RECURSION AND EFFECTIVITY IN THE DECOMPOSABILITY CONJECTURE

VASSILIOS GREGORIADES AND TAKAYUKI KIHARA

Abstract. Using ideas and results from recursion theory and effective descriptive set theory we provide a partial answer to the Decomposability Conjecture in Polish spaces of small inductive dimension. The latter extends earlier work of Motto Ros and Pawlikowski-Sabok. We also state a conjecture, which would allow our arguments to be carried out in all Polish spaces, and we give the analogous results in the transfinite case of the problem. Finally we present an approach to the later, where we utilize the technique of turning Borel-measurable functions into continuous ones.

1. Introduction

By \( \omega \) we mean the first infinite ordinal which we identify with the set of all natural numbers (starting with 0) and by \( \mathcal{N} \) we mean the Baire space \( \mathbb{N}^{\mathbb{N}} \). The Cantor space is \( 2^{\omega} \). We employ the standard notation \( \Sigma^0_m, \Pi^0_m, \Delta^0_m \) of Borel pointclasses of finite order. Given a function \( f : X \to Y \) between topological spaces we write \( f \in \operatorname{dec}(\Sigma^0_m) \) if there exists a sequence \( \langle X_i \rangle_{i \in \omega} \) of subsets of \( X \) such that \( X = \bigcup_i X_i \) and \( f \upharpoonright X_i \) is \( \Sigma^0_m \)-measurable for all \( i \in \omega \). Of course one may assume without loss of generality that the preceding sets \( X_i \) are disjoint. We also write \( f \in \operatorname{dec}(\Sigma^0_m, \Delta^0_n) \) if there exists a sequence \( \langle X_i \rangle_{i \in \omega} \) of disjoint \( \Delta^0_n \) sets such that \( f \upharpoonright X_i \) is \( \Sigma^0_m \)-measurable for all \( i \in \omega \). It is clear that \( f \in \operatorname{dec}(\Sigma^0_m, \Delta^0_n) \) exactly when there exists a sequence \( \langle Z_i \rangle_{i \in \omega} \) of (not necessarily disjoint) \( \Pi^0_{n-1} \) sets such that \( f \upharpoonright Z_i \) is \( \Sigma^0_m \)-measurable for all \( i \in \omega \). By \( f^{-1} \subseteq Y \) we mean that for all \( \Sigma^0_m \) sets \( A \subseteq Y \) the preimage \( f^{-1}[A] \) is a \( \Sigma^0_n \) subset of \( X \).

It is not hard to verify that if \( f \in \operatorname{dec}(\Sigma^0_m, \Delta^0_n) \) then condition \( f^{-1} \Sigma^0_m \subseteq \Sigma^0_n \) is also satisfied as well. The well-known Jayne-Rogers Theorem (cf. [5]) states that the converse is also true for \( n = m = 2 \), whenever \( X \) is an analytic subset of a Polish space and \( Y \) is separable metrizable. It has been conjectured that the Jayne-Rogers Theorem can be extended to all finite levels of the Borel hierarchy.

The Decomposability Conjecture (cf. [1], [13], [16]). Suppose that \( X \) is an analytic subset of a Polish space and that \( Y \) is separable metrizable. For every function \( f : X \to Y \) and every \( n \geq 2 \) it holds

\[
\text{(Special Case)} \quad f^{-1} \Sigma^0_n \subseteq \Sigma^0_n \quad \implies \quad f \in \operatorname{dec}(\Sigma^1, \Delta^0_n).
\]

More generally for all \( 2 \leq m \leq n \) we have

\[
\text{(General Case)} \quad f^{-1} \Sigma^0_m \subseteq \Sigma^0_n \quad \implies \quad f \in \operatorname{dec}(\Sigma^0_{m+1}, \Delta^0_n).
\]

The cases \( m = 2, n = 3 \) and \( m = n = 3 \) have been answered positively by Semmes for functions \( f : \mathcal{N} \to \mathcal{N} \) cf. [17]. Another interesting result is the following.
Theorem 1.1 (Motto Ros [13], Pawlikowski-Sabok cf. [16]). Suppose that $X$ is analytic, $Y$ is separable metrizable and $f : X \to Y$ satisfies $f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$ for some $n \geq 3$. Then $f \in \text{dec}(\Sigma^0_n)$.

If moreover $f$ is $\Sigma^0_{n-1}$-measurable, or more generally if $f$ has a $\Sigma^0_n$ graph, then $f \in \text{dec}(\Sigma^0_n, \Delta^0_n)$.

Kihara cf. [7] has proved a related result based on the idea of witnessing condition $f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$ in a “continuous uniform way”. (We will mention this result in the sequel after giving the necessary definitions.) In this article we build upon the ideas of [7] and we prove a result similar to Theorem 1.1 in the general case of $m, n$ and for some category of spaces, which includes the Baire space and the reals.

Theorem 1.2. Suppose that $n \geq m \geq 2$, $A \subseteq N$ is analytic, and $f : A \to N$ satisfies $f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$. Then $f \in \text{dec}(\Sigma^0_{n-m+1}, \Delta^0_{n+1})$.

If moreover $m \geq 3$ and $f$ is $\Sigma^0_{n-1}$-measurable then $f \in \text{dec}(\Sigma^0_{n-m+1}, \Delta^0_n)$.

The preceding result reduces the general case of the Decomposability Conjecture to the cases $m = 2 \leq n$.\footnote{Using the same method as [7] and results of Motto Ros-Schlicht-Selivanov [14] one can see that this result extends to spaces of small inductive dimension.} We refer to Corollary 2.6 for a detailed explanation.

Notation. We continue with the notation from above. By $\omega^{<\omega}$ we mean the set of all finite sequences in $\omega$. We fix once and for all the following encoding of finite sequences of naturals,

$$\langle \cdot \rangle : \omega^{<\omega} \to \omega : \langle u_0, u_1, \ldots, u_{n-1} \rangle \mapsto p_0^{u_0+1} \cdot p_1^{u_1+1} \cdot \ldots \cdot p_{n-1}^{u_{n-1}+1},$$

where $(p_n)_{n \in \omega}$ is the increasing enumeration of all prime numbers. For $i, n \in \omega$ and $\alpha \in N$ we put

$$\alpha_i(n) = \alpha(\langle i, n \rangle)$$

$$\bar{\sigma}(n) = \langle \alpha(0), \ldots, \alpha(n-1) \rangle.$$ 

Unless stated otherwise by $\alpha^*$ we mean the function $(n \mapsto \alpha(n+1))$.

For $A, B \subseteq \omega$ we write $A \oplus B$ for the set

$$\{\langle i, n \rangle : (i = 0 \& n \in A) \lor (i = 1 \& n \in B)\}.$$ 

By $A \oplus B \oplus C$ we mean $(A \oplus B) \oplus C$.

We will often identify sets with relations and write $P(x)$ instead of $x \in P$ for $P \subseteq X$. For every $P \subseteq X \times Y$ and every $x \in X$ we put

$$P_x = \{y \in Y : P(x, y)\}$$

$$\exists Y P = \{x \in X : \exists y \in Y P(x, y)\}.$$ 

A partial function from the set $X$ to $Y$ is a function defined on a (perhaps empty) subset of $X$. Partial functions will be denoted by $f : X \to Y$. By $f(x) \downarrow$ we mean that $f$ is defined on $x$. The domain $\text{Dom}(f)$ of $f$ is the set of all $x \in X$ for which $f(x) \downarrow$.

The first uncountable ordinal is denoted by $\omega_1$ as usual. In addition to the pointclasses of finite Borel order we consider the standard pointclasses of transfinite
order $\Sigma^0_\xi$, $\Pi^0_\xi$, $\Delta^0_\xi$, for $\xi < \omega_1$, as well as the analytical pointclasses $\Sigma^1_\xi$, $\Pi^1_\xi$ and $\Delta^1_\xi$, where $n < \omega$.

In this article we employ a variety of tools from the areas of recursion theory and effective descriptive set theory. The most notable notions include recursive functions and sets, Turing reducibility, transfinite iterations of the Turing jump, recursively presented metric spaces, and the lightface pointclasses $\Sigma^0_\xi$, $\Sigma^1_n$, $\Pi^0_\xi$, $\Pi^1_n$ and $\Delta^0_n$. For reasons of self-containedness we include a brief review of the preceding notions in the Appendix at the end of this article.

Given $A, B \subseteq \omega$ we write $A \leq_T B$ to denote that $A$ is Turing reducible in $B$. By $\omega^A_1$ we mean the least non $A$-recursive ordinal and by $A^{(\xi)}$ the $\xi$-th Turing jump of $A$ for $\xi < \omega^A_1$. We will often identify subsets of the naturals with members of the Cantor space.

We continue with more central notions and theorems, which are necessary for proving our results.

**Universal sets and good universal systems.** Suppose that $\mathcal{Y}$ and $\mathcal{Z}$ are Polish spaces and that $\Gamma$ is a pointclass. We denote by $\Gamma \upharpoonright \mathcal{Y}$ the family of all subsets of $\mathcal{Y}$ which are in $\Gamma$.

A set $G \subseteq \mathcal{Z} \times \mathcal{Y}$ parametrizes $\Gamma \upharpoonright \mathcal{Y}$ if for all $P \subseteq \mathcal{Y}$ we have that

$$P \in \Gamma \iff \exists \, z \in \mathcal{Z} \text{ such that } P = G_z.$$  

We think of $y$ as a $\Gamma$-code for $P$.

The set $G \subseteq \mathcal{Z} \times \mathcal{Y}$ is universal for $\Gamma \upharpoonright \mathcal{Y}$ if $G$ is in $\Gamma$ and parametrizes $\Gamma \upharpoonright \mathcal{Y}$.

A pointclass $\Gamma$ is $\mathcal{Z}$-parametrized if for all Polish spaces $\mathcal{Y}$ there exists some $G \subseteq \mathcal{Z} \times \mathcal{Y}$ which is universal for $\Gamma \upharpoonright \mathcal{Y}$.

A $\mathcal{Z}$-universal system for $\Gamma$ is an assignment $\mathcal{Y} \mapsto G^\mathcal{Y} \subseteq \mathcal{Z} \times \mathcal{Y}$ such that the set $G^\mathcal{Y}$ is universal for $\Gamma \upharpoonright \mathcal{Y}$ for all Polish spaces $\mathcal{Y}$. We will often take the case $\mathcal{Z} = \mathcal{N}$ and mention universal system instead of $\mathcal{N}$-universal system for $\Gamma$.

A universal system $(G^\mathcal{Y})_{\mathcal{Y}}$ for $\Gamma$ is good if for all spaces $\mathcal{X}$ of the form $\omega^k \times \mathbb{N}^t$, (where $k, t \geq 0$) and for all Polish spaces $\mathcal{Y}$ there exists a continuous function

$$S^{\mathcal{X}, \mathcal{Y}} \equiv S : \mathcal{N} \times \mathcal{X} \to \mathcal{N}$$

such that

$$G^{\mathcal{X} \times \mathcal{Y}}(\varepsilon, x, y) \iff G^\mathcal{Y}(S(\varepsilon, x), y).$$  

(1)

Pointclasses which admit good universal systems carry a smooth structure. For example if $\Gamma$ admits a good universal system and has sufficient closure properties (say closure under union and closure under continuous substitution), then $\Gamma$ is uniformly closed under union, i.e., for all Polish spaces $\mathcal{Y}$ and all $P, Q \subseteq \mathcal{Y}$ in $\Gamma$ there exists a continuous function

$$u : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$$

such that if $\alpha$ and $\beta$ are codes for $P$ and $Q$ then $u(\alpha, \beta)$ is a code for $P \cup Q$. (See the Uniform Closure Theorem 3H.2 in [11].) The following is another interesting consequence of good universal systems.

**Lemma 1.3** (Folklore). Suppose that $\Gamma$ is closed under continuous substitution and admits a good universal system. Then for all $P \subseteq \mathcal{N} \times \mathcal{Y}$ in $\Gamma$ there exists a continuous function $u : \mathcal{N} \to \mathcal{N}$ such that $u(x)$ is a code (with respect to the given good universal system) of the section $P_x$ for all $x \in \mathcal{N}$.
Proof. Let \((G^\mathcal{X})_\mathcal{X}\) be a good universal system for \(\Gamma\). Since the set \(P\) is in \(\Gamma\) there exists some \(\epsilon \in \mathcal{N}\) such that
\[
P(x, y) \iff G^{\mathcal{N} \times \mathcal{Y}}(\epsilon, x, y)
\]
\[
\iff G^\mathcal{Y}(S(\epsilon, x), y).
\]
So we take the function \(u : \mathcal{N} \to \mathcal{N} : u(x) = S(\epsilon, x)\).

The pointclasses \(\Sigma^0_n\) admit in a natural way a good universal system, which also behaves well under the recursive-theoretic aspect. The latter is a direct consequence of the Good Parametrization Theorem (cf. 3H.1 in [11]) and the properties of the lightface pointclasses \(\Sigma^0_n\) (cf. 3E.2 and 3F.6 in [11]). One can however can use the following fundamental system.

