

# A Unified Worst Case for Classical Simplex and Policy Iteration Pivot Rules

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## Abstract

We construct a family of Markov decision processes for which the policy iteration algorithm needs an exponential number of improving switches with Dantzig’s rule, with Bland’s rule, and with the Largest Increase pivot rule. This immediately translates to a family of linear programs for which the simplex algorithm needs an exponential number of pivot steps with the same three pivot rules. Our results yield a unified construction that simultaneously reproduces well-known lower bounds for these classical pivot rules, and we are able to infer that any (deterministic or randomized) combination of them cannot avoid an exponential worst-case behavior. Regarding the policy iteration algorithm, pivot rules typically switch multiple edges simultaneously and our lower bound for Dantzig’s rule and the Largest Increase rule, which perform only single switches, seem novel. Regarding the simplex algorithm, the individual lower bounds were previously obtained separately via deformed hypercube constructions. In contrast to previous bounds for the simplex algorithm via Markov decision processes, our rigorous analysis is reasonably concise.

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## 1 Introduction

Since the simplex algorithm for linear programming was proposed by Dantzig in 1951 [12], it has been a central question in discrete optimization whether it admits a polynomial time pivot rule. A positive answer to this question would yield an efficient combinatorial algorithm for solving linear programs, and thus resolve an open problem on Smale’s list of mathematical problems for the 21st century [41]. It would also resolve the polynomial Hirsch conjecture [11], which states that every two vertices of every polyhedron with  $n$  facets are connected via a path of  $\mathcal{O}(\text{poly}(n))$  edges. At this point, the best known pivot rules are randomized and achieve subexponential running times in expectation [21, 27, 31, 36].

For the most natural, memoryless and deterministic, pivot rules, exponential worst-case examples based on distorted hypercubes were constructed early on [4, 25, 30, 35, 38]. Amenta and Ziegler [3] introduced the notion of deformed products to unify several of these constructions. However, while this unification defines a class of polytopes that generalizes distorted hypercubes, it does not yield a unified exponential worst-case construction to exclude all pivot rules based on these deformed products, and neither does it yield new lower bounds for additional pivot rules.

Randomized and history-based pivot rules resisted similar approaches, and it was a major breakthrough in 2011 when Friedmann et al. were able to prove the first subexponential lower bound for several randomized pivot rules [20, 21, 26]. They introduced a new technique



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based on a connection [39] between Howard’s policy iteration algorithm [28] for Markov decision processes (MDPs) and the simplex algorithm for linear programs (LPs). The same technique was later used to prove exponential lower bounds for history-based pivot rules that had been candidates for polynomial time rules for a long time [5, 15]. While the approach via MDPs has proven powerful, the resulting analyses are often very technical (the full version of [15] with all details of the proof has 197 pages).

In this paper, we apply the MDP-based technique to classical (memoryless and deterministic) pivot rules and obtain a unified construction that excludes several pivot rules at the same time, and any combination of them, while being relatively simple.

**Our results.** We give a unified worst-case construction for the policy iteration algorithm for MDPs that simultaneously applies to three of the most classical pivot rules. The rigorous analysis of the resulting MDPs is reasonably concise. We note that the exponential lower bounds for Dantzig’s rule and the Largest Increase rule seem novel for the considered version of the policy iteration algorithm, while the result for Bland’s rule is known [37].

► **Theorem 1.** *There is a family  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  of Markov decision processes  $\mathcal{D}_n$  of size  $\mathcal{O}(n)$  such that policy iteration performs  $\Omega(2^n)$  improving switches with Dantzig’s rule, Bland’s rule, and the Largest Increase pivot rule.*

In fact, all three pivot rules apply the same set of improving switches with only slight differences in the order in which they get applied. Because of this, the result still holds if we allow to change pivot rules during the course of the algorithm.

► **Corollary 2.** *For any (deterministic or randomized) combination of Dantzig’s, Bland’s, or the Largest Increase rule, the policy iteration algorithm has an exponential running time.*

A well-known connection between policy iteration and the simplex method, allows to immediately translate our result to the simplex algorithm with the same pivot rules. In particular, we obtain an exponential lower bound construction that holds even if, in every step, the entering variable is selected independently according to Dantzig’s rule, Bland’s rule, or the Largest Increase pivot rule, i.e., even if we change pivot rules during the course of the algorithm. In other words, we obtain a lower bound for a family of pivot rules that results from combining these three rules.

► **Corollary 3.** *There is a family  $(\mathcal{L}_n)_{n \in \mathbb{N}}$  of linear programs  $\mathcal{L}_n$  of size  $\mathcal{O}(n)$  such that the simplex algorithm performs  $\Omega(2^n)$  pivot operations for any (deterministic or randomized) combination of Dantzig’s, Bland’s, or the Largest Increase pivot rule.*

**Related work.** Policy iteration for MDPs has been studied extensively for a variety of pivot rules. In its original version [28], the algorithm applies improving switches to the current policy in all states simultaneously in every step. Fearnley [18] showed an exponential lower bound for a greedy pivot rule that selects the best improvement in every switchable state. In this paper, we focus on pivot rules that only apply a single switch in each iteration. Most of the MDP constructions for randomized or history-based pivot rules [5, 15, 21, 26] consider this case, and Melekopoglou and Condon [37] gave exponential lower bounds for several such deterministic pivot rules. We emphasize that their constructions already include an exponential lower bound for Bland’s rule [8]. Since policy iteration is traditionally considered with simultaneous switches, to the best of our knowledge, no exponential lower bounds are known for Dantzig’s rule [12] and the Largest Increase rule [11] in the setting of single switches.

There is a strong connection between policy iteration and the simplex algorithm, which, under certain conditions (see below), yields that worst-case results for policy iteration carry over to the simplex method [39]. This connection was used to establish subexponential lower bounds for randomized pivot rules, namely Randomized Bland [26] and Random-Edge, RaisingTheBar and Random-Facet [21]. It also lead to exponential lower bounds for history-based rules, namely Cunningham’s rule [5] and Zadeh’s rule [15]. Conversely, lower bounds for the simplex algorithm with classical pivot rules were obtained via deformed hypercubes [3] and do not transfer to MDPs. Such results include lower bounds for Dantzig’s rule [35], the Largest Increase rule [30], Bland’s rule [4], the Steepest Edge rule [25], and the Shadow Vertex rule [38]. We provide an alternate lower bound construction for the first three of these rules via a family of MDPs. As far as we can tell, as a side product, this yields the first exponential lower bound for policy iteration with Dantzig’s rule and the Largest Increase rule.

While it remains open whether LPs can be solved in strongly polynomial time, there are several, both deterministic [32, 34] and randomized [7, 17, 33], algorithms that solve LPs in weakly polynomial time. A (strongly) polynomial time pivot rule for the simplex algorithm would immediately yield a strongly polynomial algorithm.

There have been different attempts to deal with the worst-case behavior of the simplex method from a theoretical perspective. For example, the excessive running time was justified by showing that the simplex algorithm with Dantzig’s original pivot rule is *NP-mighty* [16], which means that it can be used to solve NP-hard problems. This result was subsequently strengthened by establishing that deciding which solution is computed and whether a given basis will occur is PSPACE-complete [2, 19]. On the positive side, there are different results explaining the efficiency of the simplex method in practice, such as average-case analyses [1, 9, 44]. Spielman and Teng [42] introduced smoothed analysis as a way of bridging the gap between average-case and worst-case analysis. They showed that the simplex algorithm with the shadow vertex pivot rule [24] has a polynomial smoothed complexity, and their results were further improved later [10, 14, 29, 45].

Another approach to derive stronger lower bounds on pivot rules is to consider combinatorial abstractions of LPs, such as Unique Sink Orientations (USOs) [23]. There is still a large gap between the best known deterministic algorithm for finding the unique sink, which is exponential [43], and the almost quadratic lower bound [40]. Considering randomized rules, the Random-Facet pivot rule, which is the best known simplex rule [27], is also the best known pivot rule for acyclic USOs [22], achieving a subexponential running time in both settings.

## 2 Preliminaries

### Markov Decision Processes

A *Markov decision process* is an infinite duration one-player game on a finite directed graph  $G = (V_A, V_R, E_A, E_R, r, p)$ . The vertex set  $V = V_A \cup V_R$  of the graph is divided into *agent vertices*  $V_A$  and *randomization vertices*  $V_R$ . Every agent edge  $e \in E_A \subseteq V_A \times V$  is assigned a *reward*  $r(e) \in \mathbb{R}$ , while every randomization edge  $\hat{e} \in E_R \subseteq V_R \times V$  is assigned a *transition probability*  $p(\hat{e}) \in [0, 1]$ . Outgoing transition probabilities add to one in every randomization vertex.

