mu \in \Sigma \gamma \setminus B(0, \nu).$$n

This implies, that $\varphi_{\nu + AB} \leq \theta_A + \theta_B$.

## 8.5. Bounded $H^\infty$-calculus for $\nu + AB$

In order to prove a bounded $H^\infty$-calculus for $\nu + AB$ with $\varphi_{\nu + AB} \leq \theta_A + \theta_B$, we choose angles $\phi$ and $\eta$ with $\theta_A + \theta_B < \phi < \eta < \pi$ and $f \in H_0^\infty(\Sigma_\eta)$. We have to prove the estimate

$$\left\| \int_{\Gamma_\phi} f(\lambda)(\lambda - (\nu + AB))^{-1} \, d\lambda \right\| \leq C\|f\|_\infty.$$ (8.5.1)

Here, we choose $\nu \geq 0$ in such a way, that by Theorem 8.3.1 the operator $\nu + AB$ is sectorial. Thus the resolvent in the above integral exists and can be repesented as $(\lambda - (\nu + AB))^{-1} = -S_{\nu - \lambda}(1 + Q_{\nu - \lambda})^{-1}$. Note that, as $|\arg(-\lambda)| = \pi - \phi < \pi - \theta_A - \theta_B$, we have $-\lambda \in \Sigma_\gamma$ for some $\gamma \in (\pi - \phi, \pi - \theta_A - \theta_B)$, so that $S_{\nu - \lambda}$ and $Q_{\nu - \lambda}$ are well defined.
8. $H^\infty$-Calculus for Products of Non-Commuting Operators

As we have $S_\nu(1+Q_\mu)^{-1} = S_\mu - S_\mu Q_\mu(1+Q_\mu)^{-1} =: S_\mu + P_\mu$, we may split the integral for (8.5.1) into two parts, namely

$$\int_{\Gamma_\phi} f(\lambda)(\lambda - (\nu + AB))^{-1} \, d\lambda = -\int_{\Gamma_\phi} f(\lambda)S_{\nu^{-}\lambda} \, d\lambda - \int_{\Gamma_\phi} f(\lambda)P_{\nu^{-}\lambda} \, d\lambda =: J_1 + J_2.$$ 

The easy part is to estimate $J_2$, as we may simply take the norm into the integral. Using Lemma 8.4.2 and (8.4.4), we get

$$\left\| \int_{\Gamma_\phi} f(\lambda)P_{\nu^{-}\lambda} \, d\lambda \right\| \leq \|f\|_\infty \int_{\Gamma_\phi} \|S_{\nu^{-}\lambda}Q_{\nu^{-}\lambda}(1+Q_{\nu^{-}\lambda})^{-1}\| \, |d|\lambda| \leq C\|f\|_\infty,$$

thanks to $\beta > 0$.

In order to estimate $J_1$, we calculate for $x \in D(A)$, using the definition of $S_\mu$

$$\int_{\Gamma_\phi} f(\lambda)S_{\nu^{-}\lambda}x \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\phi} f(\lambda) \int_{\Gamma_\gamma} \frac{1}{z^2} \left( \frac{\nu - \lambda}{z} + B \right)^{-1} A(z - A)^{-1} x \, dz \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \frac{1}{z} \int_{\Gamma_\phi} f(\lambda)(\lambda - (\nu + zB))^{-1} \, d\lambda A(z - A)^{-1} x \, dz.$$

As $B \in H^\infty(X)$, by permanence properties of the bounded $H^\infty$-calculus (see e.g. [DHP03, Proposition 2.11]), the operator $\nu + zB$ also is in $H^\infty(X)$ with $\varphi^\infty_{\nu + zB} \leq \varphi^\infty_{\nu} + |\arg(z)| < \theta_B + \phi - \theta_B = \phi$. Thus we may write

$$J_1 = -\int_{\Gamma_\phi} f(\lambda)S_{\nu^{-}\lambda}x \, d\lambda = \int_{\Gamma_\gamma} \frac{1}{z} f(\nu + zB)A(z - A)^{-1} x \, dz.$$

By Cauchy’s Theorem we may rewrite this for an arbitrary number $a \in (0, 1)$ as

$$\int_{\Gamma_\gamma} \frac{1}{z^a} f(\nu + zB)A^a(z - A)^{-1} x \, dz,$$

in the same manner as in the proof of Lemma 8.4.2. The resulting integral now also looks very much like the integral we started the proof of this lemma with (we just have $f(\nu + zB)$ instead of $(\nu + zB)^{-1}$). Indeed, the method here will be exactly the same as in the proof of Lemma 8.4.2, so we will freely use the notations introduced there and only indicate the differences in the proof.