**Definition 1.4.** For every Polish space \(\mathcal{Y}\) we define the sets \(G^\mathcal{Y}_n \subseteq \mathcal{N} \times \mathcal{Y}\) recursively on \(n \geq 1\) as follows
\[
G^\mathcal{Y}_1(\epsilon, y) \iff (\exists k)[y \in N(\mathcal{Y}, \epsilon(k))]
\]
\[
G^\mathcal{Y}_{n+1}(\epsilon, y) \iff (\exists k) \neg G^\mathcal{Y}_n((\epsilon)_k, y).
\]
It is clear that \(G^\mathcal{Y}_n\) is universal for \(\Sigma^0_n \upharpoonright \mathcal{Y}\).

**Lemma 1.5** (See also 3H.1 in [11]). Suppose that \(\mathcal{Y}\) is a recursively presented metric space and that \(n \geq 1\). Then the set \(G^\mathcal{Y}_n\) belongs to \(\Sigma^0_n\). Moreover for all \(P \subseteq \mathcal{Y}\) we have that
\[
P \in \Sigma^0_n \iff P \text{ is the } \epsilon\text{-section of } G^\mathcal{Y}_n \text{ for some recursive } \epsilon \in \mathcal{N}\]
and similarly for all \(\gamma\),
\[
P \in \Sigma^0_n(\gamma) \iff P \text{ is the } \epsilon\text{-section of } G^\mathcal{Y}_n \text{ for some } \gamma\text{-recursive } \epsilon \in \mathcal{N}.
\]
Also for every space \(\mathcal{X}\) of the form \(\omega^s \times \mathcal{N}^t\), \(s, t \geq 0\) there exists a recursive function \(S^\mathcal{X} : \mathcal{N} \times \mathcal{X} \to \mathcal{N}\) such that
\[
G_n^{\mathcal{X} \times \mathcal{Y}}(\epsilon, x, y) \iff G^\mathcal{Y}_n(S^\mathcal{X, \mathcal{Y}}(\epsilon, x), y)
\]
for all \(\epsilon, x, y\).

**Proof.** This is proved by induction on \(n \geq 1\). The set \(G^\mathcal{Y}_1\) is easily \(\Sigma^0_1\) from the closure properties of the latter class, see for example 3C.3 in [11]. Also the assertion that a set \(P \subseteq \mathcal{Y}\) is \(\Sigma^0_1\) if and only if it is the \(\epsilon\)-section of \(G^\mathcal{Y}_1\) for some recursive \(\epsilon\), is exactly the definition of \(\Sigma^0_1\). Before we prove the relativized version of the latter, we need to deal with the function \(S^\mathcal{X, \mathcal{Y}}_n\).

Since every space the form \(\omega^s \times \mathcal{N}^t\) for \(s, t \geq 0\) is recursively isomorphic to either \(\omega\) or \(\mathcal{N}\) it is enough to consider one of the following cases. Let us take the case \(\mathcal{X} = \mathcal{N}\), the other one being similar. There exists a recursive function \(h = (h_1, h_2) : \omega \to \omega^2\) such that
\[
N(\mathcal{N} \times \mathcal{Y}, m) = N(\mathcal{N}, h_1(m)) \times N(\mathcal{X}, h_2(m)).
\]

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3 Compare with the \(S^\mathcal{N}_n\) Theorem A.4.
4 If \(\mathcal{Y}\) is \(\alpha\)-recursively presented then the preceding statements hold by replacing the terms “recursive”, “\(\gamma\)-recursive”, “\(\Sigma^0_n\)” and “\(\Sigma^0_n(\gamma)\)” with “\(\alpha\)-recursive”, “\((\alpha, \gamma)\)-recursive”, “\(\Sigma^0_\alpha(\alpha)\)” and “\(\Sigma^0_\alpha(\alpha, \gamma)\)” respectively.
We compute
\[ G^{N \times Y}_{N \times Y}(\varepsilon, \alpha, y) \iff (\exists k)[(\alpha, y) \in N(N \times Y, \varepsilon(k))] \]
\[ \iff (\exists k)[\alpha \in N(N, h_1(\varepsilon(k))) \& y \in N(Y, h_2(\varepsilon(k)))] . \]
Hence we define
\[ S^{N \times Y}_{N \times Y}(\varepsilon, \alpha)(k) = \begin{cases} h_2(\varepsilon(k)), & \text{if } \alpha \in N(N, h_1(\varepsilon(k))) \\ 0, & \text{else} \end{cases} . \]

Clearly the function \( S^{N \times Y}_{N \times Y} \) is continuous and since \( N(X, 0) = \emptyset \) it follows that
\[ G^{N \times Y}_{N \times Y}(\varepsilon, \alpha, x) \iff (\exists k)[x \in N(X, S^{N \times Y}_{N \times Y}(\varepsilon, \alpha)(k))] \]
\[ \iff G^{N \times Y}_{N \times Y}(S^{N \times Y}_{N \times Y}(\varepsilon, \alpha), x) . \]
Now if \( P \subseteq Y \) is \( \Sigma^0_1(\gamma) \) there exists some \( Q \subseteq N \times Y \) in \( \Sigma^0_1 \) such that \( P = Q_\gamma \).
We choose some recursive \( \alpha \in N \) such that \( Q \) is the \( \alpha \)-section of \( G^{N \times Y}_{N \times Y} \) and we compute
\[ P(y) \iff Q(\alpha, y) \]
\[ \iff G^{N \times Y}_{N \times Y}(\alpha, \gamma, y) \]
\[ \iff G^{Y}_{Y}(S^{N \times Y}_{N \times Y}(\alpha, \gamma), y) , \]
i.e., \( P \) is the \( S^{N \times Y}_{N \times Y}(\alpha, \gamma) \)-section of \( G^{Y}_{Y} \). Moreover \( \varepsilon := S^{N \times Y}_{N \times Y}(\alpha, \gamma) \) is \( \gamma \)-recursive. Conversely if \( P \) is the \( \varepsilon \)-section of \( G^{Y}_{Y} \) for some \( \gamma \)-recursive \( \varepsilon \), it follows that \( P \in \Sigma^0_1(\varepsilon) \subseteq \Sigma^0_1(\gamma) \). This finishes the case \( n = 1 \).
The inductive step is treated similarly. What needs a slightly different treatment is the definition of \( S^{N \times Y}_{n+1} \); we have that
\[ G^{N \times Y}_{n+1}(\varepsilon, \alpha, y) \iff (\exists k)[G^{N \times Y}_{n}(\varepsilon_k, \alpha, y)] \]
\[ \iff (\exists k)[G^{Y}_{n}(S^{N \times Y}_{n}(\varepsilon_k, \alpha), y)] . \]
So we take
\[ S^{N \times Y}_{n+1}(\varepsilon, \alpha)((k, t)) := S^{N \times Y}_{n}(\varepsilon_k(\alpha))(t) . \]

\section*{Basic results from effective descriptive set theory.}

The following constitute some deep results from effective descriptive set theory that we will use in the sequel.

\begin{theorem}[cf. [9] p. 367 and [11] 4D.6] \end{theorem} Let \( X \) and \( Y \) be recursively presented metric spaces and \( P \subseteq X \times Y \) be \( \Pi^1_1(\varepsilon) \) for some \( \varepsilon \in N \). Then there exists a partial \( \Pi^1_1(\varepsilon) \)-recursive function
\[ f : X \rightarrow Y \]
such that
\begin{enumerate}
\item \( f(x) \downarrow \iff \exists y \in \Delta^1_1(\varepsilon, x)P(x, y) \) and
\item \( P(x, f(x)) \) for all \( x \) for which \( f(x) \) is defined.
\end{enumerate}

Using that every total \( \Pi^1_1(\varepsilon) \)-recursive function is \( \Delta^1_1(\varepsilon) \)-recursive, cf. 4C.3 [11] and Theorem 1.6 one obtains the following.
Theorem 1.7 ([9] p. 367). Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are recursively presented metric spaces and that $P \subseteq \mathcal{X} \times \mathcal{Y}$ is $\Pi^1_1(\varepsilon)$ for some $\varepsilon \in \mathcal{N}$, which satisfies

$$(\forall x)(\exists y \in \Delta^1_1(\varepsilon, x)) P(x, y).$$

Then there exists a $\Delta^1_1(\varepsilon)$-recursive function $f : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$P(x, f(x))$$

for all $x \in \mathcal{X}$.

It is a well-known result that $f(x) \in \Delta^1_1(x)$ for a $\Delta^1_1$-recursive $f$, and hence in the case of the Cantor space there exists some $\xi < \omega^{CK}$ such that $f(x) \leq_T x^{(\xi)}$. The preceding $\xi$ can in fact be chosen uniformly on $x$.

Lemma 1.8 (Folklore). Suppose that $A$ is a $\Sigma^1_1(A)$ subset of $2^\omega$ for some $A \subseteq \omega$ and that $f : A \rightarrow 2^\omega$ is $\Delta^1_1(A)$-recursive on its domain. Then there exists some $\xi < \omega^A_1$ such that

$$f(x) \leq_T (x \oplus A)^{(\xi)}$$

for all $x \in X$.

Proof. For every $x \in A$, since $f$ is $\Delta^1_1(A)$-recursive we have that $f(x) \in \Delta^1_1(A, x)$. It follows that there exists some $\xi < \omega^A_1$ such that $f(x) \leq_T (x \oplus A)^{(\xi)}$. Let $(S^A, | \cdot |_{S^A})$ be the notation system of ordinals that we fix in paragraph C of the Appendix. We consider the set $P \subseteq 2^\omega$ defined by

$$P(x, i) \iff i \in S^A \land [x \notin A \lor f(x) \leq_T (x \oplus A)^{|i|_{S^A}}].$$

Then $P$ is in $\Pi^1_1(A)$ and for all $x \in 2^\omega$ there exists some $i$ such that $P(x, i)$. It follows from Theorem 1.7 that there exists a $\Delta^1_1(A)$-recursive function $h : 2^\omega \rightarrow \omega$ such that $P(x, h(x))$ for all $x \in 2^\omega$.

The image $h[2^\omega]$ is a $\Sigma^1_1(A)$ subset of $S^A$. Hence from $\Sigma^1_1$-Boundedness there exists some $i_0 \in S^A$ such that $|h(x)|_{S^A} \leq |i_0|_{S^A}$ for all $x \in 2^\omega$. So if $x$ belongs to $A$ we have that

$$f(x) \leq_T (x \oplus A)^{|h(x)|_{S^A}} \leq_T (x \oplus A)^{|i_0|_{S^A}}.$$

Therefore we take $\xi = |i_0|_{S^A}$ and we are done. $\dashv$

Another important tool is the following result of Louveau. Let us recall that a set $C$ separates $A$ from $B$ or that $A$ is separated from $B$ by $C$ if $A \subseteq C$ and $C \cap B = \emptyset$. It is clear that if $C$ separates $A$ from $B$ then the complement of $C$ separates $B$ from $A$.

Theorem 1.9 (Louveau separation cf. [9]). Suppose that $\mathcal{X}$ is a recursively presented metric space and that $A, B$ are disjoint $\Sigma^1_1$ subsets of $\mathcal{X}$. If $A$ is separated from $B$ by a $\Pi^0_\xi$ set, then $A$ is separated from $B$ by a $\Pi^0_\xi(\gamma)$ set for some $\gamma \in \Delta^1_1$.

In particular if $A$ is $\Pi^0_\xi$ and $\Delta^1_1$, then it is $\Pi^0_\xi(\gamma)$ for some $\gamma \in \Delta^1_1$. The analogue statement holds for the pointclass $\Delta^1_1(\varepsilon)$ in the place of $\Delta^1_1$ for every $\varepsilon \in \mathcal{N}$ (relativized version).
2. Borel transition in the codes and decomposability

In this section we elaborate on the techniques of \([7]\), and after establishing some intermediate results, which are interesting in their own right, we prove Theorem 1.2. We also give the analogous results in the case of transfinite levels of the Borel hierarchy.

**Definition 2.1.** Suppose that \(\Gamma\) is a given pointclass, \(X, Y\) are topological spaces, \(\emptyset \neq A \subseteq X\) and \(f : A \to Y\) is a function which satisfies \(f^{-1}\Sigma^0_m \subseteq \Sigma^0_n\) for some \(m, n \geq 1\). We say that condition \(f^{-1}\Sigma^0_m \subseteq \Sigma^0_n\) holds \(\Gamma\)-uniformly (in the codes with respect to \(G^\omega \times Y, G^\omega \times X\)) if there exists a \(\Gamma\)-measurable function \(u : \mathcal{N} \to \mathcal{N}\) such that for all \(\alpha, i\) and all \(x \in A\) we have that

\[
G^\omega \times Y(\alpha, i, f(x)) \iff G^\omega \times X(u(\alpha), i, x).
\]

If \(\Gamma = \Sigma^1_1\) we say “continuous-uniformly”, and if \(\Gamma\) is the class of all Borel sets we say “Borel-uniformly”.

The condition of continuous-uniform transition in the codes has the following interesting application.

**Theorem 2.2** (Kihara cf. \([7]\)). Suppose that for a function \(f : \mathcal{N} \to \mathcal{N}\) condition \(f^{-1}\Sigma^0_m \subseteq \Sigma^0_n\) holds \(\Gamma\)-uniformly for some \(m, n\) with \(3 \leq m \leq n < 2m - 1\). Then \(f \in \text{dec}(\Sigma^0_{n-m+1}, \Delta^0_n)\). The similar assertion holds for spaces of small inductive dimension.