A process starts in an arbitrary starting vertex. If this is an agent vertex, the agent moves along one of the outgoing edges of this vertex (we assume that all vertices have at least one outgoing edge) and collects the corresponding reward. Otherwise, it gets randomly moved along one of the outgoing edges according to the transition probabilities. The process continues in this manner ad infinitum.

An agent vertex  $s \in V_A$  whose only outgoing edge is a self-loop with reward zero is called *sink* of  $G$  if it is reachable from all vertices. A *policy* for  $G$  is a function  $\pi: V_A \rightarrow V$  with  $(v, \pi(v)) \in E_A$  for all  $v \in V_A$ , determining the behavior of the process in agent vertices. A policy  $\pi$  for  $G$  is called *weak unichain* if  $G$  has a sink  $s$  such that  $\pi$  reaches  $s$  with a probability of one from every starting vertex.

The *value* of a vertex  $v$  w.r.t. a policy  $\pi$  for a Markov decision process  $G$  is given by the expected total reward that the agent collects with policy  $\pi$  when the process starts in  $v$ . More formally, the value function  $\text{Val}_{\pi,G}: V \rightarrow \mathbb{R}$  is defined by the following system of Bellman [6] equations

$$\text{Val}_{\pi,G}(u) = \begin{cases} r((u, \pi(u))) + \text{Val}_{\pi,G}(\pi(u)), & \text{if } u \in V_A, \\ \sum_{v \in \Gamma^+(u)} p((u, v)) \text{Val}_{\pi,G}(v), & \text{if } u \in V_R, \end{cases}$$

together with  $\text{Val}_{\pi,G}(s) = 0$  if  $G$  has a sink  $s$ . The policy  $\pi$  is optimal (w.r.t. the *expected total reward criterion*) if  $\text{Val}_{\pi,G}(v) \geq \text{Val}_{\tilde{\pi},G}(v)$  for all  $v \in V_A$  and all policies  $\tilde{\pi}$  for  $G$ . Whenever the underlying process  $G$  is clear from the context, we write  $\text{Val}_{\pi}$  instead of  $\text{Val}_{\pi,G}$ .

We say that the agent edge  $(u, v) \in E_A$  is an *improving switch* for the policy  $\pi$  for process  $G$  if it satisfies  $z_{\pi,G}(u, v) := r((u, v)) + \text{Val}_{\pi,G}(v) - \text{Val}_{\pi,G}(u) > 0$ , where  $z_{\pi,G}(u, v)$  are the *reduced costs* of  $(u, v)$  with respect to  $\pi$ . Again, we usually write  $z_{\pi}$  instead of  $z_{\pi,G}$ .

If we *apply* an improving switch  $s = (u, v) \in E_A$  to a policy  $\pi$ , we obtain a new policy  $\pi^s$  which is given by  $\pi^s(u) = v$  and  $\pi^s(w) = \pi(w)$  for all  $w \in V_A \setminus \{u\}$ . The improving switch  $s$  increases the value of  $u$  without decreasing the value of any other vertex.

## Policy Iteration for Markov Decision Processes

Howard's [28] policy iteration algorithm receives as input a finite Markov decision process  $G$  and a weak unichain policy  $\pi$  for  $G$ . It then iteratively applies a set of improving switches to the current policy until there are none left. In the remainder of this paper, we consider a version of this algorithm that applies a single switch in every iteration, cf. Algorithm 1. Due to monotonicity of the vertex values, this procedure visits every policy at most once. As there are only finitely many policies, the algorithm thus terminates after a finite number of iterations for every initial policy.

■ **Algorithm 1** POLICYITERATION( $G, \pi$ ).

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**input:** a weak unichain policy  $\pi$  for a Markov decision process  $G$

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**while**  $\pi$  admits an improving switch:

$\bar{s} \leftarrow$  improving switch for  $\pi$   
 $\pi \leftarrow \pi^{\bar{s}}$

**return**  $\pi$

---

We know that the policy iteration algorithm returns an optimal policy if there is an optimal policy which is weak unichain.

► **Theorem 4** ([20]). *Let  $\pi$  be a weak unichain policy for a Markov decision process  $G$ . If  $G$  admits a weak unichain, optimal policy, then POLICYITERATION( $G, \pi$ ) only visits weak unichain policies and returns an optimal policy w.r.t. the expected total reward criterion.*

In this paper, we consider the following three *pivot rules*, i.e., rules that determine the choice of  $\text{POLICYITERATION}(G, \pi)$  in each iteration:

- *Bland's pivot rule* assigns a unique number to every agent edge of  $G$ . Then, in every iteration, it chooses the improving switch with the smallest number.
- *Dantzig's pivot rule* chooses an improving switch  $\bar{s}$  maximizing the reduced costs  $z_\pi(\bar{s})$ .
- The *Largest Increase rule* chooses an improving switch  $\bar{s}$  maximizing  $\sum_{v \in V_A} \text{Val}_{\pi^{\bar{s}}}(v)$ .

## A Connection between Policy Iteration and the Simplex Method

Given a Markov decision process, we can formulate a linear program such that the application of the simplex method is in some sense equivalent to the application of policy iteration. We refer to [20] for more details and the derivation of the following result.

► **Theorem 5** ([20]). *Let  $\pi$  be a weak unichain policy for a Markov decision process  $G$ . Assume that there is an optimal, weak unichain policy for  $G$  and that  $\text{POLICYITERATION}(G, \pi)$  with a given pivot rule takes  $N$  iterations. Then, there is an LP of linear size such that the simplex algorithm with the same pivot rule takes  $N$  iterations.*

In terms of the simplex method, Bland's pivot rule chooses the entering variable of smallest index [8], Dantzig's rule chooses an entering variable maximizing the reduced costs [12], and the Largest Increase rule greedily maximizes the objective function value.

The linear program in the previous theorem has one variable for every agent edge of the Markov decision process such that the reduced costs of a given edge equal the reduced costs of the corresponding variable, and the objective function equals the sum over all vertex values as given in the Largest Increase rule for policy iteration [5, 15, 26]. Therefore, the choices of each pivot rule in the two settings are consistent.

Additionally, we want to mention that the linear program from Theorem 5 is always non-degenerate. Therefore, we cannot reduce the number of required iterations on these programs by combining a given pivot rule with the Lexicographic pivot rule [13].

## Notation

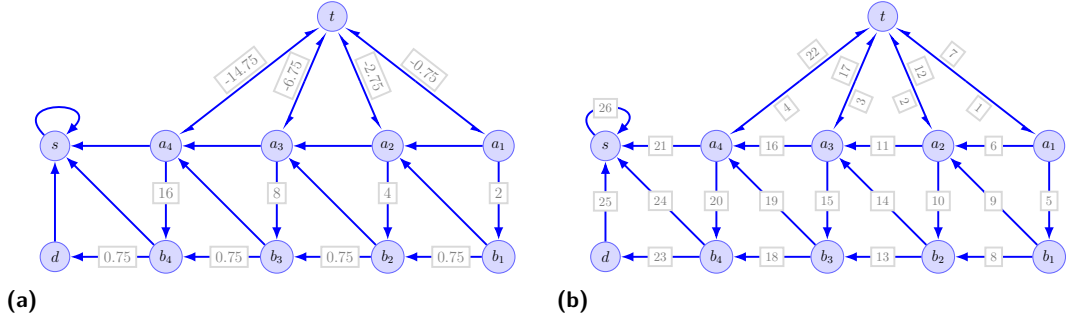
Let  $n \in \mathbb{N}$  be fixed. We write  $[n] = \{1, 2, \dots, n\}$  and  $[n]_0 = \{0, 1, \dots, n\}$ . Then, the set of all numbers that can be represented with  $n$  bits is  $[2^n - 1]_0$ .

For every  $x \in [2^n - 1]_0$  and  $i \in [n]$ , let  $x_i$  denote the  $i$ -th bit of  $x$ , i.e.,  $x = \sum_{i \in [n]} x_i 2^{i-1}$ , and let  $L(i, x) = \max\{j \in [i - 2] \mid x_j = 1 \text{ or } j = 1\}$  for  $i \geq 3$ . Finally, for  $x \in [2^n - 1]$ , we denote the *least significant set bit* of  $x$  by  $\ell_1(x) = \min\{i \in [n] : x_i = 1\}$ , and the *most significant set bit* of  $x$  by  $m_1(x) = \max\{i \in [n] : x_i = 1\}$ .