After commuting $(A^a(z - A)^{-1})^b$ with $f(\nu + zB)$ we now end up with

$$J_1 = -\int_{\Gamma_\gamma} \frac{1}{z^a} (A^a(z - A)^{-1})^b f(\nu + zB)(A^a(z - A)^{-1})^{1-b} x \, dz$$

$$-\int_{\Gamma_\gamma} \frac{1}{z^a} \left[ f(\nu + zB), (A^a(z - A)^{-1})^b \right] (A^a(z - A)^{-1})^{1-b} x \, dz$$

$$=: J_{11} + J_{12}.$$

Evaluating the commutator in $J_{12}$ this time leads to a double integral:

$$\left[ f(\nu + zB), (A^a(z - A)^{-1})^b \right]$$

$$= \frac{1}{4\pi i} \int_{\Gamma_\phi} \int_{\Gamma_\gamma} f(\lambda) \frac{\zeta a b}{(z - \zeta)^b} \left[ (\lambda - (\nu + zB))^{-1}, (\zeta - A)^{-1} \right] \, d\zeta \, d\lambda.$$
As we may again apply our commutator estimates, thanks to

\[
\left[ (\lambda - (\nu + zB))^{-1}, (\zeta - A)^{-1} \right] = \frac{1}{z}(\zeta - A)^{-1} \left[ \left( \frac{\nu - \lambda}{z} + B \right)^{-1}, A \right] (\zeta - A)^{-1},
\]

we find

\[
\left\| \left[ f(\nu + zB), (A^a(z - A)^{-1})^b \right] \right\| 
\leq C\|f\|_{\infty}^\eta \int_{\Gamma_\alpha} \int_{\Gamma_\alpha^c} \frac{1}{|z - \zeta|^b} \| (\zeta - A)^{-1} \left[ \left( \frac{\nu - \lambda}{z} + B \right)^{-1}, A \right] (\zeta - A)^{-1} \| d|\zeta| d|\lambda|
\leq C\|f\|_{\infty}^\eta |z|^{-b} \int_{\Gamma_\alpha} \frac{1}{|\nu - \lambda|^{1+\beta}} d|\lambda| \int_{\Gamma_\alpha^c} \frac{|z|^b |\zeta|^{a b}}{|z - \zeta|^b (1 + |\zeta|)^{2-\alpha}} d|\zeta| \leq C\|f\|_{\infty}^\eta |z|^{\beta - b}.
\]

Here the first integral converges thanks to \( \beta > 0 \) and the second is of exactly the same form as in (8.4.1). Now we can conclude \( \|J_{12}\| \leq C\|f\|_{\infty}^\eta \|x\| \) as before.

Also for \( J_{11} \) we may do the same calculations as we did for \( I_1 \) in the proof of Lemma 8.4.2 until we reach the line

\[
\int_1^2 \left\| \sum_{k=n_0}^{N-1} \varepsilon_k f(\nu + 2^k s e^{i\varphi} B) \hat{g}_e^{i\varphi} \left( \frac{A}{2^k s} \right) x \right\|_{L^2(\Omega;X)} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_e^{i\varphi} \left( \frac{A^*}{2^k s} \right) x^* \right\|_{L^2(\Omega;X^*)} \frac{ds}{s},
\]

where we used the \( \mathcal{R} \)-sectoriality of \( B \) before. Evidently, now the \( \mathcal{R} \)-bounded \( H^\infty \)-calculus of \( B \) does the job. Thus we end up with

\[
C_R \|f\|_{\infty}^\eta \int_1^2 \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_e^{i\varphi} \left( \frac{A}{2^k s} \right) x \right\|_{L^2(\Omega;X)} \left\| \sum_{k=n_0}^{N-1} \varepsilon_k \hat{g}_e^{i\varphi} \left( \frac{A^*}{2^k s} \right) x^* \right\|_{L^2(\Omega;X^*)} \frac{ds}{s},
\]

where \( C_R := \mathcal{R} \left( \{ f(\nu + 2^k s e^{i\varphi} B) : f \in H^\infty_0(\Sigma_0), \|f\|_{\infty}^\eta \leq 1 \} \right) \). As

\[
\{ f(\nu + 2^k s e^{i\varphi} B) : f \in H^\infty_0(\Sigma_0), \|f\|_{\infty}^\eta \leq 1 \} \subseteq \{ f(B) : f \in H^\infty_0(\Sigma_{\theta_B}), \|f\|_{\infty}^\eta \leq 1 \}
\]

and \( \theta_B > \varphi_B^{\infty} \), this \( \mathcal{R} \)-bound is finite. Now we estimate the two remaining norms as before, getting \( \|J_1\| \leq C\|f\|_{\infty}^\eta \|x\| \) for all \( x \in D(A) \) in the end. A density argument again finishes the proof. \( \square \)
References (Article 8)


References (Article 8)