Using the results from effective descriptive set theory that we stated above, we prove that condition \(f^{-1}\Sigma^0_m \subseteq \Sigma^0_n\), whenever holds, holds also Borel-uniformly:

**Theorem 2.3.** Suppose that \(X, Y\) are Polish spaces, \(A \subseteq X\) is analytic and that \(f : A \to Y\) satisfies \(f^{-1}\Sigma^0_m \subseteq \Sigma^0_n\) for some \(m, n \geq 1\). Then condition \(f^{-1}\Sigma^0_m \subseteq \Sigma^0_n\) holds Borel-uniformly in the codes with respect to \(G^\omega \times Y, G^\omega \times X\).

**Proof.** We fix some \(\varepsilon \in \mathcal{N}\) such that \(X\) and \(Y\) are \(\varepsilon\)-recursively presented, \(A\) belongs to \(\Sigma^1_1(\varepsilon)\) and \(f\) is \(\Delta^1_1(\varepsilon)\)-recursive. (Notice that the function \(f\) is in particular \(\Sigma^0_n\)-measurable.)

We define the sets \(P, Q \subseteq \mathcal{N} \times \omega \times X\) by

\[
P(\alpha, i, x) \iff x \in A \& G^\omega \times Y(\alpha, i, f(x))
\]

\[
Q(\alpha, i, x) \iff x \in A \& \neg G^\omega \times Y(\alpha, i, f(x)).
\]

Since \(Y\) is \(\varepsilon\)-recursively presented by using the relativized version of Lemma 1.5 the set \(G^\omega \times Y\) is \(\Sigma^0_m(\varepsilon)\). It follows that the sets \(P, Q\) are \(\Sigma^1_1(\varepsilon)\).

We now fix some \(\alpha \in \mathcal{N}\) for the discussion. It is clear that the sections \(P_\alpha, Q_\alpha\) are disjoint \(\Sigma^1_1(\varepsilon, \alpha)\) subsets of \(\omega \times X\).

For all \(i \in \omega\) the section

\[
P_\alpha, i = \{ x \in A : G^\omega \times Y(\alpha, i, f(x)) \}
\]

is the preimage of a \(\Sigma^0_n\) subset of \(Y\) under \(f\). From our hypothesis about \(f\) there exists a \(\Sigma^0_n\) set \(R_{\alpha, i} \subseteq X\) such that \(P_\alpha, i = R_{\alpha, i} \cap A\). The set

\[
R_\alpha = \{ (i, x) \in \omega \times X : x \in R_{\alpha, i} \}
\]

is also a \(\Sigma^0_n\) subset of \(\omega \times X\) and it is easy to check that

\[
P_\alpha \subseteq R_\alpha \quad \text{and} \quad R_\alpha \cap Q_\alpha = \emptyset.
\]
Therefore \( P_\alpha, Q_\alpha \) are disjoint \( \Sigma^0_1(\varepsilon, \alpha) \) subsets of \( \omega \times \mathcal{X} \) and \( P_\alpha \) is separated from \( Q_\alpha \) by a \( \Sigma^0_1 \) set. It follows from Louveau’s Theorem 1.9 (relativized version in the context of \( \varepsilon \)-recursively presented metric spaces) that \( P_\alpha \) is separated from \( Q_\alpha \) by some \( \Sigma^0_1(\varepsilon, \gamma) \) subset of \( \omega \times \mathcal{X} \), for some \( \gamma \in \Delta^0_1(\varepsilon, \alpha) \). From Lemma 1.5 the preceding \( \Sigma^0_n(\varepsilon, \gamma) \) set is the \( \beta \)-section of \( G^\omega_n \times \mathcal{X}^\omega \) for some \( (\varepsilon, \gamma) \)-recursive \( \beta \in \mathcal{N} \). The latter \( \beta \) is obviously in \( \Delta^0_1(\varepsilon, \alpha) \).

Overall we have that for all \( \alpha \in \mathcal{N} \) there exists \( \beta \in \Delta^0_1(\varepsilon, \alpha) \) such that

\[
P_\alpha \subseteq G^\omega_n \times \mathcal{X}^\omega \quad \text{and} \quad Q_\alpha \cap \bigcup_{n,\beta} G^\omega_n \times \mathcal{X}^\omega = \emptyset.
\]

We now define \( U \subseteq \mathcal{N} \times \mathcal{N} \)

\[
U(\alpha, \beta) \iff (\forall i, x) \{ [P(\alpha, i, x) \rightarrow G^\omega_n \times \mathcal{X}^\omega(\beta, i, x)] \land \neg[G^\omega_n \times \mathcal{X}^\omega(\beta, i, x)] \}.
\]

It is clear that \( U \) is a \( \Pi^1_1(\varepsilon) \) subset of \( \mathcal{N} \times \mathcal{N} \) and from the preceding we have that for all \( \alpha \) there exists some \( \beta \in \Delta^0_1(\varepsilon, \alpha) \) such that \( U(\alpha, \beta) \). It follows from Theorem 1.7 that there exists a Borel-measurable function \( u : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \) such that \( U(\alpha, u(\alpha)) \) for all \( \alpha \).

We claim that \( u \) is the required function. If \( G^\omega_n \times \mathcal{Y}(\alpha, f(x)) \) holds for some \( x \in \mathcal{A} \), then the triple \( (\alpha, i, x) \) belongs to \( P \) and since \( U(\alpha, u(\alpha)) \) holds, we have that \( G^\omega_n \times \mathcal{X}(u(\alpha), i, x) \). Conversely if \( G^\omega_n \times \mathcal{X}(u(\alpha), i, x) \) holds for some \( x \in \mathcal{A} \), since \( U(\alpha, u(\alpha)) \) holds as well, we have that the triple \( (\alpha, i, x) \) does not belong to \( Q \). Using that \( x \in \mathcal{A} \) we have that \( G^\omega_m \times \mathcal{Y}(\alpha, i, f(x)) \).

Having established the Borel-uniform transition in the codes regarding condition \( f^{-1} \Sigma^0_n \subseteq \Sigma^0_n \), it is natural to ask which part of Theorem 2.2 can be carried out in this Borel-uniform setting. The answer turns out to be Theorem 1.2. In order to prove this result we will need first to extend a result of [7] on canceling out Turing jumps. We use the following two results from the Turing degree theory.

1. (The Friedberg Jump Inversion Theorem cf. [3].) Let \( \xi \) be less than \( \omega_1^{CK} \). For every \( B \subseteq \omega \), there exists \( D \subseteq \omega \) such that \( D(\xi) \equiv_T B \oplus 0^{(\xi)} \).

2. (The Shore-Slaman Join Theorem cf. [18].) Let \( \xi \) be less than \( \omega_1^{CK} \). For any \( A, D \subseteq \omega \), if \( A \not\leq_T D(\xi) \) for every \( \zeta < \xi \), then there exists a set \( C \subseteq \omega \) such that \( C \geq_T D \) and \( C(\xi) \leq_T A \oplus C \).

Lemma 2.4 (The Cancellation Lemma). Suppose that \( A, B \subseteq \omega \), and \( \eta, \xi, \xi_0 < \omega_1^{CK} \) are given.\(^5\) If we have

\[
(A \oplus C)(\eta) \leq_T (B \oplus C^{(\xi_0)}(\xi))
\]

for all \( C \subseteq \omega \), then there exists \( \theta \) with \( \theta + \eta \leq \xi \) such that

\[
A \leq_T (B \oplus 0^{(\xi_0)})(\theta).
\]

Similarly we have the relativized version: if for some \( X \subseteq \omega \) and \( \eta, \xi, \xi_0 < \omega_1^{X} \) we have

\[
(A \oplus C \oplus X)(\eta) \leq_T (B \oplus (C \oplus X)^{(\xi_0)})(\xi)
\]

\(^5\)The case \( \xi_0 = 0 \) is implicit from [7].
for all \( C \subseteq \omega \), then there exists \( \theta \) with \( \theta + \eta \leq \xi \) such that

\[
A \oplus X \leq_T (B \oplus X(\xi_0))^{(\theta)}. \tag{6}
\]

Proof. Otherwise, \( A \not\leq_T (B \oplus \emptyset(\xi_0))^{(\theta)} \) for every \( \theta < \zeta \), where \( \zeta \) is the smallest ordinal such that \( \zeta + \eta > \xi \). By the Friedberg Jump Inversion Theorem, we have \( D \) such that \( B \oplus \emptyset(\xi_0) \equiv_T D_{(\xi_0)} \). Then, \( A \not\leq_T D^{(\xi_0+\theta)} \) for every \( \theta < \zeta \). By the Shore-Slaman Join Theorem, there is \( C \geq_T D \) such that \( C^{(\xi_0+\zeta)} \leq_T A \oplus C \). Note that \( C \geq_T D \) implies \( C^{(\xi_0)} \geq_T B \). Then,

\[
(A \oplus C)^{(\eta)} \geq_T C^{(\xi_0+\zeta+\eta)} >_T C^{(\xi_0+\zeta)} \geq_T (B \oplus C^{(\xi_0)})^{(\xi)}.
\]

This contradicts our assumption.

Regarding the relativized version one uses the corresponding result about the Friedberg Jump Inversion: for every \( X \subseteq \omega \), \( \xi_0 < \omega^1 \) and \( B \subseteq \omega \) there exists \( D \subseteq \omega \) such that \( (D \oplus X)^{(\xi_0)} \equiv_T B \oplus X^{(\xi_0)} \). Then we apply the Shore-Slaman Join Theorem with \( A \oplus X, D \oplus X \) and \( X \) in the place of \( A, D \) and \( \emptyset \) respectively. Notice also that the resulting \( C \) will be \( \geq_T X \) and so \( C \equiv_T C \oplus X \). –i

As it is remarked in [7] a necessary and sufficient condition for a function \( f : X \subseteq 2^\omega \rightarrow 2^\omega \) to be (countably) decomposable to \( \Sigma_0^{k+1} \)-measurable functions is \( f(x) \leq_T (x \oplus C)^{(k)} \) for all \( x \in X \) for some fixed \( C \subseteq \omega \). We have now reached the point, where we can verify the latter condition under the necessary hypothesis.

Lemma 2.5. Suppose that \( A \) is an analytic subset of \( 2^\omega \) and that \( f : A \rightarrow 2^\omega \) satisfies \( f^{-1} \Sigma_0^m \subseteq \Sigma_0^n \) for some \( n \geq m \geq 1 \). Then there exists some \( D \in 2^\omega \) such that for all \( x \in A \) we have

\[
f(x) \leq_T (x \oplus D)^{(n-m)}.
\]

Proof. From Theorem 2.3 there exists a Borel-measurable function \( u : N \rightarrow N \) such that

\[
G^{\omega \times N}(\alpha, i, f(x)) \iff G^{\omega \times N}_n(u(\alpha), i, x)
\]

for all \( x \in A \) and all \( \alpha, i \). We define the set \( P \subseteq 2^\omega \times \omega \times 2^\omega \) by

\[
P(C, i, y) \iff i \in (y \oplus C)^{(m)},
\]

so that every section \( P_{C, y} \) is the set \((y \oplus C)^{(m)}\). From (a) of Lemma F.2 in the Appendix we have that the set \( P \) is \( \Sigma^0_m \), hence from Lemma 1.5 there exists a recursive function \( S_{m}^{\omega \times N} \equiv S : N \times 2^\omega \rightarrow N \) and a recursive \( \varepsilon \in N \) such that

\[
P(C, i, y) \iff G^{\omega \times 2^\omega}(\varepsilon, C, i, y) \iff G^{\omega \times 2^\omega}_n(S(\varepsilon, C), i, y)
\]

for all \( C, i, y \). In particular for all \( x \in A \) and all \( C, i \) we have that

\[
P(C, i, f(x)) \iff G^{\omega \times 2^\omega}(S(\varepsilon, C), i, f(x)) \iff G^{\omega \times 2^\omega}_n(u(S(\varepsilon, C)), i, x).
\]

The preceding equivalences show that the set \((f(x) \oplus C)^{(m)}\) is a \( \Sigma^0_n(u(S(\varepsilon, C)), x) \) subset of \( \omega \), for all \( C \) and all \( x \in A \). It follows from (b) of Lemma F.2 that

\[
(f(x) \oplus C)^{(m)} \leq_T (x \oplus u(S(\varepsilon, C)))^{(n)}
\]

for all \( C \) and all \( x \in A \). (Here we view \( u(S(\varepsilon, C)) \in N \) as member of the Cantor space by identifying it with the set \( \{i, j : u(S(\varepsilon, C))(i) = j\} \subseteq \omega \)–the pointclass \( \Sigma^0_n(u(S(\varepsilon, C)), x) \) remains unaffected.)