Let  $G = (V_A, V_R, E_A, E_R, r, p)$  be a Markov decision process. For  $v \in V_A \cup V_R$ , we write  $\Gamma_G^+(v) = \{w \in V_A \cup V_R : (v, w) \in E_A \cup E_R\}$ . If the underlying process is clear from the context, we just write  $\Gamma^+(v)$ .

### 3 An Exponential Lower Bound for Bland's pivot rule

In this section, we consider a family  $(\mathcal{B}_n = (V_{\mathcal{B}_n}, E_{\mathcal{B}_n}, r_{\mathcal{B}_n}))_{n \in \mathbb{N}}$  of Markov decision processes, which do not involve any randomization. Consider Figure 1a for a drawing of  $\mathcal{B}_4$ . Every process  $\mathcal{B}_n$  consists of  $n$  separate levels, together with a global *transportation vertex*  $t$ , a sink  $s$ , and a *dummy vertex*  $d$ . Each level  $\ell \in [n]$  comprises two vertices, called  $a_\ell$  and  $b_\ell$ . For convenience, we sometimes denote the sink by  $a_{n+1}$  and the dummy vertex by  $b_{n+1}$ .



■ **Figure 1** Two drawings of the Markov decision process  $\mathcal{B}_4$ . In (a), edge labels denote rewards and unlabeled edges have a reward of zero. In (b), edge labels define the Bland numbering  $\mathcal{N}_{\mathcal{B}_4}$ .

■ **Table 1** Edge names and the definition of the Bland numbering  $\mathcal{N}_{\mathcal{B}_n}$ , where  $i \in [n]$ .

$e \in E_{\mathcal{B}_n}$	$\mathcal{N}_{\mathcal{B}_n}(e)$
$(t, a_i)$	travel( $i$ )
$(a_i, b_i)$	enter( $i$ )
$(a_i, a_{i+1})$	skip( $i$ )
$(a_i, t)$	board( $i$ )
$(b_i, b_{i+1})$	stay( $i$ )
$(b_i, a_{i+1})$	leave( $i$ )

In vertex  $a_\ell$ , the agent can either *enter* level  $\ell$  by going to vertex  $b_\ell$ , *skip* this level by going to vertex  $a_{\ell+1}$ , or *board* the transportation vertex by going to  $t$ . From the transportation vertex, the agent *travels* to one of the vertices  $a_i$  with  $i \in [n]$ . In  $b_\ell$ , the agent can decide between *leaving* the set  $\bigcup_{i \in [n+1]} \{b_i\}$  by going to  $a_{\ell+1}$  and *staying* in this set by going to  $b_{\ell+1}$ . We will simply say that the agent leaves level  $\ell$  or stays in level  $\ell$ , respectively.

Finally, when the agent reaches the dummy vertex  $d$ , it must go to the sink, and the only outgoing edge of the sink  $s$  is the self-loop  $(s, s)$ .

The function  $r_{\mathcal{B}_n}$  grants the agent a reward of  $2^\ell$  for entering level  $\ell$ , a reward of 0.75 for staying in level  $\ell$ , and a (negative) reward of  $(-2^\ell + 1.25)$  for boarding  $t$  from  $a_\ell$ ; all other rewards are zero.

The *Bland numbering*  $\mathcal{N}_{\mathcal{B}_n} : E_{\mathcal{B}_n} \rightarrow |E_{\mathcal{B}_n}|$  of the edges of  $\mathcal{B}_n$  is defined in Table 1, together with  $\mathcal{N}_{\mathcal{B}_n}((d, s)) = 6n + 1$  and  $\mathcal{N}_{\mathcal{B}_n}((s, s)) = 6n + 2 (= |E_{\mathcal{B}_n}|)$ . This table also contains alternative names for the edges, which match the description above and which we will use to simplify the exposition. Consider Figure 1b for the Bland numbering of  $\mathcal{B}_4$ .

In the following, consider  $\mathcal{B}_n$  for some arbitrary but fixed  $n \in \mathbb{N}$ . The aim of this section is to show that POLICYITERATION with Bland's pivot rule, cf. Algorithm 2, applies  $\Omega(2^n)$  improving switches when given  $\mathcal{B}_n$ , a suitable initial policy, and  $\mathcal{N}_{\mathcal{B}_n}$  as input.

■ **Algorithm 2** BLAND( $G, \pi, \mathcal{N}$ ).

---

**input:** Markov decision process  $G$ , weak unichain policy  $\pi$ , edge numbering  $\mathcal{N}$

---

**while**  $\pi$  admits an improving switch:

$\bar{s} \leftarrow$  the improving switch  $s$  for  $\pi$  that minimizes  $\mathcal{N}(s)$

$\pi \leftarrow \pi^{\bar{s}}$

**return**  $\pi$

---

More precisely, we will see that the algorithm visits all of the following policies.

► **Definition 6.** The policy  $\pi_0$  for  $\mathcal{B}_n$  such that  $\text{travel}(1)$  is active, and  $\text{skip}(i)$  and  $\text{leave}(i)$  are active for all  $i \in [n]$  is the *canonical policy* for 0. For  $x \in [2^n - 1]$ , the policy  $\pi_x$  for  $\mathcal{B}_n$  is the *canonical policy* for  $x$  if it satisfies the following conditions:

- <1> The policy travels from  $t$  to the least significant set bit, i.e.,  $\text{travel}(\ell_1(x))$  is active.
- <2> It collects no reward above the most significant set bit, i.e.,  $\text{leave}(m_1(x))$ ,  $\text{skip}(i)$ , and  $\text{leave}(i)$  are active for all  $m_1(x) < i \leq n$ .
- <3> Every set bit  $x_i = 1$  determines the behavior of the policy down to the next, less significant set bit or, if  $i = \ell_1(x)$ , down to the first bit:
  - <a>  $\text{enter}(i)$  is active.
  - <b> if  $i = 2$ , then  $\text{leave}(1)$  is active. If additionally  $x_1 = 0$ , then  $\text{skip}(1)$  is active.
  - <c> if  $i \geq 3$  and  $x_{i-1} = 1$ , then  $\text{leave}(i-1)$  is active.
  - <d> if  $i \geq 3$  and  $x_{i-1} = 0$ :
    - <d<sub>1</sub>>  $\text{stay}(i-1)$ ,  $\text{skip}(i-1)$ , and  $\text{leave}(i-2)$  are active.
    - <d<sub>2</sub>> if  $L(i, x) < i-2$ , then for all  $j \in \{L(i, x)+1, \dots, i-2\}$ , the edges  $\text{board}(j)$  and  $\text{stay}(j-1)$  are active; if  $L(i, x) = 1$  and  $x_1 = 0$ , then  $\text{board}(1)$  is active. ┘

Consider Figure 2a and Figure 2d for examples of canonical policies. Note that canonical policies exist and are unique as the definition contains precisely one condition on every agent vertex with more than one outgoing edge. Further, the  $2^n$  canonical policies are pairwise different as  $\text{enter}(i)$  is active in  $\pi_x$  if and only if  $x_i = 1$ .

We will now analyze the behavior of  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ , i.e., we choose the canonical policy for zero as our initial policy. Since this policy visits every vertex except the sink only once, it is weak unichain.

► **Observation 7.** *The canonical policy  $\pi_0$  is a weak unichain policy for  $\mathcal{B}_n$ .*

Thus, according to Theorem 5, the following result will allow us to transfer our results for the policy iteration algorithm to the simplex method.

► **Lemma 8.** *Let the policy  $\pi_*$  for  $\mathcal{B}_n$  be determined as follows:  $\text{stay}(n)$  and  $\text{travel}(1)$  are active,  $\text{enter}(i)$  is active for all  $i \in [n]$ , and  $\text{leave}(j)$  is active for all  $j \in [n-1]$ . Then,  $\pi_*$  is weak unichain and optimal for  $\mathcal{B}_n$ .*

**Proof.** Since  $\pi_*$  visits every vertex, besides the sink, only once, it is weak unichain. For optimality, note that  $t$  travels to  $a_1$  and that, when starting in a vertex  $a_\ell$ , policy  $\pi_*$  enters level  $\ell$  and all levels above and collects the reward of  $\text{stay}(n)$ . The policy is thus clearly optimal among the set of policies that do not use boarding edges.