---

6Regarding the relativized version one should put on the right hand side of the inequality in the hypothesis the term \((B \oplus X \oplus (C \oplus X)^{(\xi_0)})^{(\xi)}\), but the latter however has the same Turing degree as \((B \oplus (C \oplus X)^{(\xi_0)})^{(\xi)}\).
The function $C \mapsto u \circ S(\varepsilon, C)$ is Borel-measurable and thus it is $\Delta^1_1(X)$-recursive for some $X \in 2^\omega$. Without loss of generality we may assume that $\mathcal{A}$ is $\Sigma_1^1(X)$. By applying Lemma 1.8 we obtain some $\xi_0 < \omega^X$ such that $u(S(\varepsilon, C)) \leq_T (C \oplus X)^{(\xi_0)}$ for all $C$. Therefore we have that

\[(f(x) \oplus C)^{(m)} \leq_T (x \oplus (C \oplus X)^{(\xi_0)})^{(n)}\]

for all $C$ and all $x \in \mathcal{A}$. By replacing $C$ with $C \oplus X$ we have that

\[(f(x) \oplus C \oplus X)^{(m)} \leq_T (x \oplus (C \oplus X)^{(\xi_0)})^{(n)}\]

for all $C$ and all $x \in \mathcal{A}$. From the Cancellation Lemma for all $x \in \mathcal{A}$ there exists some $k_x \leq n - m$ such that

\[f(x) \leq_T f(x) \oplus X \leq_T (x \oplus X^{(\xi_0)})^{(k_x)} \leq_T (x \oplus X^{(\xi_0)})^{(n - m)}\]

We take $D = X^{(\xi_0)} \subseteq \omega$ and we have the result.

We are now ready to prove our result in the Introduction.

**Proof of Theorem 1.2.** The assertion is obvious for $m = 1$, so we assume that $m \geq 2$. Since $\mathcal{N}$ is homeomorphic to a $G_4$ subset of $2^\omega$, we may identify the analytic set $\mathcal{A} \subseteq \mathcal{N}$ with an analytic subset of $2^\omega$ and also view $f$ as a function from $\mathcal{A}$ to $2^\omega$.

From Lemma 2.5 there exists some $D \in 2^\omega$ such that that $f(x) \leq_T (x \oplus D)^{(n - m)}$ for all $x \in 2^\omega$. We consider the function

\[g : 2^\omega \to 2^\omega : g(x) = (x \oplus D)^{(n - m)}\]

From Lemma F.2 in the Appendix it is clear that $g$ is $\Sigma^0_{n-m+1}$-measurable.

Let us denote by $\{e\}^A$ the $e$th $A$-recursive function from $\omega$ to $2$. For all $e \in \omega$ we define the set $\mathcal{B}_e \subseteq 2^\omega$ as follows

\[x \in \mathcal{B}_e \iff x \in \mathcal{A} \land (\forall i)[\{e\}^{g(x)}(i) = f(x)(i)].\]

In other words $x \in \mathcal{B}_e$ means that $f(x) \leq_{T,e} (x \oplus D)^{(n - m)}$. Since

\[f(x) \leq_T g(x)\]

holds for all $x \in \mathcal{A}$, we have that

\[\mathcal{A} = \bigcup_e \mathcal{B}_e.\]

Using that $g$ is $\Sigma^0_{n-m+1}$-measurable it follows from Remark A.3 that the relations $R^e_0, R^e_1 \subseteq 2^\omega \times \omega$ defined by

\[R^e_0(x, i) \iff \{e\}^{g(x)}(i) = 0 \quad \text{and} \quad R^e_1(x, i) \iff \{e\}^{g(x)}(i) = 1,\]

are $\Sigma^0_{n-m+1}$ for all $e$. From this it follows that $f \upharpoonright (\mathcal{B}_e \cap \mathcal{A})$ is $\Sigma^0_{n-m+1}$-measurable.

Now we deal with the complexity of $\mathcal{B}_e$. We remark that

\[(2) \quad x \in \mathcal{B}_e \iff x \in \mathcal{A} \land (\forall i)(\exists k \leq 2)[\{e\}^{g(x)}(i) = k]\]

& $\quad (\forall i)[\{e\}^{g(x)}(i) = 1 \implies f(x)(i) = 1]$ & $\{e\}^{g(x)}(i) = 0 \implies f(x)(i) = 0\}$.

We notice that $n - m + 1 \leq n - 1$ because $m \geq 2$. Using the preceding relations $R^e_0, R^e_1$ we have that the first line on the right-hand-side of $(2)$ defines a $\Pi^0_{n-m+2} \subseteq \Pi^0_n$ subset of $\mathcal{A}$. Since $f$ is in particular $\Sigma^0_n$-measurable there exist $\Pi^0_n$ relations $P^0_0, P^0_1 \subseteq 2^\omega \times \omega$ with

\[P^0_0(x, i) \iff f(x)(i) = 0 \quad \text{and} \quad P^0_1(x, i) \iff f(x)(i) = 1\]
for all $x \in A$. Hence, by using again $R_i^0, R_i^1$, the second and the third lines of (2) define a $\Pi^0_{\max(n-m+1,n)} = \Pi^0_n$ subset of $A$. Overall the set $B_{e}$ is a $\Pi^0_n$ subset of $A$.

Now let us assume that $m \geq 3$ and that $f$ is $\Sigma^0_{n-1}$-measurable. It follows that $n-m+2 \leq n-1$ and so the first line on the right-hand-side of (2) defines a $\Pi^0_{\max(n-m+2,n)} \subseteq \Pi^0_{n-1}$ set. Moreover the relation $P^f$ is $\Delta^0_{n-1}$. Hence the second and the third lines of (2) define a $\Pi^0_{\max(n-m+1,n)} = \Pi^0_{n-1}$ set. It follows that $B_e$ is $\Pi^0_{n-1}$.

As we mentioned in the Introduction, Theorem 1.2 reduces the general case of the Decomposability Conjecture to the cases $m = 2 \leq n$.

**Corollary 2.6** (See also [13] Corollary 5.11). Suppose that $A$ is an analytic subset of $N$ and $n \geq 2$. If for all functions $f : A \to N$ the implication

$$f^{-1}\Sigma^0_2 \subseteq \Sigma^0_n \implies f \in \mathit{dec}(\Sigma^0_{n-1}, \Delta^0_n)$$

holds, then for all naturals $m$ with $2 \leq m \leq n$ and all functions $f : A \to N$ the implication

$$f^{-1}\Sigma^0_m \subseteq \Sigma^0_n \implies f \in \mathit{dec}(\Sigma^0_{n-m+1}, \Delta^0_n)$$

holds as well.

**Proof.** Let $\emptyset \neq A \subseteq N$ be analytic and suppose that $f : A \to N$ satisfies $f^{-1}\Sigma^0_m \subseteq \Sigma^0_n$ for some $3 \leq m \leq n$ (the case $m = 2$ is covered by our hypothesis). In particular $f$ satisfies condition $f^{-1}\Sigma^0_2 \subseteq \Sigma^0_n$. So from our hypothesis there exists a sequence $(B_i)_{i \in \omega}$ of $\Pi^0_{n-1}$ subsets of $N$ such that the restriction $f \mid B_i \cap A$ is $\Sigma^0_{n-1}$-measurable for all $i \in \omega$. (Here we use that the $\Pi^0_1$ subsets of $A$ are exactly the sets of the form $B \cap A$ for some $\Pi^0_2$ subset $B$ of $N$.)

Put $g_i = f \mid B_i \cap A$ for all $i$. It is clear that every function $g_i$ satisfies condition $g_i^{-1}\Sigma^0_m \subseteq \Sigma^0_n$ and that the sets $B_i \cap A$ are analytic. Moreover every function $g_i$ is $\Sigma^0_{n-1}$-measurable. Using that $m \geq 3$ it follows from the second assertion of Theorem 1.2 that for all $i$ there exists a sequence $(C^i_j)_{j \in \omega}$ of $\Pi^0_{n-1}$ subsets of $N$ such that $g_i \mid \mathit{Dom}(g_i) \cap C^i_j = f \mid A \cap B_i \cap C^i_j$ is $\Sigma^0_{n-m+1}$-measurable. Clearly every $A \cap B_i \cap C^i_j$ is a $\Pi^0_{n-1}$ subset of $A$ and so $f \in \mathit{dec}(\Sigma^0_{n-m+1}, \Delta^0_n)$. \hfill $\sqcup$

The preceding Corollary suggests that the cases $m = 2 \leq n$ in the Decomposability Conjecture is the “correct” extension of the Jayne-Rogers Theorem.

We proceed with a conjecture about extending Theorem 1.2 to all Polish spaces. The idea is to replace Turing jumps with the lightface Borel pointclasses. The problem is to prove the analogue of the Cancellation Lemma. Below by $\Gamma \cap \omega$ we mean the family of all subsets of $\omega$, which belong to $\Gamma$.

**Conjecture 2.7** (The Cancellation Conjecture for the Kleene pointclasses). Suppose that $X$ and $Y$ are $\varepsilon$-recursively presented Polish spaces with $\varepsilon \in 2^\omega$, $(x,y) \in X \times Y$, $u : N \to N$ is $\Delta^1_{\varepsilon}(\alpha)$-recursive for some $\alpha \in 2^\omega$ with $\varepsilon \leq_\Gamma \alpha$, and that $1 \leq \eta, \xi < \omega^\omega_1$. If for all $C \subseteq \omega$ we have that

$$\Sigma^0_\eta(y, \alpha, C) \cap \omega \subseteq \Sigma^0_\xi(x, \alpha, u(C)) \cap \omega,$$

then there exists some $D \subseteq \omega$ and some $\theta < \omega_1$ with $\theta + \eta \leq \xi$ such that

$$y \in \Sigma^0_{\theta+1}(x, D).$$
The Cantor space satisfies Conjecture 2.7, and as expected this is derived from the Cancellation Lemma. To see the latter notice that the hypothesis in the preceding conjecture is stronger than the one in the Cancellation Lemma. (This follows from Lemma F.2.) Moreover if \( y \leq T(x \oplus D)^{(\theta)} \) then it is easy to verify that \( y \in \Sigma_{0+1}^{\theta}(x,D) \).

**Remark 2.8.** Theorem 1.2 holds for all functions \( f : X \rightarrow Y \), where \( X, Y \) satisfy Conjecture 2.7.

**The transfinite case.** Here we show that with the necessary modifications the preceding results can be extended to the Borel pointclasses of transfinite order. We employ the set of Borel codes \( BC = \cup_{\xi} BC_{\xi} \) mentioned in paragraph F of the Appendix, as well as the parametrization function \( \pi^X \).

**Definition 2.9.** Given a Polish space \( X \) we define the sets \( B^X_\xi \subseteq N \times X \) by

\[
B^X_\xi(\alpha, x) \iff \alpha \in BC_{\xi} \& x \in \pi^X(\alpha).
\]

It is easy to verify that \( B^X_\xi \) parametrizes \( \Sigma_{1+1}^0 \upharpoonright X \). In the transfinite case we will need to replace the system \( (G^X_\alpha)_{X} \) with \( (B^X_\xi)_{X} \).

**Theorem 2.10.** Suppose that \( X, Y \) are Polish spaces, \( A \subseteq X \) is analytic and that \( f : A \rightarrow Y \) satisfies \( f^{-1}\Sigma^0_\eta \subseteq \Sigma^0_\xi \) for some \( \eta, \xi \geq 1 \). Then condition \( f^{-1}\Sigma^0_\eta \subseteq \Sigma^0_\xi \) holds Borel-uniformly in the codes with respect to \( B^Y_{\eta \times X} \).

*Proof.* We repeat the proof of Theorem 2.3 and we notice that the corresponding sets \( P, Q \) defined by

\[
P(\alpha, i, x) \iff x \in A \& B^\omega_{\eta \times Y}(\alpha, i, f(x))
\]

\[
Q(\alpha, i, x) \iff x \in A \& -B^\omega_{\eta \times Y}(\alpha, i, f(x))
\]

are analytic sets—and hence \( \Sigma^0_1(\varepsilon) \) for some \( \varepsilon \in N \). To see the this, we remark that the sets \( BC_{\xi} \) are Borel, and that for all Polish spaces \( Z \) and all \( \xi < \omega_1 \) there exists a Borel set \( C_{\xi} \subseteq N \times Z \) such that for all \( \alpha \in BC_{\xi} \) and all \( z \in Z \) we have that

\[
z \in \pi^Z(\alpha) \iff C_{\xi}(\alpha, z).
\]

Moreover we notice that the sections \( P_{\alpha, i} \) are \( \Sigma^0_{\eta} \). The rest of the proof remains the same.

In order to extend the rest of our results to the transfinite case we need to know that this parametrization \( B^X_\xi \) is good in the sense of universal systems, i.e., equivalence (1) holds also for \( B^X_{\eta \times Y} \) and \( B^Y_{\xi} \) for a suitable continuous function. The following is the analogue of Lemma 1.5 for the system \( (B^X_\xi)_{X} \). For the proof refer to paragraph G of the Appendix. We point out that the good transition can in fact be witnessed by the same continuous function.

**Lemma 2.11** (The good parametrization Lemma for \( B^X_\xi \)). For every Polish space \( Y \) and every space \( X \) of the form \( \omega^s \times N^t \), \( s, t \geq 0 \) there exists a recursive function \( R^X,Y : N \times X \rightarrow N \) such that for all and all \( \xi < \omega_1 \) and all \( \varepsilon, x, y \),

\[
B^X_{\eta \times Y}(\varepsilon, x, y) \iff B^Y_{\xi}(R^X,Y(\varepsilon, x), y).
\]
The arguments used in the proof of Lemma 2.5 yield

Lemma 2.12. Suppose that $\mathcal{A}$ is an analytic subset of $2^\omega$ and that $f : \mathcal{A} \to 2^\omega$ satisfies $f^{-1} \Sigma^0_\eta \subseteq \Sigma^0_\xi$ for some $1 \leq \eta \leq \xi < \omega_1$. Then there exists some $D \in 2^\omega$ such that for all $x \in \mathcal{A}$ we have

$$f(x) \leq_T (x \oplus D)(\theta_x),$$

for some $\theta_x < \omega_1$ with $\theta_x + \eta \leq \xi$.

Proof. We repeat the proof of Lemma 2.5 by replacing the parametrizing sets $G^\omega \times N, G^\omega \times N$ with $B^\omega \times N, B^\omega \times N$ respectively and by taking the $\theta_x$ which comes from the Cancellation Lemma instead of $k_x \leq n - m$.

Finally we give the transfinite version of Theorem 1.2.

Theorem 2.13. Suppose that $2 \leq \eta \leq \xi < \omega_1$, $\mathcal{A} \subseteq N$ is analytic, and $f : \mathcal{A} \to N$ satisfies $f^{-1} \Sigma^0_\eta \subseteq \Sigma^0_\xi$. Then there exists a sequence $(A_i)_{i \in \omega}$ of $\Delta^0_{\xi+1}$ subsets of $\mathcal{A}$ such that for all $i \in \omega$ the restriction $f \upharpoonright A_i$ is $\Sigma^0_{\theta_i+1}$-measurable for some countable $\theta_i$ with $\theta_i + \eta \leq \xi$.

If moreover $\eta \geq 3$ and $f$ is $\Sigma^0_\zeta$-measurable for some $\zeta < \xi$ then the preceding sets $A_i$ can be chosen to be $\Delta^0_{\zeta}$.

Proof. This is essentially the same as the proof of Theorem 1.2. The main difference is that we need to refine the partition. We consider the set $D$ as in Lemma 2.12 and for all $\theta$ with $\theta + \eta \leq \xi$ we define the function

$$g_\theta : 2^\omega \to 2^\omega : g_\theta(x) = (x \oplus D)(\theta).$$

Moreover for all $e \in \omega$ and $\theta$ with $\theta + \eta \leq \xi$ we put

$$B_{e,\theta} = \{ x \in \mathcal{A} : f(x) \leq_T (x \oplus D)(\theta) \}.$$

From Lemma 2.12 it follows that $\mathcal{A} = \cup_{e,\theta} B_{e,\theta}$. The sets $B_{e,\theta}$ are $\Pi^0_\zeta$ subsets of $\mathcal{A}$ and since $g$ is $\Sigma^0_{\theta+1}$-measurable we have that $f \upharpoonright B_{e,\theta}$ is $\Sigma^0_{\theta+1}$-measurable as well (Remark A.3).

If $f$ is $\Sigma^0_\zeta$-measurable for some $\zeta$ then $B_{e,\theta}$ is a $\Pi^0_{\max\{\theta+2,\zeta\}}$ subset of $\mathcal{A}$, and so for $\eta \geq 3$ we have $\theta + 2 < \theta + \eta \leq \xi$. It follows that $B_{e,\theta}$ is a $\Delta^0_\zeta$ subset of $\mathcal{A}$ when $3 \leq \eta$ and $\zeta < \xi$.

3. CONTINUOUS TRANSITION IN THE CODES

The question about witnessing condition $f^{-1} \Sigma^0_m \subseteq \Sigma^0_n$ in a continuous way in the codes appears in the area of computable analysis, most notably in [2], [15] and [7]. The preceding condition is critical for obtaining the best upper bound of the complexity of the decomposition in [7]. In this section we will present a partial answer to the preceding question. More specifically we will see that one can modify the standard universal system $(G^N_\chi)_{\chi}$ that we work with, so that the transition in the codes is done in a continuous way under the modified universal system. Unfortunately the latter universal system does not seem to be a good one, rendering thus no obvious way for applying the techniques of [7] in order to compute the (conjectured) best upper bound of the complexity of the decomposition even in the cases $n \leq 2m - 1$. 
Theorem 3.1. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Polish spaces and that $f: \mathcal{X} \to \mathcal{Y}$ satisfies $f^{-1}\Sigma^0_n \subseteq \Sigma^0_n$ for some $m, n \geq 1$. Then there exists a continuous surjection $g_1: \mathcal{N} \to \mathcal{N}$ and a $\Sigma^0_n$ set $H \subseteq \mathcal{N} \times \omega \times \mathcal{X}$ such that

$$G_{\omega \times \mathcal{Y}}(g_1(\varepsilon), i, f(x)) \iff H(\varepsilon, i, x)$$

for all $(\varepsilon, i, x) \in \mathcal{N} \times \omega \times \mathcal{X}$.

Proof. From Theorem 2.3 there exists a Borel-measurable function $\tau: \mathcal{N} \to \mathcal{N}$ such that

$$G_{\omega \times \mathcal{Y}}(\beta, i, f(x)) \iff G_{\omega \times \mathcal{X}}(\tau(\beta), i, x)$$

for all $(\beta, i, x) \in \mathcal{N} \times \omega \times \mathcal{X}$.

We consider basic neighborhoods $N_s$ of the Baire space and we define $Q \subseteq \mathcal{N} \times \omega \times \mathcal{N}$ by

$$Q(\beta, s) \iff \tau(\beta) \in N_s.$$ 

Clearly $Q$ is Borel. From (the relativized version of) 4A.7 in [11] the sets $Q$ and $\neg Q$ are the injective recursive images of closed subsets of the Baire space. Since the graph of a continuous function is a closed set there exist closed sets $C_0, C_1 \subseteq \mathcal{N} \times \omega \times \mathcal{N}$ such that

$$Q(\beta, s) \iff (\exists! \alpha) C_0(\beta, s, \alpha)$$

and

$$\neg Q(\beta, s) \iff (\exists! \alpha) C_1(\beta, s, \alpha),$$

for all $\beta$ and $s$, where $\exists!$ stands for "there exists a unique".

Now we define $F \subseteq \mathcal{N} \times \mathcal{N} \times 2^\omega$ by

$$F(\beta, \gamma, \delta) \iff (\forall s) \left\{ [\delta(s) = 0 \& C_0(\beta, s, (\gamma)_s)] \lor [\delta(s) = 1 \& C_1(\beta, s, (\gamma)_s)] \right\} \land \gamma(t) = 0 \text{ for all } t \text{ which do not fall of the form } \langle s, i \rangle.$$ 

It is clear that $F$ is a closed set and that for all $\beta \in \mathcal{N}$ there exists exactly one pair $(\gamma, \delta) \in \mathcal{N} \times 2^\omega$ such that $F(\beta, \gamma, \delta)$, namely

$$\delta(s) = \begin{cases} 0, & \text{if } \tau(\beta) \in N_s, \\ 1, & \text{if } \tau(\beta) \notin N_s, \end{cases}$$

$$(\gamma)_s = \begin{cases} \text{the unique } \alpha \text{ for which } C_0(\beta, s, \alpha), & \text{if } \tau(\beta) \in N_s, \\ \text{the unique } \alpha \text{ for which } C_1(\beta, s, \alpha), & \text{if } \tau(\beta) \notin N_s, \end{cases}$$

$$\gamma(t) = 0 \text{ if } (\forall s, i)[t \neq \langle s, i \rangle].$$

for all $s \in \omega$.

It is also clear that $F$ is a Polish space and so there exists a continuous surjection $g: \mathcal{N} \to F$.

We define the functions

$$h: F \to \mathcal{N}: h(\beta, \gamma, \delta) = \tau(\beta), \quad \text{and} \quad \pi: \mathcal{N} \to \mathcal{N}: \pi = h \circ g,$$

so that the following diagram commutes.
We claim that the function $h$ is continuous. From this it follows that the function $\pi$ is continuous as well. To see the former we notice that

$$h(\beta, \gamma, \delta) \in N_s \iff \tau(\beta) \in N_s \iff Q(\beta, s) \iff \delta(s) = 0 \& C_0(\beta, s, (\gamma)_s) \iff \neg[\delta(s) = 1 \& C_1(\beta, s, (\gamma)_s)]$$

for all $(\beta, \gamma, \delta) \in F$ and all $s \in \omega$. The preceding equivalences show that the set $h^{-1}[N_s]$ is in fact clopen for all $s$.

We define the function $g_1 = \text{pr}_1 \circ g : \mathcal{N} \to \mathcal{N}$, where $\text{pr}_1$ is the projection $(\beta, \gamma, \delta) \in F \mapsto \beta \in \mathcal{N}$. It is clear that $g_1$ is continuous, surjective and that for all $\varepsilon, \beta, \gamma, \delta$ with $g(\varepsilon) = (\beta, \gamma, \delta)$ we have that

$$\pi(\varepsilon) = h(g(\varepsilon)) = h(\beta, \gamma, \delta) = \tau(\beta) = \tau(g_1(\varepsilon)),$$

i.e., we have that $\pi = \tau \circ g_1$. Thus the preceding diagram is refined to the following (commutative) diagram

Now we define $H \subseteq \mathcal{N} \times \omega \times \mathcal{Y}$ as follows

$$H(\varepsilon, i, y) \iff G_{n}^{\omega \times Y}(\pi(\varepsilon), i, y).$$

Since $\pi$ is continuous and $G_{n}^{\omega \times Y}$ is a $\Sigma^0_n$ set it follows that $H$ is a $\Sigma^0_n$ set as well. Finally we compute

$$G((g_1(\varepsilon), i, f(x))) \iff G_{n}^{\omega \times Y}(\tau(g_1(\varepsilon)), i, x) \iff G_{n}^{\omega \times Y}(\pi(\varepsilon), i, x) \iff H(\varepsilon, i, x)$$

It is a well-known result of classical descriptive set theory that for every Borel measurable function $f : \mathcal{X} \to \mathcal{Y}$ ($\mathcal{X}$, $\mathcal{Y}$ Polish) there exists a zero-dimensional Polish topology on $\mathcal{X}$ which refines the original topology, has the same Borel sets and the function $f$ is continuous with respect to this new topology, cf. [6] Section 13.A. Here we gave in essence an effective proof of this fact for the case of the function $\tau$. The new topology on $\mathcal{N}$ is the one induced by the bijection $p : \beta \in \mathcal{N} \mapsto (\beta, \gamma, \delta) \in F$. This topology refines the original one because the function $(\beta, \gamma, \delta) \mapsto \beta$ is continuous, and the Borel structure remains the same because the function $p$ is Borel measurable. Moreover the function $\tau$ is continuous under this new topology since $\tau = h \circ p$ and $h$ is continuous. For more information on turning Borel sets into clopen in an effective way the reader can refer to [4].
for all \((\varepsilon, i, x) \in \mathcal{N} \times \omega \times \mathcal{X}\), and the proof is complete. 

**Corollary 3.2.** Suppose that \(\mathcal{X}\) and \(\mathcal{Y}\) are Polish spaces and that \(f : \mathcal{X} \to \mathcal{Y}\) satisfies \(f^{-1}\Sigma^0_\mathcal{m} \subseteq \Sigma^0_\mathcal{m}\) for some \(m, n \geq 1\). Then there exists a universal system \((J^Z_m)\) for \(\Sigma^0_\mathcal{m}\) and a continuous function \(\pi : \mathcal{N} \to \mathcal{N}\) such that

\[ J^\omega \times \Sigma^0_\mathcal{m}(\varepsilon, i, f(x)) \iff G^\omega \times \mathcal{X}(\pi(\varepsilon), i, x) \]

for all \((\varepsilon, i, x) \in \mathcal{N} \times \omega \times \mathcal{X}\).

**Proof.** From Theorem 3.1 there exists a continuous surjection \(g_1 : \mathcal{N} \to \mathcal{N}\) and a \(\Sigma^0_\mathcal{m}\) set \(H \subseteq \mathcal{N} \times \omega \times \mathcal{X}\) such that

\[ G^\omega \times \mathcal{X}(g_1(\varepsilon), i, f(x)) \iff H(\varepsilon, i, x) \]

for all \((\varepsilon, i, x) \in \mathcal{N} \times \omega \times \mathcal{X}\).

We define

\[ J^Z_m(\varepsilon, z) \iff G^Z_m(g_1(\varepsilon), z) \]

for all \((\varepsilon, z) \in \mathcal{N} \times Z\) and all Polish spaces \(Z\). Since the function \(g_1\) is continuous, \(J^Z_m\) is a \(\Sigma^0_\mathcal{m}\) set, and, since \(g_1\) is surjective, it follows that \(J^Z_m\) parametrizes \(\Sigma^0_\mathcal{m} \upharpoonright Z\). Thus \((J^Z_m)\) is a universal system for \(\Sigma^0_\mathcal{m}\).