Further, we have  $r_{\mathcal{B}_n}(\text{board}(\ell)) = -2^\ell + 1.25 = -(\sum_{i=1}^{\ell-1} 2^i + 0.75)$ . That is, the costs of  $\text{board}(\ell)$  equal the maximum reward that can be collected in the first  $\ell-1$  levels. Thus, we cannot increase vertex values by using boarding edges, which yields that  $\pi_*$  is optimal. ◀

The following technical result will be helpful in the upcoming proofs.

► **Lemma 9.** *Let  $x \in [2^n - 1]_0$  and  $i \in [n]$ . Then,  $\text{travel}(i)$  is not improving for  $\pi_x$ .*

**Proof.** All vertex values with respect to  $\pi_0$  are zero, and  $r_{\mathcal{B}_n}(\text{travel}(i)) = 0$ . Thus, the claim holds for  $x = 0$ , so we assume  $x \in [2^n - 1]$  in the following.

Let the vertices  $a_k$  and  $a_\ell$  either correspond to successive set bits, i.e.,  $x_k = x_\ell = 1$  and  $x_j = 0$  for all  $k < j < \ell$ , or let  $k = m_1(x)$  and  $\ell = n+1$ . Either way, Definition 6 implies that  $\pi_x$  includes a path from  $a_k$  to  $a_\ell$ , which does not contain any boarding edge.

Hence, we have  $\text{Val}_{\pi_x}(a_\alpha) \geq \text{Val}_{\pi_x}(a_\beta) \geq 0$  for all set bits  $x_\alpha = x_\beta = 1$  with  $\alpha \leq \beta$ . Since the transportation vertex chooses the least significant set bit in  $\pi_x$ , this yields that  $\text{travel}(i)$  is not improving if  $x_i = 1$ .

Further, Definition 6 yields that  $x_j = 1$  if and only if  $\text{enter}(j)$  is active in  $\pi_x$ . Thus, when starting in some vertex  $a_i$  with  $x_i = 0$ , policy  $\pi_x$  either boards  $t$  from  $a_i$  or it skips levels until reaching a node that boards  $t$ , a level corresponding to a set bit, or the sink. In all four cases,  $\text{travel}(i)$  is not improving. This completes the proof.  $\blacktriangleleft$

We will show in two steps that, when using the initial policy  $\pi_0$ , BLAND visits all of the other canonical policies. Firstly, given the canonical policy for an arbitrary even integer  $x$ , we see that the algorithm applies improving switches until reaching the canonical policy  $\pi_{x+1}$ .

► **Lemma 10.** *Let  $x \in [2^n - 2]_0$  be even. Then,  $\text{BLAND}(\mathcal{B}_n, \pi_x, \mathcal{N}_{\mathcal{B}_n})$  visits  $\pi_{x+1}$ .*

**Proof.** According to Lemma 9, no travel edges are improving for  $\pi_x$ , so the Bland numbering  $\mathcal{N}_{\mathcal{B}_n}$  yields that the algorithm applies the switch  $\text{enter}(1)$  to  $\pi_x$  if it is improving. This edge is improving for  $\pi_0$ , and one can easily check that its application results in the canonical policy  $\pi_1$ . Hence, it suffices to consider  $x \neq 0$  in the following. This yields  $\ell_1 := \ell_1(x) > 1$  as  $x$  is even. Influenced by condition <3> from Definition 6, we consider two cases.

Firstly, if  $\ell_1 = 2$ , conditions <1> and <b> state that  $\text{travel}(2)$ ,  $\text{skip}(1)$  and  $\text{leave}(1)$  are active in  $\pi_x$ . Hence,  $\text{enter}(1)$  is improving for  $\pi_x$  and gets applied by BLAND. The edge  $\text{travel}(1)$  becomes improving and gets applied next as it minimizes  $\mathcal{N}_{\mathcal{B}_n}$ .

Secondly, if  $\ell_1 \geq 3$ , conditions <1> and <d> yield that  $\pi_x$  includes the paths  $(a_1, t, a_{\ell_1})$  and  $(b_1, b_2, \dots, b_{\ell_1-2}, a_{\ell_1-1}, a_{\ell_1}) =: P$ . Hence,  $\text{Val}_{\pi_x}(a_1) \leq \text{Val}_{\pi_x}(a_{\ell_1}) \leq \text{Val}_{\pi_x}(b_1)$ . Therefore, as  $\text{enter}(1)$  has a positive reward, it is improving and gets applied to  $\pi_x$ . The new policy walks from  $a_1$  to  $b_1$  and then follows the path  $P$ , so  $a_1$  has a higher value than  $a_{\ell_1}$ . Since  $\text{travel}(\ell_1)$  is active in  $\pi_x$ , the edge  $\text{travel}(1)$  is improving and gets applied next.

Let  $\pi$  denote the policy resulting from the application of  $\text{enter}(1)$  and  $\text{travel}(1)$  to  $\pi_x$ . It now suffices to show that  $\pi$  satisfies the conditions of Definition 6 for  $x + 1$ .

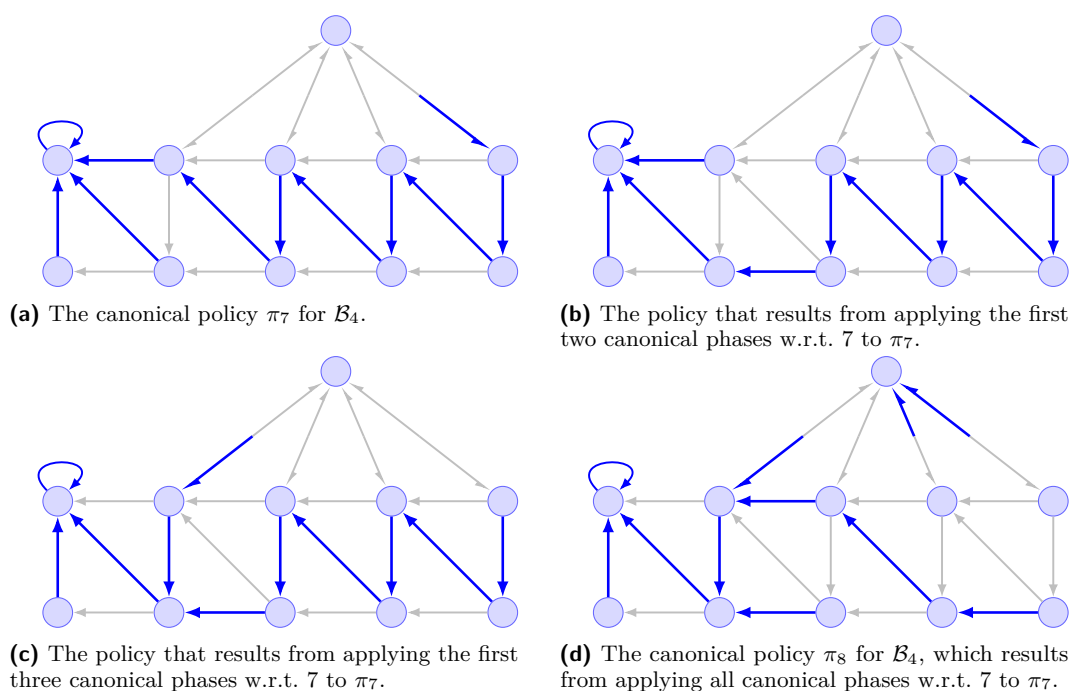
As  $x + 1$  is odd, we have  $\ell_1(x + 1) = 1$ , so  $\pi$  satisfies the first condition. Further, the second condition remains satisfied as both applied switches are below the most significant set bit. Finally, the third condition now requires that  $\text{enter}(1)$  is active – instead of  $\text{skip}(1)$  if  $\ell_1 = 2$ , or  $\text{board}(1)$  if  $\ell_1 \geq 3$  – and otherwise contains the same requirements. Hence,  $\pi$  is the canonical policy for  $x + 1$ .  $\blacktriangleleft$

Secondly, we need to show that the algorithm also transforms the canonical policy  $\pi_x$  for an arbitrary odd number  $x$  into the next canonical policy  $\pi_{x+1}$ . We will see that the algorithm does this by applying the following sequence of improving switches.