Since the set \(H \subseteq \mathcal{N} \times \omega \times \mathcal{X}\) is \(\Sigma^0_\mathcal{m}\) there exists some \(\beta \in \mathcal{N}\) such that \(H \) is the \(\beta\)-section of \(G^Z_m\), where \(Z = \mathcal{N} \times \omega \times \mathcal{X}\). It is not hard to verify that the function \(\pi : \mathcal{N} \to \mathcal{N} : \pi(\varepsilon) = S(\beta, \varepsilon)\) satisfies the required properties.

**Remark 3.3.** The universal system \((J^Z_m)\) of the preceding theorem depends on the choice of \(f\). Moreover it does not seem to be a good universal system. The reason we conjecture this lies in the following equivalences. It is clear from the definition that

\[ J^X \times Z_m(\varepsilon, x, z) \iff G^X \times Z_m(g_1(\varepsilon), x, z) \iff G^Z_m(S(g_1(\varepsilon), x, z) \]

for some continuous function \(S\). In order to infer that \((J^Z_m)\) is a good universal system we would need a continuous function \(\varphi : \mathcal{N} \times \mathcal{X} \to \mathcal{N}\) such that \(g_1 \circ \varphi = \text{id}\), i.e., \(g_1\) should have a continuous inverse. In this case we would define the function

\[ S^X \times Z_{J, m} \equiv S_f : \mathcal{N} \times \mathcal{X} \to \mathcal{N} : S_f(\varepsilon, x) = \varphi(S(g_1(\varepsilon), x)) \]

and we would have

\[ J^X \times Z_m(\varepsilon, x, z) \iff G^Z_m(S(g_1(\varepsilon), x, z) \iff G^Z_m(g_1(\varphi(S(g_1(\varepsilon), x))), z) \iff J^Z_m(\varphi(S(g_1(\varepsilon), x))), z) \iff J^Z_m(S_f(\varepsilon, x), z). \]

However the function \(g_1\) does not seem to admit a continuous inverse, for otherwise by going back to the proof of Theorem 3.1 the bijection \((\beta, \gamma, \delta) \in F \mapsto \beta\) would also have a continuous inverse. This would imply that the function \(\tau\) in Theorem 2.3 is in fact continuous, which is not implicit from the proof.
Appendix

We attach this Appendix, where we review some of the basic facts of recursion theory and effective descriptive set theory, as well as some proofs of results used in the main text.

A. Recursive functions on the naturals. We present the definition of a recursive function in the sense of Kleene, cf. [11] Chapter 3. All functions here are defined on finite products of $\omega$ and take values in $\omega$. First we consider the following basic functions:

- **the successor function**: $\text{Succ} : \omega \to \omega : x \mapsto x + 1$;
- **the projections**: $\text{pr}_j^n : \omega^n \to \omega : (x_1, \ldots, x_n) \mapsto x_j, 1 \leq j \leq n$;
- **the constants**: $c^n_k : \omega^n \to \omega : \vec{x} \mapsto k$.

Moreover we consider the following constructive schemes:

- **composition**: given functions $h : \omega^n \to \omega$ and $f_1, \ldots, f_n : \omega^k \to \omega$ we define $f : \omega^k \to \omega : f(\vec{x}) = h(f_1(\vec{x}), \ldots, f_n(\vec{x}))$;
- **primitive recursion**: given functions $g : \omega^m \times \omega \to \omega$ and $h : \omega \times \omega^m \times \omega \to \omega$ we define $f : \omega \times \omega^m \to \omega : f(0, \vec{x}) = g(\vec{x})$ and $f(n + 1, \vec{x}) = h(n, \vec{x}, f(n, \vec{x}))$, (we allow the possibility $m = 0$ for the non-parametrized version);
- **minimalisation**: given a function $g : \omega^m \times \omega \to \omega$ such that for all $\vec{x} \in \omega^m$ there exists $y \in \omega$ with $g(\vec{x}, y) = 0$, we define $f : \omega^m \to \omega : f(\vec{x}) = \text{the least } y \text{ such that } g(\vec{x}, y) = 0$.

The family of recursive functions is the least family which contains all basic functions and is closed under the preceding constructing schemes. For example the function $h : \omega^2 \to \omega : h(y, w) = \text{Succ}(w) = \text{Succ}(\text{pr}_2^n(y, w))$ is recursive, and therefore the addition function

$$+: \omega^2 \to \omega : 0 + x = x = \text{pr}_1^1(x), (n + 1) + x = \text{Succ}(n + x) = h(n, n + x)$$

is recursive as well. In fact many “natural” number theoretic functions such that multiplication $x \cdot y$, exponentiation $x^y$ and the function $i \mapsto i$th prime number, are recursive.

As it is expected a function $f : \omega^n \to \omega$ is recursive exactly when there is a finite sequence of functions $f_i : \omega^{m_i} \to \omega$, $i = 1, \ldots, n$ such that: a) each $f_i$ is either a basic function or is obtained by applying the constructive schemes to basic functions and $f_1, \ldots, f_{i-1}$; b) $f_n = f$. (For a proof refer to 3F.1 [11].) It follows that there are only countably many recursive functions.

The notion of a recursive function can be relativised: given some $\alpha : \omega \to \omega$ one defines the family of $\alpha$-recursive functions as the least family which contains the function $\alpha$ and the basic functions, and is closed under the preceding constructive schemes. The theory of recursive functions has a natural translation to the theory of $\alpha$-recursive functions — the reader can refer to the relativization principle below.

A set $P \subseteq \omega^n$ is recursive if its characteristic function $\chi_P : \omega^n \to \omega$ is recursive, and is semirecursive if $P = \exists^e Q$ for some recursive $Q \subseteq \omega^n \times \omega$. 
A partial function \( f : \omega^m \to \omega \) is recursive if the domain of \( f \), \( \{ \bar{x} \in \omega^k : f(x) \downarrow \} \) is a semirecursive set and there exists a semirecursive \( R^f \subseteq \omega^m \times \omega \) such that for all \( \bar{x} \in \omega^m \) with \( f(\bar{x}) \downarrow \) we have that
\[
f(\bar{x}) = y \iff R^f(\bar{x}, y).
\]
In case where \( f \) is in fact a total function, i.e., it is defined on the whole set \( \omega^k \) this definition coincides with the preceding one because of

**Theorem A.1** (Kleene). A function \( f : \omega^m \to \omega \) is recursive exactly when its graph \( \text{Gr}(f) \subseteq \omega^m \times \omega \) is a semirecursive set.

Similarly one defines the notion of \( \alpha \)-recursive and \( \alpha \)-semirecursive set, as well as \( \alpha \)-recursive partial function for \( \alpha : \omega \to \omega \).

**The Kleene normal form and the \( S^m_n \) Theorem.** The family of recursive partial functions admits an enumeration in a natural way.

**Theorem A.2** (The Kleene Normal Form Theorem). There exist a recursive partial function \( U : \omega \to \omega \) and recursive\(^8\) relations \( T_n(s, e, \bar{x}, y) \), where \( s, e, y \in \omega, \bar{x} \in \omega^n, n \geq 1 \), such that the arbitrary partial function \( f : \omega^n \to \omega \) is recursive exactly when there exists \( e \in \omega \) with the following properties:
\[
f(\bar{x}) \downarrow \iff (\exists y, t) T_n(\bar{x}(t), e, \bar{x}, y)
\]
\[
f(\bar{x}) \downarrow \iff f(\bar{x}) = U(\langle \text{least } (t, y) \text{ s.t. } T_n(\bar{x}(t), e, \bar{x}, y) \rangle_1).
\]

The preceding line means that if we choose the least natural encoded by some pair \((t, y)\) such that \( T_n(\bar{x}(t), e, \bar{x}, y) \) holds, then \( U(y) \) is defined and is equal to \( f(\bar{x}) \).

 Following Kleene we denote the partial function
\[
\bar{x} \mapsto U(\langle \text{least } (t, y) \text{ s.t. } T_n(\bar{x}(t), e, \bar{x}, y) \rangle_1)
\]
on the set \( \{ \bar{x} \in \omega^n : (\exists y, t) T_n(\bar{x}(t), e, \bar{x}, y) \} \) by \( \{e\}^{n,\alpha} \) or simply by \( \{e\}^\alpha \), when \( n \) is easily understood from the context. It is clear that the family
\[
\{0\}^{n,\alpha}, \{1\}^{n,\alpha}, \ldots, \{e\}^{n,\alpha}, \ldots,
\]
is an enumeration of all \( \alpha \)-recursive partial functions from \( \omega^n \) to \( \omega \). We fix the preceding \( U \) and \( T_n \) throughout the rest of this article.

**Remark A.3.** For all \( \xi < \omega_1 \) and all \( \Sigma^0_\xi \)-measurable functions \( g : 2^\omega \to 2^\omega \) the relation \( R \subseteq \omega \times 2^\omega \times \omega^2 \) defined by
\[
R(e, x, n, k) \iff \{e\}^{g(x)}(n) = k
\]
is \( \Sigma^0_\xi \). To see the latter we notice that \( R(e, x, n, k) \) holds exactly when
\[
(\exists t, y) [T_1(g(x))(t), e, n, y) \& (\forall (t', y') < (t, y)) \neg T_1(g(x))(t'), e, n, y') \& k = U(y)]
\]
and moreover we have
\[
(-) T_1(g(x))(t), e, n, y) \iff (\exists u \in 2^{<\omega}) [(-) T_1(u, e, n, y) \& u \subseteq g(x)].
\]

\(^8\)In fact the function \( U \) as well as the relations \( T_n \) are primitive recursive, i.e., \( U \) belongs to the least family of functions, which contains the basic functions and is closed under the composition and the primitive recursion scheme – the notion of a primitive recursive relation is defined as usual through characteristic functions.
Theorem A.4 (The $S^m_n$ Theorem, Kleene). For all naturals $m,n \geq 1$ there exist injective (primitive) recursive total functions $S^m_n : \omega \times \omega^m \to \omega$ such that for all $\alpha \in N$, $e,y \in \omega$, $\vec{z}, \vec{x} \in \omega^n$ and $\vec{x} \in \omega^n$,

$$\{e\}^{m+n,\alpha}(\vec{z}, \vec{x}) = \{S^m_n(e, \vec{z})\}^{n,\alpha}(\vec{x}).$$

B. The Turing jump. Given $A,B \subseteq \omega$ we say that $A$ is Turing reducible to $B$, in symbols $A \leq_T B$ if there exists some $e \in \omega$ such that $\chi_A = \{e\}^B$, i.e., the characteristic function of $A$ is a $B$-recursive function. (Here as usual we identify $B$ with $\chi_B$.) The relation $\leq_T$ is reflexive and transitive, so that the relation $\equiv_T$, defined by

$$A \equiv_T B \iff A \leq_T B \& B \leq_T A,$$

is an equivalence relation. The equivalence class of $A$ is the Turing degree of $A$. It is easy to verify that all recursive sets have the same Turing degree as the one of the empty set: in fact we have that $\emptyset \leq_T B$ for all $B$, and $B \leq_T \emptyset$ exactly when $B$ is recursive.

The Turing jump $A'$ of a set $A \subseteq \omega$ is the set of all $e \in \omega$ such that the partial function $\{e\}^A : \omega \to \omega$ is defined on $e$, i.e.,

$$e \in A' \iff \{e\}^A(e) \downarrow \iff (\exists y,t) T_n(\overline{A}(t),e,c,y).$$

We state some of the basic properties of the Turing jump: $A <_T A'$, (in particular the set $\emptyset$ is not recursive), $A \leq_T B$ implies $A' \leq_T B'$, and $A'$ is $A$-semirecursive, for all $A,B \subseteq \omega$.

The $n$th Turing jump is defined by recursion:

$$A^{(0)} = A$$

$$A^{(n+1)} = (A^{(n)})'.$$

The latter iteration extends to ordinal numbers, but first one should provide the suitable setting.

C. Recursive ordinals and notation systems. Recursion theory on the ordinals has been studied among others by Church, Kleene, Markwald and Spector. Below we denote by $\{e\}$ the $e$-th partial recursive function from $\omega$ to $\omega$.

For every $e$ we define

$$\text{Field}(e) = \{n : \{e\}(\langle n, n \rangle) = 0\}$$

and the relation $\leq_e \subseteq \omega^2$ by

$$m \leq_e n \iff n,m \in \text{Field}(e) \& \{e\}(\langle m, n \rangle) = 0.$$ 

The set Lin of codes of recursive linear orderings on a subset of $\omega$ is

$$\{e \in \omega : \{e\} \text{ is total and } \leq_e \text{ is a linear ordering on } \text{Field}(e)\}.$$ 

Spector’s $W$ is the set

$$\{e \in \text{Lin} : \leq_e \text{ has no strictly decreasing sequences}\},$$

cf. [19]. An ordinal $\xi$ is recursive if $\xi = | \leq_e |$ for some $e \in W$.

Similarly one defines $\leq^A_e$, Lin$^A$, W$^A$ and consequently $A$-recursive ordinals for $A \subseteq \omega$, by replacing the term “recursive” with “$A$-recursive”.

A notation system of ordinals (cf. [8] and [12] for modern day notation) is a pair $(S, | \cdot |_S)$ such that $S$ is a subset of the naturals and $| \cdot |_S$ is a function on $S$ to the class of ordinals with the following properties:
(1) there exists a recursive partial function $K : \omega \to \omega$ which is defined on every natural in $S$ and satisfies
\[ K(x) = 0 \iff |x|_S = 0 \]
\[ K(x) = 1 \iff |x|_S \text{ is a successor ordinal} \]
\[ K(x) = 2 \iff |x|_S \text{ is a limit ordinal.} \]

(2) there exists a recursive partial function $P : \omega \to \omega$ such that if $x$ is in $S$ and $|x|_S$ is a successor ordinal than $P(x)$ is defined, $P(x) \in S$ and $|x|_S = |P(x)|_S + 1$, and

(3) there exists a partial recursive partial function $Q : \omega^2 \to \omega$ such that if $x$ is in $S$ and $|x|_S$ is a limit ordinal then for all $t$, $Q(x, t)$ is defined, belongs to $S$, $|Q(x, t)|_S < |Q(x, t + 1)|_S$ and $|x|_S = \lim_t |Q(x, t)|_S$. An ordinal $\xi$ is constructive if $\xi = |x|_S$ for some $x$ and some notation system $(S, | \cdot |_S)$. In the latter case we say that $(S, | \cdot |_S)$ names $\xi$ or that $\xi$ receives a notation in $(S, | \cdot |_S)$. By transfinite induction one can check that the class of constructive ordinals is closed from below. It is a central result of Kleene [8] that there exists a notation system which names all constructive ordinals. It follows that constructive ordinals are countably many.

As usual the preceding notions relativize. One defines the $A$-constructive ordinals for $A \subseteq \omega$ by replacing the term “recursive” with “$A$-recursive”.

For all $A \subseteq \omega$ we fix a notation system
\[(S^A, | \cdot |_{S^A})\]
which names all $A$-constructive ordinals. When $A = \emptyset$ we just write $(S, | \cdot |_S)$.

The Church-Kleene ordinal $\omega^A_{CK}$ is the least ordinal which does not receive a notation in $(S, | \cdot |_S)$, and the relativized Church-Kleene ordinal $\omega^A_1$ is the least ordinal which does not receive a notation in $(S^A, | \cdot |_{S^A})$. The next result gives an elegant characterization of the latter ordinals.

**Theorem C.1** (Markwald [10], Spector [19]). A countable ordinal is recursive ($A$-recursive) exactly when it is constructive (resp. $A$-constructive).

In particular
\[ \omega^A_{CK} = \sup\{| \leq e : e \in W\} \quad \text{and} \quad \omega^A_1 = \sup\{| \leq^A \leq e : e \in W^A\}. \]

Hence $\omega^A_{CK}$ is the least non-recursive ordinal, and it can be considered as the recursive analogue of the least uncountable ordinal $\omega_1$.

**D. Transfinite iterations of the Turing jump.** Using the notation system of ordinals that we fixed above, one defines the transfinite iterations of the Turing jump. Let $A \subseteq \omega$ and $K, P, Q$ be the corresponding functions of the notation system $(S^A, | \cdot |_{S^A})$. For all $\xi < \omega^A_1$ we choose some $i_\xi \in S^A$ such that $\xi = |i_\xi|_{S^A}$. By recursion we define
\[ A^{(\xi)} = (A^{(\eta)})', \]
for $\xi = \eta + 1 < \omega^A_1$, and for limit $\xi < \omega^A_1$ we put
\[ (t, e) \in A^{(\xi)} \iff e \in A^{(|Q(i_\xi, t)|_{S^A})}. \]

It is clear that the set $A^{(\xi)}$ depends not only on the notation system that we chose, but also on the notation of the ordinal $\xi$. On the other hand the Turing degree of $A^{(\xi)}$ depends only on $\xi$ and $A$. (This is an application of Spector’s Uniqueness
Theorem using also the fact that the notation system in the latter theorem is “universal”, cf. Theorems 7.1 and 9.1 in [12]. Hence for given \( A, B \subseteq \omega \) the validity of the statement “\( A^{(0)} \leq_T B^{(\xi)} \)” is independent of the notation system and the functions \( \eta \in \omega^A \mapsto i_\eta \in \mathcal{S}^A, \xi \in \omega^B \mapsto i_\xi \in \mathcal{S}^B \) that we chose.

A set \( A \subseteq \omega \) is hyperarithmetic or simply HYP if there exists some \( \xi < \omega^1_{CK} \) such that \( A \leq_T \emptyset^{(\xi)} \), and hyperarithmetic in \( B \) or HYP\((B)\), where \( B \subseteq \omega \), if \( A \leq_T B^{(\xi)} \) for some \( \xi < \omega^1_{CK} \).

E. Basic elements from effective descriptive set theory. Suppose that \((\mathcal{X}, d)\) is a complete and separable metric space. A sequence \((x_n)_{n \in \omega}\) in a complete separable metric space \((\mathcal{X}, d)\) is a recursive presentation of \((\mathcal{X}, d)\) if the set \(\{x_n : n \in \omega\}\) is dense in \(\mathcal{X}\) and the relations \(P_\prec, P_\leq \subseteq \omega^3\) defined by

\[
P_\prec(i, j, k, m) \iff d(x_i, x_j) < k \cdot (m + 1)^{-1}
\]

\[
P_\leq(i, j, k, m) \iff d(x_i, x_j) \leq k \cdot (m + 1)^{-1}
\]

are recursive. We say that \((\mathcal{X}, d)\) admits a recursive presentation, or that \((\mathcal{X}, d)\) is recursively presented, if there is a sequence \((x_n)_{n \in \omega}\) in \(\mathcal{X}\) which is a recursive presentation of \((\mathcal{X}, d)\). The Baire space, \(\omega\), and other natural Polish spaces such as the reals are recursively presented.

For every complete space \((\mathcal{X}, d)\) with a recursive presentation \((x_n)_{n \in \omega}\) and all \(s = \langle n, m, k \rangle\) we put

\[
N(\mathcal{X}, s) = \{x \in \mathcal{X} : d(x, x_n) < k \cdot (m + 1)^{-1}\}.
\]

In the other cases of \(s\) we let \(N(\mathcal{X}, s)\) be the empty set. Clearly the family \(\{N(\mathcal{X}, s) : s \in \omega\}\) forms a basis for the topology of \(\mathcal{X}\). When referring to a recursively presented metric space we will often omit the metric and the recursive presentation.

We fix the following scheme for passing to products of two metrizable spaces: if \((X, d), (Y, p)\) are given we define the metric

\[
\tau : (X \times Y) \times (X \times Y) \to \mathbb{R} : \tau((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + p(y_1, y_2).
\]

It is then easy to check that recursive presentations are carried to finite products.

For every recursively presented metric space \(\mathcal{X}\) we say that \(A \subseteq \mathcal{X}\) is semirecursive if there is a recursive function \(f : \omega \to \omega\) such that

\[
A = \bigcup_{s \in \omega} N(\mathcal{X}, f(s)).
\]

In the case where \(\mathcal{X}\) is \(\omega^k\) this coincides with the definition that we mention above. It is also worth noting that the definition of semirecursive sets does not depend on the way we have encoded \(N(\mathcal{X}, s)\) cf. 3C.12 in [11].

The Kleene pointclasses. The Kleene pointclasses are

\[
\Sigma^0_1 = \text{all semirecursive sets, } \Sigma^1_1 = \exists^\omega \neg \Sigma^0_1,
\]

\[
\Sigma^0_{n+1} = \exists^\omega \neg \Sigma^0_n, \quad \Sigma^0_{n+1} = \exists^\omega \neg \Sigma^0_n,
\]

\[
\Pi^0_n = \neg \Sigma^0_n, \quad \Pi^1_n = \neg \Sigma^1_n,
\]

\[
\Delta^0_n = \Sigma^0_n \cap \Pi^0_n, \quad \Delta^1_n = \Sigma^1_n \cap \Pi^1_n.
\]

Suppose that \(\mathcal{X}\) and \(\mathcal{Y}\) are recursively presented metric spaces, \(y \in \mathcal{Y}\) and \(A \subseteq \mathcal{X}\). We say that \(A\) is \(y\)-semirecursive if there is a semirecursive \(P \subseteq \mathcal{Y} \times \mathcal{X}\) such that \(A = P_y\). Using recursion as above one builds the relativized Kleene pointclasses \(\Sigma^0_n(y), \Pi^0_n(y), \Delta^0_n(y), \Sigma^1_n(y), \Pi^1_n(y)\) and \(\Delta^1_n(y)\).
One can easily verify that the boldface analogue of a Kleene pointclass is the union along the Baire space of the relativized Kleene pointclasses, i.e.,

$$\Gamma = \cup_{\varepsilon \in \mathbb{N}} \Gamma(\varepsilon),$$

where $\Gamma$ is a Kleene pointclass.

**Points in $\Gamma$.** Suppose that $\Gamma$ is a pointclass and that $\mathcal{X}$ is a recursively presented metric space. A point $x \in \mathcal{X}$ is in $\Gamma$ or it is a $\Gamma$ point if the relation $U^x \subseteq \omega$ defined by

$$U^x(s) \iff x \in N(\mathcal{X}, s)$$

is in $\Gamma$. We use the notation $x \in \Gamma$ to denote that $x$ is a $\Gamma$ point.

We say that $x$ is a recursive point if $x \in \Sigma_0^1$ and that $x$ is recursive in $y$ if $x \in \Sigma_0^1(y)$. In case where where $\mathcal{X} = \mathcal{Y} = 2^\omega$ this is the same is saying that $x$ is a recursive subset of $\omega$ and $x \leq_T y$ respectively. (See 3D.19 in [11].)

Other cases of special interest is when $\Gamma = \Delta_1^1$ and $\Gamma = \Delta_1^1(y)$. As expected when $\mathcal{X} = \mathcal{Y} = 2^\omega$, we have that $x \in \Delta_1^1$ exactly when $x \in \text{HYP}$ and similarly $x \in \Delta_1^1(y)$ exactly when $x \in \text{HYP}(y)$, see the comments following Lemma F.2.

**$\Gamma$-recursive functions.** Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are recursively presented metric spaces. For every function $f : \mathcal{X} \to \mathcal{Y}$ we associate the relation $R(f) \subseteq \mathcal{X} \times \omega$ defined by

$$R(f)(x, s) \iff f(x) \in N(\mathcal{Y}, s).$$

Given a pointclass $\Gamma$ we say that $f$ is $\Gamma$-recursive if the preceding $R(f)$ is in $\Gamma$. In case where $\mathcal{X} = \mathcal{Y} = \omega$, $\Sigma_0^1$-recursive functions are exactly the recursive ones.

The preceding notion is the effective analogue of classical measurability. To be more specific a function $f$ is $\Gamma$-measurable exactly when it is $\Gamma(\varepsilon)$-recursive for some $\varepsilon$, where $\Gamma$ is $\Sigma_0^1$, or $\Sigma_1^1$. In particular $f$ is $\Sigma_1^1$-measurable (i.e., Borel-measurable) exactly when $f$ is $\Sigma_1^1(\varepsilon)$-recursive (which is the same as $\Delta_1^1(\varepsilon)$-recursive) for some $\varepsilon$.

A partial recursive function $f : \mathcal{X} \to \mathcal{Y}$ is $\Gamma$-recursive on $A \subseteq \text{Dom}(f)$ if there exists some $P(f) \in \Gamma$ such that

$$P(f)(x, s) \iff f(x) \in N(\mathcal{Y}, s)$$

for all $x \in A$, i.e., $P(f)$ agrees with $R(f)$ on $A \times \omega$.

**The Relativization Principle.** The preceding effective notions are given in the context of recursively presented metric spaces. It is not hard to see that not all complete separable spaces fall in the latter category. This is however not as restrictive as it seems. A sequence $(x_n)_{n \in \omega}$ in a complete separable metric space $(\mathcal{X}, d)$ is an $\varepsilon$-recursive presentation if the set $\{x_n : n \in \omega\}$ is dense in $\mathcal{X}$ and the corresponding relations $P_{<\varepsilon}$, $P_{\leq\varepsilon}$ are $\varepsilon$-recursive. Every complete separable space is $\varepsilon$-recursively presented for some $\varepsilon \in 2^\omega$: we just pick a dense sequence $(x_n)_{n \in \omega}$ and we let $\varepsilon$ be the characteristic function of $P_{<\varepsilon} \oplus P_{\leq\varepsilon}$, where $P_{<\varepsilon}$ and $P_{\leq\varepsilon}$ are viewed as subsets of naturals.

By replacing the term “recursive” with “$\varepsilon$-recursive” one can repeat all preceding definitions, also the relativized ones. For example given two $\varepsilon$-recursively presented metric spaces $\mathcal{X}$, $\mathcal{Y}$ we say that the set $A \subseteq \mathcal{X}$ is $\Sigma_0^1(\varepsilon)$ if it can be written as the union of basic neighborhoods of the form $N(\mathcal{X}, f(s))$, where $f : \omega \to \omega$ is $\varepsilon$-recursive, and $A$ is $\Sigma_0^1(\varepsilon, y)$ if it is the $y$-section of a $\Sigma_0^1(\varepsilon)$ subset of $\mathcal{Y} \times \mathcal{X}$. (Notice that $\mathcal{Y} \times \mathcal{X}$ is $\varepsilon$-recursively presented.) If $\mathcal{X}$ and $\mathcal{Y}$ are in fact recursively
presented then our present definition of $\Sigma^0_1(\varepsilon, y)$ coincides with the usual one, i.e., $A$ is a $\Sigma^0_1(\varepsilon, y)$ exactly when it is the $(\varepsilon, y)$-section of a $\Sigma^0_1$ subset of $\mathcal{N} \times Y \times X$. The theory of the recursively presented metric spaces is carried to this relativized setting. The latter is the Relativization Principle, for more information the reader can refer to Section 3I in [11].