► **Definition 11.** Let  $x \in [2^n - 3]$  be odd and write  $\ell := \ell_0(x) > 1$ . Then, the *canonical phases* with respect to  $x$  are:

1. If  $x_{\ell+1} = 1$ , activate  $\text{leave}(\ell)$ .
2. If  $x_{\ell+1} = 1$  or  $\ell > m_1(x)$ , activate  $\text{stay}(\ell - 1)$ .
3. Activate  $\text{enter}(\ell)$  and  $\text{travel}(\ell)$ .
4. If  $\ell \geq 3$ , activate  $\text{board}(j)$  for all  $j \in [\ell - 2]$  in increasing order.
5. Activate  $\text{skip}(\ell - 1)$ .
6. If  $\ell \geq 4$ , activate  $\text{stay}(j)$  for all  $j \in [\ell - 3]$  in decreasing order.
7. If  $\ell = 2$ , activate  $\text{leave}(1)$ .  $\blacktriangleright$





■ **Figure 2** An example that illustrates how the canonical phases transform one canonical policy into the next one. Active edges are depicted in a bold blue color, while inactive edges are slightly transparent. Note that  $\pi_7$  enters the first three levels, which correspond to the set bits in the binary representation of 7; analogously,  $\pi_8$  only enters the fourth level.

The following Lemma shows that if the algorithm applies the canonical phases to the corresponding canonical policy, it reaches the next canonical policy.<sup>1</sup> Consider Figure 2 for an example.

► **Lemma 12.** *Let  $\pi_x$  be the canonical policy for some odd  $x \in [2^n - 3]$ . Applying the canonical phases with respect to  $x$  to  $\pi_x$  results in the canonical policy  $\pi_{x+1}$ .*

Finally, we show that BLAND actually applies the canonical phases when given the corresponding canonical policy.

► **Lemma 13.** *Let  $x \in [2^n - 3]$  be odd. Then,  $\text{BLAND}(\mathcal{B}_n, \pi_x, \mathcal{N}_{\mathcal{B}_n})$  visits  $\pi_{x+1}$ .*

According to Lemma 10, BLAND transforms every even canonical policy  $\pi_x$  into  $\pi_{x+1}$ , and by Lemma 13, the same holds for odd canonical policies. Since the initial policy is canonical for zero, this yields that  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$  visits all canonical policies  $\pi_i$  with  $i \in [2^n - 1]$ . Since these are pairwise different, this proves the main result of this section.

► **Theorem 14.** *There is an initial policy such that the policy iteration algorithm with Bland's pivot rule performs  $\Omega(2^n)$  improving switches when applied to  $\mathcal{B}_n$ .*

We close this section with two technical observations that help us later.

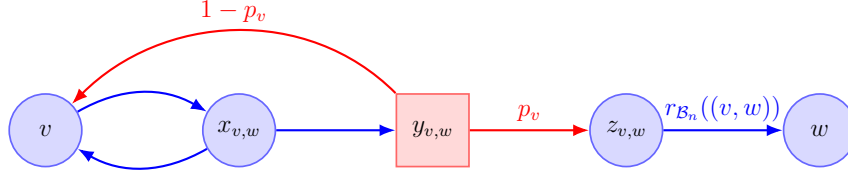
► **Observation 15.** *For every  $i \in [n]$ , whenever  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$  applies the improving switch  $\text{skip}(i)$ , this edge has higher reduced costs than  $\text{board}(i)$ ; whenever it applies  $\text{enter}(i)$ , this edge has higher reduced costs than  $\text{skip}(i)$  and  $\text{board}(i)$ .*

► **Observation 16.** *At any point during the execution of  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ , at most one of the edges  $\text{travel}(i)$  with  $i \in [n]$  is improving.*

<sup>1</sup> All missing proofs can be found in the full version.

#### 4 A Combined Exponential Bound

In this section, we consider a family  $(\mathcal{D}_n = (V_n^A, V_n^R, E_n^A, E_n^R, r_{\mathcal{D}_n}, p_{\mathcal{D}_n}))_{n \in \mathbb{N}}$  of Markov decision processes such that each process  $\mathcal{D}_n$  results from the process  $\mathcal{B}_n = (V_{\mathcal{B}_n}, E_{\mathcal{B}_n}, r_{\mathcal{B}_n})$  of the previous section by replacing every edge, besides the sink-loop, with the construction given in Figure 3; note that the construction uses a probability  $p_v \in (0, 1]$  for every  $v \in V_{\mathcal{B}_n} \setminus \{s\}$ , which we will choose later.



■ **Figure 3** The construction that replaces every edge  $(v, w) \in E_{\mathcal{B}_n} \setminus \{(s, s)\}$  in  $\mathcal{D}_n$ . Circular vertices are agent vertices, square ones are randomization vertices. Edge labels denote rewards and probabilities, where  $p_v \in (0, 1]$ . Note that, since  $v$  and  $w$  remain in the process, we have  $V_{\mathcal{B}_n} \subseteq V_n^A$ .

In the following, consider  $\mathcal{D}_n$  for some arbitrary but fixed  $n \in \mathbb{N}$ . The aim of this section is to show that policy iteration with Bland’s rule, with Dantzig’s rule, and with the Largest Increase rule performs  $\Omega(2^n)$  improving switches to a suitable initial policy for  $\mathcal{D}_n$ .

Before we can analyze the behavior of BLAND on  $\mathcal{D}_n$ , we need to specify the Bland numbering  $\mathcal{N}_{\mathcal{D}_n} : E_n^A \rightarrow |E_n^A|$  for  $\mathcal{D}_n$ . It is constructed as follows: starting from the numbering  $\mathcal{N}_{\mathcal{B}_n}$ , replace every edge  $(v, w) \in E_{\mathcal{B}_n} \setminus \{(s, s)\}$  by the edges  $(x_{v,w}, y_{v,w})$  and  $(v, x_{v,w})$ . Then, insert all edges of the form  $(x_{u,\cdot}, u)$  with  $u \in V_{\mathcal{B}_n} \setminus \{s\}$  at the beginning of the numbering (the internal order of these edges can be chosen arbitrarily). We do not need to specify the Bland numbers of edges that are the unique outgoing edge of a vertex.

Now that we have a Bland numbering, we want to transfer our results from the previous section to the new Markov decision process  $\mathcal{D}_n$ . The following definition extends policies for  $\mathcal{B}_n$  to policies for  $\mathcal{D}_n$ .

► **Definition 17.** Let  $\pi$  and  $\pi'$  be policies for  $\mathcal{B}_n$  and  $\mathcal{D}_n$ , respectively, and let  $v \in V_{\mathcal{B}_n} \setminus \{s\}$ . Assume there is a  $w \in \Gamma_{\mathcal{B}_n}^+(v)$  such that  $(v, x_{v,w})$ ,  $(x_{v,w}, y_{v,w})$ , and  $(x_{v,w}, v)$  are active in  $\pi'$  for all  $u \in \Gamma_{\mathcal{B}_n}^+(v) \setminus \{w\}$ . Then, we say that  $v$  is  $(w)$ -oriented w.r.t.  $\pi'$ . We call  $\pi'$  the *twin policy* of  $\pi$  if every vertex  $v \in V_{\mathcal{B}_n} \setminus \{s\}$  is  $\pi(v)$ -oriented w.r.t.  $\pi'$ . ◻

Let  $\pi'_0$  denote the twin policy of the canonical policy  $\pi_0$  for  $\mathcal{B}_n$ . We could start by showing that  $\text{BLAND}(\mathcal{D}_n, \pi'_0, \mathcal{N}_{\mathcal{D}_n})$  visits the twin policy of every policy that  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$  visits. Note that this would immediately imply the desired exponential number of improving switches. However, we prefer to gather some general results first, which then allows for a more unified treatment of the three pivot rules.

Starting in a  $w$ -oriented vertex  $v \in V_{\mathcal{B}_n} \setminus \{s\}$ , the agent reaches vertex  $w$  with probability one (due to  $p_v > 0$ ), while collecting a reward of  $r_{\mathcal{B}_n}((v, w))$ . This immediately yields the following result.

► **Observation 18.** Let  $\pi$  be a policy for  $\mathcal{B}_n$  with twin policy  $\pi'$  for  $\mathcal{D}_n$ . Then, for every vertex  $v \in V_{\mathcal{B}_n}$ , we have  $\text{Val}_{\pi, \mathcal{B}_n}(v) = \text{Val}_{\pi', \mathcal{D}_n}(v)$ .

By the same argument, twin policies of weak unichain policies are weak unichain, and the proof idea of Lemma 8 carries over.