F. Borel codes. We recall the following coding of the pointclasses $\Sigma^0_{\xi}$. The references about the following notions and results are to [9] and [11]. Define the sets $BC_{\xi} \subseteq N$, $\xi < \omega_1$ recursively

$$\alpha \in BC_{\xi} \iff \alpha(0) = 0,$$

$$\alpha \in BC_{\xi} \iff \alpha(0) = 1 \& (\forall n)(\exists \zeta < \xi)[(\alpha^*)_n \in BC_{\xi}].$$

The set of Borel codes is

$$BC = \cup_{\xi < \omega} BC_{\xi}.$$ 

For $\alpha \in BC$ we put

$$|\alpha| = \text{the least } \xi \text{ such that } \alpha \in BC_{\xi}.$$

Given a Polish space $\mathcal{X}$ we define the functions $\pi^{\mathcal{X}}_{\xi} : BC_{\xi} \to \Sigma^0_{\xi} \upharpoonright \mathcal{X}$ by recursion,

$$\pi^X_0(\alpha) = \mathcal{X} \setminus N(\mathcal{X}, \alpha^*(1)),$$

$$\pi^X_{\xi}(\alpha) = \bigcup_n \mathcal{X} \setminus \pi^X_{((\alpha^*)_n)},$$

whereas by $\Sigma^0_{\xi} \upharpoonright \mathcal{X}$ we mean the family $\{\mathcal{X} \setminus N(\mathcal{X}, s) : s \in \omega\}$. An easy induction shows that for all $1 \leq \zeta \leq \xi$ we have that $BC_{\zeta} \subseteq BC_{\xi}$ and $\pi^{\mathcal{X}}_{\xi} \upharpoonright BC_{\zeta} = \pi^{\mathcal{X}}_{\xi}$. We now define

$$\pi^{\mathcal{X}} : BC \to \text{Borel}(\mathcal{X}) : \pi^{\mathcal{X}}(\alpha) = \pi^X_{|\alpha|}(\alpha)$$

$$= \pi^X_{\xi}(\alpha) \quad \text{(when } \xi > |\alpha| \geq 1).$$

**Definition F.1** (Well-known). Suppose that $\mathcal{X}$ is recursively presented. The set $P \subseteq \omega \times 2^\omega$ defined by

$$P(i, A) \iff i \in A^{(\xi)}$$

is $\Sigma^0_{\xi}$;

(b) for all $A \in \Sigma^0_{\xi}(B)$ it holds $A \leq_T B^{(\xi)}$; and

(c) the function $A \in 2^\omega \mapsto A^{(\xi)}$ is $\Sigma^0_{\xi+1}$-recursive.
In resemblance to the classical case it holds that
\[ \Delta_1^1 = \bigcup_{\xi<\omega_1} \Sigma_0^0 = \bigcup_{\xi<\omega_1} \Sigma_0^0 = \Delta_1^0 \]
for \( A \subseteq \omega \), cf. [11] 7B.6. In particular for sets \( A \subseteq \omega \) the following are equivalent:
\begin{itemize}
  \item \( A \) is \( \Delta_1^1 \);
  \item there exists \( \xi < \omega_1^A \) such that \( A \leq_T \emptyset^{(\xi)} \);
  \item there exists \( \xi < \omega_1^A \) such that \( A \leq_T \emptyset^{(\xi)} \).
\end{itemize}
The similar assertion holds for the relation \( A \in \Delta_1^1 (B) \) for \( B \subseteq \omega \).

G. The system \((B^X_\xi)_X\) is good. Here we prove Lemma 2.11. First we need the following auxiliary result.

**Lemma G.1.** For every recursive function \( g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) there exists a recursive function \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) which satisfies the following conditions:
\begin{align*}
  &f(\varepsilon, \alpha)(0) = \varepsilon(0) \tag{3} \\
  &\varepsilon(0) = 0 \implies f(\varepsilon, \alpha)^* = g(\varepsilon, \alpha) \tag{4} \\
  &\varepsilon(0) = 1 \implies (f(\varepsilon, \alpha)^*)_n = f((\varepsilon^*)_n, \alpha) \tag{5}
\end{align*}
for all \( n, \varepsilon, \alpha \).

It follows that
\[ \varepsilon \in BC_\xi \iff f(\varepsilon, \alpha) \in BC_\xi \tag{6} \]
for all \( \varepsilon, \alpha \) and all \( \xi < \omega_1 \).

**Proof.** We will define by induction on \( t \) the value \( f(\varepsilon, \alpha)(t) \) for all \( \varepsilon, \alpha \) using Kleene’s Second Recursion Theorem. This is possible because of the property of the encoding of finite sequences that we chose,
\[ \max \{ u_0, \ldots, u_{n-1} \} < \langle u_0, \ldots, u_{n-1} \rangle \]
for \( u_0, \ldots, u_{n-1} \in \omega \). We define the partial function
\[ \varphi : \omega \times \omega \times \mathcal{N} \times \mathcal{N} \rightarrow \omega \]
as follows
\[ \varphi(e, k, \varepsilon, \alpha) \downarrow \iff \text{if } \varepsilon(0) = 1 \text{ and } k = 1 + \langle n, t \rangle \text{ for some } n, t < k, \]
then \( \{e\}(t, (\varepsilon^*)_n, \alpha) \downarrow \),
and when \( \varphi(e, k, \varepsilon, \alpha) \downarrow \) we have that
\[ \varphi(e, k, \varepsilon, \alpha) = \begin{cases} 
\varepsilon(0), & \text{if } k = 0 \\
g(\varepsilon, \alpha)(m), & \text{if } k = m + 1 \text{ and } \varepsilon(0) = 0 \\
\{e\}(t, (\varepsilon^*)_n, \alpha), & \text{if } k = 1 + \langle n, t \rangle \text{ for some } n, t < k \text{ and } \varepsilon(0) = 1 \\
0, & \text{else.}
\end{cases} \]
It is clear that \( \varphi \) is partial recursive on its domain. From Kleene’s Second Recursion Theorem (cf. [11] 7A.2 and 7A.4) there exists some \( e^* \in \omega \) such that whenever \( \varphi(e^*, k, \varepsilon, \alpha) \downarrow \) then \( \{e^*\}(k, \varepsilon, \alpha) \downarrow \) and
\[ \varphi(e^*, k, \varepsilon, \alpha) = \{e^*\}(k, \varepsilon, \alpha). \]
One can verify that the function
\[ f : \mathcal{N} \times \mathcal{N} \to \mathcal{N} : f(\varepsilon, \alpha)(k) = \varphi(\varepsilon^*, k, \varepsilon, \alpha) = \{e^*\}(k, \varepsilon, \alpha). \]
satisfies all required properties. We omit the details. \( \square \)

Now we proceed to the main proof. As in the proof of Lemma 1.5 we only consider the case \( \mathcal{X} = \mathcal{N} \). We take the functions \( h = (h_1, h_2) \) and \( S_1^N \cdot y \) of the latter proof and we define
\[ g : \mathcal{N} \times \mathcal{N} \to \mathcal{N} : g(\varepsilon, \alpha) = S(\varepsilon^*, \alpha). \]
Clearly \( g \) is recursive. Let \( f \) be the recursive function which is obtained from Lemma G.1 for this choice of \( g \). We claim that \( B_{N \times \mathcal{Y}}(\varepsilon, \alpha, y) \iff B_{\mathcal{Y}}(f(\varepsilon, \alpha), y) \)
for all \( \varepsilon, \alpha, y \) and all \( \xi < \omega_1 \), i.e., we can take \( R_{N \times \mathcal{Y}} \) to be the preceding function \( f \).

This is proved by induction on \( \xi \). The equivalence
\[ B_{0}^{\mathcal{N} \times \mathcal{Y}}(\varepsilon, \alpha, y) \iff B_{0}^{\mathcal{Y}}(f(\varepsilon, \alpha), y) \]
is proved using the properties (3) and (4) from above.

To complete the inductive step we compute
\[ B_{\xi}^{\mathcal{N} \times \mathcal{Y}}(\varepsilon, \alpha, y) \iff \varepsilon \in BC_{\xi} \& (\alpha, y) \in \pi_{\mathcal{N} \times \mathcal{Y}}(\varepsilon) \]
\[ \iff \varepsilon(0) = 1 \]
\[ & \& (\forall n)[(\varepsilon^*)_n \in \bigcup_{\zeta < \xi} BC_{\zeta}] \& (\exists n)[(\alpha, y) \notin \pi_{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n)] \]
\[ \iff \varepsilon(0) = 1 \& (\forall n)[(\varepsilon^*)_n \in \bigcup_{\zeta < \xi} BC_{\zeta}] \]
\[ & \& (\exists n)(\exists \zeta < \xi)[(\varepsilon^*)_n \in BC_{\zeta} \& \neg B_{\xi}^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n, \alpha, y)] \]
\[ \text{ (from the definition of } B_{\xi}^{\mathcal{N} \times \mathcal{Y}} \text{—see the comments below) } \]
\[ \iff \varepsilon(0) = 1 \& (\forall n)[f((\varepsilon^*)_n, \alpha) \in \bigcup_{\zeta < \xi} BC_{\zeta}] \]
\[ & \& (\exists n)(\exists \zeta < \xi)[f((\varepsilon^*)_n, \alpha) \in BC_{\zeta} \& \neg B_{\xi}^{\mathcal{Y}}(f((\varepsilon^*)_n, \alpha), y)] \]
\[ \text{ (from (6) and the inductive hypothesis) } \]
\[ \iff \varepsilon(0) = 1 \& (\forall n)[f((\varepsilon^*)_n, \alpha) \in \bigcup_{\zeta < \xi} BC_{\zeta}] \]
\[ & \& (\exists n)[y \notin \pi_{\mathcal{Y}}(f((\varepsilon^*)_n, \alpha))] \]
\[ \text{ (from the definition of } B_{\xi}^{\mathcal{Y}} \text{ as above) } \]
\[ \iff \varepsilon(0) = 1 \& (\forall n)[(f(\varepsilon, \alpha^*)_n) \in \bigcup_{\zeta < \xi} BC_{\zeta}] \]
\[ & \& (\exists n)[y \notin \pi_{\mathcal{Y}}((f(\varepsilon, \alpha^*)_n))] \]
\[ \text{ (from (5)) } \]
\[ \iff f(\varepsilon, \alpha)(0) = 1 \& (\forall n)[(f(\varepsilon, \alpha^*)_n) \in \bigcup_{\zeta < \xi} BC_{\zeta}] \]
\[
\& (\exists n)[y \notin \pi^Y((f(\varepsilon, \alpha)^*)_n)] \\
\iff f(\varepsilon, \alpha) \in BC_\varepsilon \& y \in \cup_n \mathcal{Y} \setminus \pi^Y_{f(\varepsilon, \alpha)^*_n}(f(\varepsilon, \alpha)^*_n)
\]

(from (3))

\[
\iff f(\varepsilon, \alpha) \in BC_\varepsilon \& y \in \pi^Y_{f(\varepsilon, \alpha)}(f(\varepsilon, \alpha))
\]

\[
\iff f(\varepsilon, \alpha) \in BC_\varepsilon \& y \in \pi^Y_{f(\varepsilon, \alpha)}(f(\varepsilon, \alpha))
\]

(since \(\xi \geq |f(\varepsilon, \alpha)| \geq 1\))

\[
\iff f(\varepsilon, \alpha) \in BC_\varepsilon \& y \in \pi^Y(f(\varepsilon, \alpha))
\]

\[
\iff B_\varepsilon^Y(f(\varepsilon, \alpha), y)
\]

and the inductive step is complete. The third of the preceding equivalences needs perhaps some more explanation. From the definition of \(B_\xi^{\mathcal{N} \times \mathcal{Y}}\) it clear that

\[
\neg B_\xi^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n, \alpha, y) \iff (\varepsilon^*)_n \not\in BC_\varepsilon \lor [(\varepsilon^*)_n \in BC_\varepsilon \& (\alpha, y) \notin \pi^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n)].
\]

Let us prove the required equivalence. For the direct implication we choose some \(n\) and \(\xi\) such that \((\varepsilon^*)_n \in BC_\varepsilon\) and \((\alpha, y) \not\in \pi^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n)\). From the preceding equivalence it follows that \(\neg B_\xi^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n, \alpha, y)\) holds. For the converse direction we notice that if \((\varepsilon^*)_n\) is in \(BC_\varepsilon\) and \(\neg B_\xi^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n, \alpha, y)\) holds, then from the preceding equivalence we have that \((\alpha, y) \not\in \pi^{\mathcal{N} \times \mathcal{Y}}((\varepsilon^*)_n)\). This finishes the proof.

References