► **Observation 19.** *The twin policy of every weak unichain policy for  $\mathcal{B}_n$  is a weak unichain policy for  $\mathcal{D}_n$ . The twin policy of the optimal policy for  $\mathcal{B}_n$  is optimal for  $\mathcal{D}_n$ .*

By Theorem 4, this guarantees the correctness of  $\text{POLICYITERATION}(\mathcal{D}_n, \pi'_0)$ . Further, Theorem 5 will allow us to carry our results over to the simplex method.

Since twin policies are central in our analysis, it comes in handy that only a certain type of edges might be improving for them.

► **Observation 20.** *Let  $\pi'$  be the twin policy of some policy for  $\mathcal{B}_n$ . Then, all improving switches for  $\pi'$  are of the form  $(x_{v,w}, y_{v,w}) \in E_n^A$  for some  $(v, w) \in E_{\mathcal{B}_n}$ .*

**Proof.** Since every vertex  $u \in V_{\mathcal{B}_n} \setminus \{s\}$  is oriented w.r.t  $\pi'$ , edges of the form  $(x_{u,\cdot}, u)$  or  $(u, x_{u,\cdot})$  are either active or their application creates a zero-reward cycle of length two. Hence, none of these edges is improving for  $\pi'$ . ◀

The following Lemma shows how the probabilities  $(p_v)_{v \in V_{\mathcal{B}_n} \setminus \{s\}}$  affect the reduced costs of these potentially improving edges. Further, it yields a connection between the improving switches for a policy for  $\mathcal{B}_n$  and those for its twin policy.

► **Lemma 21.** *Let  $\pi$  be a policy for  $\mathcal{B}_n$  with twin policy  $\pi'$ , and let  $(v, w) \in E_{\mathcal{B}_n} \setminus \{(s, s)\}$ . Then,  $z_{\pi', \mathcal{D}_n}(x_{v,w}, y_{v,w}) = p_v \cdot z_{\pi, \mathcal{B}_n}(v, w)$ . In particular,  $(x_{v,w}, y_{v,w})$  is improving for  $\pi'$  if and only if  $(v, w)$  is improving for  $\pi$ .*

**Proof.** For convenience, we write  $x$ ,  $y$ , and  $z$  instead of  $x_{v,w}$ ,  $y_{v,w}$ , and  $z_{v,w}$ . If  $(v, w)$  is active in  $\pi$ , vertex  $v$  is  $w$ -oriented w.r.t.  $\pi'$ . Thus,  $(x, y)$  is active in  $\pi'$  as well. Hence, both edges are not improving as they have reduced costs of  $z_{\pi'}(x, y) = z_{\pi}(v, w) = 0$ .

Now assume that  $(v, w)$  is inactive in  $\pi$ , which yields that  $(x, v)$  is active in  $\pi'$ . We obtain

$$\begin{aligned} z_{\pi', \mathcal{D}_n}(x, y) &= \text{Val}_{\pi'}(y) - \text{Val}_{\pi'}(x) = \text{Val}_{\pi'}(y) - \text{Val}_{\pi'}(v) \\ &= p_v \text{Val}_{\pi'}(z) + (1 - p_v) \text{Val}_{\pi'}(v) - \text{Val}_{\pi'}(v) \\ &= p_v (\text{Val}_{\pi'}(w) + r_{\mathcal{B}_n}((v, w)) - \text{Val}_{\pi'}(v)) = p_v z_{\pi, \mathcal{B}_n}(v, w), \end{aligned} \tag{1}$$

where we used Observation 18 for the last equality. The equivalence holds since  $p_v > 0$ . ◀

Note that we can transform a given twin policy with three switches into a different one by changing the orientation of an agent vertex  $v \in V_{\mathcal{B}_n} \setminus \{s\}$ . The following Lemma shows that, if applied consecutively, these switches all have the same reduced costs.

► **Lemma 22.** *Let  $(v, w) \in E_{\mathcal{B}_n} \setminus \{(s, s)\}$  and let the policy  $\pi$  for  $\mathcal{D}_n$  be the twin policy of some weak unichain policy for  $\mathcal{B}_n$ . If the edge  $(x_{v,w}, y_{v,w})$  is improving for  $\pi$ , we have*

$$z_{\pi}(x_{v,w}, y_{v,w}) = z_{\pi'}(v, x_{v,w}) = z_{\pi''}(\pi(v), v),$$

where  $\pi'$  denotes the policy that results from applying  $(x_{v,w}, y_{v,w})$  to  $\pi$  and  $\pi''$  denotes the policy that results from applying  $(v, x_{v,w})$  to  $\pi'$ .

It is essential for the proofs of Lemma 10 and Lemma 13 that  $\text{BLAND}(\mathcal{B}_n, \mathcal{N}_{\mathcal{B}_n})$  prefers switches in vertices appearing early in the *vertex numbering*  $\mathcal{N}_V: V_{\mathcal{B}_n} \rightarrow |V_{\mathcal{B}_n}|$  given by  $(t, a_1, b_1, a_2, b_2, \dots, a_n, b_n, d, s)$ , i.e., let  $\mathcal{N}_V(t) = 1$ ,  $\mathcal{N}_V(a_1) = 2$ , and so on. Using the following definition, we can observe a similar behavior of policy iteration with Dantzig's rule, cf. Algorithm 3, on  $\mathcal{D}_n$ .

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► **Definition 23.** The edge  $e \in E_n^A$  belongs to vertex  $v \in V_{\mathcal{B}_n} \setminus \{s\}$  if

$$e \in B(v) := \bigcup_{w \in \Gamma_{\mathcal{B}_n}^+(v)} \{(x_{v,w}, y_{v,w}), (v, x_{v,w}), (x_{v,w}, v)\}. \quad \lrcorner$$

We obtain the following bounds on the reduced costs.

► **Lemma 24.** Let  $v \in V_{\mathcal{B}_n} \setminus \{s\}$  and  $e \in B(v)$  be arbitrary. Let  $\pi$  be a weak unichain policy for  $\mathcal{D}_n$  such that all vertex values w.r.t.  $\pi$  are non-negative. If  $e$  is improving for  $\pi$ , then its reduced costs are bounded by  $p_v \cdot 0.25 \leq z_\pi(e) \leq p_v \cdot 2^{n+2}$ .

**Proof.** Since  $e \in B(v)$ , we have  $e \in \{(x_{v,w}, y_{v,w}), (v, x_{v,w}), (x_{v,w}, v)\}$  for some  $w \in \Gamma_{\mathcal{B}_n}^+(v)$ . For convenience, we write  $x$ ,  $y$ , and  $z$  instead of  $x_{v,w}$ ,  $y_{v,w}$ , and  $z_{v,w}$ .

Firstly, assume that  $e = (x, y)$ . Then, since  $e$  is improving,  $(x, v)$  is active in  $\pi$ . As in equation (1), we obtain  $z_\pi(x, y) = p_v(\text{Val}_\pi(w) + r_{\mathcal{B}_n}((v, w)) - \text{Val}_\pi(v)) =: p_v \cdot \delta(\pi, v, w)$ .

Secondly, assume that  $e = (v, x)$ . Then,  $(x, y)$  is active in  $\pi$  as otherwise  $e$  would not be improving due to  $z_\pi(v, x) = \text{Val}_\pi(x) - \text{Val}_\pi(v) = 0$ . This yields

$$z_\pi(v, x) = \text{Val}_\pi(x) - \text{Val}_\pi(v) = \text{Val}_\pi(y) - \text{Val}_\pi(v) = p_v \cdot \delta(\pi, v, w). \quad (2)$$

Lastly, assume  $e = (x, v)$ . Then, as before,  $(x, y)$  is active in  $\pi$ . We can thus conclude from (2) that  $z_\pi(x, v) = \text{Val}_\pi(v) - \text{Val}_\pi(x) = -p_v \cdot \delta(\pi, v, w)$ .

By assumption, every vertex has a non-negative value with respect to  $\pi$ . Further, all vertex values are bounded from above by the maximum vertex value w.r.t. the optimal policy for  $\mathcal{D}_n$ . By Lemma 8, Observation 18, and Observation 19, this is  $\text{Val}_{\pi'_*}(t) = 2^{n+1} - 1.25$ .

Since the absolute value of any edge reward is at most  $2^n$ , we obtain

$$|\delta(\pi, v, w)| \leq (2^{n+1} - 1.25 + 2^n) \leq 2^{n+2}.$$

Hence, we have an upper bound of  $z_\pi(e) \leq p_v \cdot 2^{n+2}$ .

As all edge rewards are integer multiples of 0.25, also  $\text{Val}_\pi(u)$  is an integer multiple of 0.25 for every  $u \in V_{\mathcal{B}_n}$  (starting in  $u$ , policy  $\pi$  visits every edge that has a non-zero reward either exactly once or never). This yields that  $|\delta(\pi, v, w)|$  is a multiple of 0.25 as well, which concludes the proof. ◀

Note that, by Theorem 4 and as all vertex values w.r.t.  $\pi'_0$  are zero,  $\text{DANTZIG}(\mathcal{D}_n, \pi'_0)$  only visits weak unichain policies with non-negative vertex values. Therefore, if we choose the probabilities  $(p_v)_{v \in V_{\mathcal{B}_n} \setminus \{s\}}$  for increasing vertex numbers  $\mathcal{N}_V(v)$  fast enough decreasing, then the previous lemma yields that  $\text{DANTZIG}$  prefers improving switches that belong to vertices appearing early in the vertex numbering  $\mathcal{N}_V$ . We use this in the proof of Theorem 28.

The following technical result holds independently of the chosen pivot rule.

■ **Algorithm 3**  $\text{DANTZIG}(G, \pi)$ .

---

**input:** Markov decision process  $G$ , weak unichain policy  $\pi$  for  $G$

---

**while**  $\pi$  admits an improving switch:

$\bar{s} \leftarrow$  an improving switch  $s$  for  $\pi$  maximizing  $z_\pi(s)$

$\pi \leftarrow \pi^{\bar{s}}$

**return**  $\pi$

---

► **Lemma 25.** *Let  $(v, w) \in E_{\mathcal{B}_n}$  be an improving switch that gets applied to a policy  $\pi$  during the execution of  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ . Let  $\bar{\pi}'$  denote the policy for  $\mathcal{D}_n$  that results from applying the switches  $(x_{v,w}, y_{v,w})$  and  $(v, x_{v,w})$  to the twin policy  $\pi'$  of  $\pi$ . Then, the edge  $(x_{v, \pi(v)}, v)$  is improving for  $\bar{\pi}'$  and it remains improving during the execution of  $\text{POLICYITERATION}(\mathcal{D}_n, \bar{\pi}')$  until it gets applied or until an improving switch of the form  $(x_{u,\cdot}, y_{u,\cdot})$  with  $\mathcal{N}_V(u) > \mathcal{N}_V(v)$  gets applied.*

With this, we can show that a certain class of pivot rules, including Bland's, Dantzig's, and the Largest Increase rule, yield an exponential number of improving switches on  $\mathcal{D}_n$ .

► **Lemma 26.** *Assume that  $\text{POLICYITERATION}(\mathcal{D}_n, \pi'_0)$ , where  $\pi'_0$  denotes the twin policy of  $\pi_0$ , gets applied with a pivot rule that satisfies the following conditions:*

- (a) *For every improving switch  $(v, w) \in E_{\mathcal{B}_n}$  that  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$  applies to some policy  $\pi$ ,  $\text{POLICYITERATION}$  applies  $(x_{v,w}, y_{v,w})$  and  $(v, x_{v,w})$  to the twin policy of  $\pi$ .*
- (b) *While an edge of the form  $(x_{v,\cdot}, v)$  is improving for some  $v \in V_{\mathcal{B}_n}$ ,  $\text{POLICYITERATION}$  does not apply an improving switch of the form  $(x_{u,\cdot}, y_{u,\cdot})$  with  $\mathcal{N}_V(u) > \mathcal{N}_V(v)$ .*

*Then,  $\text{POLICYITERATION}(\mathcal{D}_n, \pi'_0)$  performs  $\Omega(2^n)$  improving switches.*

**Proof.** Let  $\pi$  be a policy for  $\mathcal{B}_n$  occurring during the execution of  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ , where we allow  $\pi = \pi_0$ , and let  $(v, w) \in E_{\mathcal{B}_n}$  denote the switch that  $\text{BLAND}$  applies to  $\pi$ . Let  $\pi'$  be the twin policy of  $\pi$ , and let  $\bar{\pi} = \pi^{(v,w)}$ .

By condition (a),  $\text{POLICYITERATION}$  applies  $(x_{v,w}, y_{v,w})$  and  $(v, x_{v,w})$  to  $\pi'$ . Denote the resulting policy by  $\bar{\pi}'$ .

According to Lemma 25, the edge  $(x_{v, \pi(v)}, v)$  now stays improving until it gets applied as an improving switch or until an improving switch of the form  $(x_{u,\cdot}, y_{u,\cdot})$  with  $\mathcal{N}_V(u) > \mathcal{N}_V(v)$  gets applied. With condition (b), this yields that  $(x_{v, \pi(v)}, v)$  gets applied by  $\text{POLICYITERATION}$  at some point, and that it is constantly improving until then.

Note that, as long as  $(v, x_{v, \pi(v)})$  is inactive, the policy's choice in  $x_{v, \pi(v)}$  only affects the reduced costs of its unique incidental edge  $(v, x_{v, \pi(v)})$ . This edge is not active in  $\bar{\pi}'$  and is not improving until the application of  $(x_{v, \pi(v)}, v)$ . Therefore, if we were to force the algorithm to apply  $(x_{v, \pi(v)}, v)$  to  $\bar{\pi}'$ , this would not alter the remaining behavior of the algorithm. The policy resulting from this forced switch is the twin policy of  $\bar{\pi}$ .

Hence, without changing the total number of applied improving switches (we only rearrange them), we can assume that  $\text{POLICYITERATION}(\mathcal{D}_n, \pi'_0)$  visits the twin policy of every policy visited by  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ . By Theorem 14, this yields that the algorithm needs to perform an exponential number of improving switches, which concludes the proof. ◀

Now it suffices to check the conditions given in Lemma 26 for each pivot rule.

► **Proposition 27.** *Let  $\pi'_0$  denote the twin policy of  $\pi_0$ . Then,  $\text{BLAND}(\mathcal{D}_n, \pi'_0, \mathcal{N}_{\mathcal{D}_n})$  performs  $\Omega(2^n)$  improving switches.*

**Proof.** We check the two conditions from Lemma 26. For condition (a), let  $\pi$  be a policy for  $\mathcal{B}_n$  visited by  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ , including the case  $\pi = \pi_0$ , and let  $\pi'$  be the twin policy of  $\pi$ . Assume that  $\text{BLAND}$  applies the improving switch  $(v, w) \in E_{\mathcal{B}_n}$  to  $\pi$ .

By Observation 20, all improving switches for  $\pi'$  are of the form  $(x, y)$ . According to Lemma 21, the edge  $(x_{v,w}, y_{v,w})$  is improving for  $\pi'$ . As  $(v, w)$  is the improving switch for  $\pi$  with the smallest Bland number in  $\mathcal{N}_{\mathcal{B}_n}$ , we know that, by construction of  $\mathcal{N}_{\mathcal{D}_n}$ , the algorithm applies the switch  $(x_{v,w}, y_{v,w})$  to  $\pi'$ .

Further, since  $\pi$  is weak unichain due to Theorem 4, Lemma 22 yields that  $(v, x_{v,w})$  is improving after this switch. As it is the successor of  $(x_{v,w}, y_{v,w})$  in  $\mathcal{N}_{\mathcal{D}_n}$  and as no other edge became improving due to the first switch, the algorithm applies  $(v, x_{v,w})$  next. That is, Bland's rule satisfies condition (a).

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Additionally, condition (b) holds since the edge  $(x_{v,\pi(v)}, v)$  precedes any switch of the form  $(x_{u,\cdot}, y_{u,\cdot})$  with  $\mathcal{N}_V(u) > \mathcal{N}_V(v)$  in the Bland numbering  $\mathcal{N}_{\mathcal{D}_n}$ . ◀

As motivated above, the choice of the probabilities  $(p_v)_{v \in V_{\mathcal{B}_n} \setminus \{s\}}$  in the following theorem yields that DANTZIG prefers improving switches that belong to vertices appearing early in the vertex numbering  $\mathcal{N}_V$ .

► **Theorem 28.** *Let  $p_v = 2^{-\mathcal{N}_V(v)(n+5)}$  for all  $v \in V_{\mathcal{B}_n} \setminus \{s\}$ , and let  $\pi'_0$  denote the twin policy of  $\pi_0$ . Then, DANTZIG( $\mathcal{D}_n, \pi'_0$ ) performs  $\Omega(2^n)$  improving switches.*

**Proof.** We check the two conditions from Lemma 26. For condition (b), we compute that the choice of the probabilities  $p_v$  yields that DANTZIG prefers improving switches belonging to vertices with a small vertex number.

Let  $u, v \in V_{\mathcal{B}_n} \setminus \{s\}$  with  $\mathcal{N}_V(u) > \mathcal{N}_V(v)$ . Let further  $e_u \in B(u)$  and  $e_v \in B(v)$  be improving switches for some policy  $\pi$  for  $\mathcal{D}_n$ , which gets visited by POLICYITERATION( $\mathcal{D}_n, \pi'_0$ ). Then,  $\pi$  is weak unichain and only induces non-negative vertex values, so Lemma 24 yields

$$z_\pi(e_v) \geq p_v \cdot 0.25 = 2^{-\mathcal{N}_V(v)(n+5)-2} \geq 2^{-\mathcal{N}_V(u)(n+5)+(n+5)-2} = p_u \cdot 2^{(n+3)} > z_\pi(e_u).$$

Hence, Dantzig's rule prefers switches belonging to  $v$  over those belonging to  $u$ , so it satisfies condition (b).

For condition (a), let  $\pi$  be a policy for  $\mathcal{B}_n$  visited by BLAND( $\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n}$ ), including the case  $\pi = \pi_0$ , and let  $(v, w) \in E_{\mathcal{B}_n}$  denote the switch that BLAND applies to  $\pi$ . Let  $\pi'$  be the twin policy of  $\pi$ .

By Observation 20, all improving switches for  $\pi'$  are of the form  $(x, y)$ . By construction of  $\mathcal{N}_{\mathcal{D}_n}$ , we know that BLAND prefers those switches  $(x, y)$  that belong to vertices with a small vertex number. In the proof of Proposition 27, we see that BLAND( $\mathcal{D}_n, \pi', \mathcal{N}_{\mathcal{D}_n}$ ) applies  $(x_{v,w}, y_{v,w})$  to  $\pi'$ . Since DANTZIG also prefers switches belonging to vertices with a small vertex number, we conclude that DANTZIG( $\mathcal{D}_n, \pi'_0, \mathcal{N}_{\mathcal{D}_n}$ ) applies an improving switch to  $\pi'$  that belongs to  $v$ . However, there might be multiple such switches.

Recall that all improving switches for  $\pi'$  are of the form  $(x, y)$ . If  $v = b_i$  for some  $i \in [n]$ , then only two of these (possibly improving) edges belong to  $v$ , one of which is active in  $\pi'$ . Therefore, in this case, DANTZIG applies the improving switch  $(x_{v,w}, y_{v,w})$ .

Now assume  $v = a_i$  for some  $i \in [n]$ . Then, Observation 15 and Lemma 21 yield

$$z_{\pi'}(x_{a_i, b_i}, y_{a_i, b_i}) > \max\{z_{\pi'}(x_{a_i, a_{i+1}}, y_{a_i, a_{i+1}}), z_{\pi'}(x_{a_i, t}, y_{a_i, t})\}$$

if  $(v, w) = \text{enter}(i)$ , and

$$z_{\pi'}(x_{a_i, a_{i+1}}, y_{a_i, a_{i+1}}) > z_{\pi'}(x_{a_i, t}, y_{a_i, t})$$

if  $(v, w) = \text{skip}(i)$ . Therefore, the edge  $(x_{v,w}, y_{v,w})$  has higher reduced costs than the other edges that belong to  $v$ , so DANTZIG applies it to  $\pi'$ .

Finally, if  $v = t$ , then Observation 16 and Lemma 21 yield that  $(x_{v,w}, y_{v,w})$  is the only improving switch that belongs to  $t$ . Thus, it gets applied by DANTZIG.

We conclude that, in all cases, DANTZIG applies the switch  $(x_{v,w}, y_{v,w})$  to  $\pi'$ , which is the unique edge with the highest reduced costs. According to Lemma 22, the edge  $(v, x_{v,w})$  has now the same reduced costs as  $(x_{v,w}, y_{v,w})$  had before its application. Since  $(v, x_{v,w})$  is the only edge that became improving during the last switch, DANTZIG applies this edge next. Therefore, Dantzig's rule also satisfies condition (a), which concludes the proof. ◀

■ **Algorithm 4**  $\text{LARGESTINCREASE}(G, \pi)$ .

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**input:** Markov decision process  $G$ , weak unichain policy  $\pi$  for  $G$

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**while**  $\pi$  admits an improving switch **:**

$\bar{s} \leftarrow$  an improving switch  $s$  for  $\pi$  maximizing  $\sum_{v \in V_n^A} \text{Val}_{\pi^s}(v)$

$\pi \leftarrow \pi^{\bar{s}}$

**return**  $\pi$

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Finally, we turn to policy iteration with the Largest Increase pivot rule, cf. Algorithm 4. In the most general sense, consider an arbitrary improving switch  $s = (v, w)$  for some policy  $\pi$ . Assume that no ingoing edges of  $v$  are active in  $\pi$ . Then, the reduced costs of  $s$  coincide with the increase of the sum over all vertex values, that is, we obtain the equality  $z_\pi(s) = \sum_{v \in V_n^A} \text{Val}_{\pi^s}(v) - \sum_{v \in V_n^A} \text{Val}_\pi(v)$ . Further, the induced increase of the sum is always at least as large as the reduced costs.

From this, using the structure of  $\mathcal{D}_n$ , we can conclude that  $\text{LARGESTINCREASE}$  mirrors the behavior of  $\text{DANTZIG}$  if we choose the probabilities  $p_v$  as before.

► **Theorem 29.** *Let  $p_v = 2^{-\mathcal{N}_V(v)(n+5)}$  for all  $v \in V_{\mathcal{B}_n} \setminus \{s\}$ , and let  $\pi'_0$  denote the twin policy of  $\pi_0$ . Then,  $\text{LARGESTINCREASE}(\mathcal{D}_n, \pi'_0)$  performs  $\Omega(2^n)$  improving switches.*

**Proof.** We check the two conditions from Lemma 26. For condition (a), let  $\pi$  be a policy for  $\mathcal{B}_n$  occurring during the execution of  $\text{BLAND}(\mathcal{B}_n, \pi_0, \mathcal{N}_{\mathcal{B}_n})$ , where we allow  $\pi = \pi_0$ , and let  $(v, w) \in E_{\mathcal{B}_n}$  denote the switch that  $\text{BLAND}$  applies to  $\pi$ . Let  $\pi'$  be the twin policy of  $\pi$ .

According to the proof of Theorem 28,  $\text{DANTZIG}$  applies the improving switches  $(x_{v,w}, y_{v,w})$  and  $(v, x_{v,w})$  to  $\pi'$ . By Observation 20, all improving switches for  $\pi'$  are of the form  $(x, y)$ , where  $\pi'$  does not reach  $x$  when starting in any other vertex. Therefore, since the probabilities  $(p_v)_{v \in V_{\mathcal{B}_n} \setminus \{s\}}$  are chosen as in Theorem 28, the reduced costs of each of these improving switches coincide with the induced increase of the sum over all vertex values. Hence,  $\text{LARGESTINCREASE}$  also applies the improving switch  $(x_{v,w}, y_{v,w})$  to  $\pi'$ .

This switch only increases the reduced costs of the edge  $(v, x_{v,w})$ , which, by Lemma 22, coincide with the previous reduced costs. Therefore, the induced increase of the sum over all vertex values is for  $(v, x_{v,w})$  now at least as large as it was for  $(x_{v,w}, y_{v,w})$  before. Hence,  $\text{LARGESTINCREASE}$  also applies  $(v, x_{v,w})$  next. We conclude that the Largest Increase pivot rule satisfies condition (a).

Note that the reduced costs of the edges from condition (b) again coincide with the induced increase of the sum over all vertex values. Further, by the proof of Theorem 28, we know that  $\text{Dantzig}$ 's rule prefers switches belonging to vertices with a small vertex number. Therefore, the Largest Increase rule also satisfies condition (b). ◀

Note that Theorem 1 is now a direct consequence of Theorems 27, 28, and 29. Moreover, we have seen in the proofs of these theorems that  $\text{Bland}$ 's rule,  $\text{Dantzig}$ 's rule, and the Largest Increase rule satisfy the conditions from Lemma 26. Thus, any combination of these rules also satisfies the conditions, which immediately yields Corollary 2. Finally, Corollary 3 follows from Theorem 5 together with Observation 7, Lemma 8, and Observation 19.

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